Sequence independent lifting for mixed–integer programming

Alper Atamtürk*

atamturk@ieor.berkeley.edu

Department of Industrial Engineering and Operations Research
University of California at Berkeley
Berkeley, CA 94720–1777

November 2001

Abstract: We show that (1) superadditive lifting functions lead to sequence independent lifting of inequalities for general mixed–integer programming and (2) any mixed–integer rounding inequality can be obtained via sequence independent lifting.

Lifting is a procedure for deriving strong valid inequalities for mixed–integer sets from inequalities that are valid for their lower dimensional restrictions. It is arguably one of the most effective ways of strengthening linear programming relaxations of 0–1 programming problems. Wolsey [10] and Gu et al. [6] show that superadditive lifting functions lead to sequence independent lifting of valid inequalities for monotone 0–1 programming and for monotone mixed 0–1 programming, respectively. We show that this property holds for general mixed–integer programming (MIP) as well if lower dimensional restrictions are obtained by setting integer variables to a bound.

Lifting with general integer variables is computationally harder than lifting with 0–1 variables, since the former requires the solution of nonlinear integer problems rather than linear integer problems. Here we see that nonlinearity in lifting problems is resolved easily with superadditive lifting functions. We hope that the results presented here will pave the way for efficient applications of lifting with general integer variables.

Sequential lifting

Consider a mixed–integer set \( P = \{x \in \mathbb{Z}^I \times \mathbb{R}^C : Ax \leq b\} \), where \( A \) is a rational matrix with \( m \) rows, \( b \) a rational \( m \) column vector. Let \( y \in P \), \( F \subseteq I \), \( R = I \setminus F \), and \( P_R(d) = \{x_R \in \mathbb{R}^R \times \mathbb{R}^C : A_R x_R \leq d\} \) \((\neq \emptyset)\) be the restriction of \( P \), obtained by setting \( x_F = y_F \); so \( d = b - A_F y_F \). Let \( \pi_R x_R \leq \pi_o \) be a valid inequality for \( P_R(d) \). Then, Theorem 1 can be used to lift \( \pi_R x_R \leq \pi_o \) with the projected variables \( \{x_i\}_{i \in F} \) one at a time in some sequence in order to derive a valid inequality for \( P \). For brevity, we let \( R_i = R \cup \{i\} \), \( \min_{x \in S} f(x) = +\infty \), and \( \max_{x \in S} f(x) = -\infty \) if \( S = \emptyset \).

Theorem 1. [9] \( \pi_{R \cup \{i\}} x_R + \pi_i (x_i - y_i) \leq \pi_o \) is a valid inequality for \( P_{R_i}(d + A_i y_i) \) iff \( \bar{\pi}_i \leq \pi_i \leq \bar{\pi}_i \), where

\[
\begin{align*}
\bar{\pi}_i &= \min \left\{ \frac{\pi_0 - \pi_i R^T}{x_i - y_i} : x_i > y_i, x_R \in P_{R_i}(d + A_i y_i) \right\}, \\
\bar{\pi}_i &= \max \left\{ \frac{\pi_0 - \pi_i R^T}{x_i - y_i} : x_i < y_i, x_R \in P_{R_i}(d + A_i y_i) \right\}.
\end{align*}
\]

Moreover, if \( \pi_i = \bar{\pi}_i > -\infty \) or \( \pi_i = \bar{\pi}_i < +\infty \) and \( \pi_R x_R \leq \pi_o \) defines a \( k \)-dimensional face of \( \text{conv}(P_R(d)) \), then \( \pi_R x_R + \pi_i (x_i - y_i) \leq \pi_o \) defines at least a \( k + 1 \)-dimensional face of \( \text{conv}(P_{R_i}(d + A_i y_i)) \).

(1) In order to use of Theorem 1, for a given sequence of projected variables, one needs solve two nonlinear (fractional) mixed–integer lifting problems for each variable in the sequence. However, lifting an inequality with a 0–1 variable requires the solution of a single linear mixed–integer program, since \( x_i - y_i \in \{-1,1\} \) and one of the above problems is infeasible.

(2) For a particular \( i \in F \), the later \( x_i \) is introduced to the inequality in a lifting sequence, the smaller \( \bar{\pi}_i \) is and the larger \( \bar{\pi}_i \) is. Therefore, different sequences may lead to different lifted inequalities for \( P \) and not all sequences lead to a valid lifted inequality as it might be the case that \( \bar{\pi}_i > \bar{\pi}_i \) for some \( i \in F \) in some sequence.

(3) For many structured MIP problems, lifting problems for the first variable in the lifting sequence can be solved efficiently by exploiting the special structure of the function \( \pi_R x_R \) in the objective. However, once new lifting coefficients are introduced to the inequality, the structure in the objective function may be lost.

*Supported, in part, by NSF grants DMI–9908705 and DMI–0070127.
**Bounds on the lifting coefficients**

In order to simplify the notation, after changing variables as \( z_F = x_F - y_F \) and \( z_R = x_R \), we rewrite \( P \) as \( Q = \{ z \in \mathbb{Z}^l \times \mathbb{R}^C : Az \leq d \} \). Hence, \( Q_R(d) \) is obtained by setting \( z_F = 0 \) and \( \pi_R z_R + \pi F x_F \leq \pi_o \) is valid for \( Q \) if and only if \( \pi_R x_R + \pi F (x_F - y_F) \leq \pi_o \) is valid for \( P \).

Consider the value function \( v(h) = \max \{ \pi_R z_R : z_R \in Q_R(h) \} \) and define two functions \( \Phi: \mathbb{R}^m \to \mathbb{R} \cup \{ +\infty \} \) and \( \Psi: \mathbb{R}^m \to \mathbb{R} \cup \{ +\infty \} \) for \( \pi_R x_R \leq \pi_o \) as

\[
\Phi(a) = \pi_o - v(d - a),
\]

\[
\Psi(a) = \min \left\{ \frac{\Phi(u + ka) - \Phi(u)}{k} : u \in D, k \in \mathbb{Z}_{++} \right\},
\]

where \( D = \{ a \in \mathbb{R}^m : Q_R(d - a) \neq \emptyset \} \) and \( \mathbb{Z}_{++} \) is the set of postive integer numbers. \( \Phi < +\infty \) on \( D \) and since \( \pi_R z_R \leq \pi_o \) is valid for \( Q_R(d) \), \( \Phi > -\infty \) on \( \mathbb{R}^m \). Hence, \( \Psi > -\infty \) on \( \mathbb{R}^m \) and \( \Psi < +\infty \) on \( D \), by taking \( k = 1 \) and \( u = 0 \) as \( 0 \in D \).

Suppose \( \pi_R z_R \leq \pi_o \) is lifted with variables indexed with \( E \subset F \) in some sequence. Consider now, the lifting problems associated with \( z_i, i \in F \setminus E \)

\[
\pi_i = \min \left\{ \frac{\pi_0 - \pi_R z_R - \pi E z_E}{z_i} : z_i > 0, z_{REi} \in Q_{REi}(d) \right\},
\]

\[
\bar{\pi}_i = \max \left\{ \frac{\pi_0 - \pi_R z_R - \pi E z_E}{z_i} : z_i < 0, z_{REi} \in Q_{REi}(d) \right\},
\]

which are bounded since \( \pi_R z_R + \pi E z_E \leq \pi_o \) is valid for \( Q_{REi}(d) \).

**Lemma 2.** For any \( i \in F \), \( \Psi(A_i) \leq \pi_i \leq \Phi(A_i) \) and \( -\Phi(-A_i) \leq \bar{\pi}_i \leq -\Psi(-A_i) \) hold independent of \( E \).

**Proof.** If \( A_i \in D \), minimization lifting problem has a feasible solution such that \( z_i = 1, z_E = 0 \) and objective value \( \Phi(A_i) \); else, \( \Phi(A_i) = +\infty \geq \pi_i \).

If \( -A_i \in D \), maximum lifting problem has a feasible solution such that \( z_i = -1, z_E = 0 \) and objective value \( -\Phi(-A_i) \); else, \( -\Phi(-A_i) \) is \( -\infty \leq \pi_i \).

If the minimization lifting problem is infeasible, \( \pi_i = +\infty \geq \Psi \). Otherwise, let \( \tilde{z}_{RI \cup E} \) be an optimal solution. Then

\[
\pi_i = \min \left\{ \frac{\pi_0 - \pi_R z_R - \pi E z_E}{z_i} : z_i > 0, z_{RI} \in Q_{RI}(d - A E \tilde{z}_E) \right\}
\]

\[
\geq \min \left\{ \frac{\Phi(A E \tilde{z}_E + k A_i) - \Phi(A E \tilde{z}_E)}{k} : k \in \mathbb{Z}_{++} \right\}
\]

\[
\geq \min \left\{ \frac{\Phi(A E \tilde{z}_E + k A_i) - \Phi(A E \tilde{z}_E)}{k} : k \in \mathbb{Z}_{++} \right\}
\]

The second inequality holds because \( \pi_R z_R + \pi E z_E \leq \pi_o \) is valid for \( Q_{RE} \) iff \( \pi_E z_E \leq \pi_o - v(d - A E \tilde{z}_E) = \Phi(A E \tilde{z}_E) \).

If the maximization lifting problem is infeasible, \( \bar{\pi}_i = -\infty \). Otherwise, let \( \tilde{z}_{R \cup E} \) be an optimal solution. Then

\[
\bar{\pi}_i = \max \left\{ \frac{\pi_0 - \pi_R z_R - \pi E z_E}{z_i} : z_i < 0, z_{RI} \in Q_{RI}(d - A E \tilde{z}_E) \right\}
\]

\[
\leq \max \left\{ \frac{\Phi(A E \tilde{z}_E) - \Phi(A E \tilde{z}_E - k A_i)}{k} : k \in \mathbb{Z}_{++} \right\}
\]

\[
\leq -\Psi(-A_i)
\]

Notice that since the bounds on the \( \pi_i \) and \( \bar{\pi}_i \) are independent of \( E \), they hold for any \( i \in F \), independent of lifting sequence.

**Definition 1.** A function \( f : S \subseteq \mathbb{R}^m \to \mathbb{R} \) is superadditive if \( f(a) + f(b) \leq f(a + b) \) for all \( a, b, a + b \in S \).

**Lemma 3.** \( \Psi = \Phi \) on \( D \) iff \( \Phi \) is superadditive on \( D \).

**Proof.** In general, for any \( a \in \mathbb{R}^m \), \( \Psi(a) \leq \Phi(a) - \Phi(0) \leq \Phi(a) \), which follows from the definition of \( \Psi \) by taking \( k = 1 \) and \( u = 0 \) and from \( \Phi(0) \geq 0 \) as \( \pi_R z_R \leq \pi_o \) is valid for \( Q_R(d) \).

For \( a \in D \), let \( \Psi(a) = \frac{\Phi(u + ka) - \Phi(u)}{ka} \). Since \( u^* \in D \) and \( u^* + ka \in D \) for all \( k = 0, 1, \ldots, k^* \) for the mixed–integer set \( Q_R(d) \). Then by superadditivity of \( \Phi \) on \( D \),

\[
\frac{\Phi(u^*) + k^* \Phi(a) - \Phi(u^*)}{k^*} = \Phi(a).
\]

For the other direction, if \( \Phi(a) \leq \Psi(a) \) for all \( a \in D \), then \( \Phi(a) \leq \Phi(u + a) - \Phi(u) \) for all \( a, u \in D \).

**Remark 1.** From Lemmas 2 and 3 if \( \Phi \) is superadditive on \( D \), and for \( i \in F \), \( A_i \in D \) and \( -A_i \in D \), then \( \pi_i = \Phi(A_i) \) and \( \bar{\pi}_i = -\Phi(-A_i) \). However, since \( \Phi \) is superadditive, \( \Phi(-A_i) + \Phi(A_i) \leq \Phi(0) = 0 \), and hence \( \bar{\pi}_i \geq \pi_i \). As \( \Phi \) is typically not linear, when it is superadditive, sequential lifting by Theorem 1 often does not lead to a lifted inequality with any lifting sequence if some integer variable \( x_i, i \in F \) is fixed to a value \( y_i \) that is not equal to either its lower bound or lower bound in the restriction.
Sequence independent lifting

Based on Remark 1, now we consider a nonempty restriction $P_R(d)$ of $P$ obtained by setting integer variables $x_L$ to their lower bounds and $x_U$ to their upper bounds. Thus, $F = L \cup U \subseteq I$ and $d = b - \bar{A}_L L - \bar{A}_U U$, where $l_L$ and $u_U$ are finite lower bound and upper bound vectors of $x_L$ and $x_U$, respectively. The following theorem shows that even if $\Phi$ is not superadditive, it is sufficient to use a superadditive lower bound on $\Phi$ for deriving valid lifting coefficients for all $i \in I$ from $P_R(d)$.

**Theorem 4.** Let $\phi : \mathbb{R}^m \to \mathbb{R}$ be a superadditive function such that $\phi \leq \Phi$. Then the lifted inequality

$$\sum_{i \in L} \phi(A_i)(x_i-l_i) + \sum_{i \in U} \phi(-A_i)(u_i-x_i) + \pi_R x_R \leq \pi_o$$

is valid for $P$. In addition, if $\phi(A_i) = \Phi(A_i)$ for all $i \in L$, $\phi(-A_i) = \Phi(-A_i)$ for all $i \in U$, and $\pi_R x_R \leq \pi_o$ defines a $k$–dimensional face of $conv(P_R(d))$, then the lifted inequality defines an at least $k+|L|+|U|$ dimensional face of $conv(P)$.

**Proof.** Let $z_L = x_L - l_L$, $z_U = x_U - u_U$, and $z_F = x_F$. For any $z \in Q$, $v(d - A_L z_L - A_U z_U) \geq \pi_R z_R$. So,

$$\pi_o - \pi_R z_R \geq \pi_o - v(d - A_L z_L - A_U z_U)$$

$$= \Phi(A_L z_L + A_U z_U)$$

$$\geq \sum_{i \in L} \phi(A_i z_i) + \sum_{i \in U} \phi(A_i z_i)$$

$$\geq \sum_{i \in L} \phi(A_i) z_i - \sum_{i \in U} \phi(-A_i) z_i.$$

The last inequality follows from superadditivity of $\phi$, nonnegativity and integrality of $z_i$, $i \in L$, and nonpositivity and integrality of $z_i$, $i \in U$.

For the second part of the theorem, observe that there exists an optimal solution to the lifting problem for $i \in I$ with $z_i = 1$ and $z_j = 0$, $j \in F \setminus \{i\}$ and objective value $\Phi(A_i) < +\infty$, and for $i \in L$ with $z_i = -1$ and $z_j = 0$, $j \in F \setminus \{i\}$ and objective value $-\Phi(-A_i) > -\infty$ for $i \in U$ and it is affinely independent with the points of the face of $conv(P_R(d))$ induced by $\pi_R x_R \leq \pi_o$.

**Remark 2.** Under the conditions of the second part of Theorem 4, by Lemma 2, the lifted inequality

$$\sum_{i \in L} \Phi(A_i)(x_i-l_i) + \sum_{i \in U} \Phi(-A_i)(u_i-x_i) + \pi_R x_R \leq \pi_o$$

is the unique facet-defining inequality that can be obtained by sequential lifting of $\pi_R x_R \leq \pi_o$ in any sequence.

**Remark 3.** For a 0–1 variable $z_i$ with $i \in I$, if the lifting problem is feasible, $\Phi(A_i)$ equals the lifting coefficient of $z_i$ if $z_i$ is the first lifted variable. However, this is not true for a general integer variable, unless $\Phi$ is superadditive. In general $\Phi(A_i)$ is only an upper bound on the lifting coefficient of $z_i$ in any lifting sequence. Therefore, it is remarkable that any superadditive lower bound $\phi$ on $\Phi$ leads to a valid inequality.

**Remark 4.** If $\Phi$ is superadditive on $D$, then the lifting problem for $z_i$, $i \in F$ reduces to computing $\Phi(A_i)$ (or $\Phi(-A_i)$), which is a linear mixed–integer problem rather than a nonlinear one. Also any special structure in $\pi_R x_R$ can be exploited for all projected variables, not just for the first one in the lifting sequence. This may help to compute all lifting coefficients efficiently.

**Remark 5.** For validity of the lifted inequality in Theorem 4, $\pi_R x_R \leq \pi_o$ need not be tight in $P_R(d)$. Validity of $\pi_R x_R \leq \pi_o$ for $P_R(d)$ implies that $\Phi(0) \geq 0$. On the other hand, superadditivity of $\phi$ implies that $\phi(0) \leq 0$. Therefore, if $\Phi(0) = \phi(0)$, then $v(d) = \pi_o$, i.e., $\pi_R x_R \leq \pi_o$ is tight in $P_R(d)$.

**Remark 6.** Superadditivity of $\phi$ and $\phi \leq \Phi$ is only a sufficient condition for validity of the lifted inequality in Theorem 4. Based on the matrix coefficients of the lifted variables, it is possible to relax this condition. For instance, observe in the proof of Theorem 4 that if $A_i \geq 0$ for all $i \in L$ and $A_i \leq 0$ for all $i \in U$, then it is sufficient for $\phi$ to be superadditive and $\phi \leq \Phi$ on $\mathbb{R}^m$.

**Example: Mixed–integer rounding**

Mixed–integer rounding is a general procedure for deriving valid inequalities for MIP problems. For a constraint of a MIP problem with nonnegative variables

$$\sum_{i \in I} a_i x_i + \sum_{i \in C} g_i y_i \leq b,$$  

(1)

where $I$ is the index set of integer variables and $C$ is the index set of continuous variables, the mixed–integer rounding (MIR) inequality [8] is stated as

$$\sum_{i \in I}(\lceil a_i \rceil + \frac{(f_i - f)^+}{1 - f}) x_i + \sum_{i \in C: g_i < 0} \frac{g_i}{1 - f} y_i \leq b,$$  

(2)

where $f = b - \lfloor b \rfloor$ and $f_i = a_i - \lfloor a_i \rfloor$ for $i \in I$. The MIR inequality is equivalent to Gomory mixed–integer fractional cuts [5], Balas disjunctive cuts [2], and split cuts of Cook, Kannan and Schrijver [3]; see [8, 4, 7]. Let us call the MIR inequality obtained after multiplying (1) with $\lambda > 0$, as the $\lambda$–MIR inequality.
We will show that every \( \lambda \)-MIR inequality can be obtained using a particular superadditive lower bound for the lifting function of a simple two-variable mixed-integer inequality.

Let \( z = \sum_{i \in C: g_i < 0} -g_i y_i \) and consider the mixed-integer set defined by two variables and a single constraint (and bounds)

\[
S_c = \{ w \in \mathbb{Z}, z \in \mathbb{R}_+: cw - z \leq b, l \leq w \leq u \}
\]

Observe that the LP relaxation of \( S_c \) has a fractional vertex \((b/c,0)\) if and only if \( b/c \notin \mathbb{Z} \) and \( l < b/c < u \).

Suppose \( c > 0 \) (argument is symmetric otherwise) and let \( \eta = \lfloor b/c \rfloor \) and \( r = b - \lfloor b/c \rfloor c \). Then

\[
(c-r)w - y \leq b - \eta r
\]

(3)
cuts off the fractional vertex \((b/c,0)\) and is sufficient to describe \( \text{conv}(S_c) \) together with the original inequalities of \( S_c \). Inequality (3) is the 1/c-MIR inequality for \( cw - z \leq b \).

**Proposition 5.** The \( \lambda \)-MIR inequality for (1) is a sequence independent lifting from \( S_{1/\lambda} \).

**Proof.** For inequality (3) the lifting function

\[
\Phi_c(a) = b - \eta r - \max \{ (c-r)w - z : cw - z \leq b - a,
\]

\[
l \leq w \leq u, w \in \mathbb{Z}, z \in \mathbb{R}_+ \}
\]

can be expressed explicitly as

\[
\Phi_c(a) = \begin{cases} 
(\eta - u - 1)(c-r) & \text{if } a < d - uc, \\
k(c-r) & \text{if } kc \leq a < kc + r, \\
a - (k+1)r & \text{if } kc + r \leq a < (k+1)c, \\
a - (\eta - l)r & \text{if } a \geq d - lc,
\end{cases}
\]

where \( k \in \mathbb{Z} \). If \( l > -\infty \) or \( u < +\infty \), \( \Phi_c \) is not superadditive on \( \mathbb{R} \). However, by Theorem 4, one can use any superadditive lower bound on \( \Phi_c \) to lift inequality (3). One such function is

\[
\phi_c(a) = \begin{cases} 
k(c-r) & \text{if } kc \leq a < kc + r, \\
a - (k+1)r & \text{if } kc + r \leq a < (k+1)c.
\end{cases}
\]

For any \( c > 0 \), if necessary, by introducing a fictitious variable \( w \) with coefficient \( c \) and projecting it back to 0, we may use \( \phi_c \), to lift inequality (3) to inequality

\[
\sum_{i \in I} \phi_{1/\lambda}(a_i)x_i + \sum_{i \in C: g_i < 0} g_i y_i \leq b - \eta r,
\]

(4)

where \( \lambda = 1/c \). Observe that when \( c = 1 \), we have \( \eta = [b] \), \( r = f \) and in this case the approximate lifting function \( \phi_c \) reduces to

\[
\phi_c(a) = \begin{cases} 
\lfloor a \rfloor (1-f) & \text{if } \lfloor a \rfloor \leq a < \lfloor a \rfloor + f \\
a - \lfloor a \rfloor f & \text{if } \lfloor a \rfloor + f \leq a < \lfloor a \rfloor + 1,
\end{cases}
\]

and the right hand side of (4) becomes \([b](1-f)\). \( \Box \)

See [1] for other superadditive lower–bounding lifting functions that exploit the bounds of integer variables for deriving strong inequalities for mixed–integer knapsack sets.

**Acknowledgment** I am grateful to Ilan Adler for several discussions on this topic.

**References**


