A Note on the Ichoua et al (2003) Travel Time Model

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Abstract
In this paper we exploit some properties of the travel time model proposed by Ichoua et al (2003), on which most of the current time-dependent vehicle routing literature relies. Firstly, we prove that any continuous piecewise linear travel time model, satisfying the FIFO property, can be generated by an appropriate Ichoua et al (2003) model. We also show that the model parameters can be obtained by solving a system of linear equations for each arc. Then such parameters are proved to be nonnegative which allows to interpret them as (dummy) speeds. Finally, we illustrate the procedure through a numerical example. As a by-product, we are able to link the travel time models of a road graph and the associated complete graph over which vehicle routing problems are usually formulated.

Keywords: Time-Varying Travel Times, Vehicle Routing

1. Introduction

Most of the literature on time-dependent vehicle routing relies on the stepwise speed model proposed by Ichoua, Gendreau and Potvin in 2003 (IGP model, in the following). The main point in their model is that they do not assume a constant speed over the entire length of a link. Rather, the speed changes when the boundary between two consecutive time periods is crossed. This feature guarantees that if a vehicle leaves a node $i$ for a node $j$ at a given time, any identical vehicle leaving node $i$ for node $j$ at a later time

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will arrive later at node $j$ (no-passing or first-in-first-out (FIFO) property).
In this paper we prove that any continuous piecewise linear travel time model can be generated by an appropriate IGP model, and show how to compute the model parameters. We also prove that such parameters can be interpreted as dummy speeds. These results allow us to link the travel time models of a road graph and the associated complete graph over which vehicle routing problems are usually formulated. This is quite interesting because, while the hypothesis of instantaneous speed variation over an arc is quite realistic for the arcs of the road graph (at least if the corresponding streets are not too long), it is not so intuitive that this assumption may be reasonable for the associated complete graph as well.

The literature on Time-Dependent Vehicle Routing is fairly limited and can be divided, for the sake of convenience, into four broad areas: travel time modeling and estimation; the Time-Dependent Shortest Path Problem (TDSPP); the Time-Dependent Traveling Salesman Problem (TDTSP) and its variants; and the Time-Dependent Vehicle Routing Problem (TDVRP). Here we focus on the first research stream. [1] proposed a model for time-dependent travel speeds and several approaches for estimating the parameters of this model. The modeling approach has been implemented in a commercial courier vehicle scheduling system and was judged to be "very useful" by users in a number of different metropolitan areas in the United States. [2] proposed a travel time modeling approach based on a continuous piecewise linear travel time function (the IGP model). Later, [3] investigated the assumptions that this function must satisfy to ensure that travel times satisfy the FIFO property. They also described the derivation of travel time data from modern traffic information systems. In particular, they presented a general framework for the implementation of time-varying travel times in various vehicle-routing algorithms. Finally, they reported on computational tests with travel time data obtained from a traffic information system in the city of Berlin. [4] investigated exact and approximate methods for estimating time-minimizing vehicular movements in road network models where link speeds vary over time. The assumptions made about network conditions recognize the intrinsic relationship between speed and travel duration and are substantiated by elementary methods to obtain link travel duration. The assumptions also imply a condition of FIFO consistency, which justifies the use of Dijkstra's algorithm for path-finding purposes.

This paper is organized as follows. In Section 2, we gain some insight into a constant stepwise travel speed model with constant distances and illustrate...
a procedure for deriving any continuous piecewise linear travel time model from a suitable IGP model. We also show that the model parameters can be obtained by solving a system of linear equations for each arc. In Section 3, we illustrate a numerical example, while in Section 4 we exploit the relationship between the travel time models of a road graph and the associated complete graph over which vehicle routing problems are usually formulated. Conclusions and future research issues are reported in Section 5.

2. Continuous piecewise linear travel times and the IGP model

In this section, we prove that any continuous piecewise linear travel time model, satisfying the FIFO property, can be generated by an appropriate IGP model. Let $G = (V, A)$ be a graph, where $V = \{1, \ldots, n\}$ is a set of vertices and $A$ is a set of arcs. With each arc $(r, s) \in A$ is associated a nonnegative length $L_{rs}$ which is assumed to be constant over time. The time horizon $[0, +\infty]$ is partitioned, for any arc $(r, s) \in A$, into a finite number $H_{rs}$ of time slots $[T_{rsh}, T_{rsh+1}]$ $(h = 0, \ldots, H_{rs} - 1)$, where $T_{rs0} = 0$ and $T_{rsH_{rs}} = +\infty$. During each time slot $h$ the speed is assumed to be equal to a constant $v_{rsh}$ on arc $(r, s) \in A$. In particular, the speed is constant in the long run on every arc. Given these speeds, the arc travel time functions $\tau_{rs}(t)$ can be computed through Algorithm 1 [2].

The main idea of this model is that when the vehicle traverses an arc, speed is not a constant over the entire length but it changes when the boundary between two consecutive time periods is crossed. For the sake of simplicity, from now on, we omit the arc indices $rs$ when it is clear which arc $(r, s)$ we are referring to. We also refer to $L$ and $v_0, \ldots, v_{H-1}$ as the IGP parameters.

We observe that the length traversed by a vehicle in a time interval $[t_1, t_2]$ is equal to the integral of its speed function between $t_1$ and $t_2$. Since the speed function is constant stepwise, we get:

\[
L = (T_{p+1} - t)v_p + \sum_{\ell=p+1}^{q-1} (T_{\ell+1} - T_{\ell})v_\ell + [t + \tau(t) - T_q]v_q, \quad (1)
\]

where $[T_p, T_{p+1}]$ and $[T_q, T_{q+1}]$ are the time intervals in which the start time $t$ and the arrival time $t + \tau(t)$ fall, respectively.
Algorithm 1 Computing $\tau_{rs}(t)$ according to the IGP model

INPUT: A set of speed breakpoints $\{T_{rs0}, \ldots, T_{rs(H-1)}\}$, the corresponding speed levels $\{\nu_{rs0}, \ldots, \nu_{rs(H-1)}\}$ and a start time $t \in [T_{rsh}, T_{rs(h+1)}[ \ (h \in \{0, \ldots, H_{rs} - 1\})$

OUTPUT: Travel time $\tau_{rs}(t)$

$k \leftarrow h$

$d \leftarrow L_{rs}$

$t' \leftarrow t + d/\nu_{rsk}$

$t'' \leftarrow t$

while $t' > T_{rs(k+1)}$ do

$d \leftarrow d - \nu_{rsk}(T_{rs(k+1)} - t'')$

$t'' \leftarrow T_{rs(k+1)}$

$t' \leftarrow t'' + d/\nu_{rs(k+1)}$

$k \leftarrow k + 1$

end while

return $t' - t$.

In order to compute the IGP parameters corresponding to a given continuous piecewise linear travel time model, satisfying the FIFO property, we have devised a two-step algorithm (Algorithm 2): firstly, we determine the set $\Omega$ of the potential speed breakpoints; secondly, we determine the speed levels for each interval defined by $\Omega$.

Step 1 - Determining the potential speed breakpoints

Let $\{t_k, k = 0, \ldots, K-1\}$ be the set of breakpoints of the travel time function $\tau(t)$ and let $\Gamma(t)$ be the arrival time function, i.e. $\Gamma(t) = t + \tau(t)$. In the first phase, the set $\Omega$ is determined by means of an iterative procedure composed of main while loop in which each travel time breakpoint $t_k$ is added to $\Omega$.

Moreover, for each $t_k$:

(a) $\Omega$ is iteratively enriched by the arrival time $\Gamma(t_k)$ associated to a start time equal to $t_k$, by the arrival time $\Gamma(\Gamma(t_k))$ associated to a start time equal to $\Gamma(t_k)$, etc, until no speed breakpoint less than or equal to $t_{K-1}$ can be generated;

(b) finally, $\Omega$ is iteratively enriched by the start time $\Gamma^{-1}(t_k)$ associated to an arrival time equal to $t_k$, by the start time $\Gamma^{-1}(\Gamma^{-1}(t_k))$ associated to an arrival time equal to $\Gamma^{-1}(t_k)$, etc, until no speed breakpoint greater than or equal to $t_0 = 0$ can be generated.
Let \( \{T_0, \ldots, T_{H-1}\} \) be the potential speed breakpoints of \( \Omega \), sorted in non-decreasing order.

Step 2 - Determining the speed levels of the IGP model
Here we determine the speed values and the arc length of the IGP model.

Firstly, we observe that the IGP parameters \( L \) and \( v(t) \) can be multiplied by any positive common factor without changing the travel time function. This implies that \( L \) can be any positive number. Let \( a_{h, \ell} \) be the time spent on the arc during period \([T_\ell, T_{\ell+1}] \) \((\ell = 0, \ldots, H - 1)\) if the start time is \( T_h \), i.e.

\[
a_{h, \ell} = \begin{cases} 
\min(T_{\ell+1} - T_\ell, \max(0, \Gamma(T_\ell) - T_\ell)) & \text{if } h \leq \ell \\
0 & \text{if } h > \ell 
\end{cases}
\] (2)

Then, the travel speed levels \( \nu_h, h = 0, \ldots, H - 1 \), are determined as the (unique) nonnegative solution of the following square linear system:

\[
\begin{bmatrix}
a_{00} & a_{01} & \cdots & a_{0,H-1} \\
0 & a_{11} & \cdots & a_{1,H-1} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & a_{H-1,H-1} \\
\end{bmatrix}
\begin{bmatrix}
\nu_0 \\
\nu_1 \\
\nu_2 \\
\vdots \\
\nu_{H-1} \\
\end{bmatrix}
=
\begin{bmatrix}
L \\
L \\
L \\
\vdots \\
L \\
\end{bmatrix},
\] (3)

Since the coefficient matrix of (3) is in inferior triangular form with not null diagonal entries, Algorithm 2 determines the solution of (3) as follows:

\[
\nu_h = \frac{(L - a_{h,h+1}\nu_{h+1} - \cdots - a_{h,H-1}\nu_{H-1})}{a_{hh}},
\] (4)

with \( h = H - 1, H - 2, \ldots, 0 \). We now prove the correctness of Algorithm 2.

**Theorem 1.** Given a continuous piecewise linear FIFO travel time function \( \tau(t) \), Algorithm 2 determines the corresponding IGP parameters.

**Proof.** Let \( v(t) \) be the speed function provided by Algorithm 2 when inputted by \( \tau(t) \), and let \( \tau'(t) \) be the travel time function generated by Algorithm 1 when inputted with \( v(t) \). We want to prove that \( \tau'(t) = \tau(t) \). Firstly, we observe that the \( h - \text{th} \) row of system (3) implies that \( \tau'(T_h) = \tau(T_h) \), with \( h = 0, \ldots, H - 1 \). Hence, the thesis is proved if we demonstrate that all the breakpoints of \( \tau'(t) \) belong to \( \{T_0, \ldots, T_H\} \).
Algorithm 2 Determining the IGP model corresponding to a given continuous piecewise linear FIFO travel time function

**INPUT:** A continuous piecewise linear FIFO travel time function $\tau(t)$ with breakpoints $\{t_0, \ldots, t_{K-1}\}$

**OUTPUT:** The associated IGP model, namely the set $\Omega$ of speed breakpoints and the corresponding speed levels $\{\nu_h\}, T_h \in \Omega$

**Step 1 - Determine the set of speed breakpoints $\Omega$**

$\Omega = \emptyset$

for all $t \in \{t_0, \ldots, t_{K-1}\}$ do
  if $t \notin \Omega$ then
    $\Omega \leftarrow t$
    $t' \leftarrow t$
    **while** $(\Gamma(t') \leq t_{K-1}) \land (\Gamma(t') \notin \Omega)$ **do**
      $\Omega \leftarrow \Gamma(t')$
      $t' \leftarrow \Gamma(t')$
    **end while**
    $t' \leftarrow t$
  **while** $(\Gamma^{-1}(t') \geq t_0) \land (\Gamma^{-1}(t') \notin \Omega)$ **do**
    $\Omega \leftarrow \Gamma^{-1}(t')$
    $t' \leftarrow \Gamma^{-1}(t')$
  **end while**
  end if
end for

**Step 2 - Determine the constant speed levels**

Let $\{T_0, \ldots, T_{H-1}\}$ be the potential speed breakpoints of $\Omega$, sorted in nondecreasing order.

for $(h = (H - 1) \rightarrow 0)$ do
  $\nu_h = (L - a_{h,h+1}\nu_{h+1} - \cdots - a_{h,H-1}\nu_{H-1})/a_{hh}$
end for
We prove this thesis by contradiction. Therefore we suppose that there exists a breakpoint \( t \) such that:

\[
\frac{\tau'(t) - \tau'((t - \Delta))}{\Delta} \neq \frac{\tau'(t + \Delta) - \tau'(t)}{\Delta},
\]

with \([t - \Delta, t + \Delta] \subseteq [T_p, T_{p+1}]\) and \( \Delta > 0 \). We observe that Algorithm 2 determines the set of potential speed breakpoints \( \{T_0, \ldots, T_{H-1}\} \) so that no speed change occurs in \([\Gamma'(T_h), \Gamma'(T_{h+1})]\), with \( \Gamma'(T_h) = T_h + \tau'(T_h) \). Hence:

\[
[\Gamma'(t - \Delta), \Gamma'(t + \Delta)] \subseteq [T_q, T_{q+1}].
\]

If we write Equation (1) for time instants \((t - \Delta)\) and \( t \), we obtain:

\[
L = (T_{p+1} - t + \Delta)\nu_p + \sum_{\ell=p+1}^{q-1} (T_{\ell+1} - T_{\ell})\nu_{\ell} + (t - \Delta + \tau'(t - \Delta) - T_q)\nu_q. \quad (6)
\]

\[
L = (T_{p+1} - t)\nu_p + \sum_{\ell=p+1}^{q-1} (T_{\ell+1} - T_{\ell})\nu_{\ell} + (t + \tau'(t) - T_q)\nu_q. \quad (7)
\]

By subtracting (6) from (7), we get:

\[
\frac{\nu_p}{\nu_q} - 1 = \frac{(\tau'(t) - \tau'(t - \Delta))}{\Delta}. \quad (8)
\]

Similarly, if we consider time instants \( t \) and \((t + \Delta)\), we obtain:

\[
\frac{\nu_p}{\nu_q} - 1 = \frac{(\tau'(t + \Delta) - \tau'(t))}{\Delta}. \quad (9)
\]

Since (8) and (9) imply

\[
\frac{(\tau'(t) - \tau'(t - \Delta))}{\Delta} = \frac{(\tau'(t + \Delta) - \tau'(t))}{\Delta}, \quad (10)
\]

the thesis is proved.

As a final remark, we observe that \((a_{h,h+1}\nu_{h+1} + \cdots + a_{h,H-1}\nu_{H-1})\) represents the length traversed in the time interval \([T_{h+1}, T_{H-1}]\) if the start time is \( T_h \). This implies that the right hand side of (4) is positive and \( \nu_h \) can be interpreted as a dummy speed, with \( h = 0, \ldots, H - 1 \).
3. A Numerical Example

We provide a numerical example to illustrate the previous properties. Figure 1(a) describes a continuous piecewise linear travel time function $\tau(t)$, whose breakpoints $t_k$ are reported in Table 1 along with the corresponding $\tau(t_k)$ travel times.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$t_k$</th>
<th>$\tau(t_k)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>1.5</td>
</tr>
</tbody>
</table>

Table 1: Values of $t_k$ and $\tau(t_k)$ for the numerical example

<table>
<thead>
<tr>
<th>$h$</th>
<th>$T_h$</th>
<th>$\Gamma(T_h)$</th>
<th>$\Gamma^{-1}(T_h)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>2</td>
<td>-</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>3</td>
<td>-</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>6</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>6.5</td>
<td>3</td>
</tr>
</tbody>
</table>

Table 2: Values of $T_h$ determined by Algorithm 2 for the numerical example

We use Algorithm 2 to determine the speed breakpoints (Table 2). Then, we set $L = 3$ and compute the speed levels by solving the following square linear system:

$$\begin{align*}
\nu_0 + \nu_1 + \nu_2 &= 3 \\
\nu_1 + \nu_2 + \nu_3 &= 3 \\
\nu_2 + \nu_3 + \nu_4 &= 3 \\
\nu_3 + \nu_4 + \nu_5 &= 3 \\
\nu_4 + 1.5\nu_5 &= 3
\end{align*}$$

(11)
The speed values come up to be: $\nu_0 = 1, \nu_1 = 2, \nu_2 = 1, \nu_3 = 2, \nu_4 = 1, \nu_5 = 2$. See Figure 1(b) for a graphical representation of the corresponding $v(t)$.

4. Linking the travel time models of a road graph and the associated complete graph

Vehicle routing problems are often modelled on a complete graph $G'$ in which the vertices represent the customers (and possibly additional facilities, such as a depot) and the arcs model quickest paths between pairs of customers and facilities on the underlying road graph $G$. The main point in the IGP model is that it does not assume a constant speed over the entire length of a link. Rather, the speed changes when the boundary between two consecutive time periods is crossed. This feature guarantees that FIFO property holds.

While the hypothesis of instantaneous speed variation over an arc is quite realistic for the arcs of the road graph (at least if the corresponding streets are not too long), it is not so intuitive that this assumption may be reasonable for the associated complete graph as well.

In this section, we exploit the relationships between the travel time models of the two graphs. In particular, we show that if the arc travel times of the road graph follow Ichoua et al (2003), then the arcs of the complete graph can be modeled by the same variation law (with suitable parameters). Let $p$ be a simple path $\{i = i_0, i_1, \ldots, j = i_m\}$ on the road graph $G$. We denote with $P_{ij}$ the set of simple paths on $G$, connecting customer/facility $i$ to customer/facility $j$. Let $z(p, t)$ be the traversal time of path $p$, whenever a vehicle leaves vertex $i$ at time $t$. We observe that $z(p, t)$ is the sum of continuous piecewise linear functions. For example, for $m = 2$:

$$z(p, t) = \tau_{i_0i_1}(t) + \tau_{i_1i_2}(t + \tau_{i_0i_1}(t)).$$ (12)

Hence $z(p, t)$ is continuous piecewise linear itself. On the complete graph $G'$, the time-dependent travel time $\tau'_{ij}(t)$ of arc $(i, j) \in A'$ is given by:

$$\tau'_{ij}(t) = \min_{p \in P_{ij}} z(p, t).$$ (13)

Since $P_{ij}$ is a finite set, function $\tau'_{ij}(t)$ is continuous piecewise linear too. Hence, Theorem 1 implies that $\tau'_{ij}(t)$ can be generated by an IGP model, with a suitable choice of parameters $L'_{ij}$ and $\nu'_{hij} (h \in 0, \ldots, H'_{ij})$. It is worth noting that this property holds for any choice of $L'_{ij} > 0$. In particular, $L'_{ij}$
can be chosen equal to the length of the shortest path from node \( i \) to node \( j \) on the road graph.

A straightforward consequence is that the IGP model does not suffer from the drawback pointed out by Fleischmann et al [3] who stated: ”A drawback of the models with varying speeds but constant distances is that they do not consider potential changes of the fastest paths themselves due to varying travel times, which imply changes of distances”. Another outcome is that the lower bounding procedure proposed by Cordeau et al [5] for the Time-Dependent Traveling Salesman Problem can be applied to the wider class of instances with continuous piecewise linear arc travel times: first, the IGP parameters have to be computed by solving a system of linear equations for every arc; then speeds \( \nu'_{ijh} \) are expressed as

\[
\nu'_{ijh} = \delta_{ijh} b_h u_{ij},
\]

where:

- \( u_{ij} \) is the maximum travel speed across arc \((i, j) \in A \) during \([0, T]\), i.e. \( u_{ij} = \max_{h=0,...,H-1} \nu'_{ijh} \);

- \( b_h \) belongs to \([0, 1]\) and is the best (i.e. lightest) congestion factor during interval \([T_h, T_{h+1}]\), i.e. \( b_h = \max_{(i,j) \in A} \nu'_{ijh}/u_{ij} \);

- \( \delta_{ijh} \) belongs to \([0, 1]\) and represents the degradation of the congestion factor of arc \((i, j) \) in interval \([T_h, T_{h+1}]\) with respect to the less congested arc in \([T_h, T_{h+1}]\);

finally, a lower bound can be computed by: (a) determining a time-independent Traveling Salesman Problem optimal solution w.r.t. maximum travel speeds \( u_{ij} \); (b) evaluating its traversal time w.r.t. to the most favourable congestion factor during each interval \( h \), i.e. \( v'_{ijh} \leftarrow b_h u_{ij} \).

5. Conclusions

In this paper we have shown that the travel time model proposed by Ichoua et al (2003) is quite general since any continuous piecewise linear travel time model can be generated from it with a suitable choice of its parameters. Then some light has been shed on the relevance of this model on road graphs and the associated complete graphs. As a future research topic, we would suggest the extension of the lower bounding approach [5] to other time-dependent arc and node routing problems.
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References


Figure 1: A continuous piecewise linear arc travel time function and the associated constant stepwise speed function.