Characterization of proper optimal elements with variable ordering structures

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Abstract

In vector optimization with a variable ordering structure the partial ordering defined by a convex cone is replaced by a whole family of convex cones, one associated with each element of the space. In recent publications it was started to develop a comprehensive theory for these vector optimization problems. Thereby also notions of proper efficiency were generalized to variable ordering structures. In this paper we study the relation between several types of proper optimality. We give scalarization results based on new functionals defined by elements from the dual cones which allow complete characterizations also in the nonconvex case.

Key Words: vector optimization, variable ordering structure, proper efficiency, scalarization

Mathematics subject classifications (MSC 2000): 90C29, 90C30, 90C48

1 Introduction

In vector optimization one studies optimization problems with a vector-valued objective map. For comparing elements in the objective space, i.e. elements from a set in a linear space, one can assume that a partial ordering which is compatible with the linear structure of the space is given. Hence one can assume that a convex cone introducing this partial ordering is given. More general concepts allow that preferences vary depending on the current values in the objective space. This means that an individual ordering cone is attached to each element in the objective space. This is mathematically modeled by a set-valued map on the objective space, called ordering map, with images being convex cones. The resulting binary relations are in general not transitive and not compatible with the linear structure of the space.

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Vector optimization problems with such variable ordering structures are the topic of this manuscript.

Variable ordering structures were already introduced in the seventies [39, 7] and have gained recently more interest due to several applications for instance in medical image registration, in dynamical models in economy theory or in behavioral sciences [1, 38, 19, 36, 37, 13, 30, 3]. Various scalarization approaches have been proposed for these ordering structures, like linear scalarizations based on elements from the dual space [19, 13]. Nonlinear scalarizations were introduced based on a representation of the images of the ordering map as Bishop-Phelps cones [17], using elements of the augmented dual cones of the images of the ordering map [18], or generalizing a functional known in the literature as Tammer-Weidner functional [14, 33, 34]. Optimality conditions of Fermat and Lagrange type based on scalarization results [17] and also by a more general approach [2] were proposed and also first numerical procedures were presented [23, 32, 15, 6]. Generalizations to set optimization were studied [4, 28]. Variable ordering structures are also studied in the context of vector complementarity problems, vector variational inequalities and vector equilibrium problems. For recent papers see for instance [24, 10, 31]

Next to optimal and weakly optimal elements, also strongly optimal [17], approximate optimal [33, 34, 35] and proper optimal elements [18] for variable ordering structures were defined. First scalarization results for these properly optimal elements were given in [18] based on functionals defined by elements from the augmented dual cones.

Properly optimal elements for variable ordering structures and their characterization is the main topic of this manuscript. The set of properly optimal elements are a subset of the set of optimal elements. By additional restrictions one tries to eliminate ”improper” optimal elements and to allow more satisfactory scalarization results for the properly optimal elements (cf. [29]). By these additional restrictions, those optimal elements are eliminated which can be interpreted in a finite-dimensional space as having an unbounded trade-off, and which are for that reason not of interest in applications. Moreover, it is known that, in case the set is convex, the properly efficient elements of a set in a partially ordered space are completely characterizable by linear scalarization based on elements from the quasi-interior of the dual cone. For efficient elements, the necessary and the sufficient conditions do not match.

We concentrate in this manuscript on the definitions of properly optimal elements in the sense of Henig [22], Benson [5] and Borwein [8] for variable ordering structures. Thereby, one has to differentiate between the concepts of nondominatedness and of preference w.r.t. a variable ordering structure: based on the ordering map, two different binary relations can be defined leading to two different optimality notions, the minimal and the nondominated elements. In [18] already generalizations of the proper optimality notions known from partially ordered spaces have been suggested for both concepts, the minimal and the nondominated elements. There, also scalarization results based on nonlinear scalar-valued functionals defined by elements from the augmented dual cones are provided. For that, the cones are required to have a bounded base and in the scalarization results it is required that the cones and their $\varepsilon$-conic neighborhoods satisfy some separation property.

In this paper, we study in detail the relation between the introduced proper optimality notions in case of a variable ordering structure and we complete conclusions
to their relations and some basic properties expressed in [18]. It turns out that most results known to hold in partially ordered spaces remain true. But not all results still hold: properly nondominated elements in the sense of Henig are in general not also properly nondominated in the sense of Benson. We show that the reason for this is that already under weak assumptions any nondominated element is already a Henig properly nondominated element.

Moreover, we present a new scalarization approach which is based on elements from the dual cones of the images of the ordering map. This scalarization can be used to characterize optimal elements without convexity assumptions. While the definition of the functional is based on linear functionals from the dual cones, the scalarization functional is in general nonlinear. It allows complete characterizations of (weakly/strongly/properly) optimal elements. In contrast to the scalarizations introduced in [18], some of the assumptions are weaker — and also the proofs are more direct as no theory of augmented dual cones or separation properties for special cones are required. We also give shortly characterization results for (weakly/strongly/properly) minimal elements based on a linear functional. By that we generalize the known linear scalarization results known from partially ordered spaces to vector optimization problems with a variable ordering structure. Based on the proposed characterization results, optimality conditions of Fermat and Lagrange type can be derived as well as numerical solution methods.

In Sect. 2 we give some preliminary results and collect the definitions of (weakly/strongly) minimal and nondominated elements. The different proper optimality notions and their relations are the topic of Sect. 3. In Sect. 4 we present the mentioned scalarization functional and compare it with other functionals from the literature. Moreover, we give necessary and sufficient conditions for the various optimality notions.

2 Preliminaries

In the following, let \((Y, \| \cdot \|)\) be a real normed space and let \(2^Y\) denote the set of all subsets of \(Y\). For some nonempty set \(\Omega\), we denote by \(\text{int}(\Omega)\), \(\text{cl}(\Omega)\), \(\text{cone}(\Omega)\) and \(\text{conv}(\Omega)\) the interior of \(\Omega\), the closure of \(\Omega\), the cone generated by \(\Omega\), and the convex hull of \(\Omega\). For some nonempty set \(\Omega \in Y\) and for some element \(\hat{y} \in \text{cl}(\Omega)\), \(T(\Omega, \hat{y})\) denotes the contingent cone (or the Bouligand tangent cone) to \(\hat{y}\) at \(\hat{y}\), i.e.

\[
T(\Omega, \hat{y}) := \{ h \in Y \mid \exists (\lambda_n)_{n \in \mathbb{N}} \subseteq \mathbb{R}_{++}, \exists (y_n)_{n \in \mathbb{N}} \subseteq \Omega \text{ such that } \lim_{n \to \infty} y_n = \hat{y} \text{ and } h = \lim_{n \to \infty} \lambda_n(y_n - \hat{y}) \}.
\]

Thereby, \(\mathbb{R}_{++}\) denotes the set of real positive numbers. A nonempty subset \(K\) of \(Y\) is called a cone if \(y \in K\) and \(\lambda \geq 0\) implies \(\lambda y \in K\). We say that a given cone \(K\) is nontrivial in case \(K \neq \{0_Y\}\). A base of a nontrivial cone \(K\) is a convex set \(B \subseteq K\) such that each element \(k \in K \setminus \{0_Y\}\) has a unique representation as \(k = \lambda b\) with \(\lambda > 0\) and \(b \in B\). A based cone \(K\) is necessarily pointed, i.e. it holds \(K \cap (-K) = \{0_Y\}\).

We assume that the variable ordering structure on \(Y\) is defined by a set-valued map (also called an ordering map) \(D: Y \to 2^Y\) with \(D(y)\) a nontrivial convex cone for all \(y \in Y\). Let \(A\) be a nonempty subset of \(Y\). The following definitions
of optimal elements (w.r.t. minimization) are known in the literature for variable ordering structures with a cone-valued map \( D \) [39, 11, 13, 17, 16].

**Definition 1.** Let \( \bar{y} \in A \).

(a) The element \( \bar{y} \) is a *nondominated element* of \( A \) w.r.t. \( D \) if \( \bar{y} \not\in \{y\} + D(y) \) for all \( y \in A \setminus \{\bar{y}\} \).

(b) Supposing that \( \text{int}(D(y)) \neq \emptyset \) for all \( y \in A \), \( \bar{y} \) is a *weakly nondominated element* of \( A \) w.r.t. \( D \) if \( \bar{y} \not\in \{y\} + \text{int}(D(y)) \) for all \( y \in A \).

(c) The element \( \bar{y} \) is a *strongly nondominated element* of \( A \) w.r.t. \( D \) if \( \bar{y} \not\in \{y\} - D(y) \) for all \( y \in A \setminus \{\bar{y}\} \).

(d) The element \( \bar{y} \) is a *minimal element* of \( A \) w.r.t. \( D \) if \( y \not\in \{\bar{y}\} - D(\bar{y}) \) for all \( y \in A \setminus \{\bar{y}\} \).

(e) The element \( \bar{y} \) with \( \text{int}(D(\bar{y})) \neq \emptyset \) is a *weakly minimal element* of \( A \) w.r.t. \( D \) if \( y \not\in \{\bar{y}\} - \text{int}(D(\bar{y})) \) for all \( y \in A \).

(f) The element \( \bar{y} \) is a *strongly minimal element* of \( A \) w.r.t. \( D \) if \( A \subseteq \{\bar{y}\} + D(\bar{y}) \).

If \( D(y) = K \) for all \( y \in Y \) with \( K \) some nontrivial pointed convex cone then the definitions of a (weakly/strongly) nondominated element of a set \( A \) w.r.t. \( D \) and of a (weakly/strongly) minimal element of a set \( A \) w.r.t. \( D \) coincide with the concepts of a (weakly/strongly) optimal element of \( A \) in the space \( Y \) partially ordered by the convex cone \( K \). We will denote the (weakly/strongly/properly) optimal elements w.r.t. the partial ordering introduced by some convex cone \( K \) as (weakly/strongly/properly) efficient elements of \( A \) w.r.t. \( K \).

Throughout the paper we assume that for the ordering map \( D : Y \to 2^Y \) the cones \( D(y) \) are nontrivial pointed convex cones for all \( y \in Y \).

### 3 Proper optimality

Following the definitions for properly efficient elements given by Henig [22], Benson [5] and Borwein [8] in partially ordered space the following generalizations for variable ordering structures were introduced in [18]. Note that we do not require in the definitions that the elements \( \bar{y} \in A \) are a nondominated/minimal element of \( A \) w.r.t. \( D \) as it was done in [18]. We will show later that the definitions below already imply that \( \bar{y} \) is a nondominated or a minimal element of \( A \) w.r.t. \( D \), respectively.

**Definition 2.** Let \( \bar{y} \in A \).

(a) The element \( \bar{y} \) is a *properly nondominated element in the sense of Henig* of \( A \) w.r.t. \( D \) if there is a cone-valued map \( K : Y \to 2^Y \) with \( K(y) \) a convex cone and \( D(y) \setminus \{0_Y\} \subseteq \text{int}(K(y)) \) for all \( y \in Y \) such that \( \bar{y} \) is a nondominated element of \( A \) w.r.t. \( K \), i.e.

\[
\bar{y} \not\in \{y\} + K(y) \quad \text{for all} \quad y \in A \setminus \{\bar{y}\}.
\]
(b) The element \( \bar{y} \) is a \textit{properly nondominated element in the sense of Benson} of \( A \) w.r.t. \( D \) if \( \bar{y} \) is a nondominated element of the set
\[
\{ \bar{y} \} + \text{cl}(\text{cone}(\bigcup_{a \in A} (\{a\} + D(a)) - \{\bar{y}\}))
\]
w.r.t. \( D \).

(c) The element \( \bar{y} \) is a \textit{properly nondominated element in the sense of Borwein} of \( A \) w.r.t. \( D \) if \( \bar{y} \) is a nondominated element of the set
\[
\{ \bar{y} \} + T(\bigcup_{a \in A} (\{a\} + D(a)), \bar{y})
\]
w.r.t. \( D \).

(d) The element \( \bar{y} \) is a \textit{properly minimal element in the sense of Henig} of \( A \) w.r.t. \( D \) if there is a cone-valued map \( K: Y \to 2^Y \) with \( K(y) \) a convex cone and \( D(y) \setminus \{0_Y\} \subseteq \text{int}(K(y)) \) for all \( y \in Y \) such that \( \bar{y} \) is a minimal element of \( A \) w.r.t. \( K \), i.e.
\[
y \notin \{ \bar{y} \} - K(\bar{y}) \quad \text{for all} \quad y \in A \setminus \{ \bar{y} \}.
\]

(e) The element \( \bar{y} \) is a \textit{properly minimal element in the sense of Benson} of \( A \) w.r.t. \( D \) if \( \bar{y} \) is a minimal element of the set
\[
\{ \bar{y} \} + \text{cl}(\text{cone}(A + D(\bar{y}) - \{\bar{y}\}))
\]
w.r.t. \( D \).

(f) The element \( \bar{y} \) is a \textit{properly minimal element in the sense of Borwein} of \( A \) w.r.t. \( D \) if \( \bar{y} \) is a minimal element of the set
\[
\{ \bar{y} \} + T(A + D(\bar{y}), \bar{y})
\]
w.r.t. \( D \).

The above introduced nondominatedness notions can also be written in a unified form adopting a notation from the literature [21]:
\[
\bar{y} \in Q\text{nd}(B) \iff \bar{y} \in B \land \forall y \in B \setminus \{ \bar{y} \}: (\bar{y} - y) \notin Q(y)
\]
where \( B \) is a corresponding subset of \( Y \) and \( Q: Y \to 2^Y \) is a corresponding set-valued map such that for all \( y \in Y \) the set \( Q(y) \) is a nontrivial convex cone. This definition can be used for showing relations like \( (B_1 \cap Q_2\text{nd}(B_2)) \subseteq Q_1\text{nd}(B_1) \) by just proving that \( B_1 \subseteq B_2 \) and \( Q_1(y) \subseteq Q_2(y) \) for all \( y \in B_1 \). We will use this property in the following proofs implicitly several times. This unified form can be similarly formulated for the minimality notions.

Note that in the definition of Henig proper optimality one usually requires closed pointed ordering cones (here: \( D(y) \)). In case \( D(y) \) is a pointed convex cone with \( D(y) \setminus \{0_Y\} \) an open set for all \( y \in Y \), then for \( K(y) := D(y) \) it holds
\[
D(y) \setminus \{0_Y\} = \text{int}(D(y)) = \text{int}(K(y)).
\]
Hence, in this case, the definitions above for properly minimal/nondominated elements in the sense of Henig coincide with the definitions of minimal/nondominated elements w.r.t. \( D \).

Following [20], we define by \( \text{ndGHe}(A, D)/\text{ndBe}(A, D)/\text{ndBo}(A, D) \) and by \( \text{mGHe}(A, D)/\text{mBe}(A, D)/\text{mBo}(A, D) \) the set of all properly nondominated elements and the set of all properly minimal elements in the sense of Henig/Benson/Borwein of \( A \) w.r.t. \( D \), respectively. For relating properly minimal elements in the sense of Henig w.r.t. \( D \) with properly efficient elements in the sense of Henig in a partially ordered space we need the following lemma. This lemma also gives an alternative way of defining proper minimality in the sense of Henig w.r.t. a variable ordering structure.

**Lemma 3.1.** [18, Lemma 4] Let \( \bar{y} \in A \). Then \( \bar{y} \in \text{mGHe}(A, D) \) if and only if there is a convex cone \( K \) with \( D(\bar{y}) \setminus \{0_Y\} \subseteq \text{int}(K) \) such that \( y \not\in \{\bar{y}\} - K \) for all \( y \in A \setminus \{\bar{y}\} \).

**Remark 3.2.** As a direct consequence of the definitions and of Lemma 3.1, \( \bar{y} \) is a properly minimal element in the sense of Henig/Benson/Borwein of \( A \) w.r.t. \( D \) if and only if it is a properly efficient element in the sense of Henig/Benson/Borwein of \( A \) in the space \( Y \) partially ordered by the convex cone \( K := D(\bar{y}) \).

We define by \( \text{effGHe}(A, K)/\text{effBe}(A, K)/\text{effBo}(A, K) \) the set of all properly efficient elements in the sense of Henig/Benson/Borwein of \( A \) w.r.t. a partially ordering introduced by the nontrivial pointed convex cone \( K \). In the forthcoming Lemma 3.9 we study the relation between properly efficient and properly nondominated elements.

We will also use the following sets:

\[
M := \bigcup_{a \in A} \left( \{a\} + D(a) \right) \quad \text{and} \quad M_{\bar{y}} := A + D(\bar{y}) \text{ for } \bar{y} \in A .
\] (1)

The above definitions imply that every properly nondominated/minimal element is a nondominated/minimal element of \( A \) w.r.t. \( D \), respectively. Hence this requirement, as given in the original definitions in [18], is redundant and can be omitted:

**Lemma 3.3.** Let \( \bar{y} \in A \).

(i) If \( \bar{y} \in \text{ndGHe}(A, D) \) or \( \bar{y} \in \text{ndBe}(A, D) \) or \( \bar{y} \in \text{ndBo}(A, D) \) then \( \bar{y} \) is a nondominated element of \( A \) w.r.t. \( D \).

(ii) If \( \bar{y} \in \text{mGHe}(A, D) \) or \( \bar{y} \in \text{mBe}(A; D) \) or \( \bar{y} \in \text{mBo}(A, D) \) then \( \bar{y} \) is a minimal element of \( A \) w.r.t. \( D \).

**Proof.** Let \( \bar{y} \in \text{ndBo}(A, D) \) and the set \( M \) be defined as in (1). Assume that \( \bar{y} \) is not a nondominated element of \( A \) w.r.t. \( D \). Then there exist \( \tilde{y} \in A \setminus \{\bar{y}\} \subseteq M \setminus \{\bar{y}\} \) and \( d \in D(\tilde{y}) \setminus \{0_Y\} \) such that \( \bar{y} = \tilde{y} + d \). Let \( \lambda_n := n, d_n := (1 - \frac{1}{n})d \in D(\tilde{y}) \) and \( y_n := \tilde{y} + d_n \in \{\tilde{y}\} + D(\tilde{y}) \subseteq M \) for all \( n \in \mathbb{N} \). Then it follows

\[
\lim_{n \to \infty} y_n = \lim_{n \to \infty} (\tilde{y} + d_n) = \lim_{n \to \infty} \left( \tilde{y} - \frac{1}{n}d \right) = \bar{y}
\]
and
\[
\lim_{n \to \infty} \lambda_n (y_n - \bar{y}) = \lim_{n \to \infty} n (\bar{y} + d_n - \bar{y}) = \lim_{n \to \infty} n (d_n - d) = -d \in T(M, \bar{y}).
\]

Hence, \( \bar{y} = \bar{y} + d \in \{\bar{y}\} + D(\bar{y}) \) and \( \bar{y} = \bar{y} - d \in \{\bar{y}\} + T(M, \bar{y}) \setminus \{\bar{y}\} \) being a contradiction to that \( \bar{y} \in \text{ndBo}(A, D) \).

The assertion for \( \text{mBo}(A, D) \) is a consequence of Remark 3.2 and the fact that \( \bar{y} \in \text{effBo}(A, K) \) implies that \( \bar{y} \) is an efficient element of \( A \) w.r.t. \( K \), see [26, Proposition 3.2]. The remaining conclusions follow immediately from the definition of a nondominated/minimal element, since \( D(y) \subseteq K(y) \) for all \( y \in Y, A \subseteq \{\bar{y}\} + \text{cl}(\text{cone}(M - \{\bar{y}\})) \), and \( A \subseteq \{\bar{y}\} + \text{cl}(\text{cone}(M_{\bar{y}} - \{\bar{y}\})) \) with \( M \) and \( M_{\bar{y}} \) as defined in (1).

Under weak assumptions any nondominated element is already a properly nondominated element in the sense of Henig. For that we need the following result. It is known in partially ordered spaces that a comparison between various notions of proper efficiency in vector optimization depends on the existence of a bounded base of the ordering cone, see [9] and the references therein.

**Lemma 3.4.** [12, Proposition 2.2] Let \( P \subseteq Y \) be a weakly closed nontrivial cone and \( C \subseteq Y \) be a cone with a weakly compact base such that \( P \cap C = \{0_Y\} \). Then there exists a closed pointed convex cone \( K \subseteq Y \) which has a closed bounded base such that
\[
C \setminus \{0_Y\} \subseteq \text{int}(K) \quad \text{and} \quad P \cap K = \{0_Y\}.
\]

**Theorem 3.5.** Let \( \bar{y} \in A \) and \( D(y) \) have a weakly compact base for all \( y \in A \setminus \{\bar{y}\} \). Then \( \bar{y} \in \text{ndGHe}(A, D) \) if and only if \( \bar{y} \) is a nondominated element of \( A \) w.r.t. \( D \).

**Proof.** The necessity follows by Lemma 3.3(i). If \( \bar{y} \) is a nondominated element of \( A \) w.r.t. \( D \), then it holds \( \bar{y} - y \notin D(y) \) for all \( y \in A \setminus \{\bar{y}\} \). Let \( y \in A \setminus \{\bar{y}\} \) be arbitrarily chosen. As \( D(y) \) is a cone it follows
\[
\text{cl}(\text{cone}(\{\bar{y} - y\})) \cap (D(y)) = \{0_Y\}.
\]

Using Lemma 3.4 there exists a closed pointed convex cone \( K(y) \subseteq Y \) with
\[
D(y) \setminus \{0_Y\} \subseteq \text{int}(K(y)) \quad \text{and} \quad \text{cl}(\text{cone}(\{\bar{y} - y\})) \cap (K(y)) = \{0_Y\}.
\]

Hence, we obtain \( \bar{y} \notin \{y\} + K(y) \), and we are done.

We have used here a quite natural generalization of the proper optimality concept in the sense of Henig to variable ordering structures as introduced and studied in [18]. However, Theorem 3.5 clearly implies that, at least for cone-valued maps \( D \) where each \( D(y) \) has a weakly compact base, the definition of properly nondominated elements in the sense of Henig is not appropriate. We also illustrate this with the next example. We discuss at the end of this section a possible approach for a modification of the notion.

**Example 1.** Let \( Y = \mathbb{R}^2 \), \( A = \{(y_1, y_2)^\top \in \mathbb{R}^2 \mid y_1^2 + y_2^2 \leq 1\} \), \( D(y) = \mathbb{R}^2_+ \) for all \( y \in \mathbb{R}^2 \), and \( \bar{y} := (0, -1)^\top \). Then \( \bar{y} \) is not a properly efficient element in the sense of Henig of \( A \) w.r.t. the ordering cone \( \mathbb{R}^2_+ \). In contrast, \( \bar{y} \) is a nondominated element of
A w.r.t. $\mathcal{D}$ and by Theorem 3.5 it holds $\bar{y} \in \text{ndGHe}(A, \mathcal{D})$. To see the last assertion, define a cone-valued map $\mathcal{K} : \mathbb{R}^2 \to 2^{\mathbb{R}^2}$ for some $\varepsilon > 0$ by

$$
\mathcal{K}(y) = \begin{cases} 
\text{cone}(\text{conv}(\{(1, \sqrt{1-\frac{y_1^2}{2|y_1|}}, 1\}) \cap \text{cone}(\{(1, -\varepsilon, 1\})) & \text{for all } y \in A \text{ with } y_1 < 0, \\
\text{cone}(\text{conv}(\{(1, -\varepsilon, 1\})) & \text{else}.
\end{cases}
$$

It is easy to see that $\mathcal{D}(y) \setminus \{0_Y\} \subseteq \text{int}(\mathcal{K}(y))$ for all $y \in Y$ and $\bar{y} \notin \{y\} + \mathcal{K}(y)$ for all $y \in A \setminus \{\bar{y}\}$ with $y_1 \geq 0$. Moreover, for all $y = (-a, -\sqrt{1-a^2})^T$ with $a \in (0, 1]$ it holds

$$
\bar{y} - y = a \left(\frac{1}{\sqrt{1-\frac{y_1^2}{2|y_1|}}} - 1\right) \notin \mathcal{K}(y)
$$

and thus $\bar{y} \notin \{y\} + \mathcal{K}(y)$ for all $y \in A \setminus \{\bar{y}\}$.

The following results on relations between the proper optimality notions are direct consequences from the fact that for sets $\Omega$ and elements $\bar{y} \in \Omega$, $T(\Omega, \bar{y}) \subseteq \text{cl}(\text{cone}(\Omega - \{\bar{y}\}))$, and in case $\Omega$ is starshaped w.r.t. $\bar{y}$ even equality holds, see for instance [25, Theorem 3.44 and Corollary 3.46].

**Lemma 3.6.** Let $\bar{y} \in A$ and let the sets $M$ and $M_{\bar{y}}$ be defined as in (1). Then the following holds:

(i) $\bar{y} \in \text{ndBe}(A, \mathcal{D}) \Rightarrow \bar{y} \in \text{ndBo}(A, \mathcal{D}).$

(ii) If the set $M$ is starshaped w.r.t. $\bar{y}$, then $\bar{y} \in \text{ndBe}(A, \mathcal{D}) \Leftrightarrow \bar{y} \in \text{ndBo}(A, \mathcal{D}).$

(iii) $\bar{y} \in \text{mBe}(A, \mathcal{D}) \Rightarrow \bar{y} \in \text{mBo}(A, \mathcal{D}).$

(iv) If the set $M_{\bar{y}}$ is starshaped w.r.t. $\bar{y}$, then $\bar{y} \in \text{mBe}(A, \mathcal{D}) \Leftrightarrow \bar{y} \in \text{mBo}(A, \mathcal{D}).$

Further proper minimal elements we can also prove the following results:

**Lemma 3.7.** Let $\bar{y} \in A$. Then the following holds:

(i) If $\bar{y} \in \text{mGHe}(A, \mathcal{D})$ and the set $A$ contains more than one element then $\bar{y} \in \text{mBe}(A, \mathcal{D}).$

(ii) If $\mathcal{D}(\bar{y})$ has a weakly compact base, then $\bar{y} \in \text{mBe}(A, \mathcal{D}) \Rightarrow \bar{y} \in \text{mGHe}(A, \mathcal{D}).$

**Proof.** (i) If $\bar{y} \in \text{mGHe}(A, \mathcal{D})$, then there exists a convex cone $\mathcal{K}(\bar{y})$ with $\mathcal{D}(\bar{y}) \setminus \{0_Y\} \subseteq \text{int}(\mathcal{K}(\bar{y}))$ such that $y \notin \{\bar{y}\} - \mathcal{K}(\bar{y})$ for all $y \in A \setminus \{\bar{y}\}$. Since the set $A$ contains more than one element it follows $\mathcal{K}(\bar{y}) \neq Y$ and the convex cone $\text{int}(\mathcal{K}(\bar{y})) \cup \{0_Y\}$ is pointed. Hence we can replace $\mathcal{K}(\bar{y})$ by $\text{int}(\mathcal{K}(\bar{y})) \cup \{0_Y\}$ and we can assume that the convex cone $\mathcal{K}(\bar{y})$ can be chosen to be pointed. Then the assertion follows from Remark 3.2, together with the inclusion $\text{effGHe}(A, K) \subseteq \text{effBe}(A, K)$, given in [20, Theorem 4.2] (and for the finite dimensional case already in [22, Theorem 2.1]) and noting that closedness of $K$ is not required for the proof given there while pointedness of the cone $K'$ which contains $K \setminus \{0_Y\}$ in its interior is needed.

(ii) follows from Remark 3.2, together with the inclusion $\text{effBe}(A, K) \subseteq \text{effGHe}(A, K)$ given in [27, Remark 5.3] based on Lemma 3.4 (see also [20, Theorem 4.2 and p. 9]).
The following Corollary is a direct consequence of Lemma 3.3(i) and Theorem 3.5.

**Corollary 3.8.** Let \( \bar{y} \in A \). If \( D(y) \) has a weakly compact base for all \( y \in A \setminus \{ \bar{y} \} \), then \( \bar{y} \in \text{ndBe}(A, D) \Rightarrow \bar{y} \in \text{ndGHe}(A, D) \).

For the opposite direction in the conclusion of Corollary 3.8 we refer to the following example:

**Example 2.** Let \( Y = \mathbb{R}^2 \), \( A = \{(y_1, y_2)^\top \in \mathbb{R}^2 | y_1 + y_2 \geq 0, \ y_1 - y_2 \leq 2\} \) and

\[
D(y) = \begin{cases} 
\text{cone}(\text{conv}(\{(0, 1)^\top, (1, 1)^\top\})) \quad \text{for all } y \in A \setminus \{(0, 0)^\top\}, \\
\text{cone}(\text{conv}(\{(-1, 1)^\top, (1, 1)^\top\})) \quad \text{for all } y \in (Y \setminus A) \cup \{(0, 0)^\top\}.
\end{cases}
\]

Hence, every element of \( \text{conv}(\{(0, 0)^\top, (1, -1)^\top\}) \) is a nondominated element of \( A \) w.r.t. \( D \) and by Theorem 3.5 an element of \( \text{ndGHe}(A, D) \). Furthermore, for the set \( M \) defined in (1) we have \( M = A \). Obviously the set \( M \) is convex and by Lemma 3.6(ii) it holds \( \text{ndBe}(A, D) = \text{ndBo}(A, D) \).

For all \( \bar{y} \in \text{conv}(\{(0, 0)^\top, (1, -1)^\top\}) \setminus \{(1, -1)^\top\} \) we obtain

\[
z = (2, -2)^\top \in \{\bar{y}\} + \text{cl}(\text{cone}(M - \{\bar{y}\}))
\]

\[
= \{\bar{y}\} + T(M, \bar{y})
\]

\[
= \{(y_1, y_2)^\top \in \mathbb{R}^2 | y_1 + y_2 \geq 0\}
\]

and \( \bar{y} \in \{z\} + D(z) \).

In contrast, for \( \bar{y} = (1, -1)^\top \) it holds

\[
\{\bar{y}\} + \text{cl}(\text{cone}(M - \{\bar{y}\})) = \{\bar{y}\} + T(M, \bar{y}) = A
\]

and thus \( \text{ndBe}(A, D) = \text{ndBo}(A, D) = \{(1, -1)^\top\} \) by using Lemma 3.3(i).

See Figure 1 for a diagram illustrating the relations between the different proper optimality notions based on their definitions as given in [18].

---

**Figure 1:** Diagram illustrating the results of Lemma 3.6, Lemma 3.7 and Corollary 3.8 for a set \( A \) which is not a singleton. We use the abbreviation w.c.b. for weakly compact base.

Figure 1 shows that — as one would expect — all implications known to hold in the partially ordered case (see [20]) remain true for the proper minimality notions.
For the proper nondominatedness notions, most implications remain true. Only that properly nondominated in the sense of Henig is equivalent to properly nondominated in the sense of Benson in case of cones with weakly compact bases no longer holds in case of variable ordering structures. This is obvious by Theorem 3.5 which states that, under the assumption of weakly compact bases of the cones, Henig proper nondominatedness is equivalent to nondominatedness — and is thus a too weak concept.

Between properly nondominated elements w.r.t. a variable ordering structure $D$ and properly efficient elements in a partially ordered space the following easy-to-proof relations hold:

**Lemma 3.9.**

(i) Let $\bar{y} \in A$ and $K \subseteq Y$ be a nontrivial pointed convex cone.
If $\bar{y} \in effGHe(A, K)/effBe(A, K)/effBo(A, K)$ and if $D(y) \subseteq K$ holds for all $y \in Y$, then $\bar{y} \in ndGHe(A, D)/ndBe(A, D)/ndBo(A, D)$.

(ii) Let $\bar{y} \in A$ and $K \subseteq Y$ be a nontrivial convex cone.
If $\bar{y} \in ndBe(A, D)/ndBo(A, D)$ and if $K \subseteq D(y)$ holds for all $y \in Y$, then $\bar{y} \in effBe(A, K)/effBo(A, K)$.

(iii) Let $\bar{y} \in ndGHe(A, D)$, $\mathcal{K}: Y \rightarrow 2^Y$ be a corresponding cone-valued map with $K(y)$ a convex cone and $D(y) \setminus \{0_Y\} \subseteq int(K(y))$ for all $y \in Y$ such that $\bar{y} \notin \{y\} + K(y)$ for all $y \in \{\bar{y}\}$, $K := \bigcap_{y \in A \setminus \{\bar{y}\}} D(y)$ be a nontrivial convex cone, and $K' := \bigcap_{y \in A \setminus \{\bar{y}\}} int(K(y))$.

If

$$\bigcap_{y \in A \setminus \{\bar{y}\}} int(K(y)) \subseteq int(K'),$$

then $\bar{y} \in effGHe(A, K)$.

A possible modification of the definition of Henig proper nondominatedness which might be more appropriate could be based on the concept of an $\varepsilon$-conic neighborhood as defined in [27, Def. 4.2]: For a positive real number $\varepsilon > 0$ and a nonempty cone $K$ of a normed space $(Y, \| \cdot \|)$, a cone

$$K_{\varepsilon} := cone(\{y \in K \mid \|y\| = 1\} + B_{\varepsilon})$$

(where $B_{\varepsilon} = \{y \in Y \mid \|y\| \leq \varepsilon\}$) is called an $\varepsilon$-conic neighborhood of $K$. So one could require the cones $\mathcal{K}(y)$ in the definition of a properly nondominated element in the sense of Henig to be $\varepsilon$-conic neighborhoods of the cones $D(y)$ for some $\varepsilon > 0$ where $\varepsilon$ has to be the same for all $y \in A \setminus \{\bar{y}\}$.

**4 Scalarization results**

We give in this section necessary and sufficient conditions for (properly) optimal elements w.r.t. a variable ordering structure. For that we need some preliminaries which we shortly recall. We also recall the definitions of two related nonlinear scalarization functionals from the literature. These are defined as the sum of two
nonlinear terms: the first term is defined by linear functionals from the dual cones and the second term is defined by the distance to some chosen element. The nonlinear scalarization functional discussed in this paper will consist of the first term only. Nevertheless we will see that we obtain stronger or comparable results for characterizing (properly) optimal elements under similar or weaker assumptions.

4.1 Preliminaries for the scalarization results

In the following, \((Y^*, \| \cdot \|_*)\) denotes the topological dual space of the normed space \((Y, \| \cdot \|)\) with the induced norm \(\| \cdot \|_*\). To some set \(K \subseteq Y\),

\[ K^* := \{ \ell \in Y^* \mid \ell(y) \geq 0 \text{ for all } y \in K \} \]

denotes the dual cone and

\[ K^\# := \{ \ell \in Y^* \mid \ell(y) > 0 \text{ for all } y \in K \setminus \{0_Y\} \} \]

denotes the quasi-interior of the dual cone. It holds \(\text{int}(K^\#) = \text{int}(K^*)\) (cf. [9]) and thus \(\text{int}(K^*) \subset K^\#\). Conditions for the nonemptyness of \(K^\#\) are for instance given in [25, Theorem 3.38].

We will require \(D(y)^\# \neq \emptyset\) for \(y \in Y\) for some of the presented scalarization results. This assumption is directly related to the fact that the convex cones \(D(y)\) have a base. If \(K^\# \neq \emptyset\) for some convex cone \(K\) then each \(l \in K^\#\) defines a base of \(K\) by \(B_l := \{ y \in K \mid l(y) = 1 \}\) and therefore the cone \(K\) has to be pointed. \(B_l\) is a bounded base of the closed convex cone \(K\) if and only if \(l \in \text{int}(K^\#)\) (see [9, Theorem 3.1] and the references therein). A convex cone \(K\) of a normed space \(Y\) has a convex subset \(B\) with \(0_Y \notin \text{cl}(B)\) and \(K = \text{cone}(B)\) if and only if \(K^\# \neq \emptyset\) (see for instance [9, Remark 2.1]).

If \(K\) is a closed convex cone, then

\[ K = \{ y \in Y \mid \ell(y) \geq 0 \text{ for all } \ell \in K^* \}, \tag{2} \]

and if \(K\) is a convex cone with \(\text{int}(K) \neq \emptyset\), then

\[ \text{int}(K) = \{ y \in Y \mid \ell(y) > 0 \text{ for all } \ell \in K^* \setminus \{0_Y\} \}, \tag{3} \]

see for instance [25, Lemma 3.21].

For some of the forthcoming proofs we need the following separation theorem, see for instance [25, Theorem 3.16, Theorem 3.18, Theorem 3.22]:

**Theorem 4.1.**

(i) Let \(S\) and \(T\) be nonempty convex subsets of the normed space \((Y, \| \cdot \|)\) with \(\text{int}(S) \neq \emptyset\). Then \(\text{int}(S) \cap T = \emptyset\) if and only if there are a continuous linear functional \(l \in Y^* \setminus \{0_Y\}\) and a real number \(\alpha\) with

\[ l(s) \leq \alpha \leq l(t) \text{ for all } s \in S \text{ and all } t \in T \]

and \(l(s) < \alpha\) for all \(s \in \text{int}(S)\).
(ii) Let $S$ be a nonempty closed convex subset of the normed space $(Y, \| \cdot \|)$. Then $y \in Y \setminus S$ if and only if there are a continuous linear functional $l \in Y^* \setminus \{0_Y\}$ and a real number $\alpha$ with $l(y) < \alpha \leq l(s)$ for all $s \in S$.

(iii) Let the topology give $Y$ as the topological dual space of $Y^*$. Moreover, let $S$ and $T$ be closed convex cones in $Y$ with $\text{int}(S^*) \neq \emptyset$. Then $(-S) \cap T = \{0_Y\}$ if and only if there is a continuous linear functional $l \in Y^* \setminus \{0_Y\}$ with $l(x) \leq 0 \leq l(y)$ for all $x \in -S$ and all $y \in T$ and $l(x) < 0$ for all $x \in -S \setminus \{0_Y\}$.

The scalarization functional which we study in the following is related to other nonlinear functionals introduced in the literature for characterizing optimal elements of vector optimization problems with a variable ordering structure. Therefore we shortly recall those functionals. In [17] it was assumed that the images of the cone-valued map $D$ are representable as Bishop-Phelps (BP) cones. This means that there exists a map $\ell: Y \to Y^*$ and for each $y \in Y$ there exists a norm $\| \cdot \|_y$ which is equivalent to the norm of the space such that we can write

$$D(y) := C(\ell(y)) := \{z \in Y \mid \ell(y)(z) \geq \|z\|_y\} \text{ for all } y \in Y.$$ (4)

Based on this representation, to some element $\bar{y} \in Y$ the nonlinear scalarization functional

$$y \mapsto \ell(y)(y - \bar{y}) + \|y - \bar{y}\|_y$$ (5)

was introduced. It allows a complete characterizations of (weakly) nondominated elements. Thereby it holds, due to the representation as BP cones, $\ell(y) \in D(y)^\#$ for all $y \in Y$. We will ignore the term with the norm and consider just the functional $y \mapsto \ell(y)(y - \bar{y})$ in the next section. It will turn out that we can still obtain complete characterizations of (weakly/strongly) nondominated elements.

Later, in [18], a similar functional as in (5) was studied. However, there it is based on elements from the augmented dual cones. For a closed pointed and convex cone $K \subseteq Y$ with $K^\# \neq \emptyset$ the augmented dual cone of $K$ is the set

$$K^{\alpha*} = \{(l, \alpha) \in K^\# \times \mathbb{R}_+ \mid l(y) - \alpha \|y\| \geq 0 \text{ for all } y \in K\}.$$ 

The quasi-interior of the augmented dual cone of $K$ is the set

$$K^{\alpha^*} = \{(l, \alpha) \in K^\# \times \mathbb{R}_+ \mid l(y) - \alpha \|y\| > 0 \text{ for all } y \in K \setminus \{0_Y\}\}.$$ 

There are some relations between the existence of nontrivial elements $((l, \alpha) \neq 0)$ of the augmented dual cone of some cone $K$ and that the cone $K$ is representable as BP cone, cf. [18]. For pairs $(l_y^\#, \alpha_y^\#) \in (D(y))^\#$ for all $y \in Y$ and for some element $\bar{y} \in Y$ the functional

$$y \mapsto l_y^\#(y - \bar{y}) + \alpha_y^\# \|y - \bar{y}\|$$ (6)

was introduced and also allows to give sufficient conditions for nondominated elements. In [18] also various necessary and sufficient conditions for properly optimal elements w.r.t. a variable ordering structure were given.
Under the assumption (4) and assuming that it is possible to choose \( \| \cdot \|_y = \| \cdot \| \) for all \( y \in Y \), for \((l^*_y, \alpha^*_y) = (\ell(y), 1)\) with \( \ell(y) \) from the representation as BP cone, the functionals in (5) and in (6) coincide. We discuss in the following the special case of (6) with \( \alpha^*_y = 0 \) for all \( y \in Y \). Hence we ignore the term with the norm. We derive also characterizations of properly optimal elements under the same or weaker assumptions. For the proofs we will not need assumptions on separation properties of \( \varepsilon \)-conic neighborhoods as required in [18]. This is due to the fact that we obtain the results based on linear separation results (see Theorem 4.1) and not on conic separation theorems as in [18].

### 4.2 Characterizing nondominated elements

Let a map \( \ell : Y \to Y^* \) and an element \( \bar{y} \in Y \) be given. We consider the functional \( \varphi_{\bar{y}} : Y \to \mathbb{R} \) defined by

\[
\varphi_{\bar{y}}(y) := \ell(y)(y - \bar{y}) \quad \text{for all } y \in Y.
\]

Obviously it holds \( \varphi_{\bar{y}}(\bar{y}) = 0 \). Note that while \( \ell(y) \) is a continuous linear functional for each \( y \), the functional \( \varphi_{\bar{y}} \) is in general nonlinear. This is even the case if \( \ell : Y \to Y^* \) is a linear map, as the following example shows:

**Example 3.** Let \( Y \) be the Euclidean space \( \mathbb{R}^m \) and let \( \ell : \mathbb{R}^m \to \mathbb{R}^m \) be defined by \( \ell(y) := My \) for all \( y \in \mathbb{R}^m \) with \( M := \text{diag}(1, 2, \ldots, m) \). Thus \( \ell \) is linear and each \( \ell(y) \) defines a linear map by \( z \mapsto (\ell(y))^\top z = y^\top M z \) for all \( z \in \mathbb{R}^m \). Nevertheless, \( \varphi_{\bar{y}}(y) = y^\top M(y - \bar{y}) = \sum_{i=1}^m (iy_i^2 - iy_i \bar{y}_i) \) is not a linear map.

As the functional \( \varphi_{\bar{y}} \) equals the first term of the functional studied in [17], its properties are already studied there.

The following theorem gives sufficient criteria for nondominated elements without additional assumptions as well as complete characterizations of (weakly/strongly) nondominated elements. Some of the complete characterizations require additional assumptions.

**Theorem 4.2.** Let \( \bar{y} \in A \). Then the following holds:

(i) Let \( \ell : Y \to Y^* \) be a map with \( \ell(y) \in D(y)^* \setminus \{0_{Y^*}\} \) for all \( y \in A \setminus \{\bar{y}\} \). If

\[
\ell(y)(y - \bar{y}) > 0 \quad \text{for all } y \in A \setminus \{\bar{y}\},
\]

then \( \bar{y} \) is a nondominated element of \( A \) w.r.t. \( D \).

(ii) Let additionally \( D(y) \) be closed for all \( y \in A \setminus \{\bar{y}\} \). Then \( \bar{y} \) is a nondominated element of \( A \) w.r.t. \( D \) if and only if there is a map \( \ell : Y \to Y^* \) with \( \ell(y) \in D(y)^* \setminus \{0_{Y^*}\} \) for all \( y \in A \setminus \{\bar{y}\} \) such that \( \ell(y)(y - \bar{y}) > 0 \) for all \( y \in A \setminus \{\bar{y}\} \).

(iii) Let \( \ell : Y \to Y^* \) be a map with \( \ell(y) \in D(y)^\# \) for all \( y \in A \setminus \{\bar{y}\} \). If

\[
\ell(y)(y - \bar{y}) \geq 0 \quad \text{for all } y \in A,
\]

then \( \bar{y} \) is a nondominated element of \( A \) w.r.t. \( D \).
(iv) Let additionally the topology give \( Y \) as the topological dual space of \( Y^* \), \( D(y) \) be closed and \( \text{int}(D(y)^*) \neq \emptyset \) for all \( y \in A \setminus \{\bar{y}\} \). Then \( \bar{y} \) is a nondominated element of \( A \) w.r.t. \( D \) if and only if there is a map \( \ell : Y \to Y^* \) with \( \ell(y) \in D(y)^* = \text{int}(D(y)^*) \) for all \( y \in A \setminus \{\bar{y}\} \) such that \( \ell(y)(y - \bar{y}) \geq 0 \) for all \( y \in A \).

(v) Let additionally \( \text{int}(D(y)) \neq \emptyset \) for all \( y \in A \). Then \( \bar{y} \) is a weakly nondominated element of \( A \) w.r.t. \( D \) if and only if there is a map \( \ell : Y \to Y^* \) with \( \ell(y) \in D(y)^* \setminus \{0_{Y^*}\} \) for all \( y \in A \setminus \{\bar{y}\} \) such that \( \ell(y)(y - \bar{y}) \geq 0 \) for all \( y \in A \).

(vi) If \( D(y) \) is additionally closed for all \( y \in A \), then \( \bar{y} \) is a strongly nondominated element of \( A \) w.r.t. \( D \) if and only if \( \ell(y)(y - \bar{y}) \geq 0 \) for all \( y \in A \) holds for every map \( \ell : Y \to Y^* \) with \( \ell(y) \in D(y)^* \) for all \( y \in A \setminus \{\bar{y}\} \).

Proof.

(i) Let \( \ell : Y \to Y^* \) be a map with \( \ell(y) \in D(y)^* \setminus \{0_{Y^*}\} \) and \( \ell(y)(y - \bar{y}) > 0 \) for all \( y \in A \setminus \{\bar{y}\} \). Assume to the contrary that \( \bar{y} \) is not a nondominated element of \( A \) w.r.t. \( D \). Then there exists \( \hat{y} \in A \setminus \{\bar{y}\} \) with \( \bar{y} - \hat{y} \in D(\hat{y}) \setminus \{0_{Y^*}\} \). Since \( \ell(\hat{y}) \in D(\hat{y})^* \setminus \{0_{Y^*}\} \) we obtain \( \ell(\hat{y})(\hat{y} - \bar{y}) \leq 0 \) — being a contradiction to \( \ell(y)(y - \bar{y}) > 0 \) for all \( y \in A \setminus \{\bar{y}\} \).

(ii) The sufficiency follows by (i). It remains to show the necessity of the condition. Let \( \bar{y} \) be a nondominated element of \( A \) w.r.t. \( D \) and \( y \in A \setminus \{\bar{y}\} \) arbitrarily chosen. Then it holds \( \bar{y} - y \notin D(y) \). By Theorem 4.1(ii) there exist \( l_y \in Y^* \setminus \{0_{Y^*}\} \) and \( \alpha \in \mathbb{R} \) such that

\[
l_y(\bar{y} - y) < \alpha \leq l_y(k) \quad \text{for all} \quad k \in D(y). \tag{8}
\]

Next we show that \( l_y \notin D(y)^* \setminus \{0_{Y^*}\} \). For this assume that \( l_y \notin D(y)^* \setminus \{0_{Y^*}\} \) which means that there exists \( d \in D(y) \setminus \{0_Y\} \) with \( l_y(d) < 0 \). Since \( D(y) \) is a cone it follows \( \lambda d \in D(y) \) for all \( \lambda \geq 0 \) and \( \lim_{\lambda \to \infty} l_y(\lambda d) = -\infty \), in contradiction to (8). As \( 0_Y \in D(y) \) we obtain from (8) \( l_y(\bar{y} - y) < 0 \). By setting \( \ell(y) := l_y \) for all \( y \in A \setminus \{\bar{y}\} \) we obtain a map \( \ell : Y \to Y^* \) with \( \ell(y) \in D(y)^* \setminus \{0_{Y^*}\} \) for all \( y \in A \setminus \{\bar{y}\} \) and \( \ell(y)(y - \bar{y}) > 0 \) for all \( y \in A \setminus \{\bar{y}\} \).

(iii) follows by similar arguments as in the proof of (i).

(iv) The sufficiency follows by (iii) and the necessity of the condition follows by similar arguments as in the proof of (ii) by applying Theorem 4.1(iii) on \( \text{cl}(\text{cone}(\{y - \bar{y}\})) \cap (-D(y)) = \{0_Y\} \) for arbitrarily chosen \( y \in A \setminus \{\bar{y}\} \) and by [25, Lemma 3.21(d)].

(v) The sufficiency follows by similar arguments as in the proof of (i) and by using (3). The necessity follows similarly as in the proof of (ii) by applying Theorem 4.1(i) on \( \{\bar{y}\} \cap \{y + \text{int}(D(y))\} = \emptyset \) for arbitrarily chosen \( y \in A \setminus \{\bar{y}\} \).

(vi) If \( \bar{y} \in A \) is a strongly nondominated element of \( A \) w.r.t. \( D \), then \( y - \bar{y} \in D(y) \) for all \( y \in A \setminus \{\bar{y}\} \). Hence, for every \( \ell : Y \to Y^* \) with \( \ell(y) \in D(y)^* \) for all \( y \in A \setminus \{\bar{y}\} \) we obtain \( \ell(y)(y - \bar{y}) \geq 0 \) for all \( y \in A \setminus \{\bar{y}\} \) and thus for all \( y \in A \).

If \( \ell(y)(y - \bar{y}) \geq 0 \) for all \( y \in A \) holds for every \( \ell : Y \to Y^* \) with \( \ell(y) \in D(y)^* \) for all \( y \in A \setminus \{\bar{y}\} \), then by (2) we obtain \( y - \bar{y} \in D(y) \) for all \( y \in A \setminus \{\bar{y}\} \) and we are done.

The above characterizations improve the results of [18, Theorem 4].

Remark 4.3. Note that by replacing \( y \in A \) in Theorem 4.2(i) – (iv) by

\[
y \in \{\bar{y}\} + \text{cl}(\text{cone}(\bigcup_{a \in A} \{a + D(a)\} - \{\bar{y}\})) \text{ or } y \in \{\bar{y}\} + T(\bigcup_{a \in A} \{a + D(a)\}, \bar{y})
\]

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we obtain sufficient criteria and complete characterizations of properly nondominated elements in the sense of Benson or in the sense of Borwein, respectively.

The following theorem gives additional necessary and sufficient conditions for properly nondominated elements. Some of them are based on the relation between properly nondominated and properly efficient elements as given in Lemma 3.9. Note that by Lemma 3.6(i) every sufficient condition for a properly nondominated element in the sense of Benson is also a sufficient condition for a properly nondominated element in the sense of Borwein, and every necessary condition for a properly nondominated element in the sense of Borwein is also a necessary condition for a properly nondominated element in the sense of Benson. Moreover, we get in case \( D(y) \) has a weakly compact base for all \( y \in A \) a complete characterization of properly nondominated elements in the sense of Henig. By using Theorem 3.5 this delivers another complete characterization of nondominated elements.

**Theorem 4.4.** Let \( \bar{y} \in A \). Then the following holds:

(i) Let \( \ell: Y \to Y^* \) be a map with \( \ell(y) \in D(y)^\# \) for all \( y \in A \setminus \{ \bar{y} \} \). If

\[
\ell(y)(y - \bar{y}) > 0 \quad \text{for all } y \in A \setminus \{ \bar{y} \},
\]

then \( \bar{y} \in \text{ndGHe}(A, D) \).

(ii) Let additionally \( D(y) \) have a weakly compact base for all \( y \in A \setminus \{ \bar{y} \} \). Then \( \bar{y} \in \text{ndGHe}(A, D) \) if and only if there is a map \( \ell: Y \to Y^* \) with \( \ell(y) \in D(y)^\# \) for all \( y \in A \setminus \{ \bar{y} \} \) such that \( \ell(y)(y - \bar{y}) > 0 \) for all \( y \in A \setminus \{ \bar{y} \} \).

(iii) Let additionally \( D(y) \) give \( Y^* \) as the topological dual space of \( Y^* \), the pointed convex cone \( K := \bigcap_{y \in Y} D(y) \) be closed with \( \text{int}(K^*) \neq \emptyset \), and \( A \) be convex. If \( \bar{y} \in \text{ndBo}(A, D) \) then there exists a map \( \bar{\ell} \in K^\# \) with \( \bar{\ell}(y - \bar{y}) \geq 0 \) for all \( y \in A \), i.e. \( \ell(y)(y - \bar{y}) \geq 0 \) holds for a map \( \ell: Y \to Y^* \) with \( \ell(y) := \bar{\ell} \in K^\# \) for all \( y \in A \).

(iv) Let \( D(Y) = \bigcup_{y \in Y} D(y) \) be a pointed convex cone. If there exists a map \( \bar{\ell} \in D(Y)^\# \) with \( \bar{\ell}(y - \bar{y}) \geq 0 \) for all \( y \in Y \), i.e. \( \ell(y)(y - \bar{y}) \geq 0 \) holds for a map \( \ell: Y \to Y^* \) with \( \ell(y) := \bar{\ell} \in D(Y)^\# \) for all \( y \in A \), then \( \bar{y} \in \text{ndBe}(A, D) \).

**Proof.**

(i) Let \( \ell: Y \to Y^* \) be a map with \( \ell(y) \in D(y)^\# \subset D(y)^\# \setminus \{ 0_Y \} \) and \( \ell(y)(y - \bar{y}) > 0 \) for all \( y \in A \setminus \{ \bar{y} \} \) and let \( y \in A \setminus \{ \bar{y} \} \) be arbitrarily chosen. We define the set

\[
K(y) := \{ z \in Y \mid \ell(y)(z) \geq 0 \}.
\]

Obviously \( K(y) \) is a closed convex cone. By the assumptions it follows \( \ell(y)(w) > 0 \) for all \( w \in D(y) \setminus \{ 0_Y \} \) and \( \ell(y)(\bar{y} - y) < 0 \). As \( \text{int}(K(y)) = \{ z \in Y \mid \ell(y)(z) > 0 \} \) we obtain \( D(y) \setminus \{ 0_Y \} \subset \text{int}(K(y)) \) and also \( \bar{y} \notin \{ y \} + K(y) \). By setting \( K(y) := K(y) \) for all \( y \in A \setminus \{ \bar{y} \} \) and \( K(y) := Y \) for all \( y \in (Y \setminus A) \cup \{ \bar{y} \} \) the assertion follows.

(ii) The sufficiency follows by (i). Let \( \bar{y} \in \text{ndGHe}(A, D) \). By definition there is a cone-valued map \( K: Y \to 2^{Y} \) with \( K(y) \) a convex cone and \( D(y) \setminus \{ 0_Y \} \subset \text{int}(K(y)) \) for all \( y \in Y \) such that \( \bar{y} - y \notin K(y) \) for all \( y \in A \setminus \{ \bar{y} \} \). Let \( y \in A \setminus \{ \bar{y} \} \) be arbitrarily
chosen. Using the same arguments as in the proofs of Theorem 3.5 and Theorem 4.2(ii) we may assume the cone $\mathcal{K}(y)$ to be closed and we obtain $l_y \in \mathcal{K}(y)^* \setminus \{0_y\}$ and $l_y(y - \bar{y}) > 0$. By (3) it holds

$$\mathcal{D}(y) \setminus \{0_y\} \subseteq \text{int}(\mathcal{K}(y)) = \{z \in Y \mid l(z) > 0 \text{ for all } l \in \mathcal{K}(y)^* \setminus \{0_y\}\}$$

and hence $l_y \in \mathcal{D}(y)^\#$. By setting $\ell(y) := l_y$ for all $y \in A \setminus \{\bar{y}\}$ we obtain a map $\ell: Y \to Y^*$ with $\ell(y) \in \mathcal{D}(y)^\#$ for all $y \in A \setminus \{\bar{y}\}$ and $\ell(y)(y - \bar{y}) > 0$ for all $y \in A \setminus \{\bar{y}\}$.

(iii) follows by Lemma 3.9(ii) and by applying a linear scalarization result for partially ordered spaces, see for instance [25, Theorem 5.11].

(iv) follows by Lemma 3.9(i) and by applying a linear scalarization result for partially ordered spaces. Such a result is for instance given in [25, Theorem 5.21] for properly efficient elements in the sense of Borwein, but can be proven analogously for proper efficiency in the sense of Benson. \[\square\]

**Remark 4.5.** According to the proof of part (ii) of Theorem 4.4 we can also formulate another necessary condition: If $\bar{y} \in \text{ndGHe}(A, \mathcal{D})$ and if the convex cones $\mathcal{K}(y)$ in the definition can be chosen to be closed, then there exists a map $\ell: Y \to Y^*$ with $\ell(y) \in \mathcal{D}(y)^\#$ for all $y \in A \setminus \{\bar{y}\}$ such that $\ell(y)(y - \bar{y}) > 0$ for all $y \in A \setminus \{\bar{y}\}$. One could modify the definition of a properly nondominated element in the sense of Henig by requiring that the cones $\mathcal{K}(y)$ in the definition have to be closed convex cones. In case of such a modification we thus obtain a complete characterization of properly nondominated elements in the sense of Henig without additional assumptions on the map $\mathcal{D}$.

Theorem 4.4(i) is also a direct consequence of [18, Theorem 8] by choosing $(\ell^y, \alpha^y) := (\ell(y), 0)$ for all $y \in Y$ as elements of the augmented dual cones of the cones $\mathcal{D}(y)$ in the definition of the scalarization functional used there. For completeness and because the direct proof is more elementary, we nevertheless provided the proof above.

Theorem 4.4(ii) delivers stronger necessary conditions as those proposed in [18, Theorem 10]. Here, we need no separation property to be satisfied or the linear space $Y$ to be reflexive (but instead the cones to have weakly compact bases). The necessary condition proposed in [18, Theorem 11] (there again under the assumption of a separation property which has to hold for the cones $\mathcal{D}(y)$ and their $\varepsilon$-conic neighborhoods) is closely related to our Remark 4.3.

Theorem 4.4(iv) is related to [18, Theorem 9] by choosing $\alpha^y = 0$. Note that $\bar{\ell}(y - \bar{y}) \geq 0$ for all $y \in A$ is equivalent to that $\bar{y}$ is a minimal solution of the scalar-valued optimization problem $\min_{y \in A} \bar{\ell}(y)$. This problem was considered in [13, Theorem 3.5] for some $\bar{\ell} \in \mathcal{D}(A)^\#$. There it was shown that a minimal solution of this problem is a nondominated element of $A$ w.r.t. $\mathcal{D}$. We showed that in case $\bar{\ell} \in \mathcal{D}(Y)^\#$ a minimal solution is even a properly nondominated element in the sense of Benson of $A$ w.r.t. $\mathcal{D}$.

**Example 4.** We apply the results of Theorem 4.2 and Theorem 4.4 to Example 2. Let the cones $K$ and $\mathcal{D}(Y)$ be defined as in Theorem 4.4(iii) and (iv), respectively.
Obviously it holds

\[
K = \text{cone} (\text{conv} (\{(0, 1)^T, (1, 1)^T\})),
\]

\[
K^* = \text{cone} (\text{conv} (\{(-1, 1)^T, (1, 0)^T\})),
\]

\[
D(Y) = \text{cone} (\text{conv} (\{(-1, 1)^T, (1, 1)^T\})),
\]

\[
D(Y)^* = D(Y),
\]

and

\[
D(y)^* = \begin{cases} 
K^* & \text{for all } y \in A \setminus \{(0, 0)^T\}, \\
D(Y)^* & \text{for all } y \in (Y \setminus A) \cup \{(0, 0)^T\}.
\end{cases}
\]

Moreover, \(K^\# = \text{int}(K^*), D(Y)^\# = \text{int}(D(Y)^*), \) and \(D(y)^\# = \text{int}(D(y)^*)\) for all \(y \in Y\).

In the following, by a case-by-case analysis ((a)-(e)), we determine all (weakly/ properly) nondominated elements of the set \(A\) w.r.t. \(D\).

(a) Let \(\bar{y} \in \text{int}(A)\). If \(\bar{y} \in \text{int}(\text{cone}(\text{conv} (\{(-1, 1)^T, (1, 1)^T\}))) = \text{int}(D(Y))\), then take \(y = (0, 0)^T\). It holds \(y \in A \setminus \{\bar{y}\}\) and \(\ell^T (y - \bar{y}) = -\ell^T \bar{y} < 0\) for all \(\ell \in D(y)^* \setminus \{0_{Y^*}\} = D(Y)^* \setminus \{0_{Y^*}\}\) (cf. (3)). Otherwise, it holds \(\bar{y} \in \{(a, -a)^T\} + \text{int}(K)\) for some \(a \in [0, 1]\). Then take \(y = (a, -a)^T\). Again, it holds \(y \in A \setminus \{\bar{y}\}\) and \(\ell^T (y - \bar{y}) < 0\) for all \(\ell \in D(y)^* \setminus \{0_{Y^*}\} = K^\# \setminus \{0_{Y^*}\}\). Thus for all \(\bar{y} \in \text{int}(A)\) the necessary and sufficient condition of Theorem 4.2(v) for a weakly nondominated element of \(A\) w.r.t. \(D\) is not fulfilled.

(b) Next we consider \(\bar{y} \in \partial A := A \setminus \text{int}(A)\). It holds \(\partial A = A_1 \cup A_2\) with \(A_1 = \{(y_1, y_2)^T \in \mathbb{R}^2 | y_1 + y_2 = 0, y_1 \leq 1\}\) and \(A_2 = \{(y_1, y_2)^T \in \mathbb{R}^2 | y_1 - y_2 = 2, y_1 \geq 1\}\). Let \(\bar{y} \in A_1\). We set \(\ell(y) := (1, 1)^T\) for all \(y \in A \setminus \{\bar{y}\}\). Then \(\ell(y)^T \bar{y} = 0\) and \(\ell(y)^T y \geq 0\) for all \(y \in A\). Next, let \(\bar{y} \in A_2\). Then we set \(\ell(y) := (-1, 1)^T\) for all \(y \in A \setminus \{\bar{y}\}\). Then \(\ell(y)^T \bar{y} = -2\) and \(\ell(y)^T y \geq -2\) for all \(y \in A\). It follows in both cases \(\ell(y) \in D(y)^* \setminus \{0_{Y^*}\}\) for all \(y \in A \setminus \{\bar{y}\}\) and \(\ell(y)^T (y - \bar{y}) \geq 0\) for all \(y \in A\). Hence every element \(\bar{y} \in \partial A\) is by Theorem 4.2(v) a weakly nondominated element of \(A\) w.r.t. \(D\).

(c) For all \(\bar{y} \in A_1 \setminus \text{conv} (\{(0, 0)^T, (1, -1)^T\}) \subseteq D(Y) \setminus \{(0, 0)^T\}\) choose \(y = (0, 0)^T\) and for all \(\bar{y} \in A_2 \setminus \{(1, -1)^T\} \subseteq \{(1, -1)^T\} + K \setminus \{(0, 0)^T\}\) choose \(y = (1, -1)^T\). Then it holds \(y \in A \setminus \{\bar{y}\}\) and, using similar arguments as in (a), \(\ell^T (y - \bar{y}) \leq 0\) for all \(\ell \in D(y)^* \setminus \{0_{Y^*}\}\), and \(\ell^T (y - \bar{y}) < 0\) for all \(\ell \in D(y)^\#\), respectively. Thus for all \(\bar{y} \in \partial A \setminus \text{conv} (\{(0, 0)^T, (1, -1)^T\})\) the necessary and sufficient conditions of Theorem 4.2(ii), of Theorem 4.2(iv), and of Theorem 4.4(ii) are not fulfilled.

(d) Now let \(\bar{y} \in \text{conv} (\{(0, 0)^T, (1, -1)^T\}) \setminus \{(1, -1)^T\}\), i.e. \(\bar{y} = (a, -a)^T\) with \(a \in [0, 1]\). We can define a map \(\ell: Y \to Y^*\) on the set \(A\) for instance by

\[
\ell(y) := \begin{cases} 
(0, 1)^T & \text{for all } y \in A \text{ with } y_1 \leq a, \\
(2, 1)^T & \text{for all } y \in A \text{ with } y_1 > a.
\end{cases}
\]

Then \(\bar{y} \in A\) and it holds \(\ell(y) \in D(y)^\#\) for all \(y \in A\). By an easy calculation which uses that for any \(y \in A\) we have \(y_1 + y_2 \geq 0\) and that \(y_2 > -a\) holds for any \(y \in A \setminus \{\bar{y}\}\) with \(y_1 \leq a\) we obtain \(\ell(y)^T (y - \bar{y}) > 0\) for all \(y \in A \setminus \{\bar{y}\}\).
Hence, all the necessary and sufficient conditions of Theorem 4.2(ii) and (iv) and of Theorem 4.4(ii) are fulfilled and \( y \) is a nondominated element of \( A \) w.r.t. \( D \) and an element of \( \text{ndGHe}(A, D) \). Moreover, for \( \bar{\ell} = (1, 1)^\top \) it holds \( \bar{\ell} \in K^\# \) and \( \bar{\ell}^\top (y - \bar{y}) \geq 0 \) for all \( y \in A \) and the necessary condition of Theorem 4.4(iii) for a nondominated element in the sense of Borwein (and thus of Benson) is fulfilled for \( y \). However, the sufficient condition of Theorem 4.4(iv) for \( y \in \text{ndBe}(A, D) \) is not fulfilled for any \( y \). Let \( \bar{\ell} \in \mathcal{D}(Y)^\# \) be arbitrarily chosen. Then \( \bar{\ell}_1 - \bar{\ell}_2 < 0 \). For \( y = (1, -1)^\top \in A \setminus \{\bar{y}\} \) it holds \( \bar{\ell}^\top (y - \bar{y}) = (1-a)(\bar{\ell}_1 - \bar{\ell}_2) < 0 \). In fact, we have \( \bar{y} \notin \text{ndBo}(A, D) = \text{ndBe}(A, D) \) (see Example 2).

(e) Finally, let \( \bar{y} = \{(1, -1)^\top\} \). For \( \bar{\ell} = (0, 1)^\top \) and for a map \( \ell : Y \to Y^* \) with \( \ell(y) = \bar{\ell} \) for all \( y \in A \setminus \{\bar{y}\} \) it holds \( \ell(y) \in \mathcal{D}(y)^\# \) for all \( y \in A \setminus \{\bar{y}\} \), \( \bar{\ell} \in K^\# \), and \( \bar{\ell} \in \mathcal{D}(Y)^\# \). Since \( y_2 > -1 \) holds for any \( y \in A \setminus \{\bar{y}\} \) it follows \( \bar{\ell}^\top (y - \bar{y}) = \ell(y)^\top (y - \bar{y}) > 0 \) for all \( y \in A \setminus \{\bar{y}\} \). Hence, all the necessary and sufficient conditions of Theorem 4.2(ii) and (iv) and of Theorem 4.4(ii), the necessary condition of Theorem 4.4(iii), and the sufficient condition of Theorem 4.4(iv) are fulfilled and \( \bar{y} \) is a nondominated element of \( A \) w.r.t. \( D \) and an element of \( \text{ndGHe}(A, D)/\text{ndBe}(A, D)/\text{ndBo}(A, D) \).

### 4.3 Characterizing minimal elements

For characterizing minimal elements w.r.t. a variable ordering structure under convexity assumptions it is enough to consider linear scalarization functionals as the one defined in (9). We give such results for completeness in Lemma 4.6. However, one can also characterize minimal elements w.r.t. a variable ordering structure using the nonlinear functional defined in (7), see the forthcoming Lemma 4.7.

As before let a map \( \ell : Y \to Y^* \) and an element \( \bar{y} \in Y \) be given. We study additionally the functional \( \psi_{\bar{y}} : Y \to \mathbb{R} \) defined by

\[
\psi_{\bar{y}}(y) = \ell(\bar{y})(y - \bar{y}) \quad \text{for all } y \in Y.
\]

Obviously \( \psi_{\bar{y}}(\bar{y}) = 0 \). As in the definition of \( \psi_{\bar{y}} \) the linear functional \( l := \ell(\bar{y}) \) does not depend on \( y \), the functional can also be written as \( \psi_{\bar{y}}(y) = l(\bar{y}) - l(y) \).

We stick here with the structure which is more similar to the definition of \( \varphi_{\bar{y}} \), and, what is more, resembles that of the functional used in [18]. It is well known, as a direct consequence of the definitions, that \( \bar{y} \) is a (weakly/strongly/properly) minimal element of \( A \) w.r.t. \( D \) if and only if it is a (weakly/strongly/properly) efficient element of \( A \) w.r.t. the partial ordering defined by \( K := \mathcal{D}(\bar{y}) \).

Therefore, linear scalarization results known to hold in the partially ordered case can directly be applied to the notion of minimality w.r.t. a variable ordering structure. See the next lemma:

**Lemma 4.6.** Let \( \bar{y} \in A \) and the set \( M_{\bar{y}} \) be defined as in (1). Then the following holds:

(i) Let additionally \( M_{\bar{y}} \) be convex and \( \text{int}(M_{\bar{y}}) \neq \emptyset \). If \( \bar{y} \) is a minimal element of \( A \) w.r.t. \( D \), then there is a map \( \ell : Y \to Y^* \) with \( \ell(\bar{y}) \in \mathcal{D}(\bar{y})^* \setminus \{0_Y\} \) such that \( \ell(\bar{y})(y - \bar{y}) \geq 0 \) for all \( y \in A \).
(ii) Let additionally $D(\bar{y})$ be closed. Then $\bar{y} \in A$ is a strongly minimal element of $A$ w.r.t. $D$ if and only if $\ell(\bar{y})(y - \bar{y}) \geq 0$ for all $y \in A \setminus \{\bar{y}\}$ holds for every map $\ell: Y \to Y^*$ with $\ell(\bar{y}) \in D(\bar{y})^*$.

(iii) If there exists a map $\ell: Y \to Y^*$ with $\ell(\bar{y}) \in D(\bar{y})^*$ such that $\ell(\bar{y})(y - \bar{y}) \geq 0$ for all $y \in A$, then $\bar{y} \in mBe(A, D)$ and $\bar{y} \in mBo(A, D)$.

(iv) If additionally the topology gives $Y$ as the topological dual space of $Y^*$, $D(\bar{y})$ is closed with $\text{int}(D(\bar{y})^*) \neq \emptyset$, and $M_\bar{y}$ is convex, then $\bar{y} \in mBo(A, D)$ if and only if there is a map $\ell: Y \to Y^*$ with $\ell(\bar{y}) \in D(\bar{y})^*$ such that $\ell(\bar{y})(y - \bar{y}) \geq 0$ for all $y \in A \setminus \{\bar{y}\}$.

More results in this direction for (weakly) minimal elements are given in [13, Sect. 3.1]. Based on Lemma 3.7(i), Lemma 4.6(iv) also gives a necessary condition for $\bar{y} \in mGHe(A, D)$ in case the set $A$ contains more than one element. In case $D(\bar{y})$ has a weakly compact base, then Lemma 4.6(iii) and (iv) also deliver sufficient conditions for $\bar{y} \in mGHe(A, D)$ based on Lemma 3.7(ii). Lemma 4.6(iii) is also a direct consequence of [18, Theorem 5] by choosing $(\ell^#, \alpha^y) := (\ell(\bar{y}), 0)$ as element of the augmented dual cone of the cone $D(\bar{y})$ in the definition of the scalarization functional used there. Lemma 4.6(iii), used for characterizing Henig properly minimal elements under the assumption that $D(\bar{y})$ has a weakly compact base, gives the same sufficient condition as in [18, Theorem 6(iii)] by choosing $(\ell^#, \alpha^y) := (\ell(\bar{y}), 0)$. However, in [18], the assumption of a reflexive space as well as that the cones $D(\bar{y})$ have e-conic neighborhoods with which the cones satisfy a separation property are needed. The necessary conditions for properly minimal elements w.r.t. a variable ordering structure in the sense of Benson/Borwein/Henig presented in [18, Theorem 6(i),(ii) and Theorem 7] do not require convexity of the set $M_\bar{y}$ as assumed here in Lemma 4.6(iv).

The main drawback of the results of Lemma 4.6 is the required convexity for the necessary conditions. When using the functional $\varphi_{\bar{y}}$ we get the following complete characterizations of (weakly) minimal elements, where we do not need the convexity of the set $M_\bar{y}$. This is possible as we replace the linear scalarization functional by a nonlinear one. We omit the proofs as the conclusions follow from [25, Theorem 5.5] or can be proven using the same ideas as in the proofs of Theorem 3.5 and of Theorem 4.2.

**Lemma 4.7.** Let $\bar{y} \in A$. Then the following holds:

(i) Let additionally $D(\bar{y})$ be closed. Then $\bar{y}$ is a minimal element of $A$ w.r.t. $D$ if and only if there is a map $\ell: Y \to Y^*$ with $\ell(y) \in D(\bar{y})^* \setminus \{0_Y, \} \forall y \in A \setminus \{\bar{y}\}$ for all $y \in A \setminus \{\bar{y}\}$ such that $\ell(y)(y - \bar{y}) > 0$ for all $y \in A \setminus \{\bar{y}\}$.

(ii) Let additionally the topology give $Y$ as the topological dual space of $Y^*$, $D(\bar{y})$ be closed, and $\text{int}(D(\bar{y})^*) \neq \emptyset$. Then $\bar{y}$ is a minimal element of $A$ w.r.t. $D$ if and only if there is a map $\ell: Y \to Y^*$ with $\ell(y) \in D(\bar{y})^* = \text{int}(D(\bar{y})^*)$ for all $y \in A \setminus \{\bar{y}\}$ such that $\ell(y)(y - \bar{y}) \geq 0$ for all $y \in A$.

(iii) If $\bar{y}$ is a minimal element of $A$ w.r.t. $D$ and if additionally $D(\bar{y})$ has a weakly compact base, then there exists a map $\ell: Y \to Y^*$ with $\ell(y) \in D(\bar{y})^*$ for all $y \in A \setminus \{\bar{y}\}$ such that $\ell(y)(y - \bar{y}) > 0$ for all $y \in A \setminus \{\bar{y}\}$.
(iv) If additionally \( \text{int}(\mathcal{D}(\bar{y})) \neq \emptyset \), then \( \bar{y} \) is a weakly minimal element of \( A \) w.r.t. \( \mathcal{D} \) if and only if there is a map \( \ell: Y \to Y^* \) with \( \ell(y) \in \mathcal{D}(\bar{y})^* \setminus \{0_Y^*\} \) for all \( y \in A \setminus \{\bar{y}\} \) such that \( \ell(y)(y - \bar{y}) \geq 0 \) for all \( y \in A \).

Note that for the sufficient part of (ii) we do not need the additional assumptions on \( Y, \mathcal{D}(\bar{y}) \) and \( \text{int}(\mathcal{D}(\bar{y})^*) \). The above complete characterizations improve the results of [18, Theorem 3]. Moreover, note that in Theorem 4.2 we assume \( \ell \) to be a map with \( \ell(y) \in \mathcal{D}(\bar{y})^* \setminus \{0_Y^*\} \) and \( \ell(y) \in \mathcal{D}(\bar{y})^\# \) for all \( y \in A \setminus \{\bar{y}\} \) while in the lemma above we assume \( \ell(y) \in \mathcal{D}(\bar{y})^* \setminus \{0_Y^*\} \) and \( \ell(y) \in \mathcal{D}(\bar{y})^\# \) for all \( y \in A \setminus \{\bar{y}\} \), respectively. If \( \mathcal{D}(\bar{y}) \subseteq \mathcal{D}(y) \) holds for a nondominated element \( \bar{y} \) w.r.t. \( \mathcal{D} \) and all \( y \in A \setminus \{\bar{y}\} \), and thus \( \mathcal{D}(y)^* \subseteq \mathcal{D}(\bar{y})^* \), then it is known that \( \bar{y} \) is also a minimal element of \( A \) w.r.t. \( \mathcal{D} \), cf. [13, Remark 2.1]. Analogously for minimal elements in case \( \mathcal{D}(y) \subseteq \mathcal{D}(\bar{y}) \) for all \( y \in A \setminus \{\bar{y}\} \).

In the proof of Lemma 4.7(iii) one uses that in case \( \mathcal{D}(\bar{y}) \) has a weakly compact base then one can find, by Lemma 3.4, for arbitrary \( y \in A \setminus \{\bar{y}\} \) a closed pointed convex cone \( K \) such that \( \bar{y} - y \notin K \). In case we have \( \bar{y} \in m\text{GHe}(A, \mathcal{D}) \) and the convex cone \( K \) mentioned in Lemma 3.1 can be chosen to be closed, then we thus also obtain the following necessary condition:

**Lemma 4.8.** If \( \bar{y} \in m\text{GHe}(A, \mathcal{D}) \) and the convex cone \( K \) in Lemma 3.1 can be chosen to be closed then there exists a map \( \ell: Y \to Y^* \) with \( \ell(y) \in \mathcal{D}(\bar{y})^\# \) for all \( y \in A \setminus \{\bar{y}\} \) such that \( \ell(y)(y - \bar{y}) > 0 \) for all \( y \in A \setminus \{\bar{y}\} \).

Finally we want to note that we obtain also complete characterizations of properly minimal elements in the sense of Benson or in the sense of Borwein, if we replace \( y \in A \) in Lemma 4.7(i) – (ii) by \( y \in \{\bar{y}\} + \text{cl}(\text{cone}(A + \mathcal{D}(\bar{y}) - \{\bar{y}\})) \) or \( y \in \{\bar{y}\} + T(A + \mathcal{D}(\bar{y}), \bar{y}) \), respectively.

### 5 Conclusions

The focus of this paper was on the one hand on those notions of proper optimality which were introduced in the literature on variable ordering structures so far, and on their relations. The main results of this part are summed up in the scheme in Figure 1. Based on Lemma 3.3 we also slightly simplified the original definitions for the proper optimality notions from [18]. As shown in Example 2, Henig proper nondominatedness is different to Benson proper nondominatedness even in case the cones have a weakly compact base. This is due to the fact that in case the cones have a weakly compact base Henig proper nondominatedness is equivalent to nondominatedness.

On the other hand a simpler version of the scalarization functionals studied in [17, 18] was introduced – without the norm term. Similar or stronger scalarization results are obtained. Some of them under weaker assumptions as those in [18]. Based on the proposed characterization results it is possible to obtain necessary and sufficient optimality conditions of Fermat and Lagrange type. This can be done using the same steps as in the proof of the results given in [17, Chapter 4].

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