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## Generalized Lagrangian Duals and Sums of Squares Relaxations of Sparse Polynomial Optimization Problems

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### Abstract.

Sequences of generalized Lagrangian duals and their SOS (sums of squares of polynomials) relaxations for a POP (polynomial optimization problem) are introduced. Sparsity of polynomials in the POP is used to reduce the sizes of the Lagrangian duals and their SOS relaxations. It is proved that the optimal values of the Lagrangian duals in the sequence converge to the optimal value of the POP using a method from the penalty function approaches. The sequence of SOS relaxations is transformed into a sequence of SDP (semidefinite program) relaxations of the POP, which correspond to duals of modification and generalization of SDP relaxations given by Lasserre for the POP.

### Key words.

Polynomial Optimization Problem, Sparsity, Global Optimization, Lagrangian Relaxation, Lagrangian Dual, Sums of Squares Optimization, Semidefinite Program, Semidefinite Program Relaxation

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# 1 Introduction

POPs (Polynomial optimization problems or optimization problems with polynomial objective and constraints) represent a broad range of applications in science and engineering. Convex relaxation methods that relax nonconvex constraints and objective function of POPs with convex ones has been widely used for POPs. Various convex relaxation methods exist for POPs, which include quadratic, linear and 0-1 integer programming as special cases. After a lift-and-project LP procedure for 0-1 integer linear programs by Balas-Ceria-Cornuéjols [1], the RLT (Reformulation-Linearization Technique) by Sherali-Adams [17] and an SDP (Semidefinite Programming) relaxation method by Lovász-Schrijver [10] were introduced. Many convex relaxation methods such as the RLT [19, 18] for 0-1 mixed integer polynomial programs, the SCRM (Successive Convex Relaxation Method) [7, 8] for QOPs (Quadratic Optimization Problems), SOCP (Second Order Cone Programming) relaxations [3, 4] for QOPs, and SOS (Sums of Squares) relaxations for POPs [12] have been proposed.

Recently, an important theoretical development has been made by Lasserre [9] toward achieving optimal values of POPs. According to the paper [5], his method to obtain a sequence of SDP relaxations can be considered as a primal approach which is divide into two phases. The first phase is to transform a given POP into an equivalent optimization problem by adding valid polynomial constraints over semidefinite cones to the POP. The resulting optimization problem forms a PSDP (polynomial semidefinite programming problem). In the second phase, the

PSDP is linearized to an SDP. In theory, we can add any size and/or number of valid polynomial constraints over positive semidefinite cones to the POP in the first phase. Lasserre [9] proposed a systematic way of adding valid polynomial constraints over semidefinite cones to the original POP in the first phase to generate a sequence of SDP relaxations in the second phase. It was proved that when the feasible region of the POP is compact, its optimal value can be approximated within any accuracy by the sequence of SDP relaxations. However, the size of an SDP relaxation to be solved in the sequence increases very rapidly as higher accuracy for an approximation to the optimal value of the POP is required.

It is known that practical solvability of SDPs depends on their sizes. A great deal of efforts continue to be made to resolve the difficulty of solving large scale SDPs. See the survey paper [11] and references therein. From a computational point of view, a close approximation to the optimal value of a POP is hard to achieve without efficient methods to handle large scale SDPs. Rapid increase of the size of SDP relaxations in the sequence in [9] as higher accuracy desired makes it difficult to achieve their optimal values in practice. As a result, only small-size POPs can be solved by the method in [9].

The purpose of this paper is to present generalized Lagrangian duals and their SOS (sums of squares) relaxations [12] for sparse POPs. This approach may be regarded as dual of Lasserre's SDP relaxations [9] mentioned above. For the standard Lagrangian function of a POP, it is common that nonnegative numbers are assigned to Lagrangian multipliers. Instead of selecting nonnegative numbers for Lagrangian multipliers, we choose Lagrangian multipliers to be SOS polynomials satisfying similar sparsity to associated constraint polynomials. Then, we define a generalized Lagrangian dual for a POP over such SOS polynomial multipliers, and provide a theoretical foundation for the generalized Lagrangian dual. After a sequence of sets of SOS polynomials is constructed, *e.g.* SOS polynomials of increasing degree, for Lagrangian multipliers, a sequence of Lagrangian duals is obtained. We derive a

sufficient condition for the sequence of Lagrangian duals to attain the optimal value of the POP, based on the idea of the penalty function method. For practical purposes, each Lagrangian dual in the sequence is relaxed to an SOS optimization problem, which is further converted into an equivalent SDP. Thus we have sequences of SOS relaxations and SDP relaxations of the POP. The resulting sequence of SDP relaxations corresponds to dual of the sequence of SDP relaxations obtained from the primal approach.

An advantage of the dual approach in this paper is that sparsity of objective and constraint polynomials in a POP can be exploited to reduce the size of the SDP relaxations. The size of the SDP relaxations depends on the supports of the polynomials in the dual approach, whereas the size of the SDP relaxations from the primal approach depends on the degree of the polynomials.

This paper is organized as follows. In Section 2, we introduce a POP with a compact feasible region and describe the representation of sparsity of the POP. Two types of POPs are shown for different characterizations of the compact feasible region. Section 3 includes the derivation of generalized Lagrangian duals of the two POPs over Lagrangian multipliers from a set of SOS polynomials satisfying similar sparsity to the associated constraint polynomials. We show a sufficient condition for the Lagrangian duals to attain the optimal value of the original POP. In Section 4, we present numerical methods to approximate optimal values of the generalized Lagrangian duals numerically using SOS relaxations. The numerical methods take advantage of sparsity of the original POP to produce SDP relaxations with smaller size. The relationship between the resulting SDP relaxations and the SDP relaxations in [9] is also discussed. Section 5 contains the proof of a Lemma which plays an important role in Section 3. Section 6 is devoted to concluding discussions.

Throughout the paper, we use the following notation: Let  $\mathbb{R}^n$  and  $\mathbb{Z}_+^n \subset \mathbb{R}^n$  denote the  $n$ -dimensional Euclidean space and the set of  $n$ -dimensional nonnegative integer vectors, respectively. Let  $f_j : \mathbb{R}^n \rightarrow \mathbb{R}$  be a real valued polynomial in  $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$  ( $j = 0, 1, 2, \dots, m$ ). We denote each polynomial  $f_j(\mathbf{x})$  as  $f_j(\mathbf{x}) = \sum_{\mathbf{a} \in \mathcal{F}_j} c_j(\mathbf{a}) \mathbf{x}^{\mathbf{a}}$ , where a nonempty finite subset  $\mathcal{F}_j$  of  $\mathbb{Z}_+^n$  denotes a support of the polynomial  $f_j(\mathbf{x})$ ,  $c_j(\mathbf{a}) \in \mathbb{R}$  and  $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$  for every  $\mathbf{a} = (a_1, a_2, \dots, a_n) \in \mathcal{F}_j$  ( $j = 0, 1, 2, \dots, m$ ) and  $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ . Let  $r_j$  denote the degree of each polynomial  $f_j(\mathbf{x})$  ( $j = 0, 1, 2, \dots, m$ );  $r_j = \max\{\sum_{i=1}^n a_i : \mathbf{a} \in \mathcal{F}_j\}$ .

## 2 Polynomial optimization problems and sparsity

We consider the POP (polynomial optimization problem):

$$\text{minimize } f_0(\mathbf{x}) \quad \text{subject to } f_j(\mathbf{x}) \geq 0 \quad (j = 1, 2, \dots, m). \quad (1)$$

Let us focus on the support  $\mathcal{F}_j$  of the polynomial  $f_j(\mathbf{x})$  ( $j = 0, 1, 2, \dots, m$ ) to describe sparsity of the POP (1). A polynomial  $f(\mathbf{x})$  of degree  $r$  or its support  $\mathcal{F}$  is called sparse if the number of elements in the support  $\mathcal{F}$  is much smaller than the number of elements in the support  $\mathcal{G}(\xi) \equiv \{\mathbf{a} \in \mathbb{Z}_+^n : \sum_{i=1}^n a_i \leq \xi\}$  of a general fully dense polynomial of degree  $\xi$ . In particular, if the number of indices in  $I_+(\mathcal{F}) \equiv \{i : a_i > 0 \text{ for some } \mathbf{a} \in \mathcal{F}\}$  is much smaller than  $n$ , then  $f(\mathbf{x})$  is sparse. We present two examples below.

**Example 2.1.** A box constraint POP. Let  $m = n$  and  $f_j(\mathbf{x}) = 1 - x_j^2$  ( $j = 1, 2, \dots, n$ ). In this case, we have  $\mathcal{F}_j = \{\mathbf{0}, 2\mathbf{e}^j\}$  ( $j = 1, 2, \dots, n$ ). Each  $\mathcal{F}_j$  has two elements and  $I_+(\mathcal{F}_j) = \{j\}$ . Here  $\mathbf{e}^j$  denotes the  $j$ th unit coordinate vector of  $\mathbb{R}^n$  with 1 in the  $j$ th component and 0 elsewhere.

**Example 2.2.** Let  $m = n$ ,  $f_1(\mathbf{x}) = \alpha_1 - x_1^2$  and  $f_j(\mathbf{x}) = \alpha_j - (\beta_{j-1}x_{j-1} - x_j)^2$  ( $j = 2, 3, \dots, n$ ), where  $\alpha_j$  ( $j = 1, 2, \dots, n$ ) and  $\beta_j$  ( $j = 1, 2, \dots, n-1$ ) denote positive numbers. Then,  $\mathcal{F}_1 = \{\mathbf{0}, 2\mathbf{e}^1\}$ ,  $\mathcal{F}_j = \{\mathbf{0}, 2\mathbf{e}^{j-1}, \mathbf{e}^{j-1} + \mathbf{e}^j, 2\mathbf{e}^j\}$  ( $j = 2, 3, \dots, n$ ). Each  $\mathcal{F}_j$  has at most four elements and  $I_+(\mathcal{F}_j)$  contains at most two indices.

Let  $F$  denote the feasible region of the POP (1);

$$F = \{\mathbf{x} \in \mathbb{R}^n : f_j(\mathbf{x}) \geq 0 \ (j = 1, 2, \dots, m)\}.$$

Throughout the paper, we assume that  $F$  is nonempty and bounded. Then, the POP (1) has a finite optimal value  $\zeta^*$  at an optimal solution  $\mathbf{x}^* \in F$ ;

$$\zeta^* = \min\{f_0(\mathbf{x}) : \mathbf{x} \in F\} = f_0(\mathbf{x}^*) \text{ and } \mathbf{x}^* \in F.$$

In what it follows, we further need a bound  $\rho > 0$  to be known explicitly for the feasible region. We are concerned with the following cases:

- $F \subset C_\rho \equiv \{\mathbf{x} \in \mathbb{R}^n : \rho^2 - x_i^2 \geq 0 \ (i = 1, 2, \dots, n)\}$  or
- $F \subset B_\rho \equiv \{\mathbf{x} \in \mathbb{R}^n : \rho^2 - \mathbf{x}^T \mathbf{x} \geq 0\}$ .

If  $F \subset C_\rho$  holds then the POP (1) is equivalent to

$$\text{minimize } f_0(\mathbf{x}) \text{ subject to } f_j(\mathbf{x}) \geq 0 \ (j = 1, 2, \dots, m) \text{ and } \mathbf{x} \in C_\rho. \quad (2)$$

Example 2.1 is a special case of the POP (2) where we take  $m = 0$  and  $\rho = 1$ . If  $F \subset B_\rho$  is satisfied, then we have  $F \subset C_\rho$ ; hence the two POPs (1) and (2) are equivalent to

$$\text{minimize } f_0(\mathbf{x}) \text{ subject to } f_j(\mathbf{x}) \geq 0 \ (j = 1, 2, \dots, m) \text{ and } \mathbf{x} \in B_\rho. \quad (3)$$

In the next section, we present a (generalized) Lagrangian function, a Lagrangian relaxation, a Lagrangian dual and its SOS relaxation for each of the POPs (2) and (3). For both POPs, the Lagrangian dual attains the optimal value  $\zeta^*$  of the original POP (1) under a moderate assumption. But, only the SOS relaxation derived from the POP (3) is guaranteed to attain  $\zeta^*$ , while the SOS relaxation from the POP (2) inherits more sparsity of the original POP (1) than the one from the POP (3).

## 3 Generalized Lagrangian duals

### 3.1 Lagrangian functions

Let  $\bar{\Sigma}$  denote the set of sums of squares of polynomials in  $\mathbf{x} \in \mathbb{R}^n$ ;

$$\bar{\Sigma} = \left\{ \sum_{i=1}^k \chi_i(\mathbf{x})^2 : \begin{array}{l} \chi_i \text{ is a polynomial in } \mathbf{x} \in \mathbb{R}^n \ (i = 1, 2, \dots, k) \\ \text{and } k \text{ is any finite positive integer} \end{array} \right\}.$$

We define two types of (generalized) Lagrangian functions  $L_B : B_\rho \times \bar{\Sigma}^m \rightarrow \mathbb{R}$  for POP (3) and  $L_C : \mathbb{R}^n \times \bar{\Sigma}^{m+n} \rightarrow \mathbb{R}$  for POP (2):

$$\begin{aligned} L_B(\mathbf{x}, \boldsymbol{\varphi}) &= f_0(\mathbf{x}) - \sum_{j=1}^m \varphi_j(\mathbf{x}) f_j(\mathbf{x}) \\ L_C(\mathbf{x}, \boldsymbol{\varphi}, \boldsymbol{\psi}) &= L_B(\mathbf{x}, \boldsymbol{\varphi}) - \sum_{i=1}^n \psi_i(\mathbf{x}) (\rho^2 - x_i^2) \\ &= f_0(\mathbf{x}) - \sum_{j=1}^m \varphi_j(\mathbf{x}) f_j(\mathbf{x}) - \sum_{i=1}^n \psi_i(\mathbf{x}) (\rho^2 - x_i^2). \end{aligned}$$

Here  $\bar{\Sigma}^\ell$  denotes the Cartesian product of  $\ell$ -tuples of  $\bar{\Sigma}$ ;

$$\bar{\Sigma}^\ell = \{(\varphi_1, \varphi_2, \dots, \varphi_\ell) : \varphi_j \in \bar{\Sigma} \ (j = 1, 2, \dots, \ell)\} \quad (\ell = m \text{ or } m+n).$$

Let  $(\boldsymbol{\varphi}, \boldsymbol{\psi}) \in \bar{\Sigma}^{m+n}$ . Then

$$\left. \begin{aligned} L_C(\mathbf{x}, \boldsymbol{\varphi}, \boldsymbol{\psi}) &\leq f_0(\mathbf{x}) \text{ if } \mathbf{x} \in F \cap C_\rho, \\ L_B(\mathbf{x}, \boldsymbol{\varphi}) &\leq f_0(\mathbf{x}) \text{ if } \mathbf{x} \in F \cap B_\rho, \\ L_C(\mathbf{x}, \boldsymbol{\varphi}, \boldsymbol{\psi}) &\leq L_B(\mathbf{x}, \boldsymbol{\varphi}) \text{ if } \mathbf{x} \in B_\rho. \end{aligned} \right\} \quad (4)$$

### 3.2 Lagrangian relaxations and duals

We introduce a (generalized) Lagrangian relaxation of the POP (2):

$$L_C^*(\boldsymbol{\varphi}, \boldsymbol{\psi}) = \inf \{L_C(\mathbf{x}, \boldsymbol{\varphi}, \boldsymbol{\psi}) : \mathbf{x} \in \mathbb{R}^n\}$$

for each fixed  $(\boldsymbol{\varphi}, \boldsymbol{\psi}) \in \bar{\Sigma}^{m+n}$ , and a (generalized) Lagrangian relaxation of the POP (3):

$$L_B^*(\boldsymbol{\varphi}) = \inf \{L_B(\mathbf{x}, \boldsymbol{\varphi}) : \mathbf{x} \in B_\rho\}$$

for each fixed  $\boldsymbol{\varphi} \in \bar{\Sigma}^m$ . Let  $(\boldsymbol{\varphi}, \boldsymbol{\psi}) \in \bar{\Sigma}^{m+n}$ . By (4), we see that

$$\left. \begin{aligned} L_C^*(\boldsymbol{\varphi}, \boldsymbol{\psi}) &\leq \zeta^* = \min\{f_0(\mathbf{x}) : \mathbf{x} \in F\} \text{ if } F \subset C_\rho, \\ L_C^*(\boldsymbol{\varphi}, \boldsymbol{\psi}) &\leq L_B^*(\boldsymbol{\varphi}) \leq \zeta^* \text{ if } F \subset B_\rho. \end{aligned} \right\} \quad (5)$$

For every  $(\boldsymbol{\Sigma}, \boldsymbol{\Xi}) \subset \bar{\Sigma}^{m+n}$ , we define a (generalized) Lagrangian dual of the POP (2):

$$\text{maximize } L_C^*(\boldsymbol{\varphi}, \boldsymbol{\psi}) \text{ subject to } (\boldsymbol{\varphi}, \boldsymbol{\psi}) \in \boldsymbol{\Sigma} \times \boldsymbol{\Xi}. \quad (6)$$

For every  $\boldsymbol{\Sigma} \subset \bar{\Sigma}^m$ , we define a (generalized) Lagrangian dual of the POP (3):

$$\text{maximize } L_B^*(\boldsymbol{\varphi}) \text{ subject to } \boldsymbol{\varphi} \in \boldsymbol{\Sigma}. \quad (7)$$

Let  $L_C^*(\boldsymbol{\Sigma} \times \boldsymbol{\Xi})$  and  $L_B^*(\boldsymbol{\Sigma})$  denote the optimal values of (6) and (7), respectively;

$$L_C^*(\boldsymbol{\Sigma} \times \boldsymbol{\Xi}) = \sup_{(\boldsymbol{\varphi}, \boldsymbol{\psi}) \in \boldsymbol{\Sigma} \times \boldsymbol{\Xi}} L_C^*(\boldsymbol{\varphi}, \boldsymbol{\psi}) \text{ and } L_B^*(\boldsymbol{\Sigma}) = \sup_{\boldsymbol{\varphi} \in \boldsymbol{\Sigma}} L_B^*(\boldsymbol{\varphi}).$$

It follows from (5) that

$$\left. \begin{aligned} L_C^*(\Sigma \times \Xi) &\leq \zeta^* \text{ if } F \subset C_\rho, \\ L_C^*(\Sigma \times \Xi) &\leq L_B^*(\Sigma) \leq \zeta^* \text{ if } F \subset B_\rho \end{aligned} \right\} \quad (8)$$

hold for every  $(\Sigma, \Xi) \in \bar{\Sigma}^{m+n}$ .

Assume that  $F \subset C_\rho$ . Then, the two POPs (1) and (2) are equivalent. If we restrict  $(\varphi, \psi)$  to the nonnegative orthant  $\mathbb{R}_+^{m+n}$  of  $\mathbb{R}^{m+n}$ ,  $L_C(\mathbf{x}, \varphi, \psi)$  becomes the standard Lagrangian function for the POP (2). It is well-known that a positive duality gap exists in general between the standard Lagrangian dual  $L_C^*(\mathbb{R}_+^{m+n})$  and the optimal value  $\zeta^*$  of the POP (2). As we take a larger set  $\Sigma \times \Xi \subset \bar{\Sigma}^{m+n}$ , the duality gap between  $L_C^*(\Sigma \times \Xi)$  and  $\zeta^*$  is expected to decrease. In the next section, we derive a sufficient condition on  $\Sigma \times \Xi \subset \bar{\Sigma}^{m+n}$  for  $L_C^*(\Sigma \times \Xi)$  to attain the optimal value  $\zeta^*$  of the POP (2). We regard  $(\varphi, \psi) \in \bar{\Sigma}^{m+n}$  as ‘‘a penalty parameter’’ in the derivation, and

$$\Phi_C(\mathbf{x}, \varphi, \psi) = - \sum_{j=1}^m \varphi_j(\mathbf{x}) f_j(\mathbf{x}) - \sum_{i=1}^n \psi_i(\mathbf{x}) (\rho^2 - x_i^2) \quad (9)$$

(the terms added to the objective function  $f_0(\mathbf{x})$ )

in the construction of the Lagrangian function  $L_C(\mathbf{x}, \varphi, \psi)$ )

as ‘‘a penalty function’’ with a choice of penalty parameters  $(\varphi, \psi) = (\varphi^p, \psi^p) \in \bar{\Sigma}^{m+n}$  ( $p \in \mathbb{Z}_+$ ) such that

$$\begin{aligned} \text{if } \mathbf{x} \in F \text{ then } \Phi(\mathbf{x}, \varphi^p, \psi^p) &\rightarrow 0 \text{ as } p \rightarrow \infty, \\ \text{if } \mathbf{x} \notin F \text{ then } \Phi(\mathbf{x}, \varphi^p, \psi^p) &\rightarrow \infty \text{ as } p \rightarrow \infty. \end{aligned}$$

Additional properties of the penalty function  $\Phi_C(\mathbf{x}, \varphi^p, \psi^p)$  are described in Lemma 3.2. It is shown that the Lagrangian function  $L_C(\mathbf{x}, \varphi, \psi) = f_0(\mathbf{x}) + \Phi_C(\mathbf{x}, \varphi, \psi)$  with the penalty parameter  $(\varphi, \psi) = (\varphi^p, \psi^p) \in \bar{\Sigma}^{m+n}$  has a global minimizer  $\mathbf{x}^p$  over  $\mathbb{R}^n$  with the objective value  $f_0(\mathbf{x}^p) \rightarrow \zeta^*$  as  $p \rightarrow \infty$  and that  $\{\mathbf{x}^p\}$  has an accumulation point in the optimal solution set of the POP (2).

### 3.3 Main theorem

We need some additional notation and symbols. Take a positive number  $\gamma$  such that

$$\left. \begin{aligned} |f_j(\mathbf{x})/\gamma| &\leq 1 \text{ if } \|\mathbf{x}\|_\infty \leq \sqrt{2}\rho, \\ |f_j(\mathbf{x})/\gamma| &\leq \|\mathbf{x}/\rho\|_\infty^r \text{ if } \|\mathbf{x}\|_\infty \geq \sqrt{2}\rho \end{aligned} \right\} \quad (10)$$

( $j = 0, 1, 2, \dots, m$ ). Define

$$\left. \begin{aligned} \varphi_j^p(\mathbf{x}) &= (1 - f_j(\mathbf{x})/\gamma)^{2p} \quad (j = 1, 2, \dots, m, p \in \mathbb{Z}_+), \\ \varphi^p(\mathbf{x}) &= (\varphi_1^p(\mathbf{x}), \varphi_2^p(\mathbf{x}), \dots, \varphi_m^p(\mathbf{x})) \quad (p \in \mathbb{Z}_+), \\ \psi_i^p(\mathbf{x}) &= ((m+2)\gamma/\rho^2) (x_i/\rho)^{2r(p+1)} \quad (i = 1, 2, \dots, n, p \in \mathbb{Z}_+), \\ \psi^p(\mathbf{x}) &= (\psi_1^p(\mathbf{x}), \psi_2^p(\mathbf{x}), \dots, \psi_n^p(\mathbf{x})) \quad (p \in \mathbb{Z}_+). \end{aligned} \right\} \quad (11)$$

Here  $r = \max\{r_j : j = 0, 1, 2, \dots, m\}$  denotes the maximum of the degree  $r_j$  of the polynomial  $f_j(\mathbf{x})$  ( $j = 0, 1, 2, \dots, m$ ). Obviously  $(\varphi^p, \psi^p) \in \bar{\Sigma}^{m+n}$  ( $p \in \mathbb{Z}_+$ ).

**Theorem 3.1.** Assume that  $\Xi \times \Sigma \subset \overline{\Sigma}^{m+n}$  contains an infinite subsequence of  $\{(\varphi^p, \psi^p) (p \in \mathbb{Z}_+)\}$ . Then  $L_C^*(\Sigma \times \Xi) = \zeta^*$  if  $F \subset C_\rho$ , and  $L_C^*(\Sigma \times \Xi) = L_B^*(\Sigma) = \zeta^*$  if  $F \subset B_\rho$ .

To prove the theorem, we use the following lemma whose proof is given in Section 5.

**Lemma 3.2.** Suppose that  $F \subset C_\rho$ . Let  $p^*$  be the smallest nonnegative integer such that  $(m+2)n - 2^{r(p^*+1)} \leq 0$ .

(a) If  $\sqrt{2}\rho \leq \|\mathbf{x}\|_\infty$ , then

$$L_C(\mathbf{x}, \varphi^p, \psi^p) \equiv f_0(\mathbf{x}) + \Phi_C(\mathbf{x}, \varphi^p, \psi^p) \geq \|\mathbf{x}/\rho\|_\infty^{2rp} \geq \|\mathbf{x}/\rho\|_\infty^2$$

for every  $p \geq p^*$ .

(b) If  $\tilde{\mathbf{x}} \notin F$  and  $\kappa \in \mathbb{R}$ , then there exist  $\delta > 0$  and  $\tilde{p} \geq p^*$  such that

$$L_C(\mathbf{x}, \varphi^p, \psi^p) \equiv f_0(\mathbf{x}) + \Phi_C(\mathbf{x}, \varphi^p, \psi^p) \geq \kappa$$

for every  $p \geq \tilde{p}$  and  $\mathbf{x} \in U_\delta(\tilde{\mathbf{x}})$ . Here  $U_\delta(\tilde{\mathbf{x}}) = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x} - \tilde{\mathbf{x}}\| < \delta\}$ .

(c) If  $\hat{\mathbf{x}} \in F$  and  $\epsilon > 0$ , then there exist  $\delta > 0$  and  $\tilde{p} \geq p^*$  such that

$$-\epsilon \leq \Phi_C(\mathbf{x}, \varphi^p, \psi^p)$$

for every  $\mathbf{x} \in U_\delta(\hat{\mathbf{x}})$  and  $p \geq \tilde{p}$ .

(d)  $\lim_{p \rightarrow \infty} L_C^*(\varphi^p, \psi^p) = \zeta^*$ .

**Proof of Theorem 3.1:** By (8), we only need to show that if  $F \subset C_\rho$  then  $L_C^*(\Sigma \times \Xi) = \zeta^*$ . Suppose that  $F \subset C_\rho$ . Since the sequence  $\{(\varphi^p, \psi^p) (p \in \mathbb{Z}_+)\}$  lies in the set  $\Sigma \times \Xi$  by the assumption, we see that

$$\limsup_{p \rightarrow \infty} L_C^*(\varphi^p, \psi^p) \leq L_C^*(\Sigma \times \Xi) \leq \zeta^*.$$

By (d) of Lemma 3.2,

$$\limsup_{p \rightarrow \infty} L_C^*(\varphi^p, \psi^p) = \lim_{p \rightarrow \infty} L_C^*(\varphi^p, \psi^p) = \zeta^*.$$

Therefore  $L_C^*(\Sigma \times \Xi) = \zeta^*$ . ■

### 3.4 Construction of $\Sigma \times \Xi$ satisfying the assumption of Theorem 3.1

For every nonempty subset  $\mathcal{A}$  of  $\mathbb{Z}_+^n$ , we define

$$\Sigma(\mathcal{A}) = \left\{ \sum_{i=1}^k \chi_i(\mathbf{x})^2 : \begin{array}{l} \chi_i \text{ is a polynomial in } \mathbf{x} \in \mathbb{R}^n \text{ with a support in } \mathcal{A} \\ (i = 1, 2, \dots, k) \text{ and } k \text{ is any finite positive integer} \end{array} \right\}.$$

Suppose that  $\mathcal{A}_j^q$  ( $j = 1, 2, \dots, m$ ,  $q \in \mathbb{Z}_+$ ) and  $\mathcal{B}_i^q$  ( $i = 1, 2, \dots, n$ ,  $q \in \mathbb{Z}_+$ ) are nonempty finite subsets of  $\mathbb{Z}_+^n$  such that

$$\varphi_j^{\lambda(q)} \in \Sigma(\mathcal{A}_j^q) \quad \text{and} \quad \psi_i^{\lambda(q)}(\mathbf{x}) \in \Sigma(\mathcal{B}_i^q) \quad \text{if } q \geq q^* \quad (12)$$

for some  $q^* \in \mathbb{Z}_+$  and some mapping  $\lambda$  from  $\mathbb{Z}_+$  into itself satisfying

$$\lambda(q) \leq \lambda(q+1) \quad (q \in \mathbb{Z}_+) \quad \text{and} \quad \lim_{q \rightarrow \infty} \lambda(q) = \infty.$$

Let

$$\begin{aligned} \Sigma^q &= \prod_{j=1}^m \Sigma(\mathcal{A}_j^q) \quad (q \in \mathbb{Z}_+), \quad \Xi^q = \prod_{i=1}^n \Sigma(\mathcal{B}_i^q) \quad (q \in \mathbb{Z}_+), \\ \Sigma^\infty &= \bigcup_{q \in \mathbb{Z}_+} \Sigma^q \quad \text{and} \quad \Xi^\infty = \bigcup_{q \in \mathbb{Z}_+} \Xi^q. \end{aligned}$$

By construction, we see that

$$(\varphi^{\lambda(q)}, \psi^{\lambda(q)}) \in \Sigma^q \times \Xi^q \quad (q \geq q^*).$$

Hence  $\Sigma^\infty \times \Xi^\infty$  contains an infinite subsequence  $\{(\varphi^{\lambda(q)}, \psi^{\lambda(q)}) \mid (q \geq q^*)\}$ . Therefore  $\Sigma \times \Xi = \Sigma^\infty \times \Xi^\infty$  satisfies the assumption of Theorem 3.1.

We give some examples of  $\mathcal{A}_j^q$  ( $j = 1, 2, \dots, m$ ,  $q \in \mathbb{Z}_+$ ) and  $\mathcal{B}_i^q$  ( $i = 1, 2, \dots, n$ ,  $q \in \mathbb{Z}_+$ ) satisfying the assumption (12).

**Example 3.3.** For every  $j = 1, 2, \dots, m$ ,  $i = 1, 2, \dots, n$ ,  $q \in \mathbb{Z}_+$ , let

$$\begin{aligned} \mathcal{A}_j^0 &= \{\mathbf{0}\}, \quad \mathcal{A}_j^1 = \{\mathbf{0}\} \bigcup \mathcal{F}_j, \quad \mathcal{A}_j^{q+1} = \{\mathbf{a} + \mathbf{b} : \mathbf{a} \in \mathcal{A}_j^q, \mathbf{b} \in \mathcal{A}_j^1\} \quad (q \geq 1), \\ \mathcal{B}_i^q &= \{k\mathbf{e}^i : k = 0, 1, 2, \dots, (q+1)r\}. \end{aligned}$$

**Example 3.4.** For every  $j = 1, 2, \dots, m$ ,  $i = 1, 2, \dots, n$ ,  $q \in \mathbb{Z}_+$ , let

$$\begin{aligned} \mathcal{A}_j^0 &= \{\mathbf{0}\}, \quad \mathcal{A}_j^1 = \{\mathbf{0}\} \bigcup \{\mathbf{e}^k : k \in I_+(\mathcal{F}_j)\}, \\ \mathcal{A}_j^{q+1} &= \{\mathbf{a} + \mathbf{b} : \mathbf{a} \in \mathcal{A}_j^q, \mathbf{b} \in \mathcal{A}_j^1\} \quad (q \geq 1), \\ \mathcal{B}_i^q &= \{k\mathbf{e}^i : k = 0, 1, 2, \dots, q+1\} \quad (q \in \mathbb{Z}_+). \end{aligned}$$

Here  $I_+(\mathcal{F}_j) = \{i : a_i > 0 \text{ for some } \mathbf{a} \in \mathcal{F}_j\}$ .

**Example 3.5.** For every  $j = 1, 2, \dots, m$ ,  $i = 1, 2, \dots, n$ ,  $q \in \mathbb{Z}_+$ , let

$$\begin{aligned} \mathcal{A}_j^0 &= \mathcal{B}_i^0 = \{\mathbf{0}\}, \quad \mathcal{A}_j^1 = \mathcal{B}_i^1 = \{\mathbf{0}\} \bigcup \{\mathbf{e}^k : k = 1, 2, \dots, n\}, \\ \mathcal{A}_j^{q+1} &= \mathcal{B}_i^{q+1} = \{\mathbf{a} + \mathbf{b} : \mathbf{a} \in \mathcal{A}_j^q, \mathbf{b} \in \mathcal{A}_j^1\} \quad (q \geq 1). \end{aligned}$$



In all the examples, both  $\mathcal{A}_j^q$  and  $\mathcal{B}_i^q$  expand monotonically as  $q$  increases, and for any  $\bar{p} \in \mathbb{Z}_+$  there exists  $\bar{q}$  such that

$\varphi_j^p(\mathbf{x}) \in \Sigma(\mathcal{A}_j^q)$  for all  $p \leq \bar{p}$  and  $q \geq \bar{q}$  and that  $\psi_j^p(\mathbf{x}) \in \Sigma(\mathcal{B}_i^q)$  for all  $p \leq \bar{p}$  and  $q \geq \bar{q}$ . It should be noted that if  $\mathcal{F}_j$  is sparse then  $\mathcal{A}_j^q$  remains sparse in Example 3.3. The choice of  $\mathcal{A}_j^q$  in Example 3.4 may be also reasonable when the number of the indices  $I_+(\mathcal{F}_j)$  is smaller than  $n$ . When we take  $\mathcal{A}_j^q$  and  $\mathcal{B}_i^q$  as in Example 3.5, we have

$$\begin{aligned} \Sigma(\mathcal{A}_j^q) &= \Sigma(\mathcal{B}_i^q) \\ &= \left\{ \sum_{i=1}^k \chi_i(\mathbf{x})^2 : \begin{array}{l} \chi_i \text{ is a polynomial in } \mathbf{x} \in \mathbb{R}^n \text{ with degree } \leq q \\ (i = 1, 2, \dots, k) \text{ and } k \text{ is any finite positive integer} \end{array} \right\}; \end{aligned}$$

Hence the sparsity of the original POP (1) is destroyed in the Lagrangian duals (6) and (7). We remark here that the choice of  $\mathcal{A}_j^q$  as in Example 3.5 for the Lagrangian dual (7) leads to the dual of Lasserre's SDP relaxation [9] applied to the POP (3). This is discussed briefly in Section 4.5.

## 4 Numerical methods for generalized Lagrangian duals

In the following five subsections, we discuss numerical methods for generalized Lagrangian duals. The first three subsections include how  $L_C^*(\Sigma^\infty \times \Xi^\infty)$  can be approximated numerically, and the fourth section is for the approximation of  $L_B^*(\Sigma^\infty)$ . We briefly mention relationship between our methods and Lasserre's SDP relaxation [9] applied to the POP (2) in the last subsection. Throughout this section, we assume that  $F \subset C_\rho$ , so that  $L_C^*(\Sigma^\infty \times \Xi^\infty) = \zeta^*$ .

### 4.1 Approximation of generalized Lagrangian duals

We introduce a sequence of subproblems of the Lagrangian dual (6).

$$\text{maximize } L_C^*(\varphi, \psi) \quad \text{subject to } (\varphi, \psi) \in \Sigma^q \times \Xi^q \quad (13)$$

( $q \in \mathbb{Z}_+$ ).

**Lemma 4.1.**

(a)  $L_C^*(\Sigma^q \times \Xi^q) \leq L_C^*(\Sigma^\infty \times \Xi^\infty)$  ( $q \in \mathbb{Z}_+$ ).

(b) For any  $\epsilon > 0$ , there exists a nonnegative integer  $\bar{q}$  such that

$$L_C^*(\Sigma^\infty \times \Xi^\infty) - \epsilon \leq L_C^*(\Sigma^q \times \Xi^q) \quad (q \geq \bar{q}).$$

*Proof:* By construction, we know that  $\Sigma^q \times \Xi^q \subset \Sigma^\infty \times \Xi^\infty$  ( $q \in \mathbb{Z}_+$ ). Thus, (a) follows. Let  $\epsilon > 0$ . By (d) of Lemma 3.2, there exists  $\bar{p}$  such that

$$L_C^*(\Sigma^\infty \times \Xi^\infty) - \epsilon \leq L_C^*(\varphi^p, \psi^p) \quad (p \geq \bar{p}).$$

Take  $\bar{q} \in \mathbb{Z}_+$  such that  $\lambda(q) \geq \bar{p}$  ( $q \geq \bar{q}$ ). Then

$$(\boldsymbol{\varphi}^{\lambda(q)}, \boldsymbol{\psi}^{\lambda(q)}) \in \boldsymbol{\Sigma}^q \times \boldsymbol{\Xi}^q \quad \text{and } \lambda(q) \geq \bar{p} \text{ for every } q \geq \bar{q}.$$

Therefore we obtain that

$$L_C^*(\boldsymbol{\Sigma}^\infty \times \boldsymbol{\Xi}^\infty) - \epsilon \leq L_C^*(\boldsymbol{\varphi}^{\lambda(q)}, \boldsymbol{\psi}^{\lambda(q)}) \leq L_C^*(\boldsymbol{\Sigma}^q \times \boldsymbol{\Xi}^q) \text{ for every } q \geq \bar{q}.$$

■

## 4.2 Sums of square relaxations

Let  $q \in \mathbb{Z}_+$  be fixed throughout this subsection. We can rewrite the problems in (13) as

$$\left. \begin{array}{l} \text{maximize} \quad \zeta \\ \text{subject to} \quad L_C(\mathbf{x}, \boldsymbol{\varphi}, \boldsymbol{\psi}) - \zeta \geq 0 \ (\forall \mathbf{x} \in \mathbb{R}^n), \\ \quad \quad \quad (\boldsymbol{\varphi}, \boldsymbol{\psi}) \in \boldsymbol{\Sigma}^q \times \boldsymbol{\Xi}^q. \end{array} \right\} \quad (14)$$

Note that  $\mathbf{x} \in \mathbb{R}^n$  is not a vector variable but it serves as an index vector for infinite number of inequality constrains  $L_C(\mathbf{x}, \boldsymbol{\varphi}, \boldsymbol{\psi}) - \zeta \geq 0$  ( $\forall \mathbf{x} \in \mathbb{R}^n$ ). Replacing the inequality constraints  $L_C(\mathbf{x}, \boldsymbol{\varphi}, \boldsymbol{\psi}) - \zeta \geq 0$  ( $\forall \mathbf{x} \in \mathbb{R}^n$ ) by a sum of squares condition  $L_C(\mathbf{x}, \boldsymbol{\varphi}, \boldsymbol{\psi}) - \zeta \in \bar{\Sigma}$  in (14), we obtain an SOSOP (sums of squares optimization problem).

$$\left. \begin{array}{l} \text{maximize} \quad \zeta \\ \text{subject to} \quad L_C(\mathbf{x}, \boldsymbol{\varphi}, \boldsymbol{\psi}) - \zeta = \varphi_0(\mathbf{x}) \ (\forall \mathbf{x} \in \mathbb{R}^n), \\ \quad \quad \quad (\boldsymbol{\varphi}, \boldsymbol{\psi}) \in \boldsymbol{\Sigma}^q \times \boldsymbol{\Xi}^q, \ \varphi_0(\mathbf{x}) \in \bar{\Sigma}. \end{array} \right\} \quad (15)$$

Let  $\zeta_C^q$  denote the optimal value of the SOSOP (15);

$$\zeta_C^q = \sup \left\{ \zeta : \begin{array}{l} L_C(\mathbf{x}, \boldsymbol{\varphi}, \boldsymbol{\psi}) - \zeta = \varphi_0(\mathbf{x}) \ (\forall \mathbf{x} \in \mathbb{R}^n), \\ (\boldsymbol{\varphi}, \boldsymbol{\psi}) \in \boldsymbol{\Sigma}^q \times \boldsymbol{\Xi}^q, \ \varphi_0(\mathbf{x}) \in \bar{\Sigma} \end{array} \right\}.$$

If  $(\zeta, \boldsymbol{\varphi}, \boldsymbol{\psi}, \varphi_0)$  is a feasible solution of the SOSOP (15), then  $(\zeta, \boldsymbol{\varphi}, \boldsymbol{\psi})$  is a feasible solution of the problem (14). It follows that  $\zeta_C^q \leq L_C^*(\boldsymbol{\Sigma}^q \times \boldsymbol{\Xi}^q)$ . Although neither  $\zeta_C^q = L_C^*(\boldsymbol{\Sigma}^q \times \boldsymbol{\Xi}^q)$  nor the convergence of  $\zeta_C^q$  to  $L_C^*(\boldsymbol{\Sigma}^\infty \times \boldsymbol{\Xi}^\infty)$  as  $q \rightarrow \infty$  is guaranteed, we can solve the SOSOP (15) as we show in the next subsection while the problem (14) is intractable in general.

## 4.3 Reduction to SDPs

Let us fix  $q \in \mathbb{Z}_+$  throughout this subsection. We show how to solve the SOSOP (15) as an SDP (semidefinite program). If we rewrite the constraint  $(\boldsymbol{\varphi}, \boldsymbol{\psi}) \in \boldsymbol{\Sigma}^q \times \boldsymbol{\Xi}^q$  for each component, we have

$$\varphi_j(\mathbf{x}) \in \Sigma(\mathcal{A}_j^q) \ (j = 1, 2, \dots, m) \quad \text{and} \quad \psi_i(\mathbf{x}) \in \Sigma(\mathcal{B}_i^q) \ (i = 1, 2, \dots, n).$$

Notice that finite supports  $\mathcal{A}_j^q$  and  $\mathcal{B}_i^q$  are given for generating variable polynomials  $\varphi_j(\mathbf{x})$  and  $\psi_i(\mathbf{x})$  ( $j = 1, 2, \dots, m$ ,  $i = 1, 2, \dots, n$ ). But, no finite support is specified for the

variable polynomial  $\varphi_0(\mathbf{x})$ . The first step for constructing an SDP is to find an appropriate finite set  $\mathcal{G} \subset \mathbb{Z}_+^n$  so that  $\varphi_0(\mathbf{x})$  can be chosen from  $\Sigma(\mathcal{G})$ .

To choose such a  $\mathcal{G} \subset \mathbb{Z}_+^n$ , we focus on the support of the left hand side polynomial  $L_C(\mathbf{x}, \boldsymbol{\varphi}, \boldsymbol{\psi}) - \zeta$  of the equality constraint in the SOSOP (15). From the support  $\mathcal{F}_0$  of the objective polynomial function  $f_0(\mathbf{x})$ , the support of each term  $\varphi_j(\mathbf{x})f_j(\mathbf{x})$

$$\widehat{\mathcal{A}}_j^q \equiv \{\mathbf{a} + \mathbf{b} + \mathbf{c} : \mathbf{a} \in \mathcal{A}_j^q, \mathbf{b} \in \mathcal{A}_j^q, \mathbf{c} \in \mathcal{F}_j\}$$

( $j = 1, 2, \dots, m$ ) and the support of term  $\phi_i(\mathbf{x})(\rho^2 - x_i^2)$

$$\widehat{\mathcal{B}}_i^q \equiv \{\mathbf{a} + \mathbf{b} + \mathbf{c} : \mathbf{a} \in \mathcal{B}_i^q, \mathbf{b} \in \mathcal{B}_i^q, \mathbf{c} \in \{\mathbf{0}, 2\mathbf{e}^i\}\}$$

( $i = 1, 2, \dots, n$ ), we know that the support of  $L_C(\mathbf{x}, \boldsymbol{\varphi}, \boldsymbol{\psi}) - \zeta$  becomes

$$\mathcal{F}_L = \mathcal{F}_0 \cup \{\mathbf{0}\} \cup \left( \bigcup_{j=1}^m \widehat{\mathcal{A}}_j^q \right) \cup \left( \bigcup_{i=1}^n \widehat{\mathcal{B}}_i^q \right).$$

Here  $\{\mathbf{0}\}$  stands for the support of the term  $\zeta$ .

By Theorem 1 of [16], we can use

$$\mathcal{G}^0 = \left( \text{the convex hull of } \left\{ \mathbf{a}/2 : \begin{array}{l} \mathbf{a} \in \mathcal{F}_L, \\ \text{every } a_i \text{ is even } (i = 1, 2, \dots, n) \end{array} \right\} \right) \cap \mathbb{Z}_+^n.$$

for such a support  $\mathcal{G}$  that  $\varphi_0(\mathbf{x})$  can be chosen from  $\Sigma(\mathcal{G})$ . We can further apply a method proposed recently by the authors [6] for reducing the size of  $\mathcal{G}^0$  to obtain the smallest support  $\mathcal{G}^*$  in a class of supports including  $\mathcal{G}^0$ . See the paper [6] for more details.

**Remark 4.2.** Even when all  $\mathcal{F}_j$  ( $j = 0, 1, 2, \dots, m$ ) are sparse,  $\mathcal{G}^*$  becomes dense. This is because the method proposed in [6] does not eliminate any integer point in the simplex with vertices  $\bar{k}\mathbf{e}^i$  ( $i = 1, 2, \dots, n$ ) and  $\mathbf{0}$  contained in

$$\text{the convex hull of } \left\{ \mathbf{a}/2 : \mathbf{a} \in \mathcal{F}_L, \text{ every } a_i \text{ is even } (i = 1, 2, \dots, n) \right\},$$

which has induced  $\mathcal{G}^0$  above. Here  $\bar{k} = \min_i \max\{k : k\mathbf{e}^i \in \mathcal{B}_i^q\}$ . Hence the support  $\mathcal{G}^*$  does not benefit much from the sparsity of  $\mathcal{F}_j$  ( $j = 0, 1, 2, \dots, m$ ).

To transform the SOSOP (15) into an SDP, we need some notation and symbols. Let  $\mathcal{F} \in \mathbb{Z}_+^n$  be a nonempty finite set. Let  $|\mathcal{F}|$  denote the cardinality of  $\mathcal{F}$  and  $\mathbb{R}(\mathcal{F})$  the  $|\mathcal{F}|$ -dimensional Euclidean space whose coordinates are indexed by  $\mathbf{a} \in \mathcal{F}$ . Although the order of the coordinates is not relevant in the succeeding discussions, we may assume that the coordinates are arranged according to the lexicographical order. Each element of  $\mathbb{R}(\mathcal{F})$  is denoted as  $\mathbf{v} = (v_{\mathbf{a}} : \mathbf{a} \in \mathcal{F})$ . We use the symbol  $\mathcal{S}(\mathcal{F})_+$  for the set of  $|\mathcal{F}| \times |\mathcal{F}|$  symmetric positive semidefinite matrices with coordinates  $\mathbf{a} \in \mathcal{F}$ ; each  $\mathbf{V} \in \mathcal{S}(\mathcal{F})_+$  has elements  $V_{\mathbf{ab}}$  ( $\mathbf{a} \in \mathcal{F}, \mathbf{b} \in \mathcal{F}$ ) such that  $V_{\mathbf{ab}} = V_{\mathbf{ba}}$  and that  $\mathbf{w}^T \mathbf{V} \mathbf{w} = \sum_{\mathbf{a} \in \mathcal{F}} \sum_{\mathbf{b} \in \mathcal{F}} V_{\mathbf{ab}} w_{\mathbf{a}} w_{\mathbf{b}} \geq 0$  for every  $\mathbf{w} = (w_{\mathbf{a}} : \mathbf{a} \in \mathcal{F}) \in \mathbb{R}(\mathcal{F})$ . For every  $\mathbf{x} \in \mathbb{R}^n$ , let  $\mathbf{u}(\mathbf{x}, \mathcal{F}) = (\mathbf{x}^{\mathbf{a}} : \mathbf{a} \in \mathcal{F})$  be a column vector consisting of elements  $\mathbf{x}^{\mathbf{a}}$  ( $\mathbf{a} \in \mathcal{F}$ ).

The lemma below is well-known ([2, 6, 12, 14]).

**Lemma 4.3.** Let  $\mathcal{F}$  be a nonempty finite subset of  $\mathbb{Z}_+^n$ . A polynomial  $f(\mathbf{x})$  is contained in  $\Sigma(\mathcal{F})$  if and only if there exists a  $\mathbf{V} \in \mathcal{S}(\mathcal{F})_+$  such that

$$f(\mathbf{x}) = \mathbf{u}(\mathbf{x}, \mathcal{F})^T \mathbf{V} \mathbf{u}(\mathbf{x}, \mathcal{F}) = \sum_{\mathbf{a} \in \mathcal{F}} \sum_{\mathbf{b} \in \mathcal{F}} V_{\mathbf{ab}} \mathbf{x}^{\mathbf{a}+\mathbf{b}}. \quad (16)$$

Applying Lemma 4.3 to the polynomials  $\varphi_j(\mathbf{x}) \in \Sigma(\mathcal{A}_j^{(\lambda_j)})$  ( $j = 1, 2, \dots, m$ ),  $\psi_i(\mathbf{x}) \in \Sigma(\mathcal{B}_i^{(\mu_i)})$  ( $i = 1, 2, \dots, n$ ) and  $\varphi_0(\mathbf{x}) \in \Sigma(\mathcal{G}^*)$ , we represent as follows:

$$\begin{aligned} \varphi_j(\mathbf{x}) &= \mathbf{u}(\mathbf{x}, \mathcal{A}_j^q)^T \mathbf{V}^j \mathbf{u}(\mathbf{x}, \mathcal{A}_j^q), \quad \mathbf{V}^j \in \mathcal{S}(\mathcal{A}_j^q)_+, \\ \psi_i(\mathbf{x}) &= \mathbf{u}(\mathbf{x}, \mathcal{B}_i^q)^T \mathbf{V}^{m+i} \mathbf{u}(\mathbf{x}, \mathcal{B}_i^q), \quad \mathbf{V}^{m+i} \in \mathcal{S}(\mathcal{B}_i^q)_+, \\ \varphi_0(\mathbf{x}) &= \mathbf{u}(\mathbf{x}, \mathcal{G}^*)^T \mathbf{V}^0 \mathbf{u}(\mathbf{x}, \mathcal{G}^*), \quad \mathbf{V}^0 \in \mathcal{S}(\mathcal{G}^*)_+. \end{aligned}$$

Substituting these functions in the SOSOP (15) leads to

$$\left. \begin{array}{l} \text{maximize} \quad \zeta \\ \text{subject to} \quad \left. \begin{array}{l} f_0(\mathbf{x}) - \sum_{j=1}^m f_j(\mathbf{x}) \mathbf{u}(\mathbf{x}, \mathcal{A}_j^q)^T \mathbf{V}^j \mathbf{u}(\mathbf{x}, \mathcal{A}_j^q) \\ - \sum_{i=1}^n (\rho - x_i^2) \mathbf{u}(\mathbf{x}, \mathcal{B}_i^q)^T \mathbf{V}^{m+i} \mathbf{u}(\mathbf{x}, \mathcal{B}_i^q) \\ - \mathbf{u}(\mathbf{x}, \mathcal{G}^*)^T \mathbf{V}^0 \mathbf{u}(\mathbf{x}, \mathcal{G}^*) - \zeta = 0 \quad (\forall \mathbf{x} \in \mathbb{R}^n), \\ \mathbf{V}^j \in \mathcal{S}(\mathcal{A}_j^q)_+ \quad (j = 1, 2, \dots, m), \\ \mathbf{V}^{m+i} \in \mathcal{S}(\mathcal{B}_i^q)_+ \quad (i = 1, 2, \dots, n), \\ \mathbf{V}^0 \in \mathcal{S}(\mathcal{G}^*)_+. \end{array} \right\} \end{array} \right\}$$

Since the left hand side of the equality constraint in the problem above is a polynomial with the support

$$\mathcal{F}_C = \mathcal{F}_0 \cup \{0\} \cup \left( \bigcup_{j=1}^m \widehat{\mathcal{A}}_j^q \right) \cup \left( \bigcup_{i=1}^n \widehat{\mathcal{B}}_i^q \right) \cup \{\mathbf{a} + \mathbf{b} : \mathbf{a} \in \mathcal{G}^*, \mathbf{b} \in \mathcal{G}^*\},$$

and the coefficients are linear functions of matrix variable

$$\mathbf{V}^j \quad (j = 1, 2, \dots, m), \quad \mathbf{V}^{m+i} \quad (i = 1, 2, \dots, n), \quad \mathbf{V}^0$$

and  $\zeta$ , we can rewrite equality constraint of the problem above as

$$\sum_{\mathbf{a} \in \mathcal{F}_C} d(\mathbf{a}, \mathbf{V}, \zeta) \mathbf{x}^{\mathbf{a}} = 0,$$

where  $d(\mathbf{a}, \mathbf{V}, \zeta)$  is a linear function in the matrix variables  $\mathbf{V}^j$  ( $j = 0, 1, 2, \dots, m+n$ ) and a real variable  $\zeta$  for each  $\mathbf{a} \in \mathcal{F}_C$ . This identity needs to be satisfied for all  $\mathbf{x} \in \mathbb{R}^n$  in the problem, and the equality constraint is equivalent to a system of linear equations

$$d(\mathbf{a}, \mathbf{V}, \zeta) = 0 \quad (\mathbf{a} \in \mathcal{F}_C).$$

Consequently, we obtain the following SDP which is equivalent to the SOSOP (15).

$$\left. \begin{array}{l} \text{maximize} \quad \zeta \\ \text{subject to} \quad \left. \begin{array}{l} d(\mathbf{a}, \mathbf{V}, \zeta) = 0 \quad (\mathbf{a} \in \mathcal{F}_C), \\ \mathbf{V}^j \in \mathcal{S}(\mathcal{A}_j^q)_+ \quad (j = 1, 2, \dots, m), \\ \mathbf{V}^{m+i} \in \mathcal{S}(\mathcal{B}_i^q)_+ \quad (i = 1, 2, \dots, n), \\ \mathbf{V}^0 \in \mathcal{S}(\mathcal{G}^*)_+. \end{array} \right\} \end{array} \right\} \quad (17)$$

The numerical efficiency of solving an SDP depends mainly on its size. In the SDP (17) above, the number of equality constraint and the sizes of matrix variables are determined by the supports  $\mathcal{F}_j$  of the polynomial functions  $f_j(\mathbf{x})$  ( $j = 0, 1, \dots, m$ ) in the original POP (1) to be solved and  $q \in \mathbb{Z}_+$ . When the supports are sparse, the size of the resulting SDP becomes small. As we take a larger  $q \in \mathbb{Z}_+$ , we can expect to have a more accurate lower bound for the unknown optimal value  $\zeta^*$  of the POP (1), but the number of equality constraint and the size of the matrix variables increase.

#### 4.4 Sums of squares relaxation of $L_B^*(\Sigma^\infty)$

In this section, we derive a sequence of SOSOPs from the Lagrangian dual (7). The SOSOPs obtained here are less sparse than (15) in the previous section, but their optimal values converge to  $L_B^*(\Sigma^\infty)$ . Thus, this compensates the theoretical weakness of the SOSOP (15) whose optimal value is not guaranteed to converge to  $L_C^*(\Sigma^\infty \times \Xi^\infty)$ . Throughout this section, we assume that  $F \subset B_\rho$ ; hence  $L_C^*(\Sigma^\infty \times \Xi^\infty) = L_B^*(\Sigma^\infty) = \zeta^*$  is satisfied by Theorem 3.1.

As we have obtained the sequence of SOSOPs (15) from the Lagrangian dual (6), we can similarly derive the following sequence of SOSOPs from the Lagrangian dual (7):

$$\left. \begin{array}{l} \text{maximize} \quad \zeta \\ \text{subject to} \quad L_B(\mathbf{x}, \boldsymbol{\varphi}) - \zeta - \varphi_{m+1}(\mathbf{x})(\rho^2 - \mathbf{x}^T \mathbf{x}) = \varphi_0(\mathbf{x}) \quad (\forall \mathbf{x} \in \mathbb{R}^n), \\ \boldsymbol{\varphi} \in \Sigma^q, \varphi_{m+1} \in \Sigma(\mathcal{G}(\tau(q) - 1)), \varphi_0 \in \bar{\Sigma}. \end{array} \right\} \quad (18)$$

( $q \in \mathbb{Z}_+$ ). Here

$$\begin{aligned} \tau(q) &= \lceil (\text{the degree of } L_B(\mathbf{x}, \boldsymbol{\varphi}) \text{ with } \boldsymbol{\varphi} \in \Sigma^q) / 2 \rceil, \\ \mathcal{G}(\xi) &= \left\{ \mathbf{a} \in \mathbb{Z}_+^n : \sum_{i=1}^n a_i \leq \xi \right\} \quad (\xi = 0, 1, 2, \dots). \end{aligned}$$

Let  $\zeta_B^q$  denote the optimal value of the SOSOP (18);

$$\zeta_B^q = \sup \left\{ \zeta : \begin{array}{l} L_B(\mathbf{x}, \boldsymbol{\varphi}) - \zeta - \varphi_{m+1}(\mathbf{x})(\rho^2 - \mathbf{x}^T \mathbf{x}) = \varphi_0(\mathbf{x}), \\ \boldsymbol{\varphi} \in \Sigma^q, \varphi_{m+1} \in \Sigma(\mathcal{G}(\tau(q) - 1)), \varphi_0 \in \bar{\Sigma} \end{array} \right\}.$$

#### Theorem 4.4.

- (a)  $\zeta_B^q \leq L_B^*(\Sigma^\infty)$  ( $q \in \mathbb{Z}_+$ ).
- (b) In addition to (12), assume that  $\mathcal{A}_j^q \subset \mathcal{A}_j^{q+1}$  ( $j = 1, 2, \dots, m$ ,  $q \in \mathbb{Z}_+$ ). For any  $\epsilon > 0$ , there exists a  $\hat{q} \in \mathbb{Z}_+$  such that  $L_B^*(\Sigma^\infty) - \epsilon \leq \zeta_B^q$  ( $q \geq \hat{q}$ ).

*Proof:* (a) Let  $q \in \mathbb{Z}_+$ . Suppose that  $(\zeta, \boldsymbol{\varphi}, \varphi_{m+1}, \varphi_0) = (\bar{\zeta}, \bar{\boldsymbol{\varphi}}, \bar{\varphi}_{m+1}, \bar{\varphi}_0)$  is a feasible solution of (18). Then  $0 \leq L_B(\mathbf{x}, \boldsymbol{\varphi}) - \bar{\zeta}$  for every  $\mathbf{x} \in B_\rho$ ; hence  $\bar{\zeta} \leq L_B^*(\boldsymbol{\varphi}) \leq L_B^*(\Sigma^\infty)$ . This ensures that  $\zeta_B^q \leq L_B^*(\Sigma^\infty)$ .

(b) It follows from the assumption  $\mathcal{A}_j^q \subset \mathcal{A}_j^{q+1}$  ( $j = 1, 2, \dots, m$ ,  $q \in \mathbb{Z}_+$ ) that  $\Sigma^q \subset \Sigma^{q+1}$  ( $q \in \mathbb{Z}$ ). Let  $\epsilon > 0$ . By the construction of  $L_B^*(\Sigma^\infty)$ , there exist  $\hat{q} \in \mathbb{Z}_+$  and  $\hat{\boldsymbol{\varphi}} \in \Sigma^{\hat{q}}$  such that

$$L_B^*(\Sigma^\infty) - \epsilon/2 \leq L_B^*(\hat{\boldsymbol{\varphi}}).$$

Then

$$L_B(\mathbf{x}, \widehat{\boldsymbol{\varphi}}) - L_B^*(\boldsymbol{\Sigma}^\infty) + \epsilon$$

is a polynomial that is positive in the ball  $B_\rho$ . By Lemma 4.1 of [15],

$$L_B(\mathbf{x}, \widehat{\boldsymbol{\varphi}}) - L_B^*(\boldsymbol{\Sigma}^\infty) + \epsilon - \widehat{\varphi}_{m+1}(\mathbf{x})(\rho^2 - \mathbf{x}^T \mathbf{x}) \in \overline{\Sigma}$$

for some  $\widehat{\omega} \geq \tau(\widehat{q})$  and some  $\widehat{\varphi}_{m+1} \in \Sigma(\mathcal{G}(\widehat{\omega} - 1))$ . Choose a  $\tilde{q} \in \mathbb{Z}_+$  such that  $\widehat{q} \leq \tilde{q}$  and  $\tau(\tilde{q}) \geq \widehat{\omega}$ . Let  $\tilde{\omega} = \tau(\tilde{q})$ . Then

$$\begin{aligned} \zeta_B^{\tilde{q}} &= \sup \left\{ \zeta : \begin{array}{l} L_B(\mathbf{x}, \boldsymbol{\varphi}) - \zeta - \varphi_{m+1}(\mathbf{x})(\rho^2 - \mathbf{x}^T \mathbf{x}) = \varphi_0(\mathbf{x}), \\ \boldsymbol{\varphi} \in \boldsymbol{\Sigma}^{\tilde{q}}, \varphi_{m+1} \in \Sigma(\mathcal{G}(\tau(\tilde{q}) - 1)), \varphi_0 \in \overline{\Sigma} \end{array} \right\} \\ &\geq \sup \left\{ \zeta : \begin{array}{l} L_B(\mathbf{x}, \widehat{\boldsymbol{\varphi}}) - \zeta - \widehat{\varphi}_{m+1}(\mathbf{x})(\rho^2 - \mathbf{x}^T \mathbf{x}) = \varphi_0(\mathbf{x}), \\ \varphi_0 \in \overline{\Sigma} \end{array} \right\} \\ &\quad (\text{since } \widehat{\boldsymbol{\varphi}} \in \boldsymbol{\Sigma}^{\widehat{q}} \subset \boldsymbol{\Sigma}^{\tilde{q}} \text{ and } \widehat{\varphi}_{m+1} \in \Sigma(\mathcal{G}(\tau(\widehat{q}) - 1)) \subset \Sigma(\mathcal{G}(\tau(\tilde{q}) - 1))) \\ &\geq L_B^*(\boldsymbol{\Sigma}^\infty) - \epsilon. \end{aligned}$$

■

Comparing the SOSOP (15) with the SOSOP (18) shows that the latter problem involves the term  $\varphi_{m+1}(\mathbf{x})(\rho^2 - \mathbf{x}^T \mathbf{x})$  with  $\varphi_{m+1} \in \Sigma(\mathcal{G}(\tau(q) - 1))$  which is a fully dense polynomial with degree  $2\tau(q)$ ; hence we need to prepare a fully dense polynomial variable  $\varphi_0 \in \Sigma(\mathcal{G}(\tau(q)))$  in general. If we convert the SOSOP (18) into an SDP as converted the SOSOP (15) into the SDP (17), the polynomial variables  $\varphi_{m+1} \in \Sigma(\mathcal{G}(\tau(q) - 1))$  and  $\varphi_0 \in \Sigma(\mathcal{G}(\tau(q)))$  induce matrix variables

$$\mathbf{V}^{m+1} \in \mathcal{S}(\mathcal{G}(\tau(q) - 1))_+ \text{ and } \mathbf{V}^0 \in \mathcal{S}(\mathcal{G}(\tau(q)))_+,$$

respectively. Let  $\mathcal{F}_B$  denote the support of the polynomial

$$\begin{aligned} &L_B(\mathbf{x}, \boldsymbol{\varphi}) - \zeta - \varphi_{m+1}(\mathbf{x})(\rho^2 - \mathbf{x}^T \mathbf{x}) - \varphi_0(\mathbf{x}) \text{ with} \\ &\boldsymbol{\varphi} \in \overline{\Sigma}^q, \varphi_{m+1} \in \Sigma(\mathcal{G}(\tau(q) - 1)) \text{ and } \varphi_0 \in \Sigma(\mathcal{G}(\tau(q))). \end{aligned}$$

Then the resulting SDP formulation of the SOSOP (18) turns out to be

$$\left. \begin{array}{l} \text{maximize } \zeta \\ \text{subject to } \left. \begin{array}{l} d'(\mathbf{a}, \mathbf{V}, \zeta) = 0 \ (\mathbf{a} \in \mathcal{F}_B), \\ \mathbf{V}^j \in \mathcal{S}(\mathcal{A}_j^q)_+ \ (j = 1, 2, \dots, m), \\ \mathbf{V}^{m+1} \in \mathcal{S}(\mathcal{G}(\tau(q) - 1))_+, \\ \mathbf{V}^0 \in \mathcal{S}(\mathcal{G}(\tau(q)))_+. \end{array} \right\} \end{array} \right\} \quad (19)$$

Here  $d'(\mathbf{a}, \mathbf{V}, \zeta)$  denotes a linear function in the matrix variables  $\mathbf{V}^j$  ( $j = 0, 1, 2, \dots, m+1$ ) and a real variable  $\zeta$  for each  $\mathbf{a} \in \mathcal{F}_B$ . If the polynomials  $f_j(\mathbf{x})$  ( $j = 0, 1, 2, \dots, m$ ) are sparse, the size of the SDP (19) is larger than the size of the SDP (17); the number  $|\mathcal{F}_B|$  of linear equations in the SDP (19) is larger than the number  $|\mathcal{F}_C|$  of linear equations in the SDP (17), and the sizes of two matrix variables  $\mathbf{V}^{m+1}$  and  $\mathbf{V}^0$  in the SDP (19) are larger than the sizes of the matrix variables in the SDP (17).

## 4.5 Comparison to Lasserre's SDP relaxation

In this subsection, we briefly mention a modification of Lasserre's SDP relaxation [9] that takes account of the supports  $\mathcal{F}_j$  of the polynomials  $f_j(\mathbf{x})$  ( $j = 0, 1, \dots, m$ ) in the POP (3). The modification leads to the dual SDP of (19).

First, we describe the original Lasserre's SDP relaxation [9] applied to the POP (3) according to the interpretation given in the paper [5] on the relaxation. Let  $\lceil r/2 \rceil \leq N \in \mathbb{Z}_+$  and  $\nu_j = N - \lceil r_j/2 \rceil$  ( $j = 1, 2, \dots, m$ ). Consider the following polynomial optimization problem on positive semidefinite cones which is equivalent to the POP (1).

$$\left. \begin{array}{l} \text{minimize} \quad f_0(\mathbf{x}) \\ \text{subject to} \quad f_j(\mathbf{x})\mathbf{u}(\mathbf{x}, \mathcal{G}(\nu_j))\mathbf{u}(\mathbf{x}, \mathcal{G}(\nu_j))^T \in \mathcal{S}(\mathcal{G}(\nu_j))_+ \quad (j = 1, 2, \dots, m) \\ \quad (\rho^2 - \mathbf{x}^T \mathbf{x})\mathbf{u}(\mathbf{x}, \mathcal{G}(N-1))\mathbf{u}(\mathbf{x}, \mathcal{G}(N-1))^T \in \mathcal{S}(\mathcal{G}(N-1))_+, \\ \quad \mathbf{u}(\mathbf{x}, \mathcal{G}(N))\mathbf{u}(\mathbf{x}, \mathcal{G}(N))^T \in \mathcal{S}(\mathcal{G}(N))_+. \end{array} \right\} \quad (20)$$

We then apply a linearization to this problem to obtain an SDP relaxation of the POP (1). See the paper [5]

When the modification for the primal approach corresponding to (15) is implemented, the POP (3) is converted into a polynomial optimization problem on positive semidefinite cones:

$$\left. \begin{array}{l} \text{minimize} \quad f_0(\mathbf{x}) \\ \text{subject to} \quad f_j(\mathbf{x})\mathbf{u}(\mathbf{x}, \mathcal{A}_j^q)\mathbf{u}(\mathbf{x}, \mathcal{A}_j^q)^T \in \mathcal{S}(\mathcal{A}_j^q)_+ \quad (j = 1, 2, \dots, m), \\ \quad (\rho^2 - \mathbf{x}^T \mathbf{x})\mathbf{u}(\mathbf{x}, \mathcal{G}(\tau(q)-1))\mathbf{u}(\mathbf{x}, \mathcal{G}(\tau(q)-1))^T \in \mathcal{S}(\mathcal{G}(\tau(q)-1))_+, \\ \quad \mathbf{u}(\mathbf{x}, \mathcal{G}(\tau(q)))\mathbf{u}(\mathbf{x}, \mathcal{G}(\tau(q)))^T \in \mathcal{S}(\mathcal{G}(\tau(q)))_+. \end{array} \right\} \quad (21)$$

We obtain an SDP by applying the linearization to the problem (21). We can prove that the resulting SDP is the dual of (19) derived from the Lagrangian dual (7) of the POP (3).

Notice that  $N$  in the problem (20) corresponds to  $\tau(q)$  in the problem (21). Using Lemma 4.1 of [15] which we have used to prove for the convergence of  $\zeta_B^q$  to  $\zeta^*$ , Lasserre proved the convergence of the optimal value of the SDP derived from the problem (20) to the optimal value  $\zeta^*$  of the POP (1) as the parameter  $N$  tends to  $\infty$  in the paper [9]. An essential difference between the original and the modified Lasserre's SDP relaxations is that the modified relaxation takes account of the supports  $\mathcal{F}_j$  of the polynomials  $f_j(\mathbf{x})$  ( $j = 1, \dots, m$ ) in the POP (3). As the supports  $\mathcal{F}_j$  ( $j = 0, 1, 2, \dots, m$ ) are more sparse, the modified relaxation results in a smaller SDP relaxation. In other words, the SDP (17) derived from the Lagrangian dual (6) utilizes the sparsity of the POP (1) more effectively and therefore, its size is smaller than the SDP (19) derived from the Lagrangian dual (7) of the POP (3).

## 5 Proof of Lemma 3.2

(a) Suppose that  $\sqrt{2}\rho \leq \|\mathbf{x}\|_\infty$ . Let  $p \geq p^*$ . If  $f_j(\mathbf{x}) < 0$  then

$$f_j(\mathbf{x})\varphi_j^p(\mathbf{x}) = f_j(\mathbf{x})(1 - f_j(\mathbf{x})/\gamma)^{2p} < 0.$$

Otherwise, we obtain that

$$\begin{aligned}
0 &\leq f_j(\mathbf{x})\varphi_j^p(\mathbf{x}) \\
&= f_j(\mathbf{x})(1 - f_j(\mathbf{x})/\gamma)^{2p} \\
&\leq \gamma \|\mathbf{x}/\rho\|_\infty^r \max\{1, (f_j(\mathbf{x})/\gamma)^{2p}\} \\
&\quad (\text{by (10) and } |1 - f_j(\mathbf{x})/\gamma| \leq \max\{1, |f_j(\mathbf{x})/\gamma|\}) \\
&\leq \gamma \|\mathbf{x}/\rho\|_\infty^r (\|\mathbf{x}/\rho\|_\infty^r)^{2p} \quad (\text{by (10)}) \\
&= \gamma \|\mathbf{x}/\rho\|_\infty^{r(2p+1)}.
\end{aligned}$$

Hence

$$\sum_{j=1}^m f_j(\mathbf{x})\varphi_j^p(\mathbf{x}) \leq m\gamma \|\mathbf{x}/\rho\|_\infty^{r(2p+1)}.$$

On the other hand, we see that

$$\begin{aligned}
&\sum_{i=1}^n (\rho^2 - x_i^2)\psi_i^p(\mathbf{x}) \\
&= ((m+2)\gamma/\rho^2) \sum_{i=1}^n (\rho^2 - x_i^2) (x_i/\rho)^{2r(p+1)} \\
&= ((m+2)\gamma/\rho^2) \left( \sum_{x_i^2 \leq \rho^2} (\rho^2 - x_i^2) (x_i/\rho)^{2r(p+1)} + \sum_{\rho^2 < x_i^2} (\rho^2 - x_i^2) (x_i/\rho)^{2r(p+1)} \right) \\
&\leq (m+2)\gamma \left( n + (1 - \|\mathbf{x}/\rho\|_\infty^2) \|\mathbf{x}/\rho\|_\infty^{2r(p+1)} \right) \quad (\text{since } \{i \mid \rho^2 < x_i^2\} \neq \emptyset) \\
&\leq \gamma \left( (m+2)n - (m+2) \|\mathbf{x}/\rho\|_\infty^{2r(p+1)} \right) \quad (\text{since } \sqrt{2} \leq \|\mathbf{x}/\rho\|_\infty) \\
&\leq \gamma \left( (m+2)n - \|\mathbf{x}/\rho\|_\infty^{2r(p^*+1)} - (m+1) \|\mathbf{x}/\rho\|_\infty^{2r(p+1)} \right) \\
&\quad (\text{since } p^* \leq p \text{ and } \sqrt{2} \leq \|\mathbf{x}/\rho\|_\infty) \\
&\leq -(m+1)\gamma \|\mathbf{x}/\rho\|_\infty^{2r(p+1)} \\
&\quad (\text{by } \sqrt{2} \leq \|\mathbf{x}/\rho\|_\infty \text{ and } (m+2)n - 2^{r(p^*+1)} \leq 0).
\end{aligned}$$

Therefore

$$\begin{aligned}
&L_C(\mathbf{x}, \varphi^p, \psi^p) \\
&= f_0(\mathbf{x}) - \sum_{j=1}^m f_j(\mathbf{x})\varphi_j^p(\mathbf{x}) - \sum_{i=1}^n (\rho^2 - x_i^2)\psi_i^p(x_i) \\
&\geq f_0(\mathbf{x}) - m\gamma \|\mathbf{x}/\rho\|_\infty^{r(2p+1)} + (m+1)\gamma \|\mathbf{x}/\rho\|_\infty^{2r(p+1)} \\
&\geq f_0(\mathbf{x}) + \gamma \|\mathbf{x}/\rho\|_\infty^{2r(p+1)} \quad (\text{since } -m \|\mathbf{x}/\rho\|_\infty^{r(2p+1)} + m \|\mathbf{x}/\rho\|_\infty^{2r(p+1)} \geq 0) \\
&\geq -\gamma \|\mathbf{x}/\rho\|_\infty^r + \gamma \|\mathbf{x}/\rho\|_\infty^{2r(p+1)} \quad (\text{by (10)}) \\
&= \gamma \|\mathbf{x}/\rho\|_\infty^{2r(p+1)} \left( 1 - \|\mathbf{x}/\rho\|_\infty^{-r(2p+1)} \right) \\
&\geq \frac{1}{2} \|\mathbf{x}/\rho\|_\infty^{2r} \|\mathbf{x}/\rho\|_\infty^{2rp} \quad (\text{since } \gamma \geq 1, r \geq 1, p \geq 1 \text{ and } \|\mathbf{x}/\rho\|_\infty \geq \sqrt{2})
\end{aligned}$$



$$\geq \|\mathbf{x}/\rho\|_\infty^{2rp} \geq \|\mathbf{x}/\rho\|_\infty^2 \text{ (since } r \geq 1, p \geq 1 \text{ and } \|\mathbf{x}/\rho\|_\infty \geq \sqrt{2}).$$

Thus we have shown (a).

(b) Suppose that  $\tilde{\mathbf{x}} \notin F$  and  $\kappa > 0$ . If  $\|\tilde{\mathbf{x}}\|_\infty > \sqrt{2}\rho$ , take  $\delta > 0$  such that  $\|\mathbf{x}\|_\infty > \sqrt{2}\rho$  for every  $\mathbf{x} \in U_\delta(\tilde{\mathbf{x}})$ . Then the desired result follows from (a). Now assume that  $\|\tilde{\mathbf{x}}\|_\infty \leq \sqrt{2}\rho$ . By (10), we see that

$$|f_j(\tilde{\mathbf{x}})/\gamma| \leq 1 \quad (j = 0, 1, 2, \dots, m).$$

Since  $\tilde{\mathbf{x}} \notin F$ ,  $f_k(\tilde{\mathbf{x}}) < 0$  also holds for some  $k \in \{1, 2, \dots, m\}$ . Hence we can take  $\epsilon > 0$  and  $\delta > 0$  such that

$$f_k(\mathbf{x})/\gamma < -\epsilon \text{ and } |f_j(\mathbf{x})/\gamma| \leq 2 \quad (j = 0, 1, 2, \dots, m) \text{ for every } \mathbf{x} \in U_\delta(\tilde{\mathbf{x}}).$$

Let  $\tilde{p} \geq p^*$  be a positive integer such that

$$\epsilon(1 + \epsilon)^{2\tilde{p}} - 2 - 2m - (m + 2)n \geq \kappa/\gamma.$$

Then, for every  $\mathbf{x} \in U_\delta(\tilde{\mathbf{x}})$  and  $p \geq \tilde{p}$ ,

$$\begin{aligned} L_C(\mathbf{x}, \boldsymbol{\varphi}^p, \boldsymbol{\psi}^p) &= f_0(\mathbf{x}) - \sum_{j=1}^m f_j(\mathbf{x})\varphi_j^p(\mathbf{x}) - \sum_{i=1}^n (\rho^2 - x_i^2)\psi_i^p(\mathbf{x}) \\ &\geq -2\gamma - f_k(\mathbf{x})\varphi_k^p(\mathbf{x}) - \sum_{f_j(\mathbf{x}) \geq 0} f_j(\mathbf{x})\varphi_j^p(\mathbf{x}) - \sum_{\rho^2 - x_i^2 \geq 0} (\rho^2 - x_i^2)\psi_i^p(\mathbf{x}) \\ &= -2\gamma - f_k(\mathbf{x})(1 - f_k(\mathbf{x})/\gamma)^{2p} - \sum_{f_j(\mathbf{x}) \geq 0} f_j(\mathbf{x})(1 - f_j(\mathbf{x})/\gamma)^{2p} \\ &\quad - ((m + 2)\gamma/\rho^2) \sum_{\rho^2 - x_i^2 \geq 0} (\rho^2 - x_i^2)(x_i/\rho)^{2r(p+1)} \\ &\geq -2\gamma + \gamma\epsilon(1 + \epsilon)^{2p} - 2m\gamma - (m + 2)n\gamma \\ &\geq \gamma(\epsilon(1 + \epsilon)^{2p} - 2 - 2m - (m + 2)n) \\ &\geq \kappa. \end{aligned}$$

(c) Suppose that  $\hat{\mathbf{x}} \in F$  and  $\epsilon > 0$ . Then

$$\|\hat{\mathbf{x}}\|_\infty \leq \rho \text{ and } 0 \leq f_j(\hat{\mathbf{x}})/\gamma \leq 1 \quad (j = 1, 2, \dots, m).$$

We can take a  $\delta > 0$  such that

$$f_j(\hat{\mathbf{x}})/\gamma \leq 1.1 \quad (j = 1, 2, \dots, m) \text{ for every } \mathbf{x} \in U_\delta(\hat{\mathbf{x}}),$$

and  $\tilde{p} \geq p^*$  such that

$$\begin{aligned} 0 &\leq \eta(1 - \eta)^{2\tilde{p}} \leq \epsilon/(2m\gamma) \text{ if } 0 \leq \eta \leq 1.1, \\ 0 &\leq (1 - \xi)\xi^{r(\tilde{p}+1)} \leq \epsilon/((2m + 4)n\gamma) \text{ if } 0 \leq \xi \leq 1. \end{aligned}$$

Then, for every  $\mathbf{x} \in U_\delta(\hat{\mathbf{x}})$  and  $p \geq \tilde{p}$ ,

$$\begin{aligned}
\Phi_C(\mathbf{x}, \boldsymbol{\varphi}^p, \boldsymbol{\psi}^p) &= - \sum_{j=1}^m f_j(\mathbf{x}) \varphi_j^p(\mathbf{x}) - \sum_{i=1}^n (\rho^2 - x_i^2) \psi_i^p(\mathbf{x}) \\
&\geq - \sum_{f_j(\mathbf{x}) \geq 0} f_j(\mathbf{x}) (1 - f_j(\mathbf{x})/\gamma)^{2p} \\
&\quad - ((m+2)\gamma/\rho^2) \sum_{\rho^2 - x_i^2 \geq 0} (\rho^2 - x_i^2) (x_i/\rho)^{2r(p+1)} \\
&\geq - \sum_{f_j(\mathbf{x}) \geq 0} \gamma\epsilon/(2m\gamma) - (m+2)\gamma \sum_{\rho^2 - x_i^2 \geq 0} \epsilon/((2m+4)n\gamma) \\
&= -\epsilon \sum_{f_j(\mathbf{x}) \geq 0} 1/(2m) - \sum_{\rho^2 - x_i^2 \geq 0} \epsilon/(2n) \\
&\geq -\epsilon.
\end{aligned}$$

Thus we have shown (c).

(d) Define a compact subset  $A$  of  $\mathbb{R}^n$  by

$$A = \left\{ \mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_\infty \leq \sqrt{2}\rho \right\} \cup \left\{ \mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}/\rho\|_\infty^2 \leq \zeta^* \right\}.$$

In view of the property (a), we know that

$$\{\mathbf{x} \in \mathbb{R}^n : L_C(\mathbf{x}, \boldsymbol{\varphi}^p, \boldsymbol{\psi}^p) \leq \zeta^*\} \subset A \quad (p \geq p^*).$$

By the property (4) with  $(\boldsymbol{\varphi}, \boldsymbol{\psi}) = (\boldsymbol{\varphi}^p, \boldsymbol{\psi}^p)$ ,  $\mathbf{x} = \mathbf{x}^* \in F$  and  $\zeta^* = f_0(\mathbf{x}^*)$ , we also see that

$$L_C(\mathbf{x}^*, \boldsymbol{\varphi}^p, \boldsymbol{\psi}^p) \leq f_0(\mathbf{x}^*) = \zeta^* \quad (p \geq p^*) \quad \text{and} \quad \mathbf{x}^* \in F \subset A.$$

Hence

$$L_C^*(\boldsymbol{\varphi}^p, \boldsymbol{\psi}^p) = \inf \{L_C(\mathbf{x}, \boldsymbol{\varphi}^p, \boldsymbol{\psi}^p) : \mathbf{x} \in \mathbb{R}^n\} = \inf \{L_C(\mathbf{x}, \boldsymbol{\varphi}^p, \boldsymbol{\psi}^p) : \mathbf{x} \in A\} \quad (p \geq p^*).$$

Since  $A$  is compact,  $L_C(\mathbf{x}, \boldsymbol{\varphi}^p, \boldsymbol{\psi}^p)$  has a minimizer over  $\mathbb{R}^n$  at an  $\mathbf{x} = \mathbf{x}^p \in A$ ;

$$L_C^*(\boldsymbol{\varphi}^p, \boldsymbol{\psi}^p) = L_C(\mathbf{x}^p, \boldsymbol{\varphi}^p, \boldsymbol{\psi}^p) \quad \text{and} \quad \mathbf{x}^p \in A \quad (p \geq p^*)$$

Now assume on the contrary that the sequence  $\{L_C(\mathbf{x}^p, \boldsymbol{\varphi}^p, \boldsymbol{\psi}^p) \mid p \geq p^*\}$  does not converge to  $\zeta^*$ . Then we can take an  $\epsilon > 0$  and a subsequence  $\{L_C(\mathbf{x}^p, \boldsymbol{\varphi}^p, \boldsymbol{\psi}^p) \mid p \in J\}$  such that

$$L_C(\mathbf{x}^p, \boldsymbol{\varphi}^p, \boldsymbol{\psi}^p) < \zeta^* - \epsilon \quad (p \in J). \tag{22}$$

Since the sequence  $\{\mathbf{x}^p \mid p \in J\}$  lies in the compact set  $A$ , we can take a subsequence  $\{\mathbf{x}^p \mid p \in K\}$  with  $K \subset J$  which converges to some  $\bar{\mathbf{x}} \in A$ . Since  $L_C(\mathbf{x}^p, \boldsymbol{\varphi}^p, \boldsymbol{\psi}^p) \leq \zeta^*$  ( $p \in K$ ), we know by the property (b) that  $\bar{\mathbf{x}} \in F$ . By the property (c), there exists a  $\tilde{p} \geq p^*$  such that

$$f_0(\mathbf{x}^p) - \epsilon/2 \leq L_C(\mathbf{x}^p, \boldsymbol{\varphi}^p, \boldsymbol{\psi}^p) \leq \zeta^* \quad \text{if} \quad p \in K \quad \text{and} \quad p \geq \tilde{p}.$$

Taking the limit along the subsequence  $\{\mathbf{x}^p \mid p \in K\}$ , the left hand side  $f_0(\mathbf{x}^p) - \epsilon/2$  of the inequality above converges to  $f_0(\bar{\mathbf{x}}) - \epsilon/2 \geq \zeta^* - \epsilon/2$ . Hence

$$\zeta^* - \epsilon \leq L_C(\mathbf{x}^p, \boldsymbol{\varphi}^p, \boldsymbol{\psi}^p) \leq \zeta^* \quad \text{for every sufficiently large } p \in K.$$

This contradicts to the assumption (22). Therefore, (d) follows.

## 6 Concluding discussions

Considering two types of POPs (2) and (3) obtained from different characterizations of the feasible region of the POP (1), we have proposed a sequence of SOS relaxations from generalized Lagrangian duals of each POP. We have also provided a theoretical foundation on the convergence of the sequence of SOS relaxations to the optimal value of (3). The sequence of SOS relaxations have been transformed into a sequence of SDP relaxations to solve the POP (1) computationally. We have shown that the sequence of SDP relaxations derived here has primal-dual relationship with the one obtained after modifying Lasserre's SDP relaxation [9].

Theoretically, we have proved that the SOS relaxation of the Lagrangian dual of (3) attains the optimal value  $\zeta^*$  of the POP (3). But there remains a gap between the Lagrangian dual of (2) and its SOS relaxation in Section 4.2; the former attains  $\zeta^*$  but the latter is not guaranteed to attain  $\zeta^*$ . Thus it is interesting to prove or disprove that the SOS relaxation of the Lagrangian dual of (2) attains  $\zeta^*$ . This will be a subject of future study.

The size of the SOS relaxation or the SDP relaxation obtained from the Lagrangian dual approach by exploiting sparsity is smaller than the size of Lasserre's SDP relaxation. This is of course a nice feature, but this may not necessarily mean that the former SDP relaxation is as effective as the latter in practice. To attain an approximation to the optimal value  $\zeta^*$  of the POP (1) with as high accuracy as the one from Lasserre's SDP relaxation, we may need higher degree SOS polynomials in our dual approach, which makes the size of the resulting SDP relaxation larger.

SDP relaxations whose size are dependent on the number of elements in the support of a POP may provide numerical stability as well as computational efficiency. The coefficient matrices of the SDP relaxations in [9] include many zero elements as only the degree of polynomials in POP determines the size of SDP relaxations. The number of zero elements is expected to increase even more for sparse POPs. This may result in degenerate SDP relaxations that are difficult to solve computationally. Constructing the coefficient matrices of an SDP relaxation based on the support of the polynomials in a POP as in the dual approach may improve the degeneracy.

One of the advantages of the proposed method is that we have much flexibility in implementation of the SOS relaxation and the SDP relaxation of the (1); sets  $\Sigma$  and  $\Xi$  of Lagrangian multiplier SOS polynomials satisfying the assumption of Theorem 3.1 can be freely chosen to strengthen the resulting relaxations.

Extensive numerical experiments are necessary to investigate the issues mentioned above concerning practical performance of the proposed sequence of SOS relaxations and SDP relaxations. These include selecting reasonable degrees of SOS polynomials for Lagrangian multiplier, and various ways to improve efficiency and effectiveness of the relaxations by choosing certain sets of SOS polynomials for Lagrangian multiplier. This will also be a topic of future study.

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