Duality of linear conic problems

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Abstract

It is well known that the optimal values of a linear programming problem and its dual are equal to each other if at least one of these problems is feasible. It is also well known that for linear conic problems this property of "no duality gap" may not hold. It is then natural to ask whether there exist some other convex closed cones, apart from polyhedral, for which the "no duality gap" property holds. We show that the answer to this question is negative. We then discuss the question of a finite duality gap, when both the primal and dual problems are feasible, and pose the problem of characterizing linear conic problems without a finite duality gap.

Key words: Lagrangian duality, linear conic problems, duality gap.

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1 Introduction

Consider the linear conic problem

$$(P) \qquad \underset{x\in\mathcal{X}}{\text{Min}}\ \langle c,x\rangle \ \text{ subject to } \ Ax+b=0 \text{ and } x\in K.$$

Here $\mathcal{X}$ and $\mathcal{Y}$ are *finite* dimensional vector spaces equipped with respective scalar products (denoted $\langle\cdot,\cdot\rangle$), $A:\mathcal{X}\to\mathcal{Y}$ is a linear mapping and $K\subset\mathcal{X}$ is a closed convex cone. To a specific choice of the cone K corresponds a particular class of linear conic problems. For example, if $\mathcal{X}:=\mathbb{R}^n$ equipped with the standard scalar product $\langle\cdot,\cdot\rangle$, and $K:=\mathbb{R}^n_+$ is the nonnegative orthant, then problem (P) becomes a standard linear programming problem. If $\mathcal{X}:=\mathcal{S}^p$ is the space of $p\times p$ symmetric matrices and $K:=\mathcal{S}^p_+$ is the cone of $p\times p$ positive semidefinite symmetric matrices, then problem (P) becomes a semidefinite programming problem.

With the problem (P) is associated its (Lagrangian) dual

$$(D) \qquad \underset{y\in\mathcal{Y}}{\text{Max}}\ \langle b,y\rangle \ \text{ subject to } \ A^*y+c\in K^*,$$

where

$$K^*:=\{x^*\in\mathcal{X}:\langle x^*,x\rangle\geq 0,\ \forall\, x\in\mathcal{X}\}$$

is the dual (polar) of the cone K and $A^*:\mathcal{Y}\to\mathcal{X}$ is the adjoint of the linear mapping A. Consider the optimal values $\text{val}(P)$ and $\text{val}(D)$ of the problems (P) and (D), respectively. Note that, by the definition, $\text{val}(P)=+\infty$ if (P) is infeasible, and $\text{val}(D)=-\infty$ if (D) is infeasible. It is well known that the following, so-called weak duality, relation always holds $\text{val}(P)\geq\text{val}(D)$. It is said that there is *no duality gap* between problems (P) and (D) if $\text{val}(P)=\text{val}(D)$.

We have by the theory of linear programming that if the cone K is *polyhedral* (i.e., is defined by a finite number of linear constraints), then the following two properties hold:

(A) If either (P) or (D) is feasible, then there is no duality gap between (P) and (D).

(B) If both (P) and (D) are feasible, then there is no duality gap between (P) and (D) and the optimal values $\text{val}(P)$ and $\text{val}(D)$ are finite.

The fact: "if (P) and (D) are both feasible, then $\mathrm{val}(P)$ and $\mathrm{val}(D)$ are finite," follows immediately from the weak duality. Therefore, property (B) is a consequence of property (A).

Given a convex closed cone K, we say that property (A) or (B) holds if it holds for any vectors $c \in \mathcal{X}$, $b \in \mathcal{Y}$ and linear mapping $A : \mathcal{X} \to \mathcal{Y}$. It is natural then to ask the following question: "Are there any other classes of convex cones, apart from polyhedral, for which property (A) or (B) holds?"

We show in the next section that property (A) holds only for polyhedral cones. On the other hand property (B) can hold for nonpolyhegral cones as well. This motivates the following problem.

- Give a characterization of cones K for which property (B) holds.

We give some partial results related to the above problem. A complete characterization of such cones is an open question.

2 Preliminary results

We use the following notation and terminology throughout the paper. For a point $x \in K$ the radial cone $\mathcal{R}_K(x)$, to K at x, is given by the set of vectors $h \in \mathcal{X}$ such that $x + th \in K$ for some $t > 0$. A convex subset F of K is called a *face* of K if $x \in F$, $x_1, x_2 \in K$ and x belongs to the interval (x_1, x_2) implies that $x_1, x_2 \in F$. We say that a face F of K is nontrivial if $F \neq K$ and $F \neq \{0\}$. By $\mathrm{ri}(K)$ we denote the relative interior of K, and by $\dim \mathcal{X}$ the dimension of the space $\mathcal{X}$. Recall that if the cone K is polyhedral, then for any $x \in K$ the radial cone $\mathcal{R}_K(x)$ is polyhedral and closed.

The following result was given in [5] (see, also, [1, Proposition 2.193]).

Proposition 1 *Suppose that there exists a point $x \in K$ such that the radial cone $\mathcal{R}_K(x)$ to K at x is not closed. Then it is possible to construct a problem of the form (P) (i.e., to specify c,b and A) such that* $\mathrm{val}(P) = +\infty$, *while* $\mathrm{val}(D) = 0$.

We show next that a closed convex cone $K \subset \mathbb{R}^n$ is polyhedral iff the radial cone $\mathcal{R}_K(x)$ is closed at every point $x \in K$. This together with the above proposition imply that property (A) holds only for polyhedral cones.

Proposition 2 *Let $K \subset \mathbb{R}^n$ be a closed convex cone. Then K is polyhedral if and only if at every point $x \in K$ the radial cone $\mathcal{R}_K(x)$ is closed.*

Proof of this proposition is based on the following lemma.

Lemma 1 *Let $C \subset \mathbb{R}^n$ be a compact convex set. Then C is polyhedral if and only if at every point $x \in C$ the radial cone $\mathcal{R}_C(x)$ is closed.*

Proof. If the set C is polyhedral, then to every point $x \in C$ corresponds a neighborhood N of x such that $C \cap N$ coincides with $(x + \mathcal{R}_C(x)) \cap N$. This implies that if C is polyhedral, then $\mathcal{R}_C(x)$ is polyhedral and closed.

Conversely, suppose that $\mathcal{R}_C(x)$ is closed for every $x \in C$. We need to show then that the set C has a finite number of extreme points. To this end we use induction by n. Clearly this assertion is true for $n = 2$. We argue now by a contradiction. Suppose that the set C has an infinite number of extreme points. Since C is compact, it follows then that there exists a sequence $\{e_k\}$, of extreme points of C, converging to a point $a \in C$. By passing to a subsequence if necessary, we can assume that $h_k := (a - e_k)/\|a - e_k\|$ converges to a nonzero vector h. Since $e_k \in C$, it follows that h belongs to the topological closure of $\mathcal{R}_C(a)$. Since it is assumed that $\mathcal{R}_C(a)$ is closed, we obtain that $h \in \mathcal{R}_C(a)$, and hence $b := a + th \in C$ for some $t > 0$. Consider the hyperplane $L := \{x \in \mathbb{R}^n : \langle h, x \rangle = 0\}$ orthogonal to h, and the set $S := C \cap (b + L)$. Clearly the set S also has the property that for every $x \in S$ the radial cone $\mathcal{R}_S(x)$ is closed. By the induction assumption we have then that the set S is polyhedral. Consequently, there is a convex neighborhood $N \subset \mathbb{R}^n$ of the point b such that $S \cap N$ coincides with $(b + \mathcal{R}_S(b)) \cap N$. Finally, consider the set V equal to the convex hull of the point a and the set $S \cap N$. We have that V is a subset of C. Moreover, for all k large enough and some $t_k > 0$ we have that $a + t_k h_k \in S \cap N$, and hence $e_k \in V$ for all k large enough. We also have that if $e_k \in V$ and e_k is sufficiently close to a, such that $\|a - e_k\| < \|b\|$, then e_k cannot be an extreme point of C. This completes the proof. ∎

Proof of Proposition 2. Let K be a closed convex cone in $\mathbb{R}^n$ such that $\mathcal{R}_K(x)$ is closed for every $x \in K$. Convex closed cone K can be represented in the form $K = L + K'$, where L is the lineality subspace of K and $K' := K \cap L^\perp$ is a pointed cone. Therefore we can assume without loss of generality that the cone K is pointed, and hence the dual cone K^* has a nonempty interior. Let a be an interior point of K^*. Then $\langle a, x \rangle > 0$ for any $x \in K \setminus \{0\}$. It follows that the set $C := \{x \in K : \langle a, x \rangle = 1\}$ is bounded, and hence compact. By Lemma 1 we have then that C is polyhedral. It follows that K is polyhedral. ∎

3 Discussion of property (B)

Let us start by briefly describing some results from the conjugate duality theory of conic linear problems (Rockafellar [3, 4]). For an extended real valued function $f : \mathcal{X} \to \overline{\mathbb{R}}$ we denote by

$$\mathrm{lsc} f(x) := \min \left\{ f(x),\ \liminf_{x' \to x} f(x') \right\}$$

the lower semicontinuous hull of $f(\cdot)$, and by

$$f^*(x^*) = \sup_{x \in \mathcal{X}} \{ \langle x^*, x \rangle - f(x) \}$$

its conjugate function. By f^{**} we denote the conjugate of f^*.

With the problem (P) is associated the optimal value function

$$\vartheta(z) := \inf \{ \langle c, x \rangle : x \in K,\ Ax + b + z = 0 \}. \tag{1}$$

Clearly we have that $\vartheta(0) = \mathrm{val}(P)$. Moreover, the following properties hold: (i) the optimal value function $\vartheta(\cdot)$ is convex, (ii) $\mathrm{val}(D) = \vartheta^{**}(0)$, (iii) if $\mathrm{lsc}\,\vartheta(0) < +\infty$, then $\mathrm{val}(D) = \mathrm{lsc}\,\vartheta(0)$, and hence $\mathrm{val}(P) = \mathrm{val}(D)$, iff $\vartheta(z)$ is lower semicontinuous at $z = 0$, (iv) if problem (P) has a nonempty and bounded set of optimal solutions, then $\mathrm{val}(P) = \mathrm{val}(D)$, (v) if there exists $\bar{x} \in \mathrm{ri}(K)$ such that $A\bar{x} + b = 0$, then $\mathrm{val}(P) = \mathrm{val}(D)$ and, moreover, if $\mathrm{val}(D)$ is finite, then problem (D) has an optimal solution.

Proposition 3 *Suppose that every nontrivial face of K has dimension one. Then property* (B) *holds.*

Proof. Suppose that both problems (P) and (D) are feasible. It follows then by weak duality that both $\mathrm{val}(P)$ and $\mathrm{val}(D)$ are finite. Let $L := \{x \in \mathcal{X} : Ax + b = 0\}$ and

$$X := \{x \in K : Ax + b = 0\} = K \cap L$$

be the feasible set of (P), and let F be the minimal face of K containing X. If $F = K$, then L has a nonempty intersection with $\mathrm{ri}(K)$, and hence in that case $\mathrm{val}(P) = \mathrm{val}(D)$ by property (v). If $F = \{0\}$, then (P) has unique optimal solution $x^* = 0$, and hence again, by property (iv), $\mathrm{val}(P) = \mathrm{val}(D)$. So suppose that the face F is nontrivial. By the assumption, we have then that F has dimension one. If X is strictly included in F (i.e., $X \neq F$), then X consists from a single point and hence, by property (iv), it follows that $\mathrm{val}(P) = \mathrm{val}(D)$. So suppose that $X = F$. It follows that $0 \in X$, and hence $b = 0$. It also follows that $\mathrm{val}(P) \leq 0$. Furthermore, since (D) is feasible and $b = 0$, we have that $\mathrm{val}(D) = 0$ and hence $\mathrm{val}(P) = \mathrm{val}(D) = 0$. This completes the proof. ∎

Examples of convex closed cones such that all their nontrivial faces have dimension one are:

$$K := \left\{ x \in \mathbb{R}^{n+1} : x_{n+1} \geq (|x_1|^p + ... + |x_n|^p)^{\frac{1}{p}} \right\}, \quad \text{for } p \in (1, +\infty). \tag{2}$$

For $p = 2$ the above is the so-called Lorenz, or ice-cream, cone.

Proposition 4 *Suppose that* $\dim \mathcal{X} \leq 3$. *Then property* (B) *holds for any convex closed cone* $K \subset \mathcal{X}$.

Proof. If $\dim \mathcal{X} \leq 2$, then property (B) holds by Proposition 3. So suppose that $\dim \mathcal{X} = 3$. As in the proof of Proposition 3, let L be the affine space defined by the equations $Ax + b = 0$ and F be the minimal face of K containing the feasible set $X = K \cap L$. If $\dim F \leq 1$ or $F = K$, then we can proceed as in the proof of Proposition 3. So suppose that $\dim F = 2$. Again if X is bounded, or $X = F$, or L has a common point with $\mathrm{ri}(K)$, then $\mathrm{val}(P) = \mathrm{val}(D)$. So we only need to consider the case where L is one dimensional and the feasible set X forms a half ray. Since $\mathrm{val}(P)$ is finite, it follows then the set of optimal solutions of problem (P) either coincide with X or is formed by the unique extreme point of X. We also have then that under small perturbations of vector $b \mapsto b'$, the feasible set of the perturbed problem is either empty or is a half ray parallel to L with the extreme pint converging to the corresponding extreme point of X as b' tends to b. It follows that the associated optimal value function $\vartheta(z)$ is lower semicontinuous at $z = 0$, and hence the proof is complete. ∎

By the above proposition we have that the minimal dimension of the space $\mathcal{X}$ for which property (B) may not hold is 4. And, indeed, the following is an example of conic linear problems in $\mathbb{R}^4$ with a finite gap.

Example 1 Consider cone K given by the Cartesian product $K_1 \times K_2$ of cones $K_1 := \mathbb{R}_+$ and $K_2 \subset \mathbb{R}^3$, with K_2 being a cone of the form (2) for some $p \in (1, +\infty)$. Let b_1, b_2 be nonzero vectors in $\mathbb{R}^3$ such that the ray $R := \{tb_2 : t \geq 0\}$ forms a face of K_2, b_1 is orthogonal to b_2, and the plane $L := \{t_1 b_1 + t_2 b_2 : t_1, t_2 \in \mathbb{R}\}$ supports K_2, i.e., $K \cap L = R$. For some $a > 0$, consider the following linear conic problem:

$$\underset{(x_1,x_2)\in\mathbb{R}^2}{\text{Min}}\ x_1 \ \text{ subject to }\ x_1 + a \in K_1,\ x_1 b_1 + x_2 b_2 \in K_2. \tag{3}$$

By the above construction we have that the feasible set of the above problem is $\{(x_1, x_2) : x_1 = 0,\ x_2 \geq 0\}$, and hence its optimal value is 0.

Consider the optimal value function

$$\vartheta(z) := \inf\{x_1 : x_1 + a + z_1 \in K_1,\ x_1 b_1 + x_2 b_2 + z_2 \in K_2\} \tag{4}$$

associated with problem (3). Clearly the constraint $x_1 + a + z_1 \in K_1$ means that $x_1 \geq -a - z_1$. For $z_2 \in L$ the constraint $x_1 b_1 + x_2 b_2 + z_2 \in K_2$ defines the ray R, while for $z_2 \notin L$ this constraint is either infeasible or defines a subset of $\mathbb{R}^2$ with unbounded coordinate x_1. It follows that $\text{lsc}\vartheta(0) = -a$, and hence $\text{val}(D) = -a$. That is, there is a finite gap between problem (3) and its dual.

The construction of the above example can be extended to a general case of cones of the form $K = K_1 \times K_2$, where the cone $K_2 \subset \mathbb{R}^m$ is not polyhedral.

Example 2 Consider the cone

$$K' := \left\{ (x, t) \in \mathbb{R}^n_+ \times \mathbb{R}_{++} : \sum_{i=1}^n e^{-\frac{x_i}{t}} \leq 1 \right\}. \tag{5}$$

This cone is convex, but not closed. Its topological closure is obtained by adding to K' the set $\mathbb{R}^n_+ \times \{0\}$. The obtained cone K appears in a linear conic formulation of geometric optimization and satisfies property (B), Glineur [2]. This cone is not polyhedral, therefore by the discussion of section 2 it does not satisfy property (A).

References

[1] J.F. Bonnans and A. Shapiro, *Perturbation Analysis of Optimization Problems*, Springer, New York, 2000.

[2] F. Glineur, Proving strong duality for geometric optimization using a conic formulation, *Annals of Operations Research*, 105: 155-184, 2001.

[3] R.T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, NJ, 1970.

[4] R.T. Rockafellar, *Conjugate Duality and Optimization*, Regional Conference Series in Applied Mathematics, SIAM, Philadelphia, 1974.

[5] A. Shapiro, Optimality conditions and sensitivity analysis of cone-constrained and semi-definite programs, in: *Topics in Semidefinite and Interior-Point Methods*, Fields Inst. Commun., 18, Amer. Math. Soc., Providence, RI, pp. 1–15, 1998.