Extreme Point Solutions for Infinite Network Flow Problems*

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Abstract

We study capacitated network flow problems with supplies and demands defined on a countably infinite collection of nodes having finite degree. This class of network flow models includes, for example, all infinite horizon deterministic dynamic programs with finite action sets since these are equivalent to the problem of finding a shortest infinite path in an infinite directed network. We derive necessary and sufficient conditions for flows to be extreme points of the set of feasible flows. Under a regularity condition met by all such problems with integer data, we show that a feasible solution is an extreme point if and only if it contains no finite or infinite cycles of free arcs (an arc is free if its flow is strictly between its upper and lower bounds). We employ this characterization to establish the integrality of extreme point flows whenever demands and supplies and arc capacities are integer valued. We illustrate our results with an application to an infinite horizon economic lot-sizing problem.

1 Introduction

An important class of problems is formed by planning problems over time, e.g., production planning problems and equipment replacement problems. Many of these planning problems can quite naturally be formulated as network flow problems. When, as is usually the case, there is no natural finite planning horizon that can be specified a priori, these planning problems are formulated as problems over an infinite horizon. For examples, we refer to Bean and Smith [4, 5], Bès and Sethi [6], Romeijn et al. [10], and Schochetman and Smith [12, 13]. Such problems then give rise to network flow problems in networks with infinitely

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many nodes and arcs. In this paper, we focus on capacitated network flow problems with supplies and demands defined on a countably infinite collection of nodes having finite degree. This class of network flow models includes, for example, all infinite horizon deterministic dynamic programs with finite action sets since these are equivalent to the problem of finding a shortest infinite path in an infinite directed network.

Network flow problems form a very well-studied area. The excellent book by Ahuja, Magnanti and Orlin [1] describes the state of the art in designing algorithms for various types of network flow problems. To date, however, attention has almost exclusively been focused on network flow problems in a network containing only a finite number of nodes and arcs (a notable exception being the work by Fuchssteiner and Morisse [9], who studied network flow problems in networks with an uncountable number of nodes). Many optimization algorithms for finite network flow problems, in particular for linear and concave minimum cost network flow problems, employ a characterization of extreme points of such problems using spanning trees. Progress on developing algorithms for infinite network flow problems has been thwarted by the difficulty of extending such a characterization to the infinite dimensional case. Although Anderson and Nash [3] characterized extreme point solutions through basic and nonbasic variables for general doubly infinite linear programs, the abstract nature of these results prevented their further exploitation. The lack of and need for concrete characterizations of extreme points for the case of infinite network flow problems motivated our work.

In this paper, we extend concepts and structural properties of solutions to network flow problems from the finite to the infinite case. We derive properties of the set of all feasible solutions (i.e., flows that satisfy all flow balance and bound constraints) and establish a relationship between extreme point solutions to the network flow problem and trees in the network, generalizing an analogous property of the finite version of the problem. In particular, we provide necessary and sufficient conditions for a feasible solution to be an extreme point. Under an additional so-called non-vanishing support assumption on arc flows in extreme point solutions we show that all extreme point solutions can be uniquely characterized via a specification of a set of free variables. We find that, as in finite networks, in infinite networks in which all node demands and supplies and all arc capacities are integer valued, all extreme points have integral flows, which in turn yields that the non-vanishing support assumption is met. As an example, we study a capacitated economic lot-sizing problem with concave costs, and extend the finite-horizon characterization of all (including the optimal) extreme point solutions to such problems (see e.g. Denardo [7]) to the infinite horizon case.

The paper is organized as follows. In Section 2 we introduce the necessary notation for the network flow problem, and extend some concepts from finite graph theory to the infinite case. In Section 3 we derive structural properties of feasible flows as well as flows corresponding to extreme points of the set of all feasible flows. In Section 4 we study a production planning problem over an infinite horizon with concave costs. We end in Section 5 with some concluding remarks and directions for future research.

2 Notation and Definitions

In this section, we introduce the notation and some basic definitions used throughout the paper. While most of the definitions introduced here are commonly used for finite networks, some definitions are extended or introduced to specifically deal with infinite networks.

2.1 Network definitions

Let $G = (N, A)$ be a directed network consisting of a countable set $N = \{1, 2, 3, \ldots\}$ of nodes and a set $A \subseteq N \times N$ of arcs. Let $d \in \mathbb{R}^{|N|}$ with typical element $d(i)$ represent the demand for each node, i.e., $d(i)$ is the demand associated with node $i \in N$. For each node $i \in N$, we define $\delta^-(i)$ and $\delta^+(i)$ to be the set of incoming and outgoing arcs at node i, respectively. We assume that the in and out degree of each node $i \in N$ is finite, i.e., $|\delta^-(i)| < \infty$ and $|\delta^+(i)| < \infty$. Let $\ell, u \in \mathbb{R}^{|A|} \cup \{-\infty, \infty\}$ with typical elements $\ell(i,j)$ and $u(i,j)$ be the vectors of lower and upper bounds, respectively, for the flows on arcs in A. Without loss of generality, we assume that $\ell(i,j) < u(i,j)$ for all $(i,j) \in A$.

We define a *path* in the graph G to be a collection of arcs in G representing a sequence of nodes $i_1 - i_2 - i_3 - \cdots$ such that no node is repeated in the sequence and for each k, either $(i_k, i_{k+1}) \in A$ or $(i_{k+1}, i_k) \in A$. If the path corresponds to a finite sequence then it is called a *finite path*. A *cycle* in the graph G is defined as a finite path $i_1 - i_2 - i_3 - \cdots - i_k$ with an additional arc (i_k, i_1) or (i_1, i_k). Given a cycle C (or a path P), we shall use $A(C)$ (or $A(P)$) to denote the set of arcs in the cycle (or path). We say that two vertices $i, j \in N$ are *connected* if there exists a finite path $i_1 - i_2 - \cdots - i_k$ in G with $i_1 = i$ and $i_k = j$. The graph G is called *connected* if every pair of nodes in G is connected. In this paper, we shall assume that the graph G is connected.

A subgraph $T = (N', A')$ of G is called a *tree* if T is connected and it does not contain any cycles. Note that there must be a unique finite path connecting each pair of nodes in T. We can designate some node $r \in N'$ to be the *root* of T. In this case, the tree T is called a *rooted tree*. The designation of a node as a root node allows us to associate additional properties with the nodes in a tree. Given a rooted tree $T = (N', A')$ with root r, we define the *depth* $\lambda(i)$ of a node $i \in N'$ as the number of arcs in the unique path connecting root node r and node i. If (j,i) or (i,j) is the arc on the unique path connecting node i and node r then j is called the *parent* of i, denoted by $p(i)$, and i is called a *child* of j. For a node $i \in N'$, we define $C(i)$ as the set of children of i. The set N^i of *descendants* of a node $i \in T$ consists of node i, its children, children of its children, and so on. We say that node i is a *leaf* node if it is the only descendant of itself. We define the *subtree* $T^i = (N^i, A^i)$ at a node $i \in N'$ as the subgraph of T that is a tree rooted at node i and consists of the descendants of node i in T. We say that a set of arcs $S \subseteq A'$ is a *finite cut* in the rooted tree T if there exists some $n < \infty$ such that for all $i \in N'$ with $\lambda(i) > n$, the nodes i and r are not connected in the graph $(N', A' \setminus S)$. Note that any finite cut contains only a finite number of arcs, i.e., $|S| < \infty$. We denote the *forward arcs* in a finite cut S by $S^+ = \{(i,j) \in S : \lambda(i) = \lambda(j) - 1\}$ and *backward arcs* in S by $S^- = \{(i,j) \in S : \lambda(i) = \lambda(j) + 1\}$. In Figure 1 we illustrate a tree and a subtree. The arcs $S = \{(5,2), (3,6)\}$ form a finite cut where as the arcs $\{(2,4), (3,6)\}$ do not form a finite cut. In the first case, $S^+ = \{(3,6)\}$ and $S^- = \{(5,2)\}$. We denote the set of finite cuts in a rooted tree T by $\mathbf{S}(T)$.

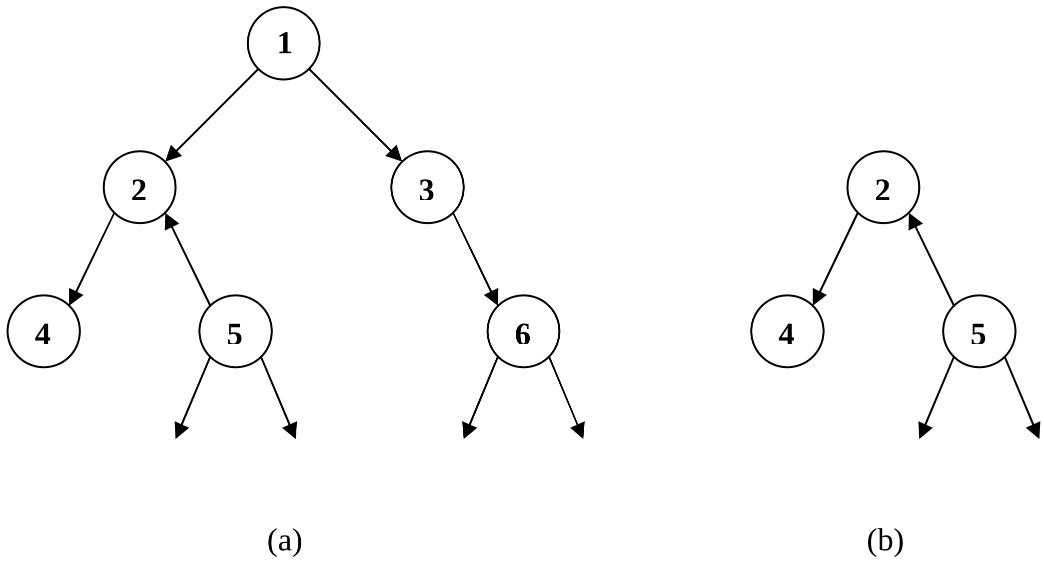

Figure 1: (a) A tree T, and (b) its subtree T^2.

2.2 Feasible flows and extreme points

A vector $x \in \mathbb{R}^{|A|}$ with typical element $x(i,j)$ is called a *feasible flow* if for each node $i \in N$, the flow balance constraint

$$\sum_{j \in N:(j,i) \in \delta^-(i)} x(j,i) - \sum_{j \in N:(i,j) \in \delta^+(i)} x(i,j) = d(i) \tag{1}$$

is satisfied, and $\ell(i,j) \leq x(i,j) \leq u(i,j)$ for all $(i,j) \in A$. We assume that $\mathbb{R}^{|A|}$ is a product space equipped with the product topology inherited from the underlying Euclidean spaces. This means that a sequence $x_1, x_2, \ldots$ where $x_n \in \mathbb{R}^{|A|}$ for all n converges to $x \in \mathbb{R}^{|A|}$ precisely when all its components $x_n(i,j)$ converge to $x(i,j)$ in the ordinary Euclidean metric on $\mathbb{R}$ for all (i,j).

A feasible flow x is an *extreme point* of the set of feasible flows if there do not exist feasible flows x^1 and x^2 such that $x^1 \neq x^2$ and $x = (x^1 + x^2)/2$. Given a feasible flow x, we define $r(i,j) = \min\{x(i,j) - \ell(i,j), u(i,j) - x(i,j)\}$ as the maximum amount by which the flow on (i,j) can be increased or decreased without violating bounds. We define the *free arc graph of* x as the graph $G(x) = (N, A(x))$ where $A(x) = \{(i,j) \in A : r(i,j) > 0\}$ is the set of free arcs, i.e., arcs whose flow can be increased as well as decreased. We define the set of arcs with their flow equal to their upper bound as $U(x) = \{(i,j) \in A : x(i,j) = u(i,j)\}$ and arcs with flow equal to their lower bound as $L(x) = \{(i,j) \in A : x(i,j) = \ell(i,j)\}$. We note that $A(x)$, $U(x)$, and $L(x)$ are pairwise disjoint and $A(x) \cup U(x) \cup L(x) = A$.

3 Extreme Point Flows

In this section we develop necessary and sufficient conditions for a feasible flow x in a graph G to be an extreme point. However, we first address the issue of the existence of an extreme point in an infinite network flow problem.

Theorem 3.1 *Given a network G with demands d and* finite *lower and upper bounds $\ell, u \in \mathbb{R}^{|A|}$, if the network flow problem is feasible then an extreme point solution exists.*

Proof: The set of solutions of the network flow problem is clearly convex. Further, since the value of each arc flow is bounded, the solution space is also compact with respect to the product topology. The Krein-Milman theorem (see Aliprantas and Border [2]) then implies that, if the solution space is non-empty, it contains an extreme point. □

Note that all results in this paper remain to hold if some or all of the flow bounds are infinite. However, in that case the set of extreme points may be empty.

3.1 Why the finite extreme point characterization fails for infinite networks

Given the result of Theorem 3.1, there always exists an extreme point solution whenever the problem is feasible. We next derive a basic condition that is necessary for a flow to be an extreme point.

Lemma 3.2 *If a feasible flow x is an extreme point then $G(x)$ contains no cycles.*

Proof: Let x be a feasible flow in G, and suppose $G(x)$ contains a cycle $C = i_1 - i_2 - \cdots - i_p - i_1$. Let $\Delta = \min_{(i,j)\in A(C)} r(i,j)$. By definition of $G(x)$ we know that $\Delta > 0$. We will create two flows x^1, x^2 such that $x^1 \neq x^2$ and $x = (x^1 + x^2)/2$. For $k = 1, \ldots, p$, if (i_k, i_{k+1}) is an arc in $A(C)$ then set $x^1(i_k, i_{k+1}) = x(i_k, i_{k+1}) + \Delta$ and $x^2(i_k, i_{k+1}) = x(i_k, i_{k+1}) - \Delta$; otherwise, set $x^1(i_{k+1}, i_k) = x(i_{k+1}, i_k) - \Delta$ and $x^2(i_{k+1}, i_k) = x(i_{k+1}, i_k) + \Delta$. For all remaining arcs $(i,j) \in A$, set $x^1(i,j) = x^2(i,j) = x(i,j)$. Clearly, both x^1 and x^2 are feasible flows, $x^1 \neq x^2$, and $x = (x^1 + x^2)/2$, so x is not an extreme point. □

We note that the converse of Lemma 3.2 holds for network flow problems over finite networks: if $G(x)$ contains no cycle then x is an extreme point. Another way of putting this is that, in the case of finite networks, an extreme point solution can always be specified by providing a set of *restricted* arcs, i.e., arcs in the network whose flow is restricted to be at their lower or upper bound. Such arcs are called *restricted arcs at lower bound* and *restricted arcs at upper bound*, respectively. The flows on all remaining arcs, i.e., all so-called *free arcs* in the free arc graph $G(x)$, are *uniquely* determined by the flows on the restricted arcs. *However, for networks with an infinite number of nodes the converse of Lemma 3.2 is not true.* We will illustrate this with three examples.

Example 1. Our first example considers an infinite network which basically consists of a pair of infinite paths connected at one of the nodes. In particular, consider the infinite graph

$G = (N, A)$ where $N = 1, 2, \ldots$ and $A = \{(2i+1, 2i-1) : i = 1, 2, \ldots\} \cup \{(1, 2)\} \cup \{(2i, 2i+2) : i = 1, 2, \ldots\}$. This network does not contain any cycles. Let all nodes be transshipment nodes, i.e., the demands are $d(i) = 0$ for all $i \in N$. Moreover, let the lower and upper bounds on all arcs be $\ell(i, j) = L$ and $u(i, j) = U$ for all $(i, j) \in A$, where $L < U$ are finite numbers. The network is illustrated in Figure 2. Note that, for any constant K such that $L < K < U$, the constant flow given by $x(i, j) = K$ for all $(i, j) \in A$ is a feasible flow with $G(x) = G$. However, the flow x is not an extreme point flow because it can be written as $x = (x^1 + x^2)/2$ where x^1 and x^2 are constant flows with values $K - \Delta$ and $K + \Delta$, respectively, for $\Delta = \min\{K - L, U - K\}$. The only two extreme points of this network flow problem are given by constant flows of L or U.

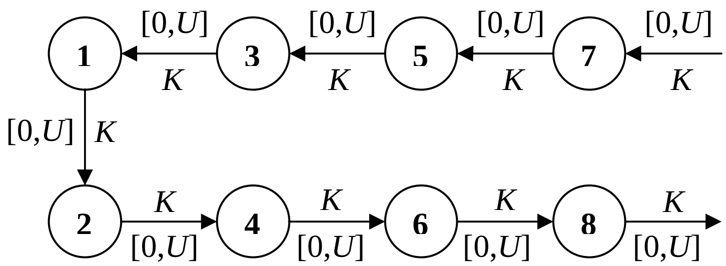

Figure 2: A network with no cycles.

Example 2. Our second example considers an infinite graph $G = (N, A)$ that is similar to the graph in Example 1. The node set is again $N = \{1, 2, \ldots\}$, but the arc set is now chosen to be $A = \{(2i - 1, 2i + 1) : i = 1, 2, \ldots\} \cup \{(1, 2)\} \cup \{(2i, 2i + 2) : i = 1, 2, \ldots\}$. This network does not contain any cycles. We choose the demands to be given by: $d(1) = -2$, and $d(2i) = d(2i + 1) = 1/2^{i+1}$ for $i \in \{1, 2, \ldots\}$. The lower and upper bounds on each arc are set to 0 and 2, respectively. The network is shown in Figure 3. By construction, any feasible flow x would satisfy $0 < x(i, j) \leq 3/2$ for all $(i, j) \in A$. Hence, for any feasible flow x we have that $G(x) = G$. However, there are only two feasible solutions that are extreme points. The first solution is given by $x(2i - 1, 2i + 1) = 1 + 1/2^i$ for $i \in \{1, 2, \ldots\}$, $x(1, 2) = 1/2$, and $x(2i, 2i + 2) = 1/2^i$ for $i \in \{1, 2, \ldots\}$. The second extreme point is given by $x(2i - 1, 2i + 1) = 1/2^i$ for $i \in \{1, 2, \ldots\}$, $x(1, 2) = 3/2$, and $x(2i, 2i + 2) = 1 + 1/2^i$ for $i \in \{1, 2, \ldots\}$. Note that the total demand of all demand nodes is equal to 1, whereas the supply of the only supply node is equal to 2. Therefore, there is one unit of flow that is shipped from the supply node for which no demand exists. An extreme point in the network corresponds to a solution in which the excess unit supplied flows through either the infinite path $1 - 2 - 4 - 6 - \cdots$ or the infinite path $1 - 3 - 5 - 7 - \cdots$. Any feasible flow in which the excess unit supplied is shared by these two infinite path is not an extreme point.

Example 3. Our third example is the case of an infinite binary tree (see Figure 4). In this network, we assume that the root node has a supply of 1 unit (i.e., $d(1) = -1$), and all other nodes are transshipment nodes ($d(i) = 0$ for all $i \in N \setminus \{1\}$. Furthermore, let the lower and upper bounds on the arc flows be 0 and 2 units, respectively. Figure 4 shows a

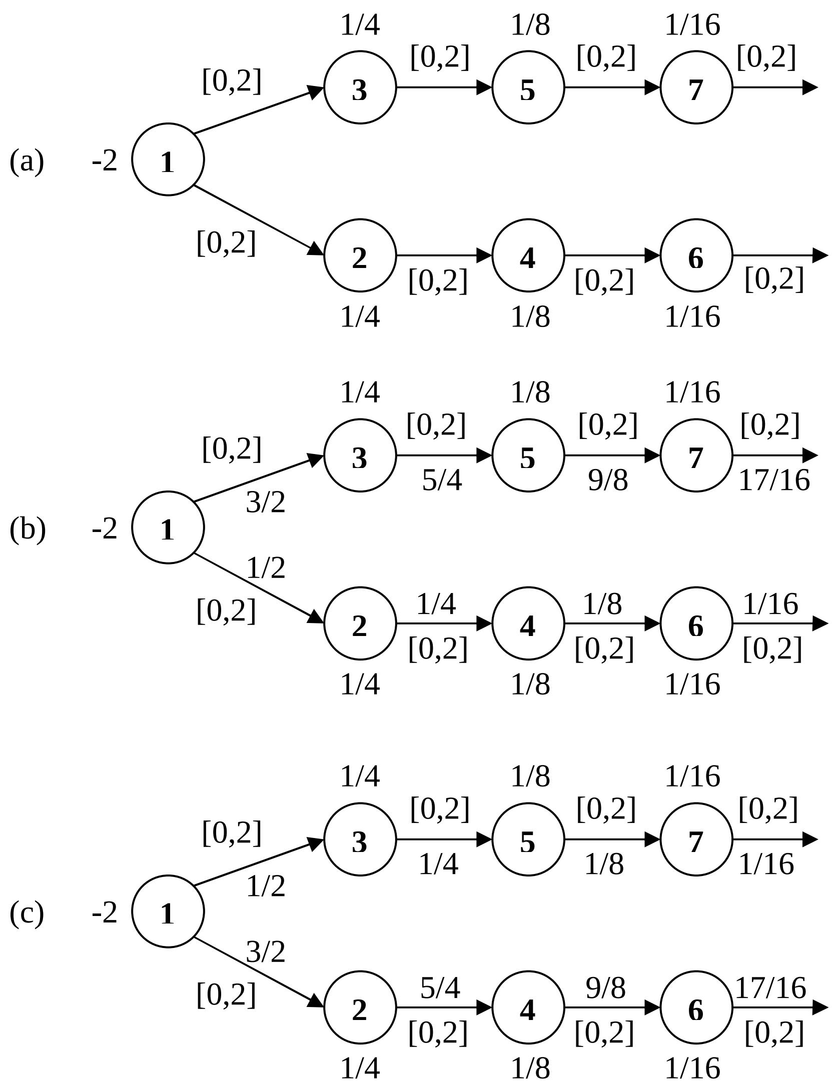

Figure 3: (a) Example network flow problem; (b) first extreme point; (c) second extreme point.

feasible solution with positive flow on all arcs in the network. Moreover, it is clear that the free arc graph corresponding to *any* feasible flow does not contain any cycles. However, the only extreme points in this network flow problem are feasible flows that carry a single unit along a single infinite path from the root node 1.

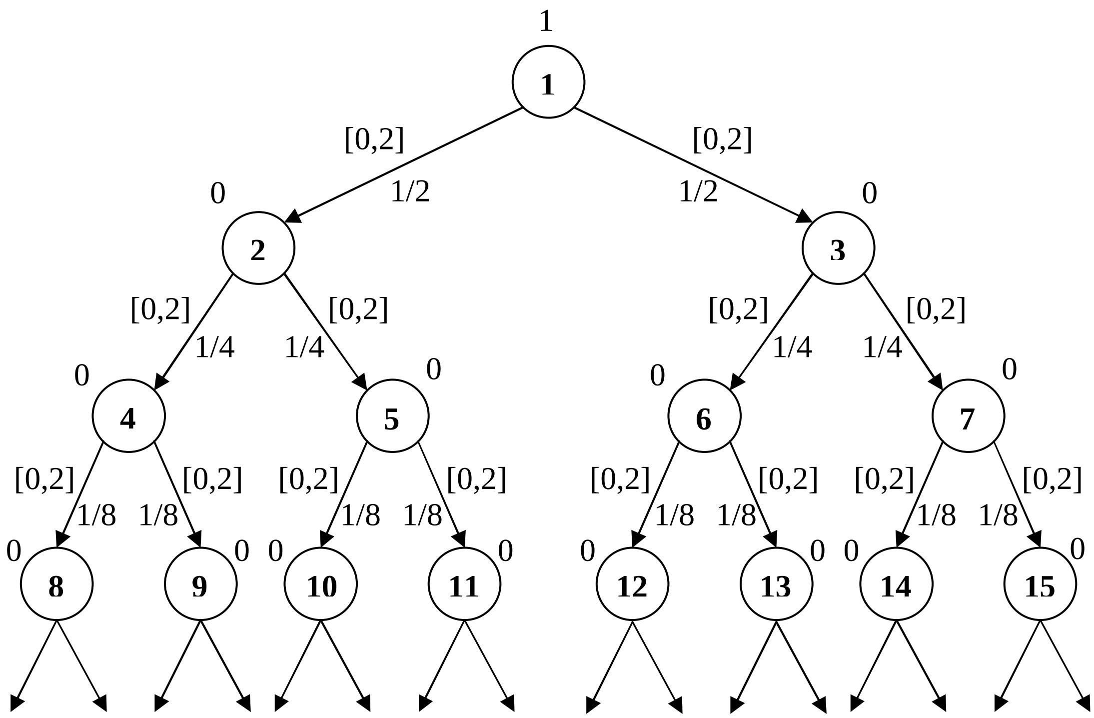

Figure 4: An infinite binary tree.

All three examples show that extreme points in infinite networks cannot in general be characterized by specifying an acyclic subgraph. The main cause of this failure to generalize the characterization of extreme points in network flow problems from finite to infinite networks is that a given acyclic free arc graph (together with flows on the restricted arcs) uniquely determines the flows on the free arcs in the finite case, but does not uniquely determine these flows in the infinite case. Hence, an extreme point is not solely determined by specifying the free and restricted arcs in the case of infinite networks. In the remainder of this section, we will provide a set of necessary and sufficient conditions for a given feasible flow to be an extreme point in an infinite network flow problem.

3.2 Characterizing extreme points in infinite networks

From Lemma 3.2, it follows that if x is an extreme point then any connected subgraph of $G(x)$ must be a tree. We note that with each node $i \in N$, we can associate a *maximal tree* $T = (N', A')$ in the graph $G(x)$ given by the node set $N' = \{i' \in N : i \text{ and } i' \text{ are connected in } G(x)\}$. It can be easily shown that the maximal trees associated with any pair of nodes

$i, j \in N$ that are connected are identical. Therefore, the graph $G(x)$ consists of one or more (and possibly infinitely many) maximal trees such that there are no arcs in $A(x)$ connecting two different maximal trees. Given a maximal tree $T = (N', A')$ in $G(x)$ rooted at a node $r \in N'$, for every node $i \in N'$, we define

$$R(i) = \inf_{S \in \mathbf{S}(T^i)} \sum_{(i',j') \in S} r(i', j').$$

The value $R(i)$ provides a measure of the maximum amount by which the flow on the arc connecting node i and its parent in T, could be changed when we are only allowed to balance that change by modifying arc flows in the subtree T^i. Note that if T^i is a finite tree, the empty cut is a valid finite cut, and we can conclude that $R(i) = 0$. Intuitively, this can be seen by observing that the flow balance constraints prohibit changing the flow on any given arc when we may only balance that change using changes in the subtree T^i. We next derive a property of the function $R(i)$ that will be used in our key result later in the section.

Lemma 3.3 *Given a rooted tree $T = (N', A')$, the function $R(i)$ for $i \in N'$ is given by*

$$R(i) = \sum_{j \in C(i)} \min\{\alpha_{ij}, R(j)\},$$

where

$$\alpha_{ij} = \begin{cases} r(i,j) & \text{if } (i,j) \in A' \\ r(j,i) & \text{otherwise.} \end{cases}$$

Proof: We observe that a finite cut $S \in \mathbf{S}(T^i)$ can be partitioned into $|C(i)|$ disjoint sets S^j $(j \in C(i))$ where S^j either contains a finite cut of the subtree T^j or contains the arc between node j and i in T. Conversely, if we choose sets S^j for $j \in C(i)$ such that S^j is either a finite cut of subtree T^j or contains only the arc between node j and node i, then $\cup_{j \in C(i)} S^j$ is a finite cut. Therefore, we can write

$$\mathbf{S}(T^i) = \left\{ \bigcup_{j \in C(i)} S^j : S^j \in \mathbf{S}(T^j) \text{ or } S^j \text{ is the arc between } i \text{ and } j \right\}.$$

If the arc between i and j is (i, j), the lemma now follows from this observation because

$$\begin{aligned} R(i) &= \inf_{S \in \mathbf{S}(T^i)} \sum_{(i',j') \in S} r(i', j') \\ &= \inf_{S \in \left\{ \cup_{j \in C(i)} S^j : S^j \in \mathbf{S}(T^j) \cup \{(i,j)\} \right\}} \sum_{(i',j') \in S} r(i', j') \\ &= \sum_{j \in C(i)} \inf_{S^j \in \mathbf{S}(T^j) \cup \{(i,j)\}} \sum_{(i',j') \in S^j} r(i', j') \\ &= \sum_{j \in C(i)} \min \left\{ r(i,j), \inf_{S^j \in \mathbf{S}(T^j)} \sum_{(i',j') \in S^j} r(i', j') \right\} \\ &= \sum_{j \in C(i)} \min\{\alpha_{ij}, R(j)\}. \end{aligned}$$

The case where the arc between i and j is (j, i) follows in an analogous fashion. $\square$

We next state some important properties of feasible flows that will be used later in this section.

Proposition 3.4 *Let x^1 and x^2 be two feasible flows and $x = (x^1 + x^2)/2$. Then,*

1. $A(x^1) \cup A(x^2) \subseteq A(x)$.
2. $L(x) = L(x^1) \cap L(x^2)$.
3. $U(x) = U(x^1) \cap U(x^2)$.

Proof: The first result follows by noting that for any arc $(i, j) \in A$ if $\ell(i, j) < x^1(i, j) < u(i, j)$ or $\ell(i, j) < x^2(i, j) < u(i, j)$ then $\ell(i, j) < x(i, j) < u(i, j)$. The second and third results follow by the observation that $x(i, j) = \ell(i, j) \iff x^1(i, j) = x^2(i, j) = \ell(i, j)$, and $x(i, j) = u(i, j) \iff x^1(i, j) = x^2(i, j) = u(i, j)$. $\square$

The following lemma is a direct consequence of the flow balance equations satisfied by feasible flows.

Lemma 3.5 *Let x and x' be two feasible flows such that $L(x) \subseteq L(x')$, $U(x) \subseteq U(x')$, and $G(x)$ contains no cycles. Let $T = (N', A')$ be a maximal tree in $G(x)$ rooted at some node $r \in N'$. Then, for any node $i \in N'$:*

- *if $(i, p(i)) \in A'$ then $|x(i, p(i)) - x'(i, p(i))| \leq \sum_{(i',j') \in S} |x(i', j') - x'(i', j')|$ for all $S \in \mathbf{S}(T^i)$*
- *if $(p(i), i) \in A'$ then $|x(p(i), i) - x'(p(i), i)| \leq \sum_{(i',j') \in S} |x(i', j') - x'(i', j')|$ for all $S \in \mathbf{S}(T^i)$.*

Proof: We deal explicitly with the case $(i, p(i)) \in A'$; the result for the other case can be proven in a similar manner. Consider a node $i \in N'$ and some $S \in \mathbf{S}(T^i)$ in the rooted tree T. Let $\bar{N}$ be the finite set of nodes in T^i that are connected to the node i after the edges in S are removed from T^i. Since $L(x) \subseteq L(x')$ and $U(x) \subseteq U(x')$, the flows x and x' can only differ over the edges in $A(x)$. Further, since both the flows satisfy the flow balance equation (1) over the nodes in $\bar{N}$, the difference between the total flow on the arcs from set $N' \setminus \bar{N}$ to $\bar{N}$ under x and x' should be equal to the difference between the total flow on the arcs from $\bar{N}$ to $N' \setminus \bar{N}$ under x and x'. That is,

$$\sum_{(i',j') \in A(x): i' \in \bar{N}, j' \in N \setminus \bar{N}} (x(i', j') - x'(i', j')) = \sum_{(i',j') \in A(x): i' \in N \setminus \bar{N}, j' \in \bar{N}} (x(i', j') - x'(i', j')).$$

Since the set $\bar{N}$ is in the tree $T^i = (N^i, A^i)$, the only nodes in $N \setminus \bar{N}$ that are connected to $\bar{N}$ in $A(x)$ are $p(i)$ and some nodes in $N^i \setminus \bar{N}$. Let

$$\begin{aligned} S^1 &= \{(i', j') \in A^i : i' \in \bar{N}, j' \in N^i \setminus \bar{N}\} \\ S^2 &= \{(i', j') \in A^i : i' \in N^i \setminus \bar{N}, j' \in \bar{N}\}. \end{aligned}$$

By our observations,

$$\begin{aligned} \{(i',j') \in A(x) : i' \in \bar{N}, j' \in N \setminus \bar{N}\} &= \{(i,p(i))\} \cup S^1 \\ \{(i',j') \in A(x) : i' \in N \setminus \bar{N}, j' \in \bar{N}\} &= S^2. \end{aligned}$$

Using this information in the equation above,

$$x(i,p(i)) - x'(i,p(i)) = \sum_{(i',j') \in S^2} (x(i',j') - x'(i',j')) - \sum_{(i',j') \in S^1} (x(i',j') - x'(i',j')).$$

It follows immediately that

$$|x(i,p(i)) - x'(i,p(i))| \leq \sum_{(i',j') \in S^1 \cup S^2} |(x(i',j') - x'(i',j'))|.$$

Finally, we note that by the definition of the set $\bar{N}$ and finite cuts, $S^1 \cup S^2 \subseteq S$, and the result follows. □

We are now ready to provide the necessary and sufficient conditions for a feasible flow in G to be an extreme point. Let x be a flow in a network $G = (N, A)$.

Condition 3.6

(a) *$G(x)$ contains no cycles.*

(b) *Every maximal tree $T = (N', A')$ in $G(x)$ can be rooted at some node $r \in N'$ such that any node $i \in N'$ with $R(i) > 0$ has at most one child $j \in C(i)$ such that $R(j) > 0$.*

We will refer to this condition as *Extreme Point Condition 3.6*. Although Extreme Point Condition 3.6(b) requires the existence of *some* root node with the desired property for each maximal tree, it is in fact equivalent to the (seemingly more restrictive) condition that the desired property holds for *all* choices of root node. That is, we can replace Extreme Point Condition 3.6(b) by

(b′) *In every maximal tree $T = (N', A')$ in $G(x)$ rooted at* any *node $r \in N'$, any node $i \in N'$ with $R(i) > 0$ has at most one child $j \in C(i)$ such that $R(j) > 0$.*

The next lemma formally proves the equivalence of Extreme Point Conditions 3.6(b) and 3.6(b′). We will later use this equivalence result to establish the necessity and sufficiency of Extreme Point Condition 3.6 for any feasible flow to be an extreme point.

Lemma 3.7 *If a feasible flow x satisfies Extreme Point Condition 3.6(b) then it also satisfies Extreme Point Condition 3.6(b′).*

Proof: Consider a maximal tree $T = (N', A')$ in $G(x)$. Suppose that T satisfies Extreme Point Condition 3.6(b) for some choice of root node $r \in N'$. We first show that Extreme Point Condition 3.6(b) is also satisfied when T is rooted at any node $r' \in C(r)$, and then generalize the result to any choice of root node in N', proving the lemma.

Note that when the root for T is changed from r to $r' \in C(r)$, the parent-child relationship is modified only for the nodes r and r'. If T is rooted at r then r' is a child of r whereas r is a child of r' if T is rooted at r'. For each node $i \in N' \setminus \{r, r'\}$, the set $C(i)$ and the associated subtree T^i are unchanged by changing the root from r to r'. By Lemma 3.3, we can conclude that the value $R(i)$ for any node $i \in N \setminus \{r, r'\}$ is unchanged by the change of root. Therefore, these nodes still have at most one child with positive $R(\cdot)$ value when the root is changed to r'. Also, since node r loses a child (node r') when the tree T is rooted at r', node r still has at most one child $j \in C(r) \setminus r'$ with $R(j) > 0$ after the change of the root. When T is rooted at r, Lemma 3.3 says that if r has a child $j \neq r'$ with $R(j) > 0$ then $R(r')$ must be equal to 0, otherwise r would have two children with positive $R(\cdot)$ values. In that case, by Lemma 3.3 we obtain that all children of r' in $C(r')$ must have $R(\cdot)$ values equal to 0. Therefore, when the tree T is rooted at r', it can have at most one child with positive $R(\cdot)$ value: either node r or a node in $C(r')$. Hence Extreme Point Condition 3.6(b) is satisfied when T is rooted at $r' \in C(r)$.

Now consider an arbitrary node $r' \in N'$. Let $r = i_1, \ldots, i_k = r'$ be the unique finite path between r and r' in T for some $k \geq 2$. Since i_1 is a child of r in T rooted at r we can apply the result above to conclude that Extreme Point Condition 3.6(b) is satisfied when T is rooted at i_1. By repeatedly applying the above result a finite number of times, we can thus conclude that Extreme Point Condition 3.6(b) is satisfied when T is rooted at r'. □

We first establish the sufficiency of Extreme Point Condition 3.6 for a feasible flow x to be an extreme point.

Proposition 3.8 *A feasible flow x is an extreme point if it satisfies the Extreme Point Condition 3.6.*

Proof: Suppose that the feasible flow x satisfies Extreme Point Condition 3.6 but it is not an extreme point. Let x^1 and x^2 be feasible flows such that $x = (x^1 + x^2)/2$ and $x^1 \neq x^2$. Using Proposition 3.4, it follows that $L(x) = L(x^1) \cap L(x^2)$ and $U(x) = U(x^1) \cap U(x^2)$. Therefore, x^1 and x^2 can only differ over the arcs in the set $A(x)$. Since $G(x)$ contains no cycles by Extreme Point Condition 3.6(a), every arc in $A(x)$ must belong to a maximal tree in $G(x)$. Let (i, j) be an arc in some maximal tree $T = (N', A')$ in $G(x)$ such that $x^1(i, j) \neq x^2(i, j)$. Since x, x^1, and x^2 are feasible flows, they each satisfy the flow balance constraint (1) at i. Therefore one of the following two must happen (otherwise the flow balance at i would not be satisfied):

(i) there is an arc $(i, \bar{\jmath}) \in A'$ such that $\bar{\jmath} \neq j$ and $x^1(i, \bar{\jmath}) \neq x^2(i, \bar{\jmath})$.

(ii) there is an arc $(\bar{\jmath}, i) \in A'$ such that $\bar{\jmath} \neq j$ and $x^1(\bar{\jmath}, i) \neq x^2(\bar{\jmath}, i)$.

Consider the maximal tree T rooted at node i. By observations above, $j, \bar{\jmath} \in C(i)$ when i is the root node. If $R(i) = 0$, then by Lemma 3.3 it follows that $R(j) = R(\bar{\jmath}) = 0$. If $R(i) > 0$ then either $R(j) = 0$ or $R(\bar{\jmath}) = 0$ by Extreme Point Condition 3.6(b′) (which must be satisfied since it is equivalent to Extreme Point Condition 3.6(b) by Lemma 3.7).

Without loss of generality, let $R(j) = 0$ and $\Delta > 0$ be such that $x^1(i, j) = x(i, j) - \Delta$ (and thus $x^2(i, j) = x(i, j) + \Delta$). Since $R(j) = 0$, by definition of $R(\cdot)$ there must exist a finite cut $S \in \mathbf{S}(T^j)$ such that $\sum_{(i',j') \in S} r(i',j') < \Delta$. Using Lemma 3.5 with flows x and x^1 over the finite cut S, we obtain that

$$\Delta = |x(i,j) - x^1(i,j)| \le \sum_{(i',j') \in S} |x(i',j') - x^1(i',j')|.$$

We note that for any arc $(i',j') \in S$

$$x(i',j') - x^1(i',j') = x^2(i',j') - x(i,j).$$

Using the fact that $\ell(i',j') \le x^1(i',j') \le u(i',j')$ and $\ell(i',j') \le x^2(i',j') \le u(i',j')$, it follows that

$$|x(i',j') - x^1(i',j')| \le \min\{x(i',j') - \ell(i',j'), u(i',j') - x(i',j')\} = r(i',j')$$

which implies that

$$\Delta \le \sum_{(i',j') \in S} r(i',j').$$

However, this contradicts the choice of S to be such that $\sum_{(i',j') \in S} r(i',j') < \Delta$. Therefore, our assumption that $x^1(i,j) \ne x^2(i,j)$ must be incorrect. Since the arc (i,j) was chosen arbitrarily in $A(x)$, it follows that x^1 and x^2 are identical on the set $A(x)$, and we conclude that x is an extreme point. □

We next prove a result which will be used to show that Extreme Point Condition 3.6 is necessary for a feasible flow to be an extreme point.

Lemma 3.9 *Let x be a feasible flow such that $G(x)$ has no cycles and let $T = (N', A')$ be a maximal rooted tree in $G(x)$. For any node $i \in N'$ with children $C(i) = \{j_1, \ldots, j_K\}$ and a value $0 \le \Delta \le R(i)$, there exist non-negative numbers $\delta_1, \ldots, \delta_K$ such that $0 \le \delta_k \le \min\{\alpha_{ij_k}, R(j_k)\}$ and $\sum_{k=1}^K \delta_k = \Delta$.*

Proof: The case where $R(i) = 0$ is trivial. From Lemma 3.3, we know that

$$\Delta \le R(i) = \sum_{k=1}^{K} \min\{\alpha_{ij_k}, R(j_k)\}.$$

Clearly, we can choose each δ_k to be between 0 and $\min\{\alpha_{ij_k}, R(j_k)\}$ to ensure that $\sum_{k=1}^K \delta_k = \Delta$. □

Proposition 3.10 *If a feasible flow x is an extreme point then it satisfies Extreme Point Condition 3.6.*

Proof: Let x be a feasible flow, and suppose that it is an extreme point. From Lemma 3.2, it follows that $G(x)$ contains no cycles. Suppose that there exists a maximal tree $T = (N', A')$ in $G(x)$ with a choice $r \in N'$ of root node such that there is a node $i \in N'$ with two children $j, \bar{\jmath} \in C(i)$ and $R(i), R(j), R(\bar{\jmath}) > 0$. We shall construct two feasible flows x^1 and x^2 such that

$x^1 \neq x^2$ and $x = (x^1 + x^2)/2$, providing contradiction to the assumption that x is an extreme point. In particular, we will let the flows x^1 and x^2 be identical to x on all arcs except the arcs between node i and nodes $j, \bar{\jmath}$, and the arcs in sets A^j and $A^{\bar{\jmath}}$. For any of these remaining arcs, say (i', j'), we set the value of the flow $x^2(i', j')$ equal to $2x(i', j') - x^1(i', j')$, so it remains to specify the flow x^1. For simplicity of the argument, we will show the construction when $(i, j), (i, \bar{\jmath}) \in A'$; the other cases can be handled in a similar manner. Let

$$\Delta = \min \{r(i, j), r(i, \bar{\jmath}), R(j), R(\bar{\jmath})\}.$$

We set

$$\begin{aligned} x^1(i, j) &= x(i, j) + \Delta \\ x^1(i, \bar{\jmath}) &= x(i, \bar{\jmath}) - \Delta \end{aligned}$$

(and thus $x^2(i, j) = x(i, j) - \Delta$ and $x^2(i, \bar{\jmath}) = x(i, \bar{\jmath}) + \Delta$). Clearly, the flow balance constraint (1) at node i and the bounds on the flow on arcs (i, j) and $(i, \bar{\jmath})$ are satisfied for both x^1 and x^2. The flow x^1 on the arcs in A^j will be specified in a recursive manner as folows. Let $j_1, \ldots, j_K$ be the children of j. Since $\Delta \leq R(j)$ by definition, Lemma 3.9 says that there exist values $0 \leq \delta_k \leq \min\{\alpha_{jj_k}, R(j_k)\}$ such that $\sum_{k=1}^{K} \delta_k = \Delta$. For each child $j_k \in C(j)$ $(k = 1, \ldots, K)$ we now set the flow on the arc connecting j with j_k. If $(j, j_k) \in A'$, we set $x^1(j, j_k) = x(j, j_k) + \delta_k$. Otherwise, $(j_k, j) \in A'$ and we set $x^1(j_k, j) = x(j_k, j) - \delta_k$. This assignment ensures that the flow balance at node j as well as the bounds on the arcs between node j and its children are satisfied. The assignment also ensures that the flow x^2 will satisfy both the flow balance and the bound constraints. Note that the values δ_k satisfy the condition $\delta_k \leq R(j_k)$, so the method used to determine the flows x^1 on arcs between j and its children can be recursively applied to find the flows on arcs between its children and their children. Also, note that the recursive application of this procedure yields the flow x^1 for all arcs in A^j so that the flow balance constraints for the nodes in N^j and the bounds on all arcs are satisfied. A similar recursion can be used to determine an appropriate set of flow values x^1 for the arcs in $A^{\bar{\jmath}}$.

The construction shows that if the flow x violates Extreme Point Condition 3.6 then there exist flows x^1 and x^2 such that $x^1 \neq x^2$ and $x = (x^1 + x^2)/2$. □

Combining Propositions 3.8 and 3.10 we conclude the major result of this section that Extreme Point Condition 3.6 is a necessary and sufficient condition for any feasible point x to be an extreme point.

Theorem 3.11 *A feasible flow x is an extreme point if and only if it satisfies Extreme Point Condition 3.6.*

3.3 A class of problems for which extreme points are characterized through the free arc graph

We next present a regularity condition under which the extreme points of an infinite network flow problem can be specified in terms of its restricted arcs.

Assumption 3.12 *For* any *extreme point x there exists a value $\theta > 0$ such that if $0 < x(i,j) < u(i,j)$ for some $(i,j) \in A$ then $\theta \leq x(i,j) \leq u(i,j) - \theta$ or, equivalently,* $\inf_{(i,j)\in A(x)} r(i,j) > 0$.

We will refer to this assumption as *Non-Vanishing Support Assumption 3.12.* The following provides a set of set of extreme point conditions that is weaker than Extreme Point Condition 3.6.

Condition 3.13

(a) $G(x)$ contains no cycles.

(b) For any node $i \in N$ there exists at most one infinite path $i - i_1 - i_2 - \cdots$ in $G(x)$.

We will refer to this condition as *Extreme Point Condition 3.13.* If we informally think of two paths to infinity as forming an "infinite cycle", then we may rephrase Extreme Point Condition 3.13 as the requirement that $G(x)$ contains no finite or infinite cycles. The following theorem shows that, under the Non-Vanishing Support Assumption 3.12, Extreme Point Condition 3.13 is a necessary and sufficient condition for a feasible flow x to be an extreme point.

Theorem 3.14 *Suppose that the network flow problem given by $G = (N, A)$, ℓ, u, and d satisfies the Non-Vanishing Support Assumption 3.12. Then, a feasible flow x is an extreme point if and only if Extreme Point Condition 3.13 is satisfied.*

Proof: We prove the theorem by showing that if the network flow problem satisfies the Non-Vanishing Support Assumption 3.12, then Extreme Point Condition 3.13(b) is equivalent to Extreme Point Condition 3.6(b′).

Suppose that the flow x satisfies conditions (a) and (b) in the theorem. Consider any maximal tree $T = (N', A')$ in $G(x)$ rooted at some node $r \in N'$. Extreme Point Condition 3.13(b) along with the assumption that any node has finite degree implies that for any node $i \in N'$, at most one subtree T^j rooted at a child $j \in C(i)$ contains infinitely many nodes as otherwise there would be two distinct infinite paths from node i. By definition of the function $R(\cdot)$, the value $R(j)$ is zero if T^j is finite because $S = \emptyset$ is a finite cut of T^j in this case. Therefore, for any node $i \in N'$, at most one child $j \in C(i)$ can have $R(j) > 0$. In other words, the flow x satisfies Extreme Point Condition 3.6(b).

Now suppose that the flow x satisfies Extreme Point Condition 3.6(b) and therefore, by Lemma 3.7, also Extreme Point Condition 3.6(b′). Consider any node $i \in N$. Since $G(x)$ contains no cycle, node i must be part of a maximal tree. Let $T = (N', A')$ be the maximal tree containing node i, and root this tree at node i. Suppose that there are two distinct infinite paths $P^1 = i - i_1 - i_2 - \cdots$ and $P^2 = i - \bar{\imath}_1 - \bar{\imath}_2 - \cdots$ in T. Then, $i_1 \in C(i)$ and $i_{k+1} \in C(i_k)$ for $k = 1, 2, \ldots$ in T rooted at i. Similarly, $\bar{\imath}_1 \in C(i)$ and $\bar{\imath}_{k+1} \in C(\bar{\imath}_k)$ for $k = 1, 2, \ldots$. We note that every finite cut $S \in \mathbf{S}(T^{i_1})$ must contain an arc from the path P^1. Since $A(P^1) \subseteq A(x)$, $r(i', j') \geq \theta$ for $(i', j') \in A(P^1)$. Therefore, by definition of $R(\cdot)$, $R(i_1) \geq \theta > 0$. Using a similar argument, $R(\bar{\imath}_1) \geq \theta > 0$. However, this contradicts the assumption that the flow satisfies Extreme Point Condition 3.6(b′) because $i_1, \bar{\imath}_1 \in C(i)$. Hence, there cannot be two distinct infinite paths starting at node i, and the flow x must satisfy Extreme Point Condition 3.13(b). □

We note that the proof of Theorem 3.14 shows that Extreme Point Condition 3.13 is sufficient for a feasible flow x to be an extreme point in general. In the special case where the network flow problem satisfies the Non-Vanishing Support Assumption 3.12 the conditions are necessary as well, as the theorem states. Note that Extreme Point Condition 3.13 is dependent only on the graph $G(x)$ and not the actual value of the flow x, as was the case with Extreme Point Condition 3.6. We use this observation to prove that if the network flow problem satisfies the Non-Vanishing Support Assumption 3.12 then any extreme point x can be expressed in terms of its restricted arcs.

Theorem 3.15 *Suppose that the network flow problem given by $G = (N, A)$, ℓ, u, and d satisfies the Non-Vanishing Support Assumption 3.12. Then, any extreme point x for the problem is uniquely characterized by the sets $L(x)$ and $U(x)$.*

Proof: Suppose that x^1 and x^2 are two extreme points such that $L(x^1) = L(x^2)$, $U(x^1) = U(x^2)$, and $x^1 \neq x^2$. Given our observation, $G(x^1)$ and $G(x^2)$ must be identical. Now consider the feasible flow $x = (x^1 + x^2)/2$. By Proposition 3.4, $L(x) = L(x^1) = L(x^2)$ and $U(x) = U(x^1) = U(x^2)$. We observe that for any feasible flow x, the graph $G(x)$ can be specified by providing the sets $L(x)$ and $U(x)$ because $L(x) \cup U(x) \cup A(x) = A$. This implies that $G(x)$, $G(x^1)$, and $G(x^2)$ are identical. Since x^1 is an extreme point, $G(x) = G(x^1)$ satisfies the conditions of Theorem 3.14. However this leads to a contradiction because Theorem 3.14 then implies that x is an extreme point. Therefore, we can not have two extreme points $x^1 \neq x^2$ such that $L(x^1) = L(x^2)$ and $U(x^1) = U(x^2)$. □

Theorem 3.15 suggests that while it seems unlikey one can develop a simplex like method for general infinite network flow problems, it may be possible to develop such a method for network flow problems satisfying the Non-Vanishing Support Assumption 3.12. In the next section, we show that *infinite network flow problems where the demand and lower and upper bound data are integral always have integral valued extreme points. Moreover, such problems satisfy the Non-Vanishing Support Assumption 3.12 with $\theta = 1$.*

3.4 Integral network flow problems

In this section, we discuss a class of network flow problems where the demands d and the arc capacities u can only take integral values. It is well-known (see for example, Ahuja et al. [1]) that in the case of finite networks, if the demands and the arc capacities are integral, then any extreme point must have integral values for arc flows. We extend this result to infinite network flow problems.

In order to prove our extension, we need to introduce some new notation. For any real number a, we define $a^f = \min\{a - \lfloor a \rfloor, \lceil a \rceil - a\}$. For a given flow $x \in \mathbb{R}_+^{|A|}$, we define its *rounding vector* $x^f \in [0, 0.5]^{|A|}$ through $x^f(i,j) = (x(i,j))^f$ for $(i,j) \in A$. We shall rely on the following basic result in our proof.

Proposition 3.16 *Suppose $a, a_1, a_2, \ldots, a_n$ are $n+1$ real numbers such that $a = \sum_{k=1}^n a_k$, then $a^f \leq \sum_{k=1}^n a_k^f$.*

We also need the following additional result regarding the rounding vector of a feasible flow to prove our main result of this section.

Lemma 3.17 *Let x be a feasible flow such that $G(x)$ contains no cycles. Let $T = (N', A')$ be a maximal tree in $G(x)$ rooted at some node $r \in N'$. Then, for any node $i \in N'$:*

- *if $(i, p(i)) \in A'$ then $x^f(i, p(i)) \leq \sum_{(i',j') \in S} x^f(i', j')$ for all $S \in \mathbf{S}(T^i)$*
- *if $(p(i), i) \in A'$ then $x^f(p(i), i) \leq \sum_{(i',j') \in S} x^f(i', j')$ for all $S \in \mathbf{S}(T^i)$.*

Proof: The proof of this result is similar to the proof of Lemma 3.5. We show the result when $(i, p(i)) \in A'$; the other case can be shown in an analogous manner. Consider a node $i \in N'$ and some $S \in \mathbf{S}(T^i)$ in the rooted tree T. Let $\bar{N}$ be the finite set of nodes in $T^i = (N^i, A^i)$ that are connected to the node i after the edges in S are removed from T^i. Summing up the flow balance equations for the nodes in set $\bar{N}$,

$$\sum_{(i',j') \in A(x): i' \in N' \setminus \bar{N}, j' \in \bar{N}} x(i', j') - \sum_{(i',j') \in A(x): i' \in \bar{N}, j' \in N' \setminus \bar{N}} x(i', j') = \sum_{i' \in \bar{N}} d(i').$$

Since the set $\bar{N}$ is in the tree T^i, the only nodes in $N \setminus \bar{N}$ that are connected to $\bar{N}$ in $A(x)$ are $p(i)$ and some nodes in $N^i \setminus \bar{N}$. Let

$$\begin{aligned} S^1 &= \{(i', j') \in A^i : i' \in \bar{N}, j' \in N^i \setminus \bar{N}\} \\ S^2 &= \{(i', j') \in A^i : i' \in N^i \setminus \bar{N}, j' \in \bar{N}\}. \end{aligned}$$

By our observations,

$$\begin{aligned} \{(i', j') \in A(x) : i' \in \bar{N}, j' \in N \setminus \bar{N}\} &= \{(i, p(i))\} \cup S^1 \\ \{(i', j') \in A(x) : i' \in N \setminus \bar{N}, j' \in \bar{N}\} &= S^2. \end{aligned}$$

Using this information in the equation above,

$$x(i, p(i)) = \sum_{i' \in \bar{N}} d(i') + \sum_{(i',j') \in S^1} x(i', j') - \sum_{(i',j') \in S^2} x(i', j').$$

Since $\sum_{i' \in \bar{N}} d(i')$, using Proposition 3.16 on the equation above, it follows that

$$x^f(i, p(i)) \leq \sum_{(i',j') \in S^1 \cup S^2} x^f(i', j').$$

Finally, we note that by the definition of the set $\bar{N}$ and finite cuts, $S^1 \cup S^2 \subseteq S$, and the result follows. □

We now provide the main result of this section, which is an extension of the well-known integrality result for finite network flow problems.

Theorem 3.18 *If the demands d as well as the lower and upper bounds ℓ and u in a network flow problem only take integer values, then any extreme point of the problem is integral.*

Proof: We shall use Lemma 3.17 above to show that any feasible flow x which has a fractional value for some arc in A does not satisfy Extreme Point Condition 3.6, so it can not be an extreme point. Hence, the extreme points must have integral values for all arcs.

From Lemma 3.2, if the graph $G(x)$ contains a cycle then x can not be an extreme point. So suppose that we have a feasible flow x such that $G(x)$ contains no cycle and for some $(i, j) \in A(x)$, $x(i, j)$ has a fractional value (i.e., $x^f(i, j) > 0$). Let $T = (N', A')$ be the maximal tree containing the arc (i, j) and choose node i as its root. Since the flow x satisfies the flow balance equation (1) at node i and $d(i)$ is an integer, there must exist another arc incident to node i (different from (i, j)) that has fractional flow. There are two possible cases:

(i) there is an arc $(i, \bar{\jmath})$ in A' such that $\bar{\jmath} \neq j$ and $x(i, \bar{\jmath})$ is fractional;

(ii) there is an arc $(\bar{\jmath}, i)$ in A' such that $\bar{\jmath} \neq j$ and $x(\bar{\jmath}, i)$ is fractional.

We deal with case (i); case (ii) can be handled similarly. We observe that if the upper bounds u only takes integer values, then for any feasible flow x, $r(i, j) \geq x^f(i, j)$ for $(i', j') \in A$. By definition of the function $R(\cdot)$, $R(j) = \inf_{S \in \mathbf{S}(T^j)} \sum_{(i',j') \in S} r(i', j')$. From our observation it follows that $R(j) \geq \inf_{S \in \mathbf{S}(T^j)} \sum_{(i',j') \in S} x^f(i', j')$. Finally, by Lemma 3.17, $R(j) \geq x^f(i, j) > 0$ as $i = p(j)$ in T. By a similar argument, we can conclude that $R(\bar{\jmath}) \geq x^f(i, \bar{\jmath}) > 0$. This means that node i has two children j and $\bar{\jmath}$ such that $R(j), R(\bar{\jmath}) > 0$ and therefore x does not satisfy Extreme Point Condition 3.6(b′), which is equivalent to Extreme Point Condition 3.6(b) by Lemma 3.7. □

The integrality of the extreme points in case the problem data are integral immediately implies that the Non-Vanishing Support Assumption 3.12 is satisfied:

Corollary 3.19 *If the demands d as well as the lower and upper bounds ℓ and u in a network flow problem only take integer values, then the Non-Vanishing Support Assumption 3.12 is satisfied with $\theta = 1$.*

In turn, this then implies that *Extreme Point Condition 3.13 characterizes all extreme points to network flow problems with integral data.*

4 An Application to the Infinite Horizon Capacitated Economic Lot-Sizing Problem

In this section we will study the extreme point structure of the infinite horizon variant of the classical capacitated economic lot-sizing problem. In this problem, the uncapacitated version of which dates back to Wagner and Whitin [14], a producer faces a deterministic demand stream for a single product over a sequence of time periods. This demand needs to be satisfied from production, which may be held in inventory until demand occurs. The problem is then to decide how much to produce in each time period to ensure that all demands are satisfied at minimum production and inventory holding costs. To formulate this problem as an optimization problem, let us denote the time periods by $t = 1, 2, \ldots$, and the (integral) demand in time period t by d_t. Furthermore, let $c_t(\cdot)$ and $h_t(\cdot)$ denote the production and inventory holding cost functions in period t, and assume that they are concave and nondecreasing in the quantity produced and held in inventory, respectively, where we assume that inventory costs are charged against end of period inventory levels. Let $\overline{p}_t$ and $\overline{I}_t$ denote the (integral and finite) production and inventory limits in period t. Finally, denote the quantity produced in period t by p_t and the quantity in inventory at the end of period t by I_t. The problem can then be formulated as:

$$\text{minimize} \sum_{t=1}^{\infty} (c_t(p_t) + h_t(I_t))$$

subject to

$$\begin{aligned} x_1 &= d_1 + I_1 & \\ I_{t-1} + x_t &= d_t + I_t & t = 2, 3, \ldots \\ p_t &\leq \overline{p}_t & t = 1, 2, \ldots \\ I_t &\leq \overline{I}_t & t = 1, 2, \ldots \\ p_t, I_t &\geq 0 & t = 1, 2, \ldots \end{aligned}$$

When viewed as a subset of the product space $\prod_{i=1}^{\infty} \mathbb{R}^2$, the feasible region is compact. Due to the concavity of the cost functions, we then know that if the lot-sizing problem has an optimal solution, an extreme point optimal solution exists by Bauer's Minimum Principle (see Roy [11]). An optimal solution, in turn, exists if there exists a feasible solution with finite cost value. It is therefore of interest to study the extreme point structure of the feasible region of the economic lot-sizing problem.

When the horizon is finite, say T periods, the first step is usually to formulate the problem as a minimum-cost network flow problem. In the most common network flow formulation (see Figure 5), each period t is represented by a demand node with demand d_t, and production is represented by a source node with supply equal to the sum of all production capacities, or demand equal to $-\sum_{t=1}^{T} \overline{p}_t$. In addition, a sink node $T + 1$ is added with demand equal to the difference between total demands and total production capacities, i.e., $\sum_{t=1}^{T} (\overline{p}_t - d_t)$. Production in period t is represented by a directed arc from the source node to node t with capacity $\overline{p}_t$, and the unused capacity is represented by a directed arc from the source node to the sink node $T + 1$ with capacity $\sum_{t=1}^{T} (\overline{p}_t - d_t)$. Inventory carried from period t to period $t + 1$ is represented by a directed arc from node t to node $t + 1$ with capacity $\overline{I}_t$. In this network, each feasible solution to the lot-sizing problem is represented by a feasible flow.

This network flow formulation of the finite-horizon economic lot-sizing problem can be used to derive properties of the extreme point solutions to this problem. It is well-known that any extreme point solution satisfies the following conditions:

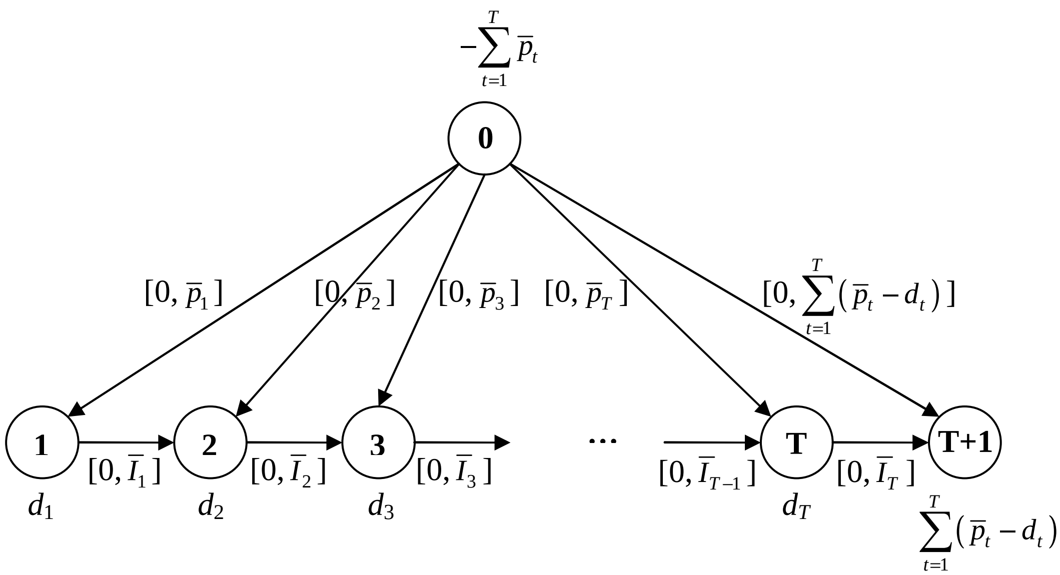

Figure 5: A network representation of the finite-horizon economic lot-sizing problem.

(i) Consider two periods, say t_1 and t_2, in which inventory is equal to zero, while $I_t > 0$ for $t = t_1 + 1, \ldots, t_2 - 1$ (such a sequence is often called a *subplan*). Then at most one of the production quantitites p_t, $t = t_1 + 1, \ldots, t_2$ satisfies both its upper and lower bound strictly, i.e., there exists at most one $t = t_1 + 1, \ldots, t_2$ such that $0 < p_t < \overline{p}_t$ (see, e.g., Florian and Klein [8]).

(ii) Since all demands and production and inventory bounds are integral, all production and inventory quantities are integral-valued as well.

The goal of this section is to generalize these two properties to the infinite-horizon case.

When attempting to generalize the network-flow approach to the infinite-horizon economic lot-sizing problem, it is immediate that we cannot use a network of the same form as in the finite-horizon case. Since all demands are integral, total demand will be infinite if we are dealing with a truly infinite-horizon problem. This implies that the total production capacity should be infinite as well. However, this means that the supply of the source node as well as the demand of the sink node are not well-defined in the straightforward generalization of the network. Another complication is that the total outdegree of the source node in this generalization would be infinite, while the results on the structure of extreme points of infinite network flow problems derived in this paper require that the in and outdegrees of all nodes in the network are finite. We therefore consider a new network-flow formulation of the problem. In this formulation, each period is still represented by a demand node, and inventory flows are represented by capacitated directed arcs between successive period nodes. However, production in each period t is represented by an infinite directed path leading to period t's node, where all nodes on this infinite path have demand 0, and all arcs have capacity $\overline{p}_t$. This representation of the problem eliminates the need for specifying the total production in advance. The infinite paths in each period t ensure that any production quantity not exceeding the capacity can be produced. See Figure 6 for an illustration of this network.

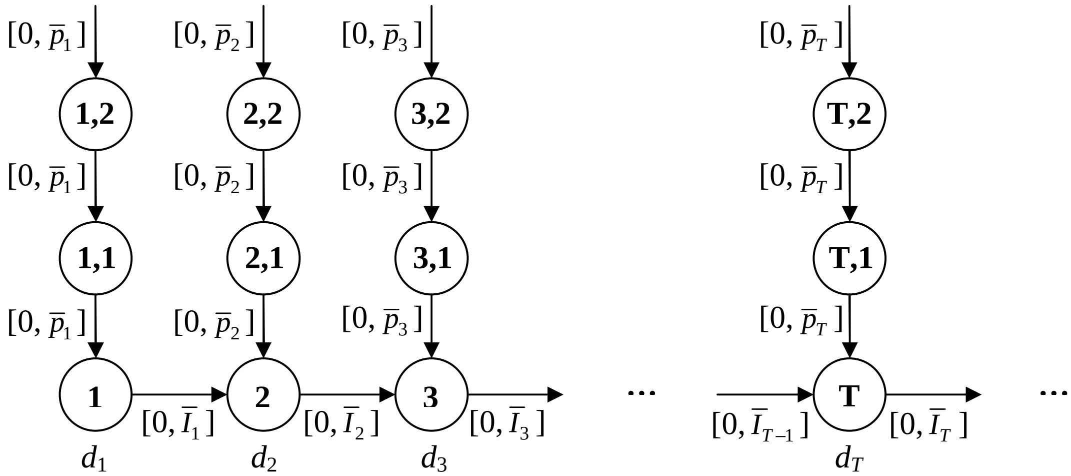

Figure 6: A network representation of the infinite-horizon economic lot-sizing problem.

All results regarding the extreme point structure of infinite-horizon network-flow problems that have been derived in this paper now apply to this representation of the infinite horizon economic lot-sizing problem. Theorem 3.18 now immediately generalizes extreme point property (ii) stated above to the infinite dimensional case. But, Corollary 3.19 also states that, since the demands are integral, we know that Extreme Point Condition 3.13 characterizes the extreme points to this problem. Recall that this condition says that, for any extremal flow, say x, we have that

(a) $G(x)$ contains no cycles;

(b) for any node i there exists at most one infinite path $i - i_1 - i_2 - \cdots$ in $G(x)$.

Since our network for the infinite horizon economic lot-sizing problem is acyclic, condition (a) does not yield any additional information. In fact, as in some earlier examples in this paper, we can immediately conclude that *any* feasible solution x corresponds to an acyclic free arc graph $G(x)$. But it is also easy to see that not all feasible flows are extreme point solutions. For instance, if all but a finite number of the demands are equal to zero, we essentially have a finite lot-sizing problem, and clearly not all feasible solutions to such a problem are extreme points. However, condition (b) generalizes property (ii) stated above to the infinite dimensional case. This can be seen as follows. Consider a feasible solution, and let t_1 and t_2 identify a subplan. Now suppose that $0 < p_t < \bar{p}_t$ for two values of $t = t_1 + 1, \ldots, t_2$, say s_1 and s_2. This means that node s_1 is in two infinite paths: $\cdots - (s_1, 2) - (s_1, 1) - s_1$ and $s_1 - \cdots - s_2 - (s_2, 1) - (s_2, 2) - \cdots$, contradicting Extreme Point Condition 3.13(b). Therefore the solution cannot be an extreme point.

Finally, we would like to remark that a similar approach to the one in this section may be used to show that any extreme point to the *uncapacitated* infinite horizon economic lot-sizing problem satisfies the standard zero-inventory ordering property that production can only take place when there is no inventory, that is, $I_{t-1}x_t = 0$ for all $t = 1, 2, \ldots$.

5 Conclusions and Future Research

In this paper, we studied the structure of the extreme points of infinite network flow problems. Our results show that, unlike in the finite dimensional case, an extreme point for an infinite network flow problem can, in general, not be uniquely represented in terms of free arcs. Nevertheless, we developed necessary and sufficient conditions for a feasible flow to be an extreme point that generalize this result. Moreover, under a regularity condition that is met by network flow problems with integral data, we show that an extreme point can be uniquely characterized by a set of free arcs corresponding to a subnetwork containing no finite or infinite cycles. Finally, we showed that, when all problem data is integral, the extreme points always have integral values, extending a result for finite network flow problems to the infinite case.

In finite dimensional network flow problems, there exists a concept of basic and nonbasic variables that partition the set of flow variables and uniquely characterize an extreme point. Moreover, such basic feasible solutions correspond to a spanning tree in the network. In our future research we expect to be able to employ our extreme point conditions to extend the concept of basic feasible solutions to the infinite dimensional case. Using such a characterization, we hope to then develop a generalization of the network simplex method for solving linear minimum cost infinite network flow problems.

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