Department of Computer Science
Series of Publications C
Report C-2004-2

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Technical report, Series of Publications C, Report C-2004-2

Helsinki, February 2004, 1 + 12 pages

Abstract

The maximum intersection problem for a matroid and a greedoid, given by polynomial-time oracles, is shown NP-hard by expressing the satisfiability of boolean formulas in 3-conjunctive normal form as such an intersection. The corresponding approximation problems are shown NP-hard for certain approximation performance bounds. Moreover, some natural parameterized variants of the problem are shown $W[P]$-hard. The results are in contrast with the maximum matroid-matroid intersection which is solvable in polynomial time by an old result of Edmonds. We also prove that it is NP-hard to approximate the weighted greedoid maximization within $2^{n^{O(1)}}$ where n is the size of the domain of the greedoid.

A preliminary version "The Complexity of Maximum Matroid-Greedoid Intersection" appeared in Proc. FCT 2001, LNCS 2138, pp. 535–539, Springer-Verlag 2001.

Computing Reviews (1998) Categories and Subject Descriptors:
F.2.2 Analysis of Algorithms and Problem Complexity: Nonnumerical Algorithms and Problems: Computations on discrete structures

General Terms:
Algorithms, Theory

Additional Key Words and Phrases:
Combinatorial Optimization, NP-Hardness, Inapproximability, Fixed-parameter Intractability

1 Introduction

A set system (S, F) where S is a finite set (the *domain* of the system) and F is a collection of subsets of S is a *matroid* if

(M1) $\emptyset \in F$;

(M2) If $Y \subseteq X \in F$ then $Y \in F$;

(M3) If $X, Y \in F$ and $|X| > |Y|$ then there is an $x \in X \setminus Y$ such that $Y \cup \{x\} \in F$.

A *greedoid* is a set system (S, F) that satisfies (M1) and (M3).

In applications a matroid or a greedoid is given by an oracle, i.e., by a deterministic algorithm that answers the question whether X belongs to F for any $X \subseteq S$.

Many combinatorial problems can be formulated using matroids or greedoids (see e.g. [7, 8]). The seminal example is the maximum matching problem in bipartite graphs. Each instance of the problem can be represented as the intersection of two matroids. For a bipartite graph $B = (V \cup V', E)$ where $V \cap V' = \emptyset$ and $E \subseteq V \times V'$, the first matroid consists of all subsets of the edge set E such that a subset contains at most one edge starting from the same node in V. The second matroid consists of similar subsets but now a subset can contain at most one edge ending at the same node in V'. Then the maximum matching corresponds to the largest set in the intersection of the two matroids.

We want to consider in the matroid-greedoid framework the computational complexity of general combinatorial problems that have infinitely many instances. Therefore we introduce families of matroids and greedoids that have uniform polynomial-time representations as follows. Let $\mathcal{F} = \{(S_h, F_h)_{h \in H}\}$ be a possibly infinite set of matroids or greedoids. Then $\mathcal{F}$ is said to be given by a *uniform polynomial-time oracle* if there is an algorithm $\mathcal{O}$, that when given h and some $X \subseteq S_h$ answers whether or not $X \in F_h$ in time polynomial in $|S_h|$.

Let $\mathcal{F} = \{(S_h, F_h)_{h \in H}\}$ and $\mathcal{G} = \{(S_h, G_h)_{h \in H}\}$ be two such families given by uniform polynomial-time oracles. Note that the index set H is the same for both, and for a given h, both have the same domain S_h.

The *maximum intersection problem* for $\mathcal{F}$ and $\mathcal{G}$ is to find, given an index $h \in H$, a set $X \in F_h \cap G_h$ such that $|X|$ is maximum. A solution algorithm of the maximum intersection problem is polynomial-time if its running time is polynomial in $|S_h|$.

Edmonds [5] gave the first polynomial-time solution for the intersection problem in the case that both $\mathcal{F}$ and $\mathcal{G}$ are families of matroids. In this paper we consider the obvious next step, namely the intersection of families of matroids and greedoids.

The following constrained version of the bipartite matching gives an example of a problem that can be represented as an intersection of a greedoid and a matroid. The problem is called the *maximum tree-constrained matching problem*. An instance of it consists of a bipartite graph $B = (V \cup V', E)$ and a rooted tree $T = (V, D, r)$ where $r \in V$ is the root. The tree T will constrain the use of V in the matching: the problem is to find a maximum-size matching in B such that the *matched* nodes of V include r and induce a connected subgraph (actually a tree rooted at r) of T.

To represent this problem as an intersection of a greedoid and a matroid we modify the construction given above for the unconstrained bipartite matching. The collection of subsets of edges ending at different nodes in V' remains as in the unconstrained case. Hence it is a matroid. The collection of subsets of edges starting from different nodes in V must now satisfy the additional tree-constraint given by T. That is, this collection only contains edge sets such that each subset W of V that is adjacent to such an edge set contains the root node r and forms a connected subgraph of T. It follows straightforwardly from the properties of connected subgraphs of a tree that this collection is a greedoid (but not necessarily a matroid). It is immediate that the largest element in the intersection of the greedoid and the matroid is a solution of our maximum tree-constrained matching problem.

A closer look also reveals that the difficulty of the problem is determined by the topology of the tree T or, more precisely, by the number of connected subgraphs of T that contain r. If the number of such subgraphs is polynomial in $|V|$ (this is the case for example if T is a path), then the maximum tree-constrained matching can be found in polynomial time: we just apply Edmond's algorithm repeatedly on all bipartite graphs that are obtained from $B = (V \cup V', E)$ by replacing V with the nodes $W \subseteq V$ in each connected subgraph of T. The largest matching found that matches the corresponding W entirely is a solution of our problem. The number of connected subgraphs can be super-polynomial (for example if T is a balanced binary tree) suggesting that our problem might not be polynomial-time solvable in general. Consistently with this observation we will show that the maximum matroid-greedoid intersection problem is NP-hard.

The paper has been organized as follows. In Section 2 we show, by reduction from 3SAT, that the maximum intersection problem for a matroid family and a greedoid family, given by uniform polynomial-time oracles, is NP-hard. In Section 3, this reduction is modified to show that the maximum matroid-greedoid intersection problem is not approximable within a factor $|S_h|^{1-\epsilon}$ for any fixed $\epsilon > 0$, and its weighted version, the maximum weight matroid-greedoid intersection problem, is not approximable within $2^{|S_h|^k}$ for any fixed $k > 0$, unless $P = NP$. Finally, in Section 4, we consider our problem in the parameterized complexity framework. We show that it is $W[P]$-hard to decide whether or not a matroid-greedoid intersection contains a set of given size.

2 NP-hardness

The hardness proofs in this section and in Section 3 reduce some NP-hard problem H to the intersection problem of a matroid family $\mathcal{F}$ and a greedoid family $\mathcal{G}$. This polynomial-time many-one reduction is non-standard as $\mathcal{F}$ and $\mathcal{G}$ are given only by oracles. An instance $h \in H$ will be reduced to (S_h, F_h, G_h). Here S_h is the domain, (S_h, F_h) a matroid and (S_h, G_h) a greedoid such that the families $\mathcal{F} = \{(S_h, F_h)_{h \in H}\}$ and $\mathcal{G} = \{(S_h, G_h)_{h \in H}\}$ are given by uniform polynomial-time oracles.

The reduction step $h \mapsto (S_h, F_h, G_h)$ is implemented by specializing the uniform oracle algorithms $\mathcal{O}$ and $\mathcal{O}'$ for the matroid family $\mathcal{F}$ and the greedoid family $\mathcal{G}$ to h, giving specialized algorithms $\mathcal{O}(h)$ and $\mathcal{O}'(h)$. Specializing simply means that an input parameter of the algorithms is fixed to the given value h. This reduction can obviously be accomplished in polynomial time. The specialized oracles $\mathcal{O}(h)$ and $\mathcal{O}'(h)$ recognize members of F_h and G_h in time polynomial in $|S_h|$. To make this a hardness proof we must also require that $|S_h|$ is polynomial in $|h|$. Then the running times of the oracles $\mathcal{O}(h)$ and $\mathcal{O}'(h)$ actually become polynomial in $|h|$.

Recall that the NP-complete problem 3-*satisfiability* (3SAT) is, given a boolean formula h in 3-conjunctive normal form (3CNF), to decide whether or not there is a truth assignment Z for the variables of h such that $h(Z) = \text{TRUE}$.

We construct the instance $(S_h, F_h), (S_h, G_h)$ of matroid-greedoid intersection that corresponds to h as follows. Let h contain n different boolean variables. Then S_h contains symbols $t_1, f_1, \ldots, t_n, f_n$. The symbols $t_1, f_1, \ldots, t_n, f_n$ will be used to encode truth assignments: t_i encodes that the ith variable is TRUE and f_i that it is FALSE.

The subset collection F_h consists of all subsets of S_h that contain at most one of the symbols t_i, f_i for $i = 1, \ldots, n$. It is immediate, that (S_h, F_h) satisfies the matroid properties (M1), (M2), and (M3).

The subset collection G_h consists of two groups. The first group A consists of all subsets X of S_h such that $|X| \leq n$ and $X \cap \{t_n, f_n\} = \emptyset$. The second group B consists of the sets that represent a truth assignment that satisfies h. Such a set is of size n and contains one element from each t_i, f_i.

To verify that (S_h, G_h) is a greedoid, first note that (M1) is obviously true. To verify (M3), let $X, Y \in G_h$ such that $|X| > |Y|$.

1. If $|X| < n$ then X and Y must belong to group A. Hence for any element $x \in X \setminus Y$, set $Y \cup \{x\}$ belongs to group A and hence to G_h.

2. If $|X| = n$ and $|X \setminus Y| = 1$ then $Y \cup (X \setminus Y) = X$, i.e., property (M3) holds.

3. In the remaining case $|X| = n$ and $|X \setminus Y| > 1$. As $X \setminus Y$ contains at least two elements and no set of G_h contains both t_n and f_n, at least one element $x \in X \setminus Y$ must be different from t_n, f_n. Then $Y \cup \{x\}$ belongs to group A.

The matroid-greedoid intersection $F_h \cap G_h$ contains a set X such that $|X| = n$ if and only if the group B in the definition of G_h is non-empty, that is, if and only if h is satisfiable. As such a set X is also the largest in $F_h \cap G_h$, we have shown:

Lemma 1 *Boolean formula h is satisfiable if and only if the maximum element in $F_h \cap G_h$ for matroid (S_h, F_h) and greedoid (S_h, G_h) is of size n where n is the number of variables of h.*

The above construction yields a matroid family $\mathcal{F} = \{(S_h, F_h)_{h \in 3\text{CNF}}\}$ and a greedoid family $\mathcal{G} = \{(S_h, G_h)_{h \in 3\text{CNF}}\}$. Both have a uniform polynomial-time oracle for checking membership in F_h and G_h: The only nontrivial task of the oracle is to verify when a truth assignment satisfies a given formula h, but this is doable in time polynomial in $|h|$ using well-known techniques. As $|h| = O(n^3)$ for a 3CNF formula h and $|S_h| = 2n$, the running time of the oracle is polynomial in $|S_h|$, too.

It follows from Lemma 1 and the discussion above that our construction is a polynomial-time reduction of 3SAT to the maximum matroid-greedoid intersection problem. Therefore we have the following.

Theorem 1 *The maximum intersection problem for a matroid family and a greedoid family that are given by uniform polynomial-time oracles is NP-hard.*

Also the *maximum weight* matroid-greedoid intersection problem is NP-hard since maximum matroid-greedoid intersection problem is its special case. In this problem one should find, given integer weights $w(x)$ for $x \in S_h$, a set $X \in F_h \cap G_h$ such that $\sum_{x \in X} w(x)$ is maximum.

3 Inapproximability

As the maximum matroid-greedoid intersection problem is a maximization problem whose exact solution turned out to be NP-hard, it is of interest to see whether or not an *approximation algorithm* with a performance guarantee is possible. An approximation algorithm would find an element in the intersection of the matroid and the greedoid which is not necessarily the largest one.

Following the standard approach (see e.g. [1, 2]), we say that maximization problem is *polynomial-time approximable within* r where r is a function from $\mathbb{N}$ to $\mathbb{Q}$ if there is a polynomial-time algorithm that finds for each instance x of the problem a feasible solution with value $c(x)$ such that

$$\frac{c_{Max}(x)}{c(x)} \leq r(|x|)$$

where $c_{Max}(x)$ is the largest possible value (the optimal value) of a feasible solution of x. The performance ratio of such an approximation algorithm is bounded by the performance guarantee r.

Theorem 2 *The maximum intersection problem for a matroid family and a greedoid family with domains $\{S_h : h \in H\}$, given by uniform polynomial-time oracles, is not polynomial-time approximable within $|S_h|^{1-\epsilon}$ for any fixed $\epsilon > 0$, unless $P = NP$.*

Proof. Assume that for some $\epsilon > 0$, the maximum matroid-greedoid intersection problem is polynomial-time approximable within $|S_h|^{1-\epsilon}$. We show that then we can solve 3SAT in polynomial time.

As in the proof of Lemma 1, let h again be a boolean formula with n variables in 3-conjunctive normal form. Now set S_h contains in addition to the truth assignment symbols $t_1, f_1, \ldots, t_n, f_n$ also some indicator elements p_i, $1 \leq i \leq I(\epsilon)$. Here the number of indicators, $I(\epsilon)$, depends on ϵ as will be shown below. The indicators are needed for padding the elements of the matroid and the greedoid such that the maximum intersection becomes for a satisfiable h sufficiently larger than for a non-satisfiable h.

We again construct a matroid (S_h, F_h) and a greedoid (S_h, G_h) as follows.

The subset collection F_h contains all subsets of S_h that do not contain both t_i and f_i for any $1 \leq i \leq n$. It is again clear, that (S_h, F_h) satisfies properties (M1) and (M2). As regards (M3), let $X, Y \in F_h$ such that $|X| > |Y|$. If there is some indicator x in $X \setminus Y$, then $Y \cup \{x\} \in F_h$. Otherwise X must contain more truth assignment symbols than Y. Then there must be index i such that either t_i or f_i, call it x, belongs to X but neither of t_i and f_i belongs to Y. Then $Y \cup \{x\} \in F_h$. Thus (S_h, F_h) is a matroid.

The subset collection G_h consists of three groups. Groups A and B are exactly same as in the construction of Lemma 1. Hence the sets in groups A and B do not contain any indicator elements. Group C consists of the sets of size n in groups A and B, padded with indicators in all possible ways. That is, if $X \in A$ or $X \in B$ such that $|X| = n$ and Q is a non-empty subset of $\{p_1, \ldots, p_{I(\epsilon)}\}$, then $X \cup Q$ belongs to group C.

To verify that (S_h, G_h) is a greedoid, property (M1) clearly holds. To verify (M3), let $X, Y \in G_h, |X| > |Y|$ and consider the following cases.

1. If $|Y| < n$ then there is a truth assignment symbol $x \in X \setminus Y$ such that $Y \cup \{x\}$ belongs to group A or to group B as shown in the proof of Lemma 1.

2. If $|Y| \geq n$ then there is an indicator $x \in X \setminus Y$ and thus $Y \cup \{x\}$ belongs to group C.

By our construction, the boolean formula h is satisfiable if and only the largest element in $F_h \cap G_h$ is of size $|S_h| - n = I(\epsilon) + n$: The matroid contains all subsets of S_h corresponding to truth assignments with all possible paddings with the padding elements. The greedoid contains all satisfying truth assignments but no unsatisfying truth assignment. Thus if h is satisfiable then the largest element in $F_h \cap G_h$ is of size $|S_h| - n$ consisting of a satisfying truth assignment and all padding elements. If h is not satisfiable then the greedoid does not contain any complete truth assignment. Since the padding elements can occur in a set $X \in G_h$ only if $|X \cap \{t_1, f_1, \ldots, t_n, f_n\}| \geq n$, the size of the largest element is at most $n - 1$.

Let now $I(\epsilon) = (2n)^{1/\epsilon} - 2n$. Thus $|S_h| = (2n)^{1/\epsilon}$. To test the satisfiability of h we use the approximation algorithm to find a approximately largest element of $F_h \cap G_h$. Let c be the size of this element. If h is not satisfiable then certainly $c < n$. On the other hand, if h is satisfiable, then the largest element of $F_h \cap G_h$ is of size $|S_h| - n$. Therefore

$$\frac{|S_h| - n}{c} \leq |S_h|^{1-\epsilon}.$$

But then

$$c \geq \frac{|S_h| - n}{|S_h|^{1-\epsilon}} \geq \frac{|S_h|}{2|S_h|^{1-\epsilon}} = \frac{|S_h|^{\epsilon}}{2} = n$$

where the second inequality follows from $|S_h| \geq 2n$. Hence $c \geq n$ if h is satisfiable and $c < n$ if it is not. We have a polynomial-time satisfiability test because $I(\epsilon)$ is a polynomial in n and hence in $|h|$ when ϵ is fixed, and therefore the matroid family $\{(S_h, F_h)_{h \in 3\text{CNF}}\}$ and the greedoid family $\{(S_h, G_h)_{h \in 3\text{CNF}}\}$ can be represented by uniform oracles whose run times are polynomial in $|S_h|$, hence in $|h|$. □

It is obvious that the maximum weight matroid-greedoid intersection problem is at least as difficult as the maximum matroid-greedoid intersection problem. The approximability gap between these two problems turns out to be exponential: a special case of the maximum weight matroid-greedoid intersection, weighted greedoid maximization, turns out to be inapproximable within $2^{|S_h|^k}$ for any fixed k. (Note that instead of the bound $2^{|S_h|^k}$, any function computable in time polynomial in $|S_h|$ would be suitable. The explicit function $2^{|S_h|^k}$ was chosen for the sake of concreteness.)

The *weighted greedoid maximization problem* for a greedoid family $\{(S_h, G_h)_{h \in H}\}$ is, given an index h and weights $w(x)$ for $x \in S_h$, to find a set $X \in G_h$ such that the weight of the set X,

$$w(X) = \sum_{x \in X} w(x),$$

is maximum. The problem is known to be NP-hard [7].

Theorem 3 *The weighted greedoid maximization problem is not polynomial-time approximable within* $2^{|S_h|^k}$ *for any fixed* $k > 0$*, unless* $P = NP$.

Proof. Assume that for some $k > 0$, the weighted greedoid maximization problem is polynomial-time approximable within $2^{|S_h|^k}$. We show that then we can solve 3SAT in polynomial time.

Let h be a boolean formula with n boolean variables. Then let S_h be the set $\{t_1, f_1, \ldots, t_n, f_n, 1\}$ where t_i and f_i correspond to true and false truth assignments for the ith boolean variable of the formula h, respectively, and 1 is an indicator element for satisfying truth assignments. The set collection G_h consists of two groups. The first group consists of all subsets of $S_h \setminus \{1\}$ of size at most $n+1$. The second group consists of the subsets of S_h that contain 1 and represent satisfying truth assignments of h and hence are of size $n+1$.

Clearly (M1) and (M3) hold and thus (S_h, G_h) is a greedoid.

We give weights to the elements of S_h as follows. The indicator 1 has weight $(n+1)2^{|S_h|^k} - n + 1$ and the symbols $t_1, f_1, \ldots, t_n, f_n$ have weight 1 each. Then the maximum weight set $X \in G_h$ has weight $n+1$ if the formula is unsatisfiable and $(n+1)2^{|S_h|^k} + 1$ otherwise. Since

$$\frac{(n+1)2^{|S_h|^k} + 1}{n+1} > 2^{|S_h|^k},$$

we could separate these two cases using the approximation algorithm and thus P would be equal to NP. □

The weighted greedoid maximization problem is a special case of the maximum weight matroid-greedoid intersection problem since we can choose the matroid's set collection F_h to be a superset of the greedoid's set collection G_h, e.g., $F_h = \{X : X \subseteq S_h\}$.

Corollary 1 *The maximum weight intersection problem for a matroid family and a greedoid family with domains* $\{S_h : h \in H\}$*, given by uniform polynomial-time oracles, is not polynomial-time approximable within* $2^{|S_h|^k}$ *for any fixed* $k > 0$*, unless* $P = NP$.

Note that the maximization problem for unweighted greedoids is trivially in P.

4 Fixed-parameter intractability

Parameterized complexity contemplates the computational complexity of decision problems when some parameters of the problems, e.g. the number of vertices in a vertex cover or the maximum length allowed for a shortest common supersequence, are fixed [4]. This is motivated by the observation that many problems have natural parameters that are quite small in practical applications of the problem. A *parameterized language*, representing the positive instances of the parameterized (decision) problem, is a set $L \subseteq \Sigma^* \times \mathbb{N}$ where Σ is the input alphabet and $\mathbb{N}$ is the set of parameters.

A parameterized language L is said to be *fixed-parameter tractable* (FPT) if there is an algorithm A that decides whether $(e, k) \in L$ for every instance $(e, k) \in \Sigma^* \times \mathbb{N}$ in time $f(k)|e|^c$, where f is an arbitrary function and c is a constant independent from k. Parameterized languages have a hierarchy (called the W-hierarchy) similar to the polynomial hierarchy

$$FPT \subseteq W[1] \subseteq W[2] \subseteq \ldots \subseteq W[P].$$

It is believed that the containments are proper [3].

A parameterized language L reduces to another such language L' by a *standard parameterized m-reduction* if there are functions f and g from $\mathbb{N}$ to $\mathbb{N}$, and a function $(e, k) \mapsto e'$ from $\Sigma^* \times \mathbb{N}$ to Σ^*, such that $(e, k) \mapsto e'$ is computable in time $g(k)|e|^c$, and $(e, k) \in L$ if and only if $(e', f(k)) \in L'$.

In our case L' will be presented as an intersection of a matroid and a greedoid, given by uniform polynomial-time oracles. Then the reduction step $(e, k) \mapsto e'$ is implemented by specializing the oracle algorithms to (e, k) such that they then recognize the matroid and the greedoid for e'. This specialization can take time $g(k)|e|^c$.

The *parameterized weighted circuit satisfiability* is a fundamental $W[P]$-complete problem. This problem asks, given a boolean circuit h and a positive integer k, to decide whether or not there exists a satisfying truth assignment of weight k, i.e., a satisfying truth assignment with exactly k variables set to TRUE.

The *parameterized intersection problem* for a matroid family $\mathcal{F}$ and a greedoid family $\mathcal{G}$ is to decide, given an index $h \in H$ and a parameter k, whether or not there exists a set $X \in F_h \cap G_h$ such that $|X| = k$.

The *dual parameterized intersection problem* for a matroid family $\mathcal{F}$ and a greedoid family $\mathcal{G}$ is to decide, given an index $h \in H$ and a parameter k, whether or not there exists a set $X \in F_h \cap G_h$ such that $|X| = |S_h| - k$.

We will show, by reduction from the parameterized weighted circuit satisfiability, that these natural parameterizations of the maximum matroid-greedoid intersection problem are $W[P]$-hard. We consider only the above versions of these problems where the solution is required to be of certain size. As the decision version of the maximum matroid-greedoid intersection problem is in NP, the (dual) parameterized matroid-greedoid intersection problems with inequality constraints on the size of the solution are fixed-parameter polynomial-time equivalent to the (dual) parameterized matroid-greedoid intersection problem (with equality constraint on the size of the solution) [4, page 51].

Theorem 4 *The parameterized intersection problem for a matroid family and a greedoid family, given by uniform polynomial-time oracles, is* $W[P]$*-hard.*

Proof. We reduce the parameterized weighted circuit satisfiability to the parameterized matroid-greedoid intersection problem as follows.

Let $\mathcal{C}$ denote the set of boolean circuits and let $h = (e, k) \in \mathcal{C} \times \mathbb{N}$ be an instance of the parameterized weighted circuit satisfiability problem. The circuit e has n variables. To construct the corresponding matroid (S_h, F_h) and greedoid (S_h, G_h), we let the set S_h consist of symbols $t_1, \ldots, t_n, 1, d_1, \ldots, d_{|e|}$ where $|e|$ is the size of the circuit e in some fixed encoding scheme. The symbol t_i denotes that the ith variable of e is TRUE. Symbol 1 is an indicator element for satisfying truth assignments of k TRUE variables. Symbols $d_1, \ldots, d_{|e|}$ are padding elements only needed to make S_h large enough such that the value of the circuit e can be computed in polynomial time in $|S_h|$.

The set collection F_h consists of all subsets of $\{t_1, \ldots, t_n, 1\}$ of size at most $k + 1$ containing a maximum of k symbols t_i. Clearly the matroid properties (M1), (M2), and (M3) hold.

The set collection G_h consists of three groups. Group A consists of the subsets of $\{t_1, \ldots, t_n\}$ of size at most k representing truth assignments of maximum k TRUE variables. Group B consists of the subsets of $\{t_1, \ldots, t_n, 1\}$ of size $k + 1$ representing truth assignments of weight k that satisfy e. Hence each member X of the group B contains element 1 and elements t_i for the k variables with value TRUE in an assignment satisfying e. Group C consists of the subsets of $\{t_1, \ldots, t_n\}$ of size $k + 1$.

It is immediate that (M1) holds. To verify (M3), let $X, Y \in G_h$ such that $|X| > |Y|$.

1. If $|Y| < k$ then there is $x \in X \setminus Y$ such that $Y \cup \{x\}$ is in group A.
2. If $|Y| = k$ then there is $x \in X \setminus Y$ such that $Y \cup \{x\}$ is in group B or in group C.

Thus (S_h, G_h) is a greedoid.

We now have families $\mathcal{F} = \{(S_h, F_h)_{h \in \mathcal{C} \times \mathbb{N}}\}$ and $\mathcal{G} = \{(S_h, G_h)_{h \in \mathcal{C} \times \mathbb{N}}\}$ that obviously can be given by uniform polynomial-time oracles. Given $h = (e, k)$ the oracles check memberships in F_h and G_h in time polynomial in $|e|$. As $|S_h| = \Theta(|e|)$, because of the padding elements, these times are polynomial in $|S_h|$, too.

The boolean circuit e has a satisfying truth assignment of weight k if and only if the group B is non-empty, that is, if and only if there is a set $X \in F_h \cap G_h$ such that $|X| = k+1$: the matroid ensures that the solution X sets at most k variables TRUE and a set $X \in G_h$ of size $k+1$ contains 1 only if the truth assignment corresponding to the k t_i's in X satisfy the circuit e. In the standard parameterized m-reduction we may thus choose $f(k) = k+1$. Moreover, the reduction $h = (e,k) \mapsto (S_h, F_h, G_h)$ is done by specialization to $h = (e,k)$ of the oracles. This can obviously be done in time $O(|e|+k) = O(|e|)$. Hence we may select $g(k)$ =constant.

Thus, the parametrized matroid-greedoid intersection problem is $W[P]$-hard.

□

Theorem 5 *The dual parameterized intersection problem for a matroid family and a greedoid family, given by uniform polynomial-time oracles, is $W[P]$-hard.*

Proof. We reduce the parameterized weighted circuit satisfiability to the dual parameterized matroid-greedoid intersection problem as follows.

Let again $h = (e,k) \in \mathcal{C} \times \mathbb{N}$ be an instance of the parameterized weighted circuit satisfiability where e has n variables.

The set S_h consists of $f_1, \ldots, f_n, 1, d_1, \ldots, d_{|e|}$. The symbol f_i denotes that the ith variable is set to be FALSE and 1 is an indicator element for satisfying truth assignments. The symbols $d_1, \ldots, d_{|e|}$ ensure that the value of the circuit can be computed in time polynomial in $|S_h|$. Unlike in the proof of Theorem 4, the padding symbols $d_1, \ldots, d_{|e|}$ are now used in the subset collections F_h and G_h.

The subset collection F_h consists of all subsets of S_h containing at most $n-k$ symbols f_i. Clearly the matroid properties (M1), (M2), and (M3) hold.

The subset collection G_h consists of four groups. The first group A' consists of all subsets of $\{d_1, \ldots, d_{|e|}\}$. The second group A consists of subsets X of $\{f_1, \ldots, f_n, d_1, \ldots, d_{|e|}\}$ such that $\{d_1, \ldots, d_{|e|}\} \subset X$ and $|e|+1 \leq |X| \leq |e|+n-k$, representing the truth assignments of maximum $n-k$ FALSE variables. The third group B consists of subsets X of the set $\{f_1, \ldots, f_n, 1, d_1, \ldots, d_{|e|}\}$ of size $|e|+n-k+1$ representing the $n-k$ FALSE variables in a truth assignment of weight k that satisfies e. The fourth group C consists of subsets X of $\{f_1, \ldots, f_n, d_1, \ldots, d_{|e|}\}$ of size $|e|+n-k+1$ representing truth assignments of $n-k+1$ FALSE variables. Thus $\{d_1, \ldots, d_{|e|}\} \subset X$.

It is immediate that (M1) holds for (S_h, G_h). To verify (M3), let $X, Y \in G_h$ such that $|X| > |Y|$.

1. If $|Y| < |e|$ then there is $x \in X \setminus Y$ such that $Y \cup \{x\}$ belongs to group A'.

2. If $|e| \leq |Y| < |e| + n - k$ then there is $x \in X \setminus Y$ such that $Y \cup \{x\}$ belongs to group A.

3. If $|Y| = |e| + n - k$ then there is $x \in X \setminus Y$ such that $Y \cup \{x\}$ is in group B or in group C.

Thus (S_h, G_h) is a greedoid.

The boolean circuit e has a satisfying truth assignment of weight k if and only if the group B is non-empty, that is, if and only if there is a set $X \in F_h \cap G_h$ such that $|X| = |S_h| - k$: the matroid ensures that the solution X sets at most $n - k$ variables FALSE and a set $X \in G_h$ of size $|S_h| - k$ contains 1 only if the truth assignment corresponding to the $n - k$ f_i's in X satisfy the circuit e. In the m-reduction we may hence choose $f(k) = k$. The rest of the proof is similar to the proof of Theorem 4.

Thus the dual parametrized matroid-greedoid intersection problem is $W[P]$-hard. □

5 Conclusions

We have shown that the maximum intersection problem for a matroid family $\{(S_h, F_h)_{h \in H}\}$ and a greedoid family $\{(S_h, G_h)_{h \in H}\}$ is NP-hard, $W[P]$-hard and inapproximable within $|S_h|^{1-\epsilon}$ for any fixed $\epsilon > 0$. We have also shown that the weighted greedoid maximization is inapproximable within $2^{|S_h|^k}$ for any fixed k, and thus the weighted maximum matroid-greedoid intersection problem is inapproximable within $2^{|S_h|^k}$ for any fixed k.

The maximum matroid-greedoid intersection problem is closely related to the maximum matroid-greedoid partition problem [5]. The *maximum partition problem* for a matroid family $\{(S_h, F_h)_{h \in H}\}$ and a greedoid family $\{(S_h, G_h)_{h \in H}\}$ is to find, given an index $h \in H$, a set $X = Y \cup Z, Y \cap Z = \emptyset, Y \in F_h, Z \in G_h$ such that $|X|$ is maximum. The NP-hardness and inapproximability of the maximum matroid-greedoid intersection problem can be transformed to show that also the maximum matroid-greedoid partition problem is NP-hard and inapproximable. The fixed-parameter (in)tractability of the maximum matroid-greedoid partition problem is still open.

References

[1] G. Ausiello, P. Crescenzi, G. Gambosi, V. Kann, A. Marchetti-Spaccamela, and M. Protasi. *Complexity and Approximation: Combinatorial Optimization Problems and Their Approximability Properties*. Springer-Verlag, 1999.

[2] P. Crescenzi, V. Kann, R. Silvestri, and L. Trevisan. Structure in approximation classes. *SIAM Journal on Computing* 28 (1999) 1759–1782.

[3] R. G. Downey, and M. R. Fellows. Fixed-parameter tractability and completeness I: basic results. *SIAM Journal on Computing* 24 (1995) 873–921.

[4] R. G. Downey, and M. R. Fellows. *Parameterized Complexity*. Springer-Verlag, 1999.

[5] J. Edmonds. Minimum partition of a matroid into independent subsets. *Journal of Research of the National Bureau of Standards* 69B (1965) 67–72.

[6] M. R. Garey, and D. S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman and Company, 1979.

[7] B. Korte, and J. Vygen. *Combinatorial Optimization: Theory and Algorithms*. Springer-Verlag, 2000.

[8] C. H. Papadimitriou, and K. Steiglitz. *Combinatorial Optimization: Algorithms and Complexity*. Dover Publications, 1998.