OPTIMIZATION OF DISCRETE CONTROL SYSTEMS WITH VARYING STRUCTURE

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Abstract

In this paper a special step discrete control problem is considered. The formulation of the problem uses a parameter to control the switching point. By using Taylor's increment methods first and second order optimality conditions (in the sense of Pontryagin's maximum principle) will be derived.

1.Introduction.

Among the various optimal control problems the discrete control problem is characteristical. This problem arises e.g. by using numerical methods for solving continous control problems. But discrete control problems have an independent means, too.

First consider an example of such a problem.

Example: (problem of a cosmic ship)

Let us project a cosmic ship consisting of N steps. The ship has a starting weight G and the weight H of the rocket part. All steps have a gazeline. If one step of the rocket is pushed away the rocket takes an additional speed Δv, which depends on the weight of this step and the weight of the remaining part of the ship. We have to distribute the weight among the steps such that the speed of the rocket at the end of the flight is maximal. To model this problem, let u(t) denote the weight of the t-th step, t=1,...,N, where t=1 is the number of the last step. Let x(t) be the weight of the head of the rocet together with the first t steps. Then we can write

$$x(t)=x(t-1)+u(t), \quad t=1,...,N. \tag{1}$$

The boundary values x(0)=G and x(N)=H denote the starting weight and weight of the rocket. The additional speed gained if a step t is pushed away is

$$\Delta v(t) = f(x(t-1), u(t)), \quad t=1,...,N.$$

This additional speed depends on the weight u(t) of the t-th step and the weight x(t-1) of the remaining part of the rocket. After pushing away all steps, the common speed will be

$$S(u) = \sum_{t=1}^{N} f(x(t-1), u(t)) \tag{2}$$

The aim of the problem is now to find controls $\{u(1); u(2), ..., u(N)\}$ for the rocket such that (2) will be maximal subject to the conditions (1) and the boundary values.

The theory of discrete optimal problems has been investigated by Gabasov R, Kirillova F, Mordukhovich B, Mansimov K, Propoy A, Kroxotka V, Minyuk S, Ashepkov T, and others.

In the paper [2] Moyseev has investigated necessary optimality conditions for the continous control problem. The discrete anolog of Moyseev's problem has been investigated by Mansimov K. [7]. To formulate the results of [7], consider the problem

$$S(u) = \sum_{t=t_0}^{t_1-1} \sum_{s=t_0}^{t_1-1} F(t, s, x(t), x(s)) \rightarrow \min \tag{3}$$

subject to

$$x(t+1) = A(t)x(t) + f(t, u(t)); \ t \in T = \{t_0, t_0+1, ..., t_1-1\}, x(t_0) = x_0. \tag{4}$$

Theorem (Mansimov K [7]): For optimality of an admissible control u(t) of the problem (3),(4) it is necessary that the following inequalities hold:

$$\sum_{t=t_0}^{t_1-1} \Delta_{v(t)} H(t) \le 0 \ \text{ for all } \ v(t) \in U, t \in T.$$

Here $H(t) = \Psi(t) f(t, u(t))$ is Pontryagin's function and $\Delta_{v(t)} H(t)$ is the increment of the function H(t).

In the case of an nonsmooth objective function

$$S_0(u) = \Phi(x(t_1)) \rightarrow \min \tag{5}$$

$$S_i(u) = \Phi_i(x(t_1)) \le 0, \quad i = 1, .., p \tag{6}$$

$$x(t+1) = f(t, x(t), u(t)); t \in T = \{t_0, t_0+1, ..., t_1-1\}, x(t_0) = x_0 \tag{7}$$

$$u(t) \in U \subset R^r, \quad t \in T \tag{8}$$

the following theorem holds.

Theorem (Mansimov K,[7]): If the admissible velocities are convex along admissible processes, then for optimality in problem (5) - (8) it is necessary that the following inequalities hold:

$$max\{\frac{\partial \Phi_i(x(t_1))}{\partial l(t_1, v)} : i \in J(u)\} \quad \text{for all } v \in U .$$

Here $l(t,v)=\sum_{\tau=t_0}^{t-1} F'(t,\tau)\Delta_v f(\tau)$ and $F(t,\tau)$ is an $n \times n$ dimensional matrix-function, being the solution of the following problem [11]:

$$F(t,\tau-1) = F'(t,\tau) f_x[\tau] \text{ , } F(t,t-1) = E$$

Second order optimality conditions for control problems can be found e.g. in Gabasov R;Tarasnko N. F [5].

In all these papers one-stage optimal control problems have been investigated. But some applied problems from economy, military work, chemistry are inherently multistage problems. This means that there are several stages each being characterized by its own equations, controls, phase coordinates, constraints, etc. Usually these stages are connected by each other by additional conditions. Here problems will are considered where these connections are given by switching points which are controlled by a given parameter. These multistage processes will be called step control systems (or discrete systems with varying structure).

For example, consider a rocket with two typs of engines that work consecutively. The work of the second engine depends on the first one. Morerover, the rocket moves from one controlling area to a second one that changes all the structure (controls, functions, conditions, etc.).

This paper investigates step discrete control systems and aims to find necessary and sufficent optimality condition of first and second order.

2. Necessary optimality condition

Consider a controlling process, which is described by the following discrete system with varying structure:

$$\text{Minimize} \quad S(u,v) = \sum_{i=1}^{3} \varphi_i(x_i(t_i)) \tag{9}$$

subject to

$$x_i(t+1) = f_i(t, x_i(t), u_i(t)), \; t \in T_i = \{t_{i-1}, t_{i-1}+1, \ldots, t_i - 1\}, \; i = 1,2,3, \tag{10}$$

$$\left.\begin{aligned} x_1(t_0) &= g_1(v_1) \\ x_i(t_{i-1}) &= g_i(x_{i-1}(t_{i-1}), v_i), \quad i = 2,3 \end{aligned}\right\} \tag{11}$$

$$u_i(t) \in U_i \subset R^r, \; t \in T_i, \; i = 1,2,3. \tag{12}$$

Here, $v_i, i = 1,2,3$ are q-dimensional controlling parameters and $V_i \subseteq R^q$, $i = 1,2,3$, i. e.

$$v_i \in V_i, \; i = 1,2,3. \tag{13}$$

In this problem, $g_1 : R^q \to R^n$ are given at least twice continuously differentiable vector-valued functions, $g_i : R^n \times R^q \to R^n$ are given at least twice continuously differentiable vector-valued functions, i=2,3, $f_i : R \times R^n \times R^r \to R^n$ are given continuous vector-valued functions, which are at least twice continuously partially differentiable with respect to x, $\varphi_i : R^n \to R$ are given at least twice continuously differentiable functions, i=1,2,3, $u_i(t)$: R $U_i \subset R^r$ are controls and $v_i \in V_i \subset R^q$ are controlling parameters. The sets U_i, V_i, are assumed to be nonempty and bounded.

In this problem we consider a discrete control problem with three steps. Equations (10) describe the systems behavior within each of the steps. Here, $u(t)$ describes the control, $x(t)$ the trajectory of the system, and t the time. Equations (11) are used to fix the move of the system from one to the next step. To do this switching parameters v_i are introduced and conditions (11) guarantee that the trajectory x_{i-1} is „continued“ by x_i. In problems without these switching points, as e.g. in one-step problems, we can apply any part of the conditions obtained by using Pontryagin's principle to derive optimality conditions. Hence, as it will be seen below, such conditions appear as part of the necessary optimality conditions for problem (9)—(13), too. But new conditions for the switching points need to be added.

A tuple $(u_1(t), u_2(t), u_3(t), v_1, v_2, v_3) \equiv (u(t), v)$ with the properties (12) and (13) is called admissible control and the corresponding solution $(x_1(t), x_2(t), x_3(t)) \equiv x(t)$ of the system (9)-(13) is called admissible trajectory. For the fixed admissible control $(u^0(t), v^o)$ we introduce the following notation:

$$H_i(t, x_i, u, \Psi_i^0) = \Psi_i^{0'}(t) \cdot f_i(t, x_i, u_i),$$

$$\Delta_{u_i} H_i[t] \equiv H_i(t, x_i^0(t), u_i(t), \Psi_i^0(t)) - H_i(t, x_i^0(t), u_i^0(t), \Psi_i^0(t)),$$

$$\frac{\partial H_i[t]}{\partial x_i} = \frac{\partial H_i(t, x_i^0(t), u_i^0(t), \psi_i^0(t))}{\partial x_i}, \ \Delta_{v_1} g_1[v_1] \equiv g_1(v_1) - g_1(v_1^0)$$

$$\Delta_{v_i} g_i(x_{i-1}^0(t_{i-1}), v_i^0) \equiv g_i(x_{i-1}^0(t_{i-1}), v_i) - g_i(x_{i-1}^0(t_{i-1}), v_i^0), i = 2,3$$

$$L_1(v_1, \Psi_1^0(t_0 - 1)) = \Psi_1^0(t_0 - 1)' g_1(v_1),$$

$$L_2(x_1(t_1), v_2, \Psi_2^0(t_1 - 1)) = \Psi_2^0(t_1 - 1)' g_2(x_1(t_1), v_2)$$

$$L_3(x_2(t_2), v_3, \Psi_3^0(t_2 - 1)) = \Psi_3^0(t_2 - 1)' g_3(x_2(t_2), v_3),$$

where the unknown functions Ψ^0_i are defined below in (16).

Theorem 1 $[12]$**:** If the sets

$$f_i(t_i x_i^0(t), U_i) = \{\alpha_i : \alpha_i = f_i(t_i x_i^0(t), u_i), u_i \in U\}, \ i = 1,2,3$$

$$g_1(V_1) = \{\alpha_4 : \alpha_4 = g_1(v_1), v_1 \in V_1\}$$

$$g_i(x_{i-1}^0(t_{i-1}), V_i) = \{\alpha_i : \alpha_i = g_i(x_{i-1}^0(t_{i-1}), v_i), v_i \in V\}, \ i = 2,3$$

are convex then for optimality of an admissible control $(u^0(t), v^0)$ in problem (9)-(13) it is necessary that the following conditions are true:

a) Discrete maximum principle for the control $u_i^0(t), i = 1,2,3$:

$$\sum_{t=t_{i-1}}^{t_i - 1} \Delta_{u_{i(t)}} H_i[t] \le 0, \text{ for all } u_i(t) \in U_i, i = 1,2,3, \ t \in T_i$$

b) Discrete maximum principle for the controlling parameter $v_i^0, i = 1,2,3$.

$$\max_{v_1 \in V_1} L_1(v_1, \psi_1^0(t_0 - 1)) = L_1(v_1^0, \psi_1^0(t_0 - 1))$$

$$\max_{v_i \in V_i} L_i(x_{i-1}^0(t_{i-1}), v_i, \psi_i^0(t_{i-1} - 1)) = L_i(x_{i-1}^0(t_{i-1}), v_i^0, \psi_i^0(t_{i-1} - 1)), i = 2,3$$

It is known that, for continuous-time systems, an optimal control satisfies Pontryagin´s maximum principle without the restrictive convexity assumption. But in the discrete maximum principle this does not hold in general unless a certain convexity is imposed a priory on the control system. A clear explanation of this phenomenon is given in Pshenichnyi’s book [15], where it is shown why discrete systems require a convexity assumption for the validity of the maximum principle, while continuous-time systems enjoy it automatically due to the so-called “hidden convexity“.

In the control problem one of the methods to get necessary optimality conditions is the increment formula. For this we have to calculate the increment formula, to find a conjugate system for the corresponding problems and to use an anolog of needle variations in the continuous case. Then the rest of the increment formula can be estimated using the step method.

Proof: Using Taylor’s formula we can write the increment of the functional at an arbitrary admissible pair $(u(t), v)$ as

$$S(u(t),v)-S(u^0(t),v^0)=\Delta S(u^0,v^0)$$

$$=\sum_{i=1}^{3}\left[\frac{\partial\varphi_i(x_i^0(t_i))}{\partial x_i}+\Psi_i^0(t_i-1)\right]'\Delta x_i(t_i)-\sum_{i=1}^{3}\sum_{t=t_{i-1}}^{t_i-1}\Delta_{\bar u_i}H_i[t]-\sum_{i=1}^{3}\sum_{t=t_{i-1}}^{t_i-1}\left[\Psi_i^0(t-1)-\frac{\partial H_i[t]}{\partial x_i}\right]'\Delta x_i(t)$$

$$-\Delta_{\bar v_1}L_1(v_1^0,\Psi_1^0(t_0-1))-\Delta_{\bar v_2}L_2(x_1^0(t_1),v_2^0,\Psi_2^0(t_1-1))-\Delta_{\bar v_3}L_3(x_2^0(t_2),v_3^0,\Psi_3^0(t_2-1))-$$

$$-\sum_{i=1}^{3}\sum_{t=t_{i-1}}^{t_i-1}\frac{\partial\Delta_{\bar u_i}H_i'[t]}{\partial x_i}\Delta x_i(t)+\frac{1}{2}\sum_{i=1}^{3}\Delta x_i'(t_i)\frac{\partial^2\varphi_i(x_i^0(t_i))}{\partial x_i^2}\Delta x_i(t_i)-\frac{1}{2}\sum_{i=1}^{3}\sum_{t=t_{i-1}}^{t_i-1}\Delta x_i'(t)\frac{\partial^2 H_i[t]}{\partial x_i^2}\Delta x_i(t)-$$

$$-\frac{\partial L_2'(x_1^0(t_1),v_2^0,\psi_2^0(t_1-1))}{\partial x_1}\Delta x_1(t_1)-\frac{\partial\Delta_{\bar v_2}L_2'(x_1^0(t_1),v_2^0,\psi_2^0(t_1-1))}{\partial x_1}\Delta x_1(t_1)-$$

$$-\frac{1}{2}\Delta x_1'(t_1)\frac{\partial^2 L_2(x_1^0(t_1),v_2^0,\psi_2^0(t_1-1))}{\partial x_1^2}\Delta x_1(t_1)-\frac{\partial L_3'(x_2^0(t_2),v_3^0,\psi_3^0(t_2-1))}{\partial x_2}\Delta x_2(t_2)-$$

$$-\frac{\partial\Delta_{\bar v_3}L_3'(x_2^0(t_2),v_3^0,\psi_3^0(t_2-1))}{\partial x_2}\Delta x_2(t_2)-\frac{1}{2}\Delta x_2'(t_2)\frac{\partial^2 L_3(x_2^0(t_2),v_3^0,\psi_3^0(t_2-1))}{\partial x_2^2}\Delta x_2(t_2)+$$

$$+\eta_1(u^0,v^0;\Delta u,\Delta v), \tag{14}$$

where by definition

$$\eta_1(u^0,v^0;\Delta u,\Delta v)=\sum_{i=1}^{3}o_i\left(\|\Delta x_i(t_i)\|^2\right)-\sum_{t=t_0}^{t_1-1}o_4\left(\|\Delta x_1(t)\|^2\right)-\sum_{t=t_1}^{t_2-1}o_5\left(\|\Delta x_2(t)\|^2\right)-\sum_{t=t_2}^{t_3-1}o_6\left(\|\Delta x_3(t)\|^2\right)-$$

$$-\frac{1}{2}\sum_{i=1}^{3}\sum_{t=t_{1-1}}^{t_i-1}\Delta x_i'(t)\frac{\partial^2\Delta_{\bar{u}_i}H_i[t]}{\partial x_i^2}\Delta x_i(t)-o_7\left(\left\|\Delta x_1(t_1)\right\|^2\right)-o_8\left(\left\|\Delta x_2(t_2)\right\|^2\right)-\frac{1}{2}\Delta x_2'(t_2)\times$$

$$\times\frac{\partial^2\Delta_{\bar{v}_2}L_2\left(x_1^0(t_1),v_2^0,\psi_2^0(t_1-1)\right)}{\partial x_2^2}\Delta x_2(t_2)-\frac{1}{2}\Delta x_3'(t_3)\frac{\partial^2\Delta_{\bar{v}_3}L_3\left(x_2^0(t_2),v_3^0,\psi_3^0(t_2-1)\right)}{\partial x_3^2}\Delta x_3(t_3). \quad (15)$$

Here $o_i(\cdot)$, $i=1,..,8$ are defined by the expansions

$$\varphi_i(\bar{x}_i(t_i))-\varphi_i(x_i^0(t_i))=\frac{\partial\varphi_i'(x_i^0(t_i))}{\partial x_i}\Delta x_i(t_i)+\frac{1}{2}\Delta x'(t_i)\frac{\partial^2\varphi_i(x_i^0(t_i))}{\partial x_i^2}\Delta x_i(t_i)+$$

$$+o_i\left(\left\|\Delta x_i(t_i)\right\|^2\right),\ i=1,2,3,$$

$$H_i\left(t,\bar{x}_i(t),\bar{u}_i(t),\psi_i^0(t)\right)-H_i\left(t,x_i^0(t),\bar{u}_i(t),\psi_i^0(t)\right)=\frac{\partial H_i'\left(t,x_i^0(t),\bar{u}_i(t),\psi_i^0(t)\right)}{\partial x_i}\Delta x_i(t)+$$

$$+\frac{1}{2}\Delta x_i'(t)\frac{\partial^2H_i\left(t,x_i^0(t),\bar{u}_i(t),\psi_i^0(t)\right)}{\partial x_i^2}\Delta x_i(t)+o_{i+3}\left(\left\|\Delta x_i(t)\right\|^2\right),i=1,2,3.,$$

$$L_2\left(\bar{x}_1(t_1),\bar{v}_2,\psi_2^0(t_1-1)\right)-L_2\left(x_1^0(t_1),\bar{v}_2,\psi_2^0(t_1-1)\right)=\frac{\partial L_2'\left(x_1^0(t_1),\bar{v}_2,\psi_2^0(t_1-1)\right)}{\partial x_1}\Delta x_1(t_1)+$$

$$+\frac{1}{2}\Delta x'(t_1)\frac{\partial^2L_2\left(x_1^0(t_1),v_2,\psi_2^0(t_1-1)\right)}{\partial x_1^2}\Delta x_1(t_1)+o_7\left(\left\|\Delta x_1(t_1)\right\|^2\right),$$

$$L_3\left(\bar{x}_2(t_2),\bar{v}_3,\psi_3^0(t_2-1)\right)-L_3\left(x_2(t_2),\bar{v}_3,\psi_3(t_2-1)\right)=\frac{\partial L_3'\left(x_2^0(t_2),\bar{v}_3,\psi_3^0(t_2-1)\right)}{\partial x_2}\Delta x_2(t_2)+$$

$$+\frac{1}{2}\Delta x_2'(t_2)\frac{\partial^2L_2\left(x_2^0(t_2),\bar{v}_3,\psi_3^0(t_2-1)\right)}{\partial x_2^2}\Delta x_2(t_2)+o_8\left(\left\|\Delta x_2(t_2)\right\|^2\right).$$

Now taking $\Psi_i^0(t)$, $i=1,2,3,$ as solutions of the following linear difference equations

$$\left.\begin{aligned}
\Psi_i^0(t-1)&=\frac{\partial H_i[t]}{\partial x_i},\ i=1,2,3,t\in T_i\\
\Psi_1^0(t_1-1)&=-\frac{\partial\varphi_1(x_1^0(t_1))}{\partial x_1}+\frac{\partial L_2\left(x_1^0(t_1),v_2^0,\psi_2^0(t_1-1)\right)}{\partial x_1}\\
\Psi_2^0(t_2-1)&=-\frac{\partial\varphi_2(x_2^0(t_2))}{\partial x_2}+\frac{\partial L_3\left(x_2^0(t_2),v_3^0,\psi_3^0(t_2-1)\right)}{\partial x_1}\\
\Psi_3^0(t_3-1)&=-\frac{\partial\varphi_3(x_3^0(t_3))}{\partial x}
\end{aligned}\right\} \quad (16)$$

the increment formula (14) reduces to a simpler one:

$$S\left(u\ (t),v\ \right)-S\left(u^0(t),v^0\right)=\Delta S\left(u^0,v^0\right)=-\sum_{i=1}^{3}\sum_{t=t_{i-1}}^{t_i-1}\Delta_{\bar{u}_i}H_i[t]$$

$$-\Delta_{\bar{v}_1}L_1\left(v_1^0,\Psi_1^0(t_0-1)\right)-\Delta_{\bar{v}_2}L_2\left(x_1^0(t_1),v_2^0,\Psi_2^0(t_1-1)\right)-\Delta_{\bar{v}_3}L_3\left(x_2^0(t_2),v_3^0,\Psi_3^0(t_2-1)\right)-$$

$$-\sum_{i=1}^{3}\sum_{t=t_{i-1}}^{t_i-1}\frac{\partial\Delta_{\bar{u}_i}H_i'[t]}{\partial x_i}\Delta x_i(t)+\frac{1}{2}\sum_{i=1}^{3}\Delta x_i'(t_i)\frac{\partial^2\varphi_i\left(x_i^0(t_i)\right)}{\partial x_i^2}\Delta x_i(t_i)-\frac{1}{2}\sum_{i=1}^{3}\sum_{t=t_{i-1}}^{t_i-1}\Delta x_i'(t)\frac{\partial^2 H_i[t]}{\partial x_i^2}\Delta x_i(t)$$

$$-\frac{\partial\Delta_{\bar{v}_2}L_2'\left(x_1^0(t_1),v_2^0,\psi_2^0(t_1-1)\right)}{\partial x_1}\Delta x_1(t_1)-\frac{1}{2}\Delta x_1'(t_1)\frac{\partial^2 L_2\left(x_1^0(t_1),v_2^0,\psi_2^0(t_1-1)\right)}{\partial x_1^2}\Delta x_1(t_1)$$

$$-\frac{\partial\Delta_{\bar{v}_3}L_3'\left(x_2^0(t_2),v_3^0,\psi_3^0(t_2-1)\right)}{\partial x_2}\Delta x_2(t_2)-\frac{1}{2}\Delta x_2'(t_2)\frac{\partial^2 L_3\left(x_2^0(t_2),v_3^0,\psi_3^0(t_2-1)\right)}{\partial x_2^2}\Delta x_2(t_2)+$$

$$+\eta_1\left(u^0,v^0;\Delta u,\Delta v\right). \tag{17}$$

Let $\left(u^{\,o}(t),v^{\,o}\right)$ be an optimal pair, and assume that the sets of admissible velocities are convex along the process $\left(u(t),v,x(t)\right)$, i. e., the sets

$f_i\left(t,x_i^0(t),U_i\right)=\left\{\alpha_i:\alpha_i=f_i\left(t,x_i^0(t),u_i\right),u\in U_i\right\},\ i=1,2,3,$

$g_1(V_1)=\left\{\alpha_4:\alpha_4=g_1(v_1),v_1\in V_1\right\},$

$g_i\left(x_{i-1}^0(t_{i-1}),V_i\right)=\left\{\alpha_{i+3}:\alpha_{i+3}=g_i\left(x_{i-1}^0(t_{i-1}),v_i\right),v\in V_i\right\},\ i=2,3,$

are convex. Let $\varepsilon\in[0,1]$ be an arbitrary number. Denote the increment of the optimal pair by

$$\Delta u_i(t;\varepsilon)=u_i(t;\varepsilon)-u_i^0(t),\ t\in T_i,\ i=1,2,3, \tag{18}$$

$\Delta v_i(\varepsilon)=v_i(\varepsilon)-v_i^0,\ i=1,2,3.$

Then, by convexity, for each $u_i(t)\in U_i,\ v_i\in V,\ \ t\in T_i,\ \ i=1,2,3$, there are $u_i(t,\varepsilon)\in U_i,\ v_i(\varepsilon)\in V_i,\ i=1,2,3$ such that

$\Delta_{u_{i(t\varepsilon)}}f_i[t]=\varepsilon\Delta_{u_{i(t)}}f_i[t],\ i=1,2,3,$

$\Delta_{v_{1(\varepsilon)}}g_1\left(v_1^0\right)=\varepsilon\Delta_{v_1}g_1\left(v_1^0\right),$

$\Delta_{v_{i(\varepsilon)}}g_i\left(x_{i-1}^0(t_{i-1}),v_i^0\right)=\varepsilon\Delta_{v_i}g_i\left(x_{i-1}^0(t_{i-1}),v_i^0\right),\ i=2,3.$

The increment (18) introduces an increment of the solution $x_i(t)$ which is denoted by $\left\{\Delta x_i(t;\varepsilon),i=1,2,3\right\}$.

Using the step methods we can prove $\left\|\Delta x_i(t;\varepsilon)\right\|\le Z_{11}\varepsilon,\ t\in T_i\cup t_i,\ i=1,2,3$. Using these estimates in (17) it can easily be seen that the necessary optimality condition $\Delta S\left(u^0,v^0\right)\ge 0$ implies the conditions of the theorem. q.e.d.

It should be noted that the system of linear difference equations (16) is the conjugate system for the problem (9)-(13).

3. Special cases

Under additional assumptions the formulas in Theorem 1 reduce to easier accessible forms. As example consider the situation where some of the controls are fixed, i. e. consider the case $\Delta u_1(t)\neq 0$, $\Delta u_i(t)=0$, $i=2,3$, $\Delta v_i=0$, $i=1,2,3$. Other cases can be treated in an analogous way.

Then the increment formula in the proof of Theorem 1 takes the form:

$$\Delta S_{\bar{u}_1}(u^0,v^0)=S((\bar{u}_1,u_2^0,u_3^0),v^0)-S(u^0,v^0)=$$

$$-\sum_{t=t_0}^{t_1-1}\Delta_{\bar{u}_1}H(t)+\frac{1}{2}\sum_{i=1}^{3}\Delta x_i'(t_i)\frac{\partial^2\varphi_i(x_i^0(t_i))}{\partial x_i^2}\Delta x_i(t_i)-\frac{1}{2}\sum_{i=1}^{3}\sum_{t=t_{i-1}}^{t_i-1}\Delta x_i'(t)\frac{\partial^2 H_i[t]}{\partial x_i^2}\Delta x_i(t)$$

$$-\frac{1}{2}\Delta x_1'(t)\frac{\partial^2 L_2(x_1^0(t_1),v_2^0,\Psi_2^0(t_1-1))}{\partial x_i^2}\Delta x_1(t_1)-\frac{1}{2}\Delta x_2'(t_2)\frac{\partial^2 L_3(x_2^0(t_2),v_3^0,\Psi_3^0(t_2-1))}{\partial x_2^2}\Delta x_2(t_2)$$

$$-\sum_{t=t_0}^{t_1-1}\frac{\partial\Delta_{\bar{u}_1}H_1'[t]}{\partial x_1}\Delta x_1(t)+\eta_1(u^0,v^0,\Delta u_1), \tag{19}$$

where by definition,

$$\eta_1(u^0,v^0,\Delta u_1)=\eta_1(u^0,v^0,(\Delta u_1,0,0),0).$$

Considering (10), (11) $\Delta x_i(t)$, $i=1,2,3$, is a solutions following linearizing equations:

$$\Delta x_1(t+1)=\frac{\partial f_1'[t]}{\partial x_1}\Delta x_1(t)+\Delta_{\bar{u}_1}f_1[t]+\alpha_1(t;\Delta u_1)$$

$$\Delta x_1(t_0)=0, \tag{20}$$

$$\Delta x_2(t+1)=\frac{\partial f_2'[t]}{\partial x_2}\Delta x_2(t)+\alpha_2(t;\Delta u_1),$$

$$\Delta x_2(t_1)=\frac{\partial g_2'(x_1^0(t_1),v_2^0)}{\partial x_2}\Delta x_1(t_1)+\alpha_3(\Delta u_1), \tag{21}$$

$$\Delta x_3(t+1)=\frac{\partial f_3'[t]}{\partial x_3}\Delta x_3(t)+\alpha_4(t;\Delta u_1),$$

$$\Delta x_3(t_2)=\frac{\partial g_3'(x_2^0(t_2),v_3^0)}{\partial x_2}\Delta x_2(t_2)+\alpha_5(\Delta u_1) \tag{22}$$

where by definition

$$\alpha_1(t,\Delta u_1)=\frac{\partial \Delta_{\bar u_1} f_1'[t]}{\partial x_1}\Delta x_1(t)+o_9(\|\Delta x_1(t)\|),\ \alpha_2(t;\Delta u_1)=o_{10}(\|\Delta x_2(t)\|),$$

$$\alpha_3(\Delta u_1)=o_{11}(\|\Delta x_1(t_1)\|),\ \alpha_4(t;\Delta u_1)=o_{12}(\|\Delta x_3(t)\|),\ \alpha_5(\Delta u_1)=o_{13}(\|\Delta x_2(t_2)\|).$$

Here $o_i(\cdot), i=9,..,13$ are determined, respectively, from the expansions:

$$f_1(t,\bar x_1(t),\bar u_1(t))-f_1(t,x_1^0(t),\bar u_1(t))=\frac{\partial f_1(t,x_1^0(t),\bar u_1(t))}{\partial x_1}\Delta x_1(t)+o_9(\|\Delta x_1(t)\|),$$

$$f_2(t,\bar x_2(t),u_2(t))-f_2(t,x_2(t),u_2^0(t))=\frac{\partial f_2'(t,x_2^0(t),u_2^0(t))}{\partial x}\Delta x_2(t)+o_{10}(\|\Delta x_2(t)\|),$$

$$g_2(\bar x_1(t_1),v_2^0(t))-g_2(x_1^0(t_1),v_2^0)=\frac{\partial g_2'(x_1^0(t_1),v_2^0)}{\partial x_1}\Delta x_1(t_1)+o_{11}(\|\Delta x_1(t_1)\|),$$

$$f_3(t,\bar x_3(t),v_3^0)-f_3(t,x_3^0(t),v_3^0)=\frac{\partial f_3'(t,x_3^0(t),v_3^0)}{\partial x_3}\Delta x_3(t)+o_{12}(\|\Delta x_3(t)\|),$$

$$g_3(\bar x_2(t_2),v_3^0)-g_3(x_2^0(t_2),v_3^0)=\frac{\partial g_3'(x_2^0(t_2),v_3^0)}{\partial x_2}\Delta x_2(t_2)+o_{13}(\|\Delta x_2(t_2)\|).$$

Interpreting systems (20)-(22) as linear nonuniform difference equations in $x_i(t)$ (cf. the discrete analog of Cauchy problem), respectively, we obtain the following result, using the representation formulas for the solutions to linear nonuniform difference equations $[11]$

$$\Delta x_1(t)=\sum_{\tau=t_0}^{t-1}F_1'(t,\tau)\Delta_{\bar u_1}f_1[\tau]+\sum_{\tau=t_0}^{t-1}F_1'[t,\tau]\alpha_1(\tau,\Delta u_1)=\sum_{\tau=t_0}^{t-1}F_1'(t,\tau)\Delta_{\bar u_1}f_1[\tau]+\alpha_6(t,\Delta u_1), \tag{23}$$

$$\Delta x_2(t)=F_2'(t,t_1-1)\frac{\partial g_2'(x_1^0(t_1),v_2^0)}{\partial x_1}\Delta x_1(t_1)+F_2'(t,t_1-1)\alpha_3(\Delta u_1)+$$

$$+\sum_{\tau=t_1}^{t-1}F_2'(t,\tau)\alpha_2(\tau,\Delta u_1), \tag{24}$$

$$\Delta x_3(t)=F_3'(t,t_2-1)\frac{\partial g_3(x_2^0(t_2),v_3^0)}{\partial x_2}\Delta x_2(t_2)+F_3'(t,t_2-1)\alpha_5(\Delta u_1)+$$

$$+\sum_{\tau=t_2}^{t-1} F_3'(t,\tau)\alpha_4(\tau,\Delta u_1). \tag{25}$$

Here $F_i(t,\tau),\ i=1,2,3,$ are $(n\times n)$-matrix functions being solutions of the following problem

$$\left.\begin{aligned} F_i(t,\tau-1) &= F_i'(t,\tau)\frac{\partial f_i[\tau]}{\partial x_i} \\ F_i(t,t-1) &= E, i=1,2,3, \end{aligned}\right\}$$

where E denotes the unit matrix. Inserting (23) into (24), we derive

$$\Delta x_2(t)=\sum_{\tau=t_0}^{t_1-1} F_2'(t,t_1-1)\frac{\partial g_2'\left(x_1^0(t_1),v_2^0\right)}{\partial x_1}F_1(t_1,\tau)\Delta_{\bar{u}_1} f_1[\tau]+\alpha_7(t;\Delta u_1), \tag{26}$$

where by definition

$$\alpha_7(t;\Delta u_1)=\sum_{\tau=t_0}^{t_1-1} F_2'(t,t_1-1)\frac{\partial g_2\left(x_1^0(t_1),v_2^0\right)}{\partial x_1}F_1(t_1,\tau)\alpha_1(\tau;\Delta u_1)+F_2'(t,t_1-1)\alpha_3(\Delta u_1)$$

$$+\sum_{\tau=t_1}^{t-1} F_2'(t,\tau)\alpha_2(\tau;\Delta u_1).$$

Now, inserting (26) into (25) the following formula is obtained:

$$\Delta x_3(t)=$$

$$\sum_{\tau=t_0}^{t_1-1} F_3'(t,t_2-1)\frac{\partial g_3\left(x_2^0(t_2),v_3^0\right)}{\partial x_2}F_2(t_2,t_1-1)\frac{\partial g_2\left(x_1^0(t_1),v_2^0\right)}{\partial x_1}F_1(t_1,\tau)\Delta_{\bar{u}_1} f_1[\tau]+\alpha_8(t;\Delta u_1), \tag{27}$$

where by definition

$$\alpha_8(t;\Delta u_1)=F_3'(t,t_2-1)\frac{\partial g_3\left(x_2^0(t_2),v_3^0\right)}{\partial x_2}\alpha_7(t_2;\Delta u_1)+F_3'(t,t_2-1)\alpha_5(\Delta u_1)+$$

$$+\sum_{\tau=t_2}^{t-1} F_3'(t,\tau)\alpha_4(\tau,\Delta u_1).$$

Using the abbreviations

$$R_1(t,\tau)=F_2'(t,t_1-1)\frac{\partial g_2\left(x_1^0(t_1),v_2^0\right)}{\partial x_1}F_1(t_1,\tau),$$

$$R_2(t,\tau)=F_3'(t,t_2-1)\frac{\partial g_3\left(x_2^0(t_2),v_3^0\right)}{\partial x_2}F_2(t_2,t_1-1)\frac{\partial g_2\left(x_1^0(t_1),v_2^0\right)}{\partial x_1}F_1(t_1,\tau)=$$

$$=F_3'(t,t_2-1)\frac{\partial g_3\left(x_2^0(t_2),v_3^0\right)}{\partial x_2}R_1(t_2,\tau),$$

the formulas (23), (26) ,(27) reduce to

$$\left.\begin{aligned}\Delta x_1(t)&=\sum_{\tau=t_0}^{t-1}F_1'(t,\tau)\Delta_{\bar{u}_1}f[\tau]+\alpha_6(t;\Delta u_1)\\ \Delta x_2(t)&=\sum_{\tau=t_0}^{t_1-1}R_1'(t,\tau)\Delta_{\bar{u}_1}f_1[\tau]+\alpha_7(t;\Delta u_1)\\ \Delta x_3(t)&=\sum_{\tau=t_0}^{t_1-1}R_2'(t,\tau)\Delta_{\bar{u}_2}f_1[\tau]+\alpha_8(t;\Delta u_1)\end{aligned}\right\}$$

4. Necessary optimality conditions using the linearizing principle

If the functions f_i, g_i have also partial derivatives with respect to u_i, v_i, respectively, and the sets U_i and V_i are convex, then another necessary optimality condition can be obtained using the linearizing maximum principle of Pontryagin. The proof of the next theorem is to a large extend similar to the proof of Theorem 1 and is omitted. For it the interested reader is referred to the thesis [12].

Theorem 2 [12](Linearizing maximum princple). If the sets U_i, V_i are convex,then ,for the optimality of the pair $(u^0(t), v)$, it is necessary that following inequalities hold:

1) $\sum_{t=t_{i-1}}^{t_i-1}\frac{\partial H_i'[t]}{\partial u_i}(u_i(t)-u_i^0(t))\le 0$

for all $u_i(t)\in U_i$, $t\in T_i$, $i=1,2,3$.

2) $\frac{\partial L_1'(v_1^0,\psi_1^0(t_0-1))}{\partial v_1}(v_1-v_1^0)\le 0$,

for all $v_1\in V_1$.

3) $\frac{\partial L_i'(x_{i-1}^0(t_{i-1}),v_i^0,\Psi_i^0(t_{i-1}-1))}{\partial v_i}(v_i-v_i^0)\le 0,$ for all $v_i\in V_i, i=2,3.$

If the sets $U_i, V_i, i=1,2,3$ are open, also using Euler's equation can be used to derive necessary optimality conditions:

Theorem 3[12](An analogue of Euler equation): If the sets U_i, V_i are open, then for optimality of the pair $\left(u^{0}(t), v\right)$, it is necessary that following equations hold

1) $\dfrac{\partial H_i[t]}{\partial u_i} = 0, \; t \in T_i, \; i = 1,2,3.$

2) $\dfrac{\partial L_1\left(v_1^0, \psi_1^0(t_0 - 1)\right)}{\partial v_1} = 0$,

3) $\dfrac{\partial L_i\left(x_{i-1}^0(t_{i-1}), v_i^0, \Psi_i^0(t_{i-1} - 1)\right)}{\partial v_i} = 0, \;$ for all $i = 2,3.$

The proof is again omitted. The interested reader can find it in the thesis [12].

5. Analysis of singular control

5.1. Necessary optimality conditions using Pontryagin's maximum principle

If the first order necessary optimality condition degenerates in the sense that we have an equation for all admissible controls, the admissible controls are singular in the sense of Pontryagin's maximum principle. Then, the first-order necessary optimality conditions should be replaced by second-order ones. To derive these conditions suitable changes in the increment formula are helpful.

Definition. An admissible control $\left(u^0(t), v^0\right)$ is called singular in the sense of Pontryagin's maximum principle if the following relations hold:

$$\sum_{t=t_{i-1}}^{t_i-1} \Delta_{u_i(t)} H_i[t] = 0, \text{ for all } u_i(t) \in U_i, t \in T_i, \; i = 1,2,3$$

$$\Delta_{v_1} L_1\left(v_1^0, \Psi_1^0(t-1)\right) = 0 \text{ for all } v_1 \in V_1$$

$$\Delta_{v_i} L_i\left(x_{i-1}^0(t_{i-1}), v_i^0, \Psi_i^0(t_{i-1} - 1)\right) = 0 \text{ for all } v_i \in V_i, i = 2,3$$

It is clear that the necessary optimality conditions for problem (9)-(13) are degenerate in the singular case. Therefore, they cannot detect nonoptimality of an admissible pair (u(t),v(t)). Thus, there is some need for new necessary optimality conditions. Singular cases for various control systems were studied by many authors (see e.g. [5,7]). However, to the best of our knowledge, they have not been studied for step systems (both with continous and discrete time).

The proof of the following theorem uses the ideas of the proof of Theorem 1. It starts with the increment formula for the objective function, too.

To explain the increment formula consider the special case of Section 3 i.e. let the controls $u_2, u_3, v_i, i = 1,2,3$ be fixed and $u_1(t) \in U_1 \ \forall\, t \in T_1$. Then, the increment formula reads as

$$\Delta_{\bar{u}_1} S(u^0, v^0) = -\sum_{t=t_0}^{t_1-1} \Delta_{u_1} H_1[t]$$

$$-\frac{1}{2}\sum_{\tau=t_0}^{t_1-1}\sum_{s=t_0}^{t_1-1} \Delta_{\bar{u}_1} f_1'[\tau] K_1(\tau,s) \Delta_{u_1} f_1[s] - \sum_{t=t_0}^{t_1-1}\left[\sum_{\tau=t_0}^{t-1} \frac{\partial \Delta_{u_1} H_1'[t]}{\partial x_1} F_1(t,\tau) \Delta_{u_1} f_1[\tau]\right] + \eta_2(u^0, v^0, \Delta u_1).$$

Here we used (n,n)-dimensional matrix functions $K_1(\tau,s)$ and $F_1(t,\tau)$ defined via

$$K_1(\tau,s) = \sum_{t=\max(\tau,s)+1}^{t_1-1} F_1'(t,\tau) \frac{\partial^2 H_1[t]}{\partial x_1^2} F_1(t,s) + \sum_{t=t_1}^{t_2-1} R_1'(t,\tau) \frac{\partial^2 H_2[t]}{\partial x_2^2} R_1(t,s) +$$

$$+ \sum_{t=t_2}^{t_3-1} R_2'(t,\tau) \frac{\partial^2 H_3[t]}{\partial x_3^2} R_2(t,s) - F_1'(t_1,\tau)\left[\frac{\partial^2 \varphi_1(x_1^0(t_1))}{\partial x_1^2} - \frac{\partial^2 L_2(x_1^0(t_1), v_2^0, \psi_2^0(t_1-1))}{\partial x_1^2}\right] F_1(t_1,s) -$$

$$- R_1'(t_2,\tau)\left[\frac{\partial^2 \varphi_2(x_2^0(t_2))}{\partial x_2^2} - \frac{\partial^2 L_3(x_2^0(t_2), v_3^0, \psi_3^0(t_2-1))}{\partial x_2^2}\right] R_1(t_2,s) - R_2'(t_3,\tau) \frac{\partial^2 \varphi_3(x_3^0(t_3))}{\partial x_3^2} R_2(t_3,s),$$

and

$$F_1(t,\tau-1) = F_1'(t,\tau) \frac{\partial f_1[\tau]}{\partial x_1}, \quad F_1(t,t-1) = E,$$

$$R_1(t,\tau) = F_2'(t,t_1-1) \frac{\partial g_2(x_1^0(t_1), v_2^0)}{\partial x_1} F_1(t_1,\tau).$$

If the controls $u_2, u_3, v_i, i = 1,2,3$ are not fixed but restricted to the respective sets, the increment formula needs to be adapted by adding terms for these functions. These terms can be expressed using similar matrix functions $K_i(\tau,s)$ and $F_i(t,\tau)$ for i=2,...6. These formulas can be found in the Appendix. The increment formula above in the general case is obtained by adding five times the first three terms on the right hand of the formula (using the correct indices).

Theorem 4 $[12]$**.** In the case of singular control and if the convexity assumptions of Theorem 1 are satisfied then the following conditions are necessary for optimality of an admissible singular control for the problem (9)-(13):

1) $$\sum_{\tau=t_{i-1}}^{t_i-1}\sum_{s=t_{i-1}}^{t_i-1} \Delta_{u_i} f_i'[\tau] K_i(\tau,s) \Delta_{u_i} f_i[s] + 2\sum_{t=t_{i-1}}^{t_i-1}\left[\sum_{\tau=t_{i-1}}^{t-1} \frac{\partial \Delta_{u_i} H_i[t]}{\partial x_i} F_i(t,\tau) \Delta_{u_i} f_i[\tau]\right] \le 0,$$

for all $u_i(t)\in U_i$, $i=1,2,3$.

2) $\Delta_{v_1} g_1'(v_1^0) K_4 \Delta_{v_1} g_1(v^0)\le 0$, for all $v_1\in V_1$

3) $\Delta_{v_i} g_i'(x_{i-1}^0(t_{i-1}) v_i^0) K_{i+3} \Delta_{v_i} g_i'(x_{i-1}^0(t_{i-1}) v_i^0)\le 0$, for all $v_i\in V_i, i=2,3$.

5.2. Necessary optimality condition using linearizing maximum principle

If the linearizing maximum principle degenerates for the problem (9)-(13) in the sense that we have an equation for all admissible controls in the conditions of Theorem 2 then the admissible controls are called quasisingular. In that case the first order necessary conditions in Theorem 2 should be replaced by second order ones. For this we need twice continuously partial differentiability of f_i with respect to controls. We drop the proof since it uses mainly the ideas of the proofs of Theorems 2 and 4.

Theorem 5[12]**:** If the sets V_i and U_i are convex, then for optimality of the quasisingular control $(u^0(t), v^0)$, it is necessary that the following inequalities hold:

1) $$\sum_{\tau=t_{i-1}}^{t_i-1}\sum_{s=t_{i-1}}^{t_i-1}(u_i(t)-u_i^0(t))'\frac{\partial f_i[\tau]}{\partial u_i}K_i(\tau,s)\frac{\partial f_i[s]}{\partial u_i}(u_i(s)-u_i^0(s))+\Delta_{u_i} f_i'[\tau]K(\tau,s)\Delta_{u_i} f_i[s]+$$
$$+2\sum_{t=t_{i-1}}^{t_i-1}\left[\sum_{\tau=t_{i-1}}^{t-1}(u_i(t)-u_i^0(t))'\frac{\partial^2 H_i[t]}{\partial u_i\partial x_i}F_i(t,\tau)\frac{\partial f_i[\tau]}{\partial u_i}(u_i(\tau)-u_i^0(\tau))\right]$$
$$+\sum_{t=t_{i-1}}^{t_i-1}(u_i(t)-u_i^0(t))'\frac{\partial^2 H_i[t]}{\partial u_i^2}(u_i(t)-u_i^0(t))\le 0,$$

for all $u_i(t)\in U_i$, $i=1,2,3$.

2) $$(v_i-v_i^0)'\frac{\partial g_1(v_1^0)}{\partial v_1}K_4\frac{\partial g_1(v_1^0)}{\partial v_1}(v_1-v_1^0)+(v_1-v_1^0)'\frac{\partial^2 L_1(v_1^0,\psi_1^0(t_0-1))}{\partial v_1^0}(v_1-v_1^0)\le 0,$$

for all $v_1\in V_1$.

3) for all $v_i\in V_i$ $i=2,3$ **:**

$$(v_i-v_i^0)'\frac{\partial g_1(x_{i-1}^0(t_{i-1}),v_i^0)}{\partial v_i}K_{i+3}\frac{\partial g_i(x_{i-1}(t_{i-1}),v_i)}{\partial v_i}(v_i-v_i^0)$$
$$+(v_i-v_i^0)'\frac{\partial^2 L_i(x_{i-1}^0(t_{i-1}),v_i^0,\Psi_i(t_{i-1}-1))}{\partial v_i^2}(v_i-v_i^0)\le 0$$

Conclusion.

In this paper have given some results for necessary optimality conditions for discrete control systems with varying structure. These problems have important applications, one of them is the control of a rocket described at the beginning of the paper. The conditions formulated in the case of regular control include conditions being based on Pontryagin's maximum principle, the linearizing maximum principle and one using Euler's equations. After that we investigated singular and quasisingular controls in the sense of Pontryagin's maximum principle.

The step control problem discussed in this paper is one possible (and from the point of view of applications interesting) generalization of control problems. Other generalizations are not investigated as e.g. problems with an unknown switching time. In the continuous-time case necessary optimality conditions can be found in [18] but in the discrete-time case they seem not to be discussed, yet.

Summing up (discrete-time) step control problems rise challenging questions and need further investigation in the future.

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Appendix

$$K_2(\tau,s)=\sum_{t=\max(\tau,s)+1}^{t_2-1}F_2'(t,\tau)\frac{\partial^2 H_2[t]}{\partial x_2^2}F_2(t,s)+\sum_{t=t_2}^{t_3-1}R_3'(t,\tau)\frac{\partial^2 H_3[t]}{\partial x_3^2}R_3(t,s)-$$

$$-R_3'(t_3,\tau)\frac{\partial^2\varphi_3(x_3^0(t_3))}{\partial x_2^2}R_3(t_3,s)-F_2'(t_2,\tau)\left[\frac{\partial^2\varphi_2(x_2^0(t_2))}{\partial x_2^2}-\frac{\partial^2 L_3(x_2^0(t_2),v_3^0,\psi_3^0(t_2-1))}{\partial x_2^2}\right]F_2(t_2,s),$$

$$K_3(\tau,s)=\sum_{t=\max(\tau,s)+1}^{t_3-1}F_3'(t,\tau)\frac{\partial^2 H_3[t]}{\partial x_3^2}F_3(t,s)-F_3'(t_3,\tau)\frac{\partial^2\varphi_3(x_3^0(t_3))}{\partial x_3^2}F_3(t_3,\tau),$$

$$K_4=\sum_{t=t_0}^{t_1-1}F_1'(t,t_0-1)\frac{\partial^2 H_1[t]}{\partial x_1^2}F_1(t,t_0-1)+\sum_{t=t_1}^{t_2-1}\Phi_1'(t)\frac{\partial^2 H_2[t]}{\partial x_2^2}\Phi_1(t)+\sum_{t=t_2}^{t_3-1}\Phi_2'(t)\frac{\partial^2 H_3[t]}{\partial x_3^2}\Phi_2(t)-$$

$$F_1'(t_1,t_0-1)\left[\frac{\partial^2\varphi_1(x_1^0(t_1))}{\partial x_1^2}-\frac{\partial^2 L_2(x_1^0(t_1),v_2^0,\psi_2^0(t_1-1))}{\partial x_1^2}\right]F_1(t_1,t_0-1)-\Phi_1'(t_2)\left[\frac{\partial^2\varphi_2(x_2^0(t_2))}{\partial x_2^2}-\right.$$

$$\left.-\frac{\partial^2 L_3(x_2^0(t_2),v_3^0,\psi_3^0(t_2-1))}{\partial x_2^2}\right]\Phi_1(t_2)-\Phi_2'(t_3)\frac{\partial^2\varphi_3(x_3^0(t_3))}{\partial x_3^2}\Phi_2(t_3),$$

$$K_5=\sum_{t=t_1}^{t_2-1}F_2'(t,t_1-1)\frac{\partial^2 H_2[t]}{\partial x_2^2}F_2(t,t_1-1)+\sum_{t=t_2}^{t_3-1}\Phi_3'(t)\frac{\partial^2 H_3[t]}{\partial x_3^2}\Phi_3(t,s)-$$

$$-F_2'(t,t_1-1)\left[\frac{\partial^2\varphi_2(x_2^0(t_2))}{\partial x_2^2}-\frac{\partial^2 L_3(x_2^0(t_2),v_3^0,\psi_3^0(t_2-1))}{\partial x_2^2}\right]F_2(t,t_1-1)-\Phi_3'(t_3)\frac{\partial^2\varphi_3(x_3^0(t_3))}{\partial x_3^2}\phi_3(t_3),$$

$$K_6=\sum_{t=t_2}^{t_3-1}F_3'(t,t_2-1)\frac{\partial^2 H_3[t]}{\partial x_3^2}F_3(t,t_2-1)-F_3'(t_3,t_2-1)\frac{\partial^2\varphi_3(x_3^0(t_3))}{\partial x_3^2}F_3(t_3,t_2-1),$$

Here by definition

$$R_1(t,\tau)=F_2(t,t_1-1)\frac{\partial g_2(x_1^0(t_1),v_2^0)}{\partial x_1}F_1(t_1,\tau),$$

$$R_2(t,\tau)=F_3(t,t_2-1)\frac{\partial g_3(x_2^0(t_2),v_3^0)}{\partial x_2}R_1(t_2,\tau),$$

$$R_3(t,\tau)=F_3(t,t_2-1)\frac{\partial g_3(x_2^0(t_2),v_3^0)}{\partial x_2}F_2(t_2,\tau),$$

$$\Phi_1(t)=F_2(t,t_1-1)\frac{\partial g_2(x_1^0(t_1),v_2^0)}{\partial x_1}F_1(t_1,t_0-1),$$

$$\Phi_2(t)=F_3(t,t_2-1)\frac{\partial g_3(x_2^0(t_2),v_3^0)}{\partial x_2}\Phi_1(t_2),$$

$$\Phi_3(t) = F_3(t, t_2 - 1)\frac{\partial g_3(x_2^0(t_2), v_3^0)}{\partial x_2} F_2(t_2, t_1 - 1),$$

$F_i(t, \tau), i = 1,2,3$ are (n, n) -matrix functions and solutions of the problem

$$F_i(t, \tau - 1) = F_i(t, \tau)\frac{\partial f_1[\tau]}{\partial x_1},$$

$$F_i(t, t - 1) = E,\ i = \overline{1,3}$$