Globally Optimal Solutions for Large Single-Row Facility Layout Problems

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Abstract

This paper is concerned with the single-row facility layout problem (SRFLP). The SRFLP asks for an optimal linear placement of rectangular facilities with varying lengths on a straight line so as to minimize the total cost associated with the (known or projected) interactions between them. First, we revisit the new formulation of this problem recently proposed by the authors, and exhibit a bijection between the feasible set of the formulation, and the set of all permutations on \( n \) objects. We then consider the semidefinite programming (SDP) relaxation arising from this new formulation. On the theoretical side, we prove that a number of triangle inequalities hold automatically for the SDP relaxation, and more interestingly, that if we consider the tighter relaxation obtained by adding the remaining triangle inequalities, then a significant number of additional facet-defining inequalities of the underlying cut polytope are automatically enforced. On the computational side, we use the SDP relaxation to obtain globally optimal layouts for large SRFLPs with up to thirty facilities, some of which have remained unsolved since 1988.

Key words: Single-Row Facility Layout, Space Allocation, Combinatorial Optimization, Semidefinite Optimization, Global Optimization.

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1 Introduction

The single-row facility layout problem (SRFLP) is concerned with the arrangement of a given number of rectangular facilities along a line. This problem is a special case of the unequal-area facility layout problem, and is also known in the literature as the one-dimensional space allocation problem, see e.g. [23]. An instance of the SRFLP consists of $n$ one-dimensional facilities, denoted $1, \ldots, n$, with given positive lengths $\ell_1, \ldots, \ell_n$, and pairwise weights $c_{ij}$. The objective is to arrange the facilities so as to minimize the total weighted sum of the center-to-center distances between all pairs of facilities. If all the facilities have the same length, the SRFLP becomes an instance of the linear ordering (or linear arrangement) problem, see e.g. [20, 21], which is itself a special case of the quadratic assignment problem, see e.g. [8]. Several applications of the SRFLP have been identified in the literature. One such application arises in the area of flexible manufacturing systems, where machines within manufacturing cells are often placed along a straight path travelled by an automated guided vehicle, see e.g. [16].

The SRFLP was first studied by Simmons [25] who proposed a branch-and-bound algorithm. Subsequently, Picard and Queyranne [23] developed a dynamic programming algorithm, and mixed integer linear programming models have also been proposed, most recently in [1]. While these algorithms are guaranteed to find the globally optimal solution, they have very high computational time and memory requirements, and are unlikely to be effective for problems with more than about twenty facilities. Several heuristic algorithms for the SRFLP have also been proposed, including the work of Heragu and Kusiak on the application of nonlinear optimization methods [17], the simulated annealing algorithms proposed independently by Romero and Sánchez-Flores [24] and Heragu and Alfa [15], and a greedy heuristic algorithm proposed by Kumar et al. [18]. However, these heuristic algorithms do not provide a guarantee of global optimality, or an estimate of the distance from optimality. Progress in obtaining such estimates was recently reported in [3], where non-trivial global lower bounds were obtained using a semidefinite programming relaxation.

Semidefinite programming (SDP) refers to the class of optimization problems where a linear function of a symmetric matrix variable $X$ is optimized subject to linear constraints on the elements of $X$ and the additional constraint that $X$ must be positive semidefinite. This includes linear programming problems as a special case, namely when the matrix variable is diagonal. A variety of algorithms for solving SDP problems, including polynomial-time interior-point algorithms, have been implemented and benchmarked, and several excellent solvers for SDP are now available. We refer the reader to the SDP webpage [12] as well as the books [9, 27] for a thorough coverage of the theory and algorithms in this area, as well as of several application areas where SDP researchers have made significant contributions. In particular, SDP has been successfully applied to problems similar to the SRFLP that also have a strong combinatorial flavour. Recent survey papers on the application of SDP to combinatorial optimization include [2, 5, 19].

The application of SDP to the SRFLP was initiated in [3], which proposed an SDP relaxation as well as a heuristic that extracts a feasible solution to the SRFLP from the optimal matrix solution to the SDP relaxation. Therefore, this SDP-based approach yields both a feasible solution to the given SRFLP instance and a guarantee of how far it is from global optimality. The SDP relaxation provides a lower bound on the optimal value of the problem, and it is the first non-trivial global lower bound in the literature for large instances of the SRFLP. Most of the results in [3] were for fairly large instances, and were obtained using the spectral bundle solver SB [13, 14] which is able to handle very large SDPs. The results reported in [3] showed that the SDP-based approach yields layouts that are consistently a few percentage points from global optimality for randomly generated instances with up to 80 facilities. More recently, the authors have experimented
with a branch-and-bound algorithm for the SRFLP that solves the SDP relaxation at each node [4]. Although this approach yielded solutions that are provably very close to global optimality (typically less than 1% gap) for randomly generated instances of the SRFLP with up to 40 facilities with a reasonable amount of computational effort, the results also suggest that branching does not provide a worthwhile improvement over the results obtained at the root node of the branch-and-bound tree. Therefore in this paper, we explore the approach of introducing triangle inequalities to tighten the relaxation at the root node only. This investigation leads to several interesting theoretical and computational results.

The paper is structured as follows. In Section 2, we recall the new formulation of this problem proposed in [3], and in Section 3, we exhibit and prove the validity of a bijection between the feasible set of the formulation, and the set of all permutations on \( n \) objects. In Section 4, we consider the semidefinite programming (SDP) relaxation and first prove that a number of triangle inequalities hold automatically for every feasible solution. More interestingly, we show that if we consider the tighter relaxation obtained by adding the remaining triangle inequalities, then a significant number of other facet-defining inequalities of the underlying cut polytope are automatically enforced. Finally, in Section 5, we report computational results from using the SDP relaxation to obtain the first globally optimal layouts for large SRFLPs with up to 30 facilities, some of which have been studied in the literature but remained unsolved since 1988 [16, 18, 1].

## 2 A Quadratic Formulation of the SRFLP and its SDP Relaxation

Let \( \pi = (\pi_1, \ldots, \pi_n) \) denote a permutation of the indices \( [n] := \{1, 2, \ldots, n\} \) of the facilities, so that the leftmost facility is \( \pi_1 \), the next facility to the right is \( \pi_2 \), and so on, with \( \pi_n \) being the last facility in the arrangement. Given a permutation \( \pi \) and two distinct facilities \( i \) and \( j \), the center-to-center distance between \( i \) and \( j \) with respect to this permutation is

\[
\frac{1}{2} \ell_i + D_\pi(i, j) + \frac{1}{2} \ell_j,
\]

where \( D_\pi(i, j) \) denotes the sum of the lengths of the facilities between \( i \) and \( j \) in the ordering defined by \( \pi \). To solve the SRFLP, we seek a permutation of the facilities which minimizes the weighted sum of the distances between all pairs of facilities. We express this objective as

\[
\min_{\pi \in \Pi_n} \sum_{i < j} c_{ij} \left[ \frac{1}{2} \ell_i + D_\pi(i, j) + \frac{1}{2} \ell_j \right]
\]

where \( \Pi_n \) denotes the set of all permutations of \( [n] \).

Simmons [25] observed that if we rewrite the objective function as

\[
\min_{\pi \in \Pi_n} \sum_{i < j} c_{ij} D_\pi(i, j) + \sum_{i < j} \frac{1}{2} c_{ij} (\ell_i + \ell_j)
\]

where the second summation is a constant independent of \( \pi \), then it is clear that the crux of the problem is to minimize \( \sum_{i < j} c_{ij} D_\pi(i, j) \) over all permutations \( \pi \). Furthermore, \( D_\pi(i, j) = D_{\pi'}(i, j) \), where \( \pi' \) denotes the permutation symmetric to \( \pi \), defined by \( \pi'_i = \pi_{n+1-i}, i = 1, \ldots, n \). This shows that we can exchange the left and right ends of the layout and obtain the same objective value. Hence, it is possible to simplify the problem by considering only the permutations for which, say, facility 1 is on the left half of the arrangement. This type of symmetry-breaking strategy is important for reducing the computational requirements of most algorithms, including those based on linear programming or dynamic programming. One noteworthy aspect of the SDP-based approach is that it implicitly accounts for these symmetries, and thus does not require the use of additional explicit symmetry-breaking constraints.
The SDP relaxation for the SRFLP proposed in [3] is obtained as follows. Define a binary ±1 variable for each pair \( ij \) of facilities with \( i < j \) such that

\[
R_{ij} := \begin{cases} 
1, & \text{if facility } i \text{ is to the right of facility } j \\
-1, & \text{if facility } i \text{ is to the left of facility } j 
\end{cases}
\]

In this definition, the order of the subscripts matters, and \( R_{ij} = -R_{ji} \). To accurately formulate the SRFLP, it is further required that the \( R_{ij} \) variables represent a valid arrangement of the \( n \) facilities. Therefore we require that if \( R_{ij} = R_{jk} \) then \( R_{ik} = R_{ij} \), a necessary transitivity condition that can be formulated as a set of quadratic constraints:

\[
R_{ij}R_{jk} - R_{ij}R_{ik} - R_{ik}R_{jk} = -1 \quad \text{for all triples } i < j < k. 
\]

This leads to the following formulation of the SRFLP:

\[
\begin{align*}
\min & \quad K - \sum_{i<j} c_{ij} \left[ \sum_{k<i} \ell_k R_{ki} R_{kj} - \sum_{i<k<j} \ell_k R_{ik} R_{kj} + \sum_{k>j} \ell_k R_{ik} R_{jk} \right] \\
\text{s.t.} & \quad R_{ij}R_{jk} - R_{ij}R_{ik} - R_{ik}R_{jk} = -1 \quad \text{for all triples } i < j < k \\
& \quad R_{ij}^2 = 1 \quad \text{for all } i < j
\end{align*}
\]

where \( K := \left( \sum_{i<j} \frac{c_{ij}}{2} \right) \left( \sum_{k=1}^{n} \ell_k \right) \). Note that if every \( R_{ij} \) variable is replaced by its negative, then there is no change whatsoever to the formulation. This is how our formulation, and the subsequent SDP relaxation, implicitly take into account the natural symmetry of the SRFLP.

It remains to express the objective function of the SRFLP in terms of the variables \( R_{ij} \). To do this, it suffices to observe that since \( k \) is between \( i \) and \( j \) if and only if \( R_{ki}R_{kj} = -1 \), the sum of the lengths of the facilities between \( i \) and \( j \) can be expressed as

\[
\sum_{k \neq i,j} \ell_k \left( 1 - \frac{R_{ki}R_{kj}}{2} \right),
\]

We can now formulate the SRFLP in the space of real symmetric matrices. Fixing an ordering of all pairs \( ij \) such that \( i < j \), we define the vector

\[
\rho := (R_{p_1}, \ldots, R_{p_{(n/2)}})^T,
\]

where \( p_k \) denotes the \( k^{th} \) pair in the ordering. Using \( \rho \), we construct the rank-one matrix \( X := \rho \rho^T \) whose rows and columns are indexed by pairs. By construction, \( X_{p_i,p_j} = R_{p_i}R_{p_j} \) for any two pairs \( p_i, p_j \), and therefore we can formulate the SRFLP as:

\[
\begin{align*}
\min & \quad K - \sum_{i<j} c_{ij} \left[ \sum_{k<i} \ell_k X_{ki,kj} - \sum_{i<k<j} \ell_k X_{ik,kj} + \sum_{k>j} \ell_k X_{ik,jk} \right] \\
\text{s.t.} & \quad X_{ij,jk} - X_{ij,ik} - X_{ik,jk} = -1 \quad \text{for all triples } i < j < k \\
& \quad \text{diag} \ (X) = e \\
& \quad \text{rank} \ (X) = 1 \\
& \quad X \succeq 0
\end{align*}
\]
where $\text{diag}(X)$ represents a vector containing the diagonal elements of $X$, $e$ denotes the vector of all ones, and $X \succeq 0$ denotes that $X$ is symmetric positive semidefinite. Removing the rank constraint yields the SDP relaxation. Note that in general the SDP problem provides only a lower bound on the optimal value of the SRFLP, and not a feasible solution, unless the optimal matrix $X^*$ happens to have rank equal to one.

Before proceeding, we observe that the formulation and SDP relaxation above are closely related to the basic SDP relaxation for the max-cut problem used by Goemans and Williamson in their ground-breaking paper [11]. The max-cut SDP relaxation can be interpreted as a relaxation of the so-called cut polytope, an important and well-known structure in the area of integer programming. The reader is referred to [10] for a wealth of results about the cut polytope. In particular, we will prove later that several facet-defining inequalities of the cut polytope hold for one of our proposed SDP relaxations.

While the SDP relaxation obtained from (3) shares some common structure with the max-cut relaxation for which Goemans and Williamson analyzed a well-known randomized rounding procedure [11], we cannot use that procedure because it does not ensure that the equality constraints (1) hold, and hence it does not guarantee that we will obtain a valid representation of a permutation. This is why a different procedure was used in [3, 4] to extract a permutation from the optimal solution to the SDP relaxation. This rounding scheme specific to the SRFLP is described in Section 5.1.

3 Bijection between $\mathcal{R}_n$ and $\Pi_n$

It was shown in [3] that the $(\binom{n}{3})$ constraints on the $R_{ij}$ variables in (1) suffice to ensure that every feasible solution corresponds to a valid permutation of the $n$ facilities. Let $\rho \in \{\pm 1\}^{\binom{n}{2}}$ denote a particular assignment of values to the $R_{ij}$ variables, and hence denote the feasible set of the formulation (2) as:

$$\mathcal{R}_n := \{ \rho \in \{\pm 1\}^{\binom{n}{2}} | R_{ij}R_{jk} - R_{ik}R_{jk} - R_{ik}R_{jk} = -1 \text{ for all triples } i < j < k \}$$

Our first theoretical result is to prove the existence of a bijection $f : \mathcal{R}_n \to \Pi_n$. We consider the particular choice

$$f(\rho) = (\pi_1, \ldots, \pi_n), \text{ where } \pi_k := \frac{P_k + n + 1}{2}$$

and

$$P_k := \sum_{j \neq k} R_{kj} = \sum_{j < k} -R_{jk} + \sum_{j > k} R_{kj} \quad \text{ for } k = 1, 2, \ldots, n. \tag{4}$$

Clearly all the $P_k$ values are integer and belong to the set

$$\{-(n-1), -(n-3), \ldots, n-3, n-1\}.$$ 

The fact that no two $P_k$ values are equal was proved in [3]. We include the proof here for completeness.

**Lemma 1** [3] If $\rho \in \mathcal{R}_n$ then the values $P_k$ defined in (4) are all distinct.

**Proof:** The proof is by contradiction. Suppose that $P_{k_1} = P_{k_2}$ for $k_1 \neq k_2$. Without loss of generality, we can assume $k_1 < k_2$. Then by (4), we have

$$R_{k_1,k_2} + \sum_{k < k_1} -R_{k,k_1} + \sum_{k_1 < k < k_2} R_{k_1,k} + \sum_{k < k_2} R_{k_1,k} = -R_{k_1,k_2} + \sum_{k < k_1} -R_{k,k_2} + \sum_{k_1 < k < k_2} -R_{k,k_2} + \sum_{k_2 < k} R_{k_2,k}.$$
Multiplying on both sides by \( R_{k_1,k_2} \), we obtain
\[
1 + \sum_{k < k_1} -R_{k,k_1} R_{k_1,k_2} + \sum_{k_1 < k < k_2} R_{k_1,k} R_{k_1,k_2} + \sum_{k_2 < k} R_{k_1,k} R_{k_1,k_2} = \\
-1 + \sum_{k < k_1} -R_{k,k_2} R_{k_1,k_2} + \sum_{k_1 < k < k_2} -R_{k,k_2} R_{k_1,k_2} + \sum_{k_2 < k} R_{k_2,k} R_{k_1,k_2}
\]
since \( R_{k,k_2}^2 = 1 \). Therefore,
\[
\sum_{k < k_1} (-R_{k,k_2} R_{k_1,k_2} + R_{k,k_1} R_{k_1,k_2}) + \sum_{k_1 < k < k_2} (-R_{k,k_2} R_{k_1,k_2} - R_{k_1,k} R_{k_1,k_2}) + \sum_{k_2 < k} (R_{k_2,k} R_{k_1,k_2} - R_{k_1,k} R_{k_1,k_2}) = 2.
\]

Now, using the quadratic constraints from (1), we have that:
\[
-R_{k,k_2} R_{k_1,k_2} + R_{k,k_1} R_{k_1,k_2} = -1 + R_{k,k_1} R_{k,k_2} \\
-R_{k,k_2} R_{k_1,k_2} - R_{k_1,k} R_{k_1,k_2} = -1 - R_{k_1,k} R_{k,k_2} \\
R_{k_2,k} R_{k_1,k_2} - R_{k_1,k} R_{k_1,k_2} = -1 + R_{k_1,k} R_{k_2,k}
\]
and thus
\[
\sum_{k < k_1} (-1 + R_{k,k_1} R_{k,k_2}) + \sum_{k_1 < k < k_2} (-1 - R_{k_1,k} R_{k,k_2}) + \sum_{k_2 < k} (-1 + R_{k_1,k} R_{k_2,k}) = 2
\]
which is equivalent to
\[
\sum_{k < k_1} R_{k,k_1} R_{k,k_2} - \sum_{k_1 < k < k_2} R_{k_1,k} R_{k,k_2} + \sum_{k_2 < k} R_{k_1,k} R_{k_2,k} = n.
\]

But since the left-hand side is bounded above by \( n - 2 \), we have a contradiction.

**Lemma 2** *The function \( f \) is onto.*

**Proof:** If \( \rho \in \mathcal{R}_n \), then all the \( P_k \) values are integer and belong to the set
\[
\mathcal{P} := \{ -(n-1), -(n-3), \ldots, n-3, n-1 \}
\]
which has exactly \( n \) elements. By Theorem 1, every element of \( \mathcal{P} \) equals exactly one \( P_k \), and hence \( (\pi_1, \pi_2, \ldots, \pi_n) \) is a permutation of \( [n] \) represented by \( \rho \).

We now want to prove that \( f \) is one-to-one. To this end, consider the system of \( n \) equations in \( \binom{n}{2} + 1 \) unknowns:
\[
P_k = \sum_{j<k} -R_{jk} + \sum_{j>k} R_{kj} = \beta_k, k = 1, \ldots, n, \tag{5}
\]
for given right-hand sides \( \beta_k \) such that \( \{ \beta_k : k = 1, \ldots, n \} = \mathcal{P} \).
Lemma 3 The system (5) has a unique solution in \( R_{ij} \).

Proof: The proof is by induction. There are two base cases to prove, \( n = 2 \) and \( n = 3 \).

For \( n = 2 \), the system has the form
\[
\begin{align*}
R_{1,2} &= \beta_1, \\
-R_{1,2} &= \beta_2
\end{align*}
\]
and since \( \mathcal{P} = \{-1, 1\} \), \( \beta_1 = -\beta_2 \), and the solution is clearly unique.

For \( n = 3 \), the system has the form
\[
\begin{align*}
R_{1,2} + R_{1,3} &= \beta_1, \\
-R_{1,2} + R_{2,3} &= \beta_2, \\
-R_{1,3} - R_{2,3} &= \beta_3,
\end{align*}
\]
where \( \mathcal{P} = \{-2, 0, 2\} \). Suppose first that \( \beta_1 = -2 \), \( \beta_2 = 0 \), and \( \beta_3 = 2 \). Then \( R_{1,2} = -1 \), \( R_{1,3} = -1 \), and \( R_{2,3} = -1 \) is the unique solution. The other cases are solved similarly.

Now consider the system (5) for \( n \geq 4 \), and let \( k_1 \) and \( k_2 \) be such that \( \beta_{k_1} = -(n - 1) \) and \( \beta_{k_2} = n - 1 \). (Such \( k_1 \) and \( k_2 \) always exist.) Then for any solution of (5), it holds that
\[
-R_{j,k_1} = -1 \quad \text{for} \quad j < k_1, R_{k_1,j} = -1 \quad \text{for} \quad j > k_1, \quad \text{and} \quad -R_{j,k_2} = 1 \quad \text{for} \quad j < k_2, R_{k_2,j} = 1 \quad \text{for} \quad j > k_2.
\]

Now consider the equations for every other \( \beta_k \neq k_1, k_2 \). If \( k > k_1 \) and \( k > k_2 \) then we can write it as
\[
-R_{k_1,k} - R_{k_2,k} + \sum_{j<k, j \neq k_1,k_2} -R_{jk} + \sum_{j>k, j \neq k_1,k_2} R_{kj} = \beta_k
\]
and since \( R_{k_1,k} = -1 \) and \( R_{k_2,k} = 1 \), the first two terms cancel each other.

Similarly, if \( k_1 < k < k_2 \) then we can write it as
\[
-R_{k_1,k} + \sum_{j<k} -R_{jk} + R_{k,k_2} + \sum_{j>k} R_{kj} = \beta_k
\]
and since \( R_{k_1,j} = -1 \) and \( R_{k,k_2} = -1 \), the two terms again cancel each other.

If \( k_2 < k < k_1 \) then we can write it as
\[
-R_{k_2,k} + \sum_{j<k} -R_{jk} + R_{k,k_1} + \sum_{j>k} R_{kj} = \beta_k
\]
and since \( R_{k_2,j} = 1 \) and \( R_{k,k_1} = 1 \), again we have cancellation.

Finally, if \( k < k_1 \) and \( k < k_2 \) then
\[
\sum_{j<k} -R_{jk} + \sum_{j>k} R_{kj} + R_{k,k_1} + R_{k,k_2} = \beta_k
\]
and since \( R_{k_1,k} = 1 \) and \( R_{k_2,k} = -1 \), we cancel again.

In summary, we have reduced to a system on \( n - 2 \) equations in \( \binom{n-2}{2} \) \pm 1 unknowns of the same form.

By the inductive hypothesis, this reduced system has a unique solution. Using the values from (6), this solution can be extended to a solution to the original system, and this extension is uniquely defined. Hence, the original system has a unique solution in \( R_{ij} \). \( \blacksquare \)
Lemma 4 The function $f$ is one-to-one.

Proof: Applying Lemma 3, if $\rho^1, \rho^2 \in \mathcal{R}_n$ and $f(\rho^1) = f(\rho^2)$, then $\rho^1 = \rho^2$. □

Lemmata 2 and 4 yield the desired result.

Theorem 1 The function $f$ is a bijection.

An explicit expression of $f^{-1} : \Pi_n \to \mathcal{R}_n$ was provided in [3]. Given $\pi = (\pi_1, \pi_2, \ldots, \pi_n) \in \Pi_n$, let

$$R_{\pi_p, \pi_q} = -1 \text{ for all } p < q.$$ 

(Note that $\pi_p > \pi_q$ may hold even if $p < q$, and if that is the case, then $R_{\pi_q, \pi_p} = 1$.) Then $\pi_1$ is the leftmost facility, with $\pi_2$ on its right, and so on, up to $\pi_n$ being the rightmost facility. This is the desired representation of $\pi$.

4 Semidefinite Programming Relaxations of $\mathcal{R}_n$

Having established a bijection between $\Pi_n$ and $\mathcal{R}_n$, we now direct our attention to SDP relaxations of $\mathcal{R}_n$. In this section, we propose two such relaxations and study some of their theoretical properties.

4.1 The Relaxation $\mathcal{X}_n$

The set

$$\mathcal{X}_{rk-1} := \left\{ X \in \mathcal{S}^{\binom{n}{2}} : \text{diag}(X) = e, X \succeq 0, \text{rank}(X) = 1, X_{ij,ik} - X_{ij,jk} - X_{ik,jk} = -1 \right\}$$

$$\forall 1 \leq i < j < k \leq n$$

is an exact lifting of $\mathcal{R}_n$ into the space of $\binom{n}{2} \times \binom{n}{2}$ symmetric matrices, in the sense that $X_{ij,kl} = R_{ij}R_{kl}$ for all $i, j, k, l$. Omitting the rank constraint (which is not convex), we recover the semidefinite relaxation of $\mathcal{R}_n$:

$$\mathcal{X}_n := \left\{ X \in \mathcal{S}^{\binom{n}{2}} : \text{diag}(X) = e, X \succeq 0, X_{ij,ik} - X_{ij,jk} - X_{ik,jk} = -1 \right\}$$

$$\forall 1 \leq i < j < k \leq n$$

This is precisely the relaxation proposed in [3].

We now prove an interesting property of $\mathcal{X}_n$ that was not proved in [3], namely that a number of triangle inequalities automatically hold for every $X \in \mathcal{X}_n$. The triangle inequalities are a well-known class of valid inequalities for the cut polytope. They model the fact that for any assignment of $\pm 1$ to the entries of $\rho$, the terms $R_{p_1}R_{p_2}$, $R_{p_1}R_{p_3}$, and $R_{p_2}R_{p_3}$ must comprise an even number of negative ones for every triple of distinct pairs $p_1, p_2, p_3$. Equivalently, every $\rho \in \mathcal{R}_n$ satisfies the following $4\binom{n}{3}$ inequalities:

$$R_{p_1}R_{p_2} + R_{p_1}R_{p_3} + R_{p_2}R_{p_3} \geq -1, R_{p_1}R_{p_2} - R_{p_1}R_{p_3} - R_{p_2}R_{p_3} \geq -1,$$

$$-R_{p_1}R_{p_2} + R_{p_1}R_{p_3} + R_{p_2}R_{p_3} \geq -1, -R_{p_1}R_{p_2} - R_{p_1}R_{p_3} - R_{p_2}R_{p_3} \geq -1,$$

for every triple of pairs $p_1, p_2, p_3$, or in terms of the matrix representation of $\mathcal{R}_n$, every matrix in $\mathcal{X}_{rk-1}$ satisfies

$$X_{p_1,p_2} + X_{p_1,p_3} + X_{p_2,p_3} \geq -1, X_{p_1,p_2} - X_{p_1,p_3} - X_{p_2,p_3} \geq -1,$$

$$-X_{p_1,p_2} + X_{p_1,p_3} + X_{p_2,p_3} \geq -1, -X_{p_1,p_2} + X_{p_1,p_3} - X_{p_2,p_3} \geq -1.$$

While these hold for $X \in \mathcal{X}_{rk-1}$, they will not (in general) hold for $X \in \mathcal{X}_n$. However, we can prove the following result.
Theorem 2 If \( X \in \mathcal{X}_n \), then for every triple of pairs \((i_1, i_2), (i_1, i_3), \text{ and } (i_2, i_3)\), where \( i_1 < i_2 < i_3 \), the entries of \( X \) satisfy

\[
X_{i_1i_2, i_1i_3} + X_{i_1i_2, i_2i_3} + X_{i_1i_3, i_2i_3} \geq -1, \quad X_{i_1i_2, i_1i_3} - X_{i_1i_2, i_2i_3} - X_{i_1i_3, i_2i_3} \geq -1,
\]

\[
- X_{i_1i_2, i_1i_3} - X_{i_1i_2, i_2i_3} + X_{i_1i_3, i_2i_3} \geq -1, \quad - X_{i_1i_2, i_1i_3} + X_{i_1i_2, i_2i_3} - X_{i_1i_3, i_2i_3} \geq -1.
\]

Proof: Since \( X \in \mathcal{R}_n \), we know \( X_{i_1i_2, i_1i_3} - X_{i_1i_2, i_1i_3} - X_{i_1i_3, i_2i_3} = -1 \), and therefore the fourth inequality trivially holds. Now,

\[
X_{i_1i_2, i_1i_3} + X_{i_1i_2, i_2i_3} + X_{i_1i_3, i_2i_3} = X_{i_1i_2, i_1i_3} + X_{i_1i_2, i_1i_3} + (1 + X_{i_1i_2, i_1i_3} - X_{i_1i_2, i_1i_3})
\]

\[
= 1 + 2X_{i_1i_2, i_1i_3}
\]

\[
\geq -1, \quad \text{since } X \succeq 0,
\]

and hence the first triangle inequality above holds. Similarly,

\[
- X_{i_1i_2, i_1i_3} - X_{i_1i_2, i_1i_3} + X_{i_1i_3, i_2i_3} = -X_{i_1i_2, i_1i_3} - X_{i_1i_2, i_1i_3} + (1 + X_{i_1i_2, i_1i_3} - X_{i_1i_2, i_1i_3})
\]

\[
= 1 - 2X_{i_1i_2, i_1i_3}
\]

\[
\geq -1, \quad \text{since } X \succeq 0,
\]

and

\[
- X_{i_1i_2, i_2i_3} + X_{i_1i_2, i_1i_3} - X_{i_1i_3, i_2i_3} = -X_{i_1i_2, i_2i_3} + (1 + X_{i_1i_2, i_2i_3} - X_{i_1i_3, i_2i_3}) - X_{i_1i_3, i_2i_3}
\]

\[
= 1 - 2X_{i_1i_2, i_1i_3}
\]

\[
\geq -1, \quad \text{since } X \succeq 0.
\]

Note that the application of an equality is slightly different over the three cases. This is due to the fact that only specific equality conditions hold among the three pairs, and this lack of symmetry will become more evident in the results that follow.

4.2 The Relaxation \( \mathcal{X}_n^\triangle \)

A standard way to tighten linear or semidefinite relaxations of binary optimization problems is to add facet-defining inequalities (such as the triangle inequalities) that are valid for the integer feasible points. For the cut polytope, there are many classes of such inequalities that are known and can be considered, see e.g. [10]. Among them are the triangle inequalities, the pentagonal inequalities, and the hexagonal inequalities.

We have already shown that \( 4\binom{n}{3} \) triangle inequalities automatically hold for all the feasible matrices of \( \mathcal{X}_n \). It is natural to improve the relaxation by adding to it all the triangle inequalities that are not implicitly enforced. If we consider the addition of the remaining triangle inequalities,
we obtain the following (tighter) relaxation:

\[
\mathcal{X}_n^\triangle := \left\{ X \in S^{\binom{n}{2}} \mid \text{diag} \left( X \right) = e, X \succeq 0, \right. \\
\left. X_{i_1i_2,i_3i_3} - X_{i_1i_2,i_2i_3} - X_{i_1i_3,i_2i_3} = -1 \forall 1 \leq i_1 < i_2 < i_3 \leq n \\
X_{p_1,p_2} + X_{p_1,p_3} + X_{p_2,p_3} \geq -1, \quad X_{p_1,p_2} - X_{p_1,p_3} - X_{p_2,p_3} \geq -1, \\
- X_{p_1,p_2} - X_{p_1,p_3} + X_{p_2,p_3} \geq -1, \\
\forall p_1, p_2, p_3 : \{p_1, p_2, p_3\} \neq \{(i_1, i_2), (i_1, i_3), (i_2, i_3)\} \text{ for some } i_1 < i_2 < i_3 \right\}
\]

We now show that optimizing over \(\mathcal{X}_n^\triangle\) is a remarkably tight relaxation of \(\mathcal{R}_n\), as there are a number of facet-defining inequalities of the cut polytope that are implicitly enforced.

First, we consider the pentagonal inequalities. For each subset of pairs \(\{p_1, \ldots, p_5\}\) corresponding to rows and columns of \(X\), there are 16 such inequalities and they can be represented as:

\[
\sum_{1 \leq i < j \leq 5} \delta_i \delta_j X_{p_i,p_j} \geq -2,
\]

where \(\delta_i \in \{-1, 1\}, i = 1, 2, 3, 4, 5\). Hence, for the elements of \(\mathcal{X}_n^{rk-1}\), there are 16\(\binom{5}{2}\) valid pentagonal inequalities in total. We prove that for \(\mathcal{X}_n^\triangle\), 90\(\binom{5}{4}\) of those inequalities automatically hold.

**Lemma 5** Suppose that \(X \in \mathcal{X}_n^\triangle\), and (from the definition of \(\mathcal{X}_n^\triangle\)) consider any five pairs \(p_1, \ldots, p_5\) that satisfy

\[
X_{p_1,p_4} - X_{p_1,p_2} - X_{p_2,p_4} = -1, \quad \quad \quad \quad \quad (7)
\]

\[
X_{p_1,p_5} - X_{p_1,p_3} - X_{p_3,p_5} = -1. \quad \quad \quad \quad \quad (8)
\]

Then for all choices of \(\delta_i \in \{-1, 1\}, i = 1, 2, 3, 4, 5\) such that \((\delta_1 \delta_2 - 1)(\delta_1 \delta_3 - 1)(\delta_1 \delta_4 + 1)(\delta_1 \delta_5 + 1) = 0\), the pentagonal inequality

\[
\sum_{1 \leq i < j \leq 5} \delta_i \delta_j X_{p_i,p_j} \geq -2
\]

holds. This gives a total of 15 pentagonal inequalities.

**Proof:** We assume throughout the proof that \(\delta_1 = 1\). (Note that we may fix \(\delta_i = 1\) for some \(i\) without loss of generality, since any choice of values with \(\delta_i = -1\) is equivalently represented by negating all the values.) Thus, the assumptions imply that at least one of the following holds:

\[
\delta_2 = 1 = \delta_3 = 1, \quad \delta_4 = -1, \quad \delta_5 = -1.
\]

First suppose that \(\delta_2 = 1\). Then using (8),

\[
\sum_{1 \leq i < j \leq 5} \delta_i \delta_j X_{p_i,p_j} = X_{p_1,p_2} + \sum_{3 \leq i,j \leq 5} \delta_i \delta_j (X_{p_1,p_j} + X_{p_2,p_j}) + \sum_{3 \leq i < j \leq 5} \delta_i \delta_j X_{p_i,p_j}
\]

\[
= (1 + X_{p_1,p_2} - X_{p_2,p_4}) + \sum_{3 \leq j \leq 5} \delta_j (X_{p_1,p_j} + X_{p_2,p_j}) + \sum_{3 \leq i < j \leq 5} \delta_i \delta_j X_{p_i,p_j}
\]

\[
= 1 + (\delta_4 + 1) X_{p_1,p_4} + (\delta_4 - 1) X_{p_2,p_4} + \sum_{j=3,5} \delta_j (X_{p_1,p_j} + X_{p_2,p_j}) + \sum_{3 \leq i < j \leq 5} \delta_i \delta_j X_{p_i,p_j}.
\]

Now we consider two subcases. If \(\delta_4 = 1\),

\[
\sum_{1 \leq i < j \leq 5} \delta_i \delta_j X_{p_i,p_j} = 1 + (\delta_4 + 1) X_{p_1,p_4} + (\delta_4 - 1) X_{p_2,p_4} + \sum_{j=3,5} \delta_j (X_{p_1,p_j} + X_{p_2,p_j}) + \sum_{3 \leq i < j \leq 5} \delta_i \delta_j X_{p_i,p_j}
\]

\[
\geq -2,
\]

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Again we have two subcases. If 
and otherwise, 
since the inequalities \( X_{p_1,p_4} + \delta_3 X_{p_1,p_5} + \delta_5 X_{p_4,p_5} \geq -1, \delta_3 X_{p_1,p_3} + X_{p_1,p_4} + \delta_3 X_{p_3,p_4} \geq -1, \) and \( \delta_3 X_{p_2,p_3} + \delta_5 X_{p_2,p_5} + \delta_3 \delta_5 X_{p_3,p_5} \geq -1 \) hold by assumption. Otherwise, \( \delta_4 = -1, \) and
\[
\sum_{1 \leq i < j \leq 5} \delta_i \delta_j X_{p_i,p_j} = 1 + (\delta_4 + 1) X_{p_1,p_4} + (\delta_4 - 1) X_{p_2,p_4} + \sum_{j=3,5} \delta_j (X_{p_1,p_j} + X_{p_2,p_j}) + \sum_{3 \leq i < j \leq 5} \delta_i \delta_j X_{p_i,p_j}
\]
Again by the assumptions on \( X. \) This proves that the inequalities with \( \delta_2 = 1 \) hold.
Now suppose that \( \delta_2 = -1 \) and \( \delta_3 = 1. \) Then
\[
\sum_{1 \leq i < j \leq 5} \delta_i \delta_j X_{p_i,p_j} = -X_{p_1,p_2} + X_{p_1,p_3} + \delta_4 X_{p_1,p_4} + \delta_5 X_{p_1,p_5} - X_{p_2,p_4} - \delta_4 X_{p_2,p_4} - \delta_5 X_{p_2,p_5}
\]
Again we have two subcases. If \( \delta_5 = 1, \)
\[
\sum_{1 \leq i < j \leq 5} \delta_i \delta_j X_{p_i,p_j} = 1 + (-X_{p_1,p_2} - X_{p_1,p_5} - X_{p_1,p_5}) + (\delta_4 X_{p_1,p_4} + X_{p_1,p_5} + \delta_4 X_{p_4,p_5})
\]
and otherwise, \( \delta_5 = -1, \) therefore
\[
1 + (\delta_4 X_{p_1,p_4} - X_{p_2,p_4} - \delta_4 X_{p_2,p_4}) + (X_{p_2,p_5} - X_{p_2,p_3} - X_{p_3,p_5})
\]
Finally, if \( \delta_2 = -1, \delta_3 = -1, \) and \( \delta_4 = 1, \) then \( \delta_5 = -1. \) Therefore,
\[
\sum_{1 \leq i < j \leq 5} \delta_i \delta_j X_{p_i,p_j} = -X_{p_1,p_2} - X_{p_1,p_3} + X_{p_1,p_4} - X_{p_1,p_5} + X_{p_2,p_3} + X_{p_2,p_4} + X_{p_2,p_5}
\]

This concludes the proof.

\[ \text{Lemma 7} \]

This concludes the proof.

\[ \text{Theorem 3} \] Suppose that \( X \in X_n^\Delta \), and consider any choice \( 1 \leq i_1 < i_2 < i_3 < i_4 \leq n \). Then for each of the following sets of row indices for \( X \), precisely 15 pentagonal inequalities hold:

\[
\{(i_1, i_2), (i_1, i_3), (i_1, i_4), (i_2, i_3), (i_2, i_4)\},
\{(i_1, i_2), (i_1, i_3), (i_1, i_4), (i_2, i_3), (i_3, i_4)\},
\{(i_1, i_2), (i_1, i_3), (i_1, i_4), (i_2, i_4), (i_3, i_4)\},
\{(i_1, i_2), (i_2, i_3), (i_2, i_4), (i_3, i_4)\},
\{(i_1, i_2), (i_1, i_4), (i_2, i_3), (i_3, i_4)\},
\{(i_1, i_3), (i_1, i_4), (i_2, i_3), (i_3, i_4)\},
\{(i_1, i_3), (i_1, i_4), (i_2, i_4), (i_3, i_4)\},
\{(i_1, i_4), (i_2, i_3), (i_2, i_4), (i_3, i_4)\},
\{(i_2, i_3), (i_2, i_4), (i_3, i_4)\}.
\]

Hence, for \( X \in X_n^\Delta \), the 90(\(^n^2\)) pentagonal inequalities described above automatically hold.

\[ \text{Proof:} \] The result follows from Lemma 5.

Next, we consider the hexagonal inequalities: They are the inequalities obtained from

\[
2 \sum_{t=2}^{6} X_{p_1,p_t} + \sum_{2 \leq s < t \leq 6} X_{p_s,p_t} \geq -4.
\]

by permutation of the pairs and switching by cuts, for every 6-tuple of pairs \( p_1, \ldots, p_6 \).

First we state two straightforward lemmata.

\[ \text{Lemma 6} \] If \( \delta_i, \delta_j, \delta_k \in \{-1, 1\} \), then either

\[ (\delta_i \delta_j + 1)(\delta_i \delta_k - 1)(\delta_j \delta_k - 1) \neq 0, \]

or precisely two of the three terms in parentheses equal zero.

\[ \text{Proof:} \] Straightforward.

\[ \text{Lemma 7} \] For all choices of \( \delta_i \in \{-1, 1\}, i = 1, 2, 3, 4, 5, 6 \), at least two of the following four conditions hold:

\[
(\delta_1 \delta_4 + 1)(\delta_1 \delta_2 - 1)(\delta_2 \delta_4 - 1) = 0, \quad (9)
(\delta_1 \delta_5 + 1)(\delta_1 \delta_3 - 1)(\delta_3 \delta_5 - 1) = 0, \quad (10)
(\delta_2 \delta_6 + 1)(\delta_2 \delta_3 - 1)(\delta_3 \delta_6 - 1) = 0, \quad (11)
(\delta_4 \delta_6 + 1)(\delta_4 \delta_5 - 1)(\delta_5 \delta_6 - 1) = 0. \quad (12)
\]

\[ \text{Proof:} \] By contradiction. Suppose that the first three conditions (9-11) do not hold simultaneously. Then

\[
(\delta_1 \delta_4 + 1)(\delta_1 \delta_2 - 1)(\delta_2 \delta_4 - 1) \neq 0 \quad \Rightarrow \quad \delta_1 \delta_4 = 1, \delta_1 \delta_2 = -1, \delta_2 \delta_4 = -1;
(\delta_1 \delta_5 + 1)(\delta_1 \delta_3 - 1)(\delta_3 \delta_5 - 1) \neq 0 \quad \Rightarrow \quad \delta_1 \delta_5 = 1, \delta_1 \delta_3 = -1, \delta_3 \delta_5 = -1;
(\delta_2 \delta_6 + 1)(\delta_2 \delta_3 - 1)(\delta_3 \delta_6 - 1) \neq 0 \quad \Rightarrow \quad \delta_2 \delta_6 = 1, \delta_2 \delta_3 = -1, \delta_3 \delta_6 = -1.
\]
Therefore, \( \delta_1 \delta_2 = -1 \), \( \delta_1 \delta_3 = -1 \), and \( \delta_2 \delta_3 = -1 \) hold simultaneously, which is a contradiction.

The remaining three cases can be eliminated similarly.

Now we prove the following lemma.

**Lemma 8** Suppose that \( X \in X_n^\wedge \), and that for the set of pairs \( \{p_1, \ldots, p_6\} \) corresponding to rows and columns of \( X \), the following hold:

\[
X_{p_1,p_4} - X_{p_1,p_2} - X_{p_2,p_4} = -1, \tag{13}
\]

\[
X_{p_1,p_5} - X_{p_1,p_3} - X_{p_3,p_5} = -1, \tag{14}
\]

\[
X_{p_2,p_6} - X_{p_2,p_3} - X_{p_3,p_6} = -1, \tag{15}
\]

\[
X_{p_4,p_6} - X_{p_4,p_5} - X_{p_5,p_6} = -1. \tag{16}
\]

Then the 32 hexagonal inequalities

\[
2 \sum_{j=6}^{j=6} \delta_i \delta_j X_{p_i,p_j} + \sum_{j \neq i, k \neq i, 1 \leq j < k \leq 6} \delta_j \delta_k X_{p_j,p_k} \geq -4 \tag{17}
\]

with \( i = 1 \) hold.

**Proof:** Let \( i = 1 \) be fixed. The proof considers all possible cases for the values of the pairwise products of the form \( \delta_i \delta_j \) for \( i \neq j \). For each case, by Lemma 7, at least two of the equations (9-12) hold. Whenever equation (9) holds, we know that equation (13) can be applied in the proof, and the same is true for equations (10), (11), and (12) together with equations (14), (15), and (16) respectively.

Our main cases are determined by the possible values of the terms appearing in equation (9); by Lemma 6, there are four such cases.

**Case 1:** \( \delta_1 \delta_4 = -1 \), \( \delta_1 \delta_2 = 1 \), \( \delta_2 \delta_4 = -1 \): So (9) holds; now consider (10), there are also four cases.

\( \delta_1 \delta_5 = -1 \), \( \delta_1 \delta_3 = 1 \), \( \delta_2 \delta_5 = -1 \): Then (10) holds, and \( \delta_1 = \delta_2 = \delta_3 \) and different from \( \delta_4 = \delta_5 \); therefore using (13) and (14):

\[
2 \sum_{t=2,3,4,5,6} \delta_1 \delta_t X_{p_1,p_t} + \sum_{2 \leq s < t \leq 6} \delta_s \delta_t X_{p_s,p_t} \\
= 2 \left[ X_{p_1,p_2} + X_{p_1,p_3} + (1 - X_{p_1,p_2} - X_{p_2,p_4}) + (1 - X_{p_1,p_3} - X_{p_3,p_5}) + \delta_1 \delta_6 X_{p_1,p_6} \right] \\
+ X_{p_2,p_3} - X_{p_2,p_4} - X_{p_2,p_5} + \delta_2 \delta_6 X_{p_2,p_6} - X_{p_3,p_4} - X_{p_3,p_5} + \delta_3 \delta_6 X_{p_3,p_6} \\
+ X_{p_4,p_5} + \delta_4 \delta_6 X_{p_4,p_6} + \delta_5 \delta_6 X_{p_5,p_6} \\
= 4 + (-X_{p_2,p_4} - X_{p_2,p_5} + X_{p_4,p_5}) + (-X_{p_2,p_4} + \delta_2 \delta_6 X_{p_2,p_6} + \delta_4 \delta_6 X_{p_4,p_4}) \\
+ (X_{p_2,p_3} - X_{p_2,p_4} - X_{p_3,p_4}) + (-X_{p_3,p_5} + \delta_3 \delta_6 X_{p_3,p_6} + \delta_5 \delta_6 X_{p_3,p_4}) \\
- 2X_{p_3,p_5} + 2\delta_1 \delta_6 X_{p_1,p_5} \\
\geq 4 + (-1) + (-1) + (-1) + (-1) + (-2) + (-2) \\
\geq -4,
\]

using the following inequalities valid for \( X \in X_n^\wedge \):

\[
-X_{p_2,p_4} - X_{p_2,p_5} + X_{p_4,p_5} \geq -1, \quad -X_{p_2,p_4} + \delta_2 \delta_6 X_{p_2,p_6} + \delta_4 \delta_6 X_{p_4,p_6} \geq -1, \quad -X_{p_2,p_4} - X_{p_3,p_4} \geq -1, \quad -X_{p_3,p_5} + \delta_3 \delta_6 X_{p_3,p_6} + \delta_5 \delta_6 X_{p_5,p_6} \geq -1.
\]
$\delta_1 \delta_5 = -1$, $\delta_1 \delta_3 = -1$, $\delta_3 \delta_5 = 1$: Again (9) and (10) hold, and $\delta_1 = \delta_2$ and different from $\delta_3 = \delta_4 = \delta_5$; therefore using (13) and (14):

$$\begin{align*}
2 \sum_{t=2,3,4,5,6} \delta_1 \delta_t X_{p_1,p_t} + \sum_{2 \leq s < t \leq 6} \delta_s \delta_t X_{p_s,p_t} &= 2[X_{p_1,p_2} - X_{p_1,p_3} + (1 - X_{p_1,p_2} - X_{p_2,p_4}) + (1 - X_{p_1,p_3} - X_{p_3,p_5}) + \delta_1 \delta_6 X_{p_1,p_6}] \\
&\quad - X_{p_2,p_3} - X_{p_2,p_4} - X_{p_2,p_5} + \delta_2 \delta_6 X_{p_2,p_6} + X_{p_3,p_4} + X_{p_3,p_5} + \delta_3 \delta_6 X_{p_3,p_6} \\
&\quad + X_{p_4,p_5} + \delta_4 \delta_6 X_{p_4,p_6} + \delta_5 \delta_6 X_{p_5,p_6} \\
&\quad = 4 + (-X_{p_1,p_3} + \delta_1 \delta_6 X_{p_1,p_6} + \delta_2 \delta_6 X_{p_2,p_6}) + (-X_{p_2,p_4} - X_{p_2,p_5} + X_{p_4,p_6}) \\
&\quad + (-X_{p_2,p_4} + \delta_2 \delta_6 X_{p_2,p_6} + \delta_4 \delta_6 X_{p_4,p_6}) + (-X_{p_2,p_3} - X_{p_2,p_4} + X_{p_3,p_5}) \\
&\quad + ((-X_{p_1,p_3} - X_{p_3,p_5}) + \delta_1 \delta_6 X_{p_1,p_6} + \delta_4 \delta_6 X_{p_4,p_6}) - 2X_{p_1,p_3} \\
&\geq 4 + (-1) + (-1) + (-1) + (-1) + ((-1 - X_{p_1,p_3}) + \delta_1 \delta_6 X_{p_1,p_6} + \delta_5 \delta_6 X_{p_5,p_6}) + (-2) \\
&\geq -3 + (-X_{p_1,p_5} + \delta_1 \delta_6 X_{p_1,p_6} + \delta_5 \delta_6 X_{p_5,p_6}) \\
&\geq -4,
\end{align*}$$

using inequalities valid for $X \in X^\bullet$

$\delta_1 \delta_5 = 1$, $\delta_1 \delta_3 = 1$, $\delta_3 \delta_5 = 1$: Again (9) and (10) hold, and $\delta_1 = \delta_2 = \delta_3 = \delta_5$ and different from $\delta_4$; therefore using (13) and (14):

$$\begin{align*}
2 \sum_{t=2,3,4,5,6} \delta_1 \delta_t X_{p_1,p_t} + \sum_{2 \leq s < t \leq 6} \delta_s \delta_t X_{p_s,p_t} &= 2[X_{p_1,p_2} + (1 + X_{p_1,p_5} - X_{p_3,p_5}) + (1 - X_{p_1,p_2} - X_{p_2,p_4}) X_{p_1,p_5} + \delta_1 \delta_6 X_{p_1,p_6}] \\
&\quad + X_{p_2,p_3} - X_{p_2,p_4} - X_{p_2,p_5} + \delta_2 \delta_6 X_{p_2,p_6} - X_{p_3,p_4} + X_{p_3,p_5} + \delta_3 \delta_6 X_{p_3,p_6} \\
&\quad - X_{p_4,p_5} + \delta_4 \delta_6 X_{p_4,p_6} + \delta_5 \delta_6 X_{p_5,p_6} \\
&\quad = 4 + (-X_{p_2,p_3} - X_{p_2,p_4} - X_{p_2,p_5} + X_{p_3,p_4}) \\
&\quad + (X_{p_1,p_5} + \delta_1 \delta_6 X_{p_1,p_6} + \delta_5 \delta_6 X_{p_5,p_6}) + (-X_{p_2,p_4} + \delta_2 \delta_6 X_{p_2,p_6} + \delta_4 \delta_6 X_{p_4,p_6}) \\
&\quad + ((\delta_1 \delta_6 X_{p_1,p_6} + \delta_3 \delta_6 X_{p_3,p_5}) + X_{p_1,p_5} - X_{p_3,p_5}) + 2X_{p_1,p_3} \\
&\geq 4 + (-1) + (-1) + (-1) + (-1) + ((-1 - X_{p_1,p_3}) + X_{p_1,p_5} - X_{p_3,p_5}) + (-2) \\
&\geq -3 + (-X_{p_1,p_5} + X_{p_1,p_5} - X_{p_3,p_5}) \\
&\geq -4,
\end{align*}$$

using inequalities valid for $X \in X^\bullet$

$\delta_1 \delta_5 = 1$, $\delta_1 \delta_3 = -1$, $\delta_3 \delta_5 = -1$: In this case, (10) does not hold, so we consider (11) or (12). There are again four cases.

$\delta_2 \delta_6 = -1$, $\delta_2 \delta_3 = 1$, $\delta_3 \delta_6 = -1$: This is impossible, since $\delta_1 \delta_2 = 1$ and $\delta_1 \delta_3 = -1$.

$\delta_2 \delta_6 = -1$, $\delta_2 \delta_3 = -1$, $\delta_3 \delta_6 = 1$: If this is the case, then (11) holds, and $\delta_1 = \delta_2 = \delta_5$ and different from $\delta_3 = \delta_4 = \delta_6$; therefore using (13) and (15):

$$\begin{align*}
2 \sum_{t=2,3,4,5,6} \delta_1 \delta_t X_{p_1,p_t} + \sum_{2 \leq s < t \leq 6} \delta_s \delta_t X_{p_s,p_t} &= 2[X_{p_1,p_2} - X_{p_1,p_3} + (1 - X_{p_1,p_2} - X_{p_2,p_4}) + X_{p_1,p_5} - X_{p_1,p_6}] - X_{p_2,p_3} - X_{p_2,p_4} \\
&\quad + X_{p_2,p_5} + (1 - X_{p_2,p_3} - X_{p_3,p_6}) + X_{p_3,p_4} - X_{p_3,p_5} + X_{p_3,p_6} - X_{p_4,p_5} \\
&\quad + X_{p_4,p_5} - X_{p_5,p_6} \\
&\quad = 3 + (-X_{p_1,p_3} + X_{p_1,p_5} - X_{p_3,p_5}) + (-X_{p_2,p_3} - X_{p_2,p_4} + X_{p_3,p_4}) \\
&\quad + (X_{p_1,p_5} - X_{p_1,p_6} - X_{p_3,p_6}) + (-X_{p_2,p_4} + X_{p_2,p_5} - X_{p_4,p_5}) \\
&\quad + ((-X_{p_1,p_3} - X_{p_1,p_6} - X_{p_3,p_6}) + (-X_{p_2,p_3} - X_{p_2,p_4} + X_{p_4,p_6}) \\
&\geq 3 + (-1) + (-1) + (-1) + ((-1 - X_{p_3,p_6}) + (-1 - X_{p_3,p_6}) + X_{p_4,p_6}) \\
&\geq -3 + (-X_{p_3,p_6} - X_{p_3,p_6} + X_{p_4,p_6}) \\
&\geq -4.
\end{align*}$$
\( \delta_2 \delta_6 = 1, \, \delta_2 \delta_3 = 1, \, \delta_3 \delta_6 = 1: \) This is impossible, since \( \delta_1 \delta_2 = 1 \) and \( \delta_1 \delta_3 = -1. \)

\( \delta_2 \delta_6 = 1, \, \delta_2 \delta_3 = -1, \, \delta_3 \delta_6 = -1: \) If this is the case, then (12) holds, and \( \delta_1 = \delta_2 = \delta_5 = \delta_6 \) and different from \( \delta_3 = \delta_4; \) therefore using (13) and (16):

\[
2 \sum_{t=2,3,4,5,6} \delta_1 \delta_t X_{p_1,p_t} + \sum_{2 \leq s < t \leq 6} \delta_s \delta_t X_{p_s,p_t} \\
= 2 \left[ X_{p_1,p_2} - X_{p_1,p_3} + (1 - X_{p_1,p_2} - X_{p_2,p_4}) + (1 - X_{p_1,p_3} - X_{p_3,p_6}) + \delta_1 \delta_6 X_{p_1,p_6} \right] \\
- X_{p_2,p_3} + X_{p_2,p_5} + X_{p_2,p_6} + X_{p_3,p_4} - X_{p_3,p_5} + \delta_2 \delta_5 X_{p_2,p_5} \\
+ X_{p_4,p_5} + X_{p_4,p_6} + \delta_5 \delta_6 X_{p_4,p_6} \\
= 3 + (X_{p_1,p_3} - X_{p_1,p_5} - X_{p_3,p_5}) + (X_{p_1,p_3} + X_{p_1,p_5} - X_{p_3,p_5}) \\
+ (-X_{p_2,p_3} + X_{p_2,p_4} + X_{p_3,p_4}) + (-X_{p_2,p_4} + X_{p_2,p_5} - X_{p_4,p_5}) \\
+ (-X_{p_1,p_4} + X_{p_1,p_5} - X_{p_2,p_4} + X_{p_2,p_5} - X_{p_4,p_5}) \\
\geq 3 + (-1) + (-1) + (-1) + ((-1 - X_{p_1,p_2}) + (-1 + X_{p_1,p_4}) - X_{p_2,p_4}) \\
\geq -3 + (-X_{p_1,p_2} + X_{p_1,p_4} - X_{p_2,p_4}) \\
\geq -4.
\]

**Case 2:** \( \delta_1 \delta_4 = -1, \, \delta_1 \delta_2 = -1, \, \delta_2 \delta_4 = 1: \) In this case, equation (9) holds.

\( \delta_1 \delta_5 = -1, \, \delta_1 \delta_3 = 1, \, \delta_3 \delta_5 = -1: \) If this is the case, then \( \delta_1 = \delta_3 \) and different from \( \delta_2 = \delta_4 = \delta_5; \) therefore using (13) and (14):

\[
2 \sum_{t=2,3,4,5,6} \delta_1 \delta_t X_{p_1,p_t} + \sum_{2 \leq s < t \leq 6} \delta_s \delta_t X_{p_s,p_t} \\
= 2 \left[ -X_{p_1,p_2} + X_{p_1,p_3} + (1 - X_{p_1,p_2} - X_{p_2,p_4}) + (1 - X_{p_1,p_3} - X_{p_3,p_6}) + \delta_1 \delta_6 X_{p_1,p_6} \right] \\
+ X_{p_2,p_3} + X_{p_2,p_5} + X_{p_2,p_6} + X_{p_3,p_4} + X_{p_3,p_5} + \delta_3 \delta_6 X_{p_3,p_6} \\
+ X_{p_4,p_5} + \delta_4 \delta_6 X_{p_4,p_6} + \delta_5 \delta_6 X_{p_4,p_6} \\
= 4 + (X_{p_3,p_4} - X_{p_3,p_5} + X_{p_4,p_5}) + (X_{p_3,p_4} + X_{p_3,p_5} - X_{p_4,p_6} - X_{p_4,p_5}) \\
+ (-X_{p_2,p_3} - X_{p_3,p_5} + X_{p_2,p_5}) + (-X_{p_2,p_3} + X_{p_1,p_4} + \delta_1 \delta_6 X_{p_1,p_6} + \delta_2 \delta_5 X_{p_2,p_5}) \\
\geq 4 + (-1) + (-1) + (-1) + ((-1 - X_{p_1,p_4}) + \delta_1 \delta_6 X_{p_1,p_6} + \delta_4 \delta_6 X_{p_4,p_6}) + (-2) \\
\geq -3 + (-X_{p_1,p_4} + \delta_1 \delta_6 X_{p_1,p_6} + \delta_4 \delta_6 X_{p_4,p_6}) \\
\geq -4.
\]

\( \delta_1 \delta_5 = -1, \, \delta_1 \delta_3 = -1, \, \delta_3 \delta_5 = 1: \) If this is the case, then \( \delta_1 \) and different from \( \delta_2 = \delta_3 = \delta_4 = \delta_5; \) therefore using (13) and (14):

\[
2 \sum_{t=2,3,4,5,6} \delta_1 \delta_t X_{p_1,p_t} + \sum_{2 \leq s < t \leq 6} \delta_s \delta_t X_{p_s,p_t} \\
= 2 \left[ -X_{p_1,p_2} - X_{p_1,p_3} + (1 - X_{p_1,p_2} - X_{p_2,p_4}) + (1 - X_{p_1,p_3} - X_{p_3,p_6}) + \delta_1 \delta_6 X_{p_1,p_6} \right] \\
+ X_{p_2,p_3} + X_{p_2,p_5} + X_{p_2,p_6} + \delta_2 \delta_6 X_{p_2,p_6} + X_{p_3,p_4} + X_{p_3,p_5} + \delta_3 \delta_6 X_{p_3,p_6} \\
+ X_{p_4,p_5} + \delta_4 \delta_6 X_{p_4,p_6} + \delta_5 \delta_6 X_{p_4,p_6} \\
= 4 + (-X_{p_1,p_2} - X_{p_1,p_3} + X_{p_2,p_4}) + (-X_{p_1,p_2} + \delta_1 \delta_6 X_{p_1,p_6} + \delta_2 \delta_6 X_{p_2,p_6}) \\
+ (-X_{p_1,p_2} + \delta_4 \delta_6 X_{p_1,p_6} + \delta_3 \delta_6 X_{p_3,p_6}) + (X_{p_4,p_5} + \delta_4 \delta_6 X_{p_4,p_6} + \delta_5 \delta_6 X_{p_5,p_6}) \\
\geq 4 + (-1) + (-1) + (-1) + ((-1 - X_{p_1,p_4}) - X_{p_1,p_3} + X_{p_3,p_4}) \\
+ (-1 + X_{p_1,p_3} - X_{p_1,p_5} - X_{p_3,p_5}) \\
\geq -2 + (-X_{p_1,p_4} - X_{p_1,p_3} + X_{p_3,p_4}) + (X_{p_1,p_5} - X_{p_1,p_3} - X_{p_3,p_5}) \\
\geq -4.
\]
\( \delta_1 \delta_5 = 1, \delta_1 \delta_4 = 1, \delta_3 \delta_5 = 1 \): If this is the case, then \( \delta_1 = \delta_3 = \delta_5 \) and different from \( \delta_2 = \delta_4 \); therefore using (13) and (14):

\[
2 \sum_{t=2,3,4,5,6} \delta_1 \delta_t X_{p_1,pt} + \sum_{2 \leq s < t \leq 6} \delta_s \delta_t X_{p_s,pt} \\
= 2 \left[ -X_{p_1, p_2} + (1 - X_{p_1, p_2} - X_{p_2, p_4}) + (1 - X_{p_1, p_2} - X_{p_2, p_4}) + \delta_6 X_{p_1, p_6} \right] \\
- X_{p_2, p_4} + X_{p_4, p_6} - X_{p_3, p_5} + \delta_2 \delta_6 X_{p_2, p_6} - X_{p_3, p_4} + X_{p_3, p_5} + \delta_3 \delta_5 X_{p_3, p_6} \\
- X_{p_4, p_5} + \delta_4 \delta_6 X_{p_4, p_6} + \delta_5 \delta_6 X_{p_5, p_6} \\
= 4 + (-X_{p_1, p_2} + X_{p_1, p_5} - X_{p_2, p_3}) + (-X_{p_1, p_2} + \delta_1 \delta_6 X_{p_1, p_6} + \delta_2 \delta_6 X_{p_2, p_6}) \\
+(X_{p_1, p_5} + \delta_1 \delta_6 X_{p_1, p_6} + \delta_3 \delta_6 X_{p_3, p_6}) + (-X_{p_3, p_4} + \delta_3 \delta_6 X_{p_3, p_6} + \delta_4 \delta_6 X_{p_4, p_6}) \\
+((-X_{p_1, p_2} + X_{p_1, p_5} - X_{p_2, p_3} - X_{p_3, p_5}) + ((-X_{p_1, p_2} + X_{p_1, p_5} - X_{p_2, p_3} - X_{p_3, p_5}) \\
+(-1 + X_{p_2, p_3} - X_{p_2, p_3} - X_{p_3, p_5}) + (X_{p_2, p_3} - X_{p_2, p_3} - X_{p_3, p_5}) \\
\geq 4 + (-1) + (-1) + (-1) + ((-1 + X_{p_2, p_3}) - X_{p_2, p_3} - X_{p_3, p_5}) \\
+(-1 + X_{p_2, p_3} - X_{p_2, p_3} - X_{p_3, p_5}) \\
\geq -4.
\]

\( \delta_1 \delta_5 = 1, \delta_1 \delta_4 = -1, \delta_3 \delta_5 = -1 \): Then (10) does not hold, so we consider (11); there are again four cases.

\( \delta_2 \delta_6 = -1, \delta_2 \delta_3 = 1, \delta_3 \delta_6 = -1 \): If this is the case, then \( \delta_1 = \delta_5 = \delta_6 \) and different from \( \delta_2 = \delta_3 = \delta_4 \); therefore using (13),(15), and (16):

\[
2 \sum_{t=2,3,4,5,6} \delta_1 \delta_t X_{p_1,pt} + \sum_{2 \leq s < t \leq 6} \delta_s \delta_t X_{p_s,pt} \\
= 2 \left[ -X_{p_1, p_2} - X_{p_1, p_3} + (1 - X_{p_1, p_2} - X_{p_2, p_4}) + X_{p_1, p_5} + X_{p_1, p_6} \right] + X_{p_2, p_3} \\
+ X_{p_2, p_4} - X_{p_2, p_5} + (1 + X_{p_1, p_2} - X_{p_3, p_5} + X_{p_3, p_4} - X_{p_3, p_5} - X_{p_3, p_6} \\
- X_{p_4, p_5} - X_{p_4, p_6} + (1 + X_{p_1, p_2} - X_{p_3, p_5} - X_{p_4, p_5}) \\
= 4 + (-X_{p_1, p_2} + X_{p_1, p_5} - X_{p_2, p_3}) + (X_{p_3, p_4} - X_{p_3, p_5} - X_{p_4, p_5}) \\
+2(-X_{p_1, p_3} - X_{p_1, p_5} - X_{p_3, p_6}) + (X_{p_1, p_5} - X_{p_4, p_5}) + (-X_{p_1, p_2} - X_{p_2, p_4}) \\
-2X_{p_1, p_2} \\
\geq 4 + (-1) + (-2) + (-1) + (-1 + X_{p_1, p_4}) + (-1 - X_{p_1, p_4}) + (-2) \\
\geq -4.
\]

\( \delta_2 \delta_6 = -1, \delta_2 \delta_3 = -1, \delta_3 \delta_6 = 1 \): This is impossible, since \( \delta_1 \delta_2 = -1 \) and \( \delta_1 \delta_3 = -1 \).

\( \delta_2 \delta_6 = 1, \delta_2 \delta_3 = 1, \delta_3 \delta_6 = 1 \): If this is the case, then \( \delta_1 = \delta_5 \) and different from \( \delta_2 = \delta_3 = \delta_4 \); therefore using (13) and (15):

\[
2 \sum_{t=2,3,4,5,6} \delta_1 \delta_t X_{p_1,pt} + \sum_{2 \leq s < t \leq 6} \delta_s \delta_t X_{p_s,pt} \\
= 2 \left[ -X_{p_1, p_2} - X_{p_1, p_3} + (1 - X_{p_1, p_2} - X_{p_2, p_4}) + X_{p_1, p_5} - X_{p_1, p_6} \right] \\
+ (1 + X_{p_2, p_4} - X_{p_3, p_5} + X_{p_2, p_4} - X_{p_2, p_5} + X_{p_2, p_4} - X_{p_3, p_4} - X_{p_3, p_5} + X_{p_3, p_6} \\
- X_{p_4, p_5} - X_{p_4, p_6} - X_{p_5, p_6} \\
= 3 + 2(-X_{p_1, p_2} - X_{p_1, p_3} + X_{p_1, p_5} - X_{p_2, p_4}) + (-X_{p_1, p_2} + X_{p_1, p_5} - X_{p_2, p_3}) \\
+(-X_{p_2, p_4} + X_{p_1, p_5} - X_{p_3, p_5}) + (-X_{p_3, p_4} + X_{p_3, p_5} - X_{p_5, p_6}) \\
+(-X_{p_1, p_2} - X_{p_1, p_3} + (X_{p_3, p_4} - X_{p_2, p_4}) \\
\geq 3 + (-2) + (-1) + (-1) + (-1 - X_{p_2, p_3}) + (-1 + X_{p_2, p_3}) \\
\geq -4.
\]

\( \delta_2 \delta_6 = 1, \delta_2 \delta_3 = -1, \delta_3 \delta_6 = -1 \): This is impossible, since \( \delta_1 \delta_2 = -1 \) and \( \delta_1 \delta_3 = -1 \).

Case 3: \( \delta_1 \delta_4 = 1, \delta_1 \delta_2 = 1, \delta_2 \delta_4 = 1 \): In this case, equation (9) holds.
$$\delta_1 \delta_5 = -1, \ \delta_1 \delta_3 = 1, \ \delta_3 \delta_5 = -1: \text{If this is the case, then } \delta_1 = \delta_2 = \delta_3 = \delta_4 \text{ and different from } \delta_5; \text{ therefore using (13) and (14)}:\n$$

$$2 \sum_{t=2,3,4,5,6} \delta_1 t X_{p_1,p_t} + \sum_{2 \leq s < t \leq 6} \delta_3 \delta_1 X_{p_s,p_t}$$

$$= 2 \left[ X_{p_1,p_2} + X_{p_1,p_3} + X_{p_1,p_4} + (1 - X_{p_1,p_3} - X_{p_1,p_5}) + \delta_1 \delta_6 X_{p_1,p_6} \right] + X_{p_2,p_3} + (1 + X_{p_1,p_4} - X_{p_1,p_3}) - X_{p_2,p_5} + \delta_2 \delta_6 X_{p_2,p_6} + X_{p_3,p_4} - X_{p_3,p_5} + \delta_3 \delta_6 X_{p_3,p_6} - X_{p_4,p_5} + \delta_4 \delta_6 X_{p_4,p_6} + \delta_5 \delta_6 X_{p_5,p_6}$$

$$= 3 + (X_{p_1,p_4} + \delta_1 \delta_6 X_{p_1,p_6} + \delta_4 \delta_6 X_{p_4,p_6}) + (X_{p_1,p_2} + \delta_1 \delta_6 X_{p_1,p_6} + \delta_2 \delta_6 X_{p_2,p_6}) + \delta_3 \delta_6 X_{p_3,p_6} + \delta_2 \delta_6 X_{p_2,p_6})$$

$$\geq 3 + (-1) + (-1) + (-1) + (-1) + (-1) + (-2)$$

$$\geq -4.$$

$$\delta_1 \delta_5 = -1, \ \delta_1 \delta_3 = -1, \ \delta_3 \delta_5 = 1: \text{If this is the case, then } \delta_1 = \delta_2 = \delta_4 \text{ and different from } \delta_3 = \delta_5; \text{ therefore using (13) and (14)}:\n$$

$$2 \sum_{t=2,3,4,5,6} \delta_1 t X_{p_1,p_t} + \sum_{2 \leq s < t \leq 6} \delta_3 \delta_1 X_{p_s,p_t}$$

$$= 2 \left[ (1 + X_{p_1,p_4} - X_{p_2,p_4}) - X_{p_1,p_3} + X_{p_1,p_4} + (1 - X_{p_1,p_3} - X_{p_1,p_5}) + \delta_1 \delta_6 X_{p_1,p_6} \right]$$

$$- X_{p_2,p_3} + X_{p_2,p_4} - X_{p_2,p_5} + \delta_2 \delta_6 X_{p_2,p_6} - X_{p_3,p_4} - X_{p_3,p_5} + \delta_3 \delta_6 X_{p_3,p_6}$$

$$- X_{p_4,p_5} + \delta_4 \delta_6 X_{p_4,p_6} + \delta_5 \delta_6 X_{p_5,p_6}$$

$$= 3 + (-X_{p_1,p_3} + X_{p_1,p_4} - X_{p_3,p_4}) + (X_{p_1,p_4} + \delta_1 \delta_6 X_{p_1,p_6} + \delta_4 \delta_6 X_{p_4,p_6})$$

$$+ (X_{p_1,p_2} + \delta_2 \delta_6 X_{p_2,p_6} + \delta_3 \delta_6 X_{p_3,p_6}) + (-X_{p_2,p_5} + \delta_2 \delta_6 X_{p_2,p_6} + \delta_5 \delta_6 X_{p_5,p_6})$$

$$+ ((X_{p_1,p_4} - X_{p_2,p_4}) - X_{p_1,p_3} - X_{p_2,p_3}) + ((X_{p_1,p_4} - X_{p_1,p_3} - X_{p_3,p_5} - X_{p_4,p_5}))$$

$$\geq 3 + (-1) + (-1) + (-1) + ((-1 + X_{p_1,p_2}) - X_{p_1,p_3} - X_{p_2,p_3})$$

$$+ ((-1 + X_{p_3,p_4} - X_{p_3,p_5} - X_{p_4,p_5})$$

$$\geq -3 + (X_{p_1,p_4} - X_{p_3,p_5} - X_{p_4,p_5})$$

$$\geq -3 + (X_{p_1,p_2} - X_{p_1,p_3} - X_{p_2,p_3}) + (X_{p_3,p_4} - X_{p_3,p_5} - X_{p_4,p_5})$$

$$\geq -4.$$

$$\delta_1 \delta_5 = 1, \ \delta_1 \delta_3 = 1, \ \delta_3 \delta_5 = 1: \text{If this is the case, then } \delta_1 = \delta_2 = \delta_3 = \delta_4 = \delta_5; \text{ therefore using (13) and (14)}:\n$$

$$2 \sum_{t=2,3,4,5,6} \delta_1 t X_{p_1,p_t} + \sum_{2 \leq s < t \leq 6} \delta_3 \delta_1 X_{p_s,p_t}$$

$$= 2 \left[ X_{p_1,p_2} + X_{p_1,p_3} + X_{p_1,p_4} + X_{p_1,p_5} + \delta_1 \delta_6 X_{p_1,p_6} \right] + X_{p_2,p_3} + (1 + X_{p_1,p_4} - X_{p_1,p_3}) + X_{p_2,p_5} + \delta_2 \delta_6 X_{p_2,p_6} + X_{p_3,p_4} + X_{p_3,p_5} + \delta_3 \delta_6 X_{p_3,p_6}$$

$$+ X_{p_4,p_5} + \delta_4 \delta_6 X_{p_4,p_6} + \delta_5 \delta_6 X_{p_5,p_6}$$

$$= 3 + (X_{p_1,p_2} + X_{p_1,p_3} + X_{p_2,p_3}) + (X_{p_1,p_4} + X_{p_1,p_5} + X_{p_4,p_5})$$

$$+ ((X_{p_1,p_4} + \delta_1 \delta_6 X_{p_1,p_6} + \delta_4 \delta_6 X_{p_4,p_6}) + (X_{p_1,p_2} + \delta_1 \delta_6 X_{p_1,p_6} + \delta_2 \delta_6 X_{p_2,p_6}) + \delta_3 \delta_6 X_{p_3,p_6})$$

$$+ (X_{p_2,p_3} + \delta_2 \delta_6 X_{p_2,p_6} + \delta_3 \delta_6 X_{p_3,p_6}) + (X_{p_1,p_4} + X_{p_1,p_5} + X_{p_4,p_5})$$

$$\geq 3 + (-1) + (-1) + (-1) + (-1) + (-1) + (-2)$$

$$\geq -4.$$

$$\delta_1 \delta_5 = 1, \ \delta_1 \delta_3 = -1, \ \delta_3 \delta_5 = -1: \text{Then (10) does not hold, so we consider (11); there are again four cases.}$$

$$\delta_2 \delta_6 = -1, \ \delta_2 \delta_3 = 1, \ \delta_3 \delta_6 = -1: \text{This is impossible, since } \delta_1 \delta_2 = 1 \text{ and } \delta_1 \delta_3 = -1.$$
\[ \delta_2 \delta_6 = -1, \ \delta_2 \delta_3 = -1, \ \delta_3 \delta_6 = 1: \text{If this is the case, then } \delta_1 = \delta_2 = \delta_4 = \delta_5 \text{ and different from } \delta_3 = \delta_6; \text{ therefore using (13) and (15):} \]
\[
2 \sum_{t=2,3,4,5,6} \delta_1 \delta_4 X_{p_1,pt} + \sum_{2 \leq s < t \leq 6} \delta_s \delta_t X_{p_s,pt} \\
= 2 \left[ (1 + X_{p_1,p_2} - X_{p_2,p_4}) - X_{p_1,p_3} + X_{p_1,p_4} + X_{p_1,p_5} - X_{p_1,p_6} \right] - X_{p_2,p_3} + X_{p_2,p_4} + X_{p_2,p_5} + X_{p_2,p_6} - X_{p_3,p_4} - X_{p_3,p_5} + X_{p_3,p_6} + X_{p_4,p_5} - X_{p_4,p_6} - X_{p_5,p_6} \\
= 3 + (X_{p_1,p_1} + X_{p_1,p_5} + X_{p_4,p_4}) + (X_{p_1,p_4} - X_{p_1,p_6} - X_{p_4,p_4}) \\
+ (X_{p_1,p_5} + X_{p_1,p_6} - X_{p_3,p_5}) \\
+ (X_{p_1,p_4} - X_{p_1,p_5} - X_{p_3,p_3}) + (-X_{p_2,p_4} - X_{p_2,p_3}) \\
\geq 3 + (-1) + (-1) + (-1) + (-1) + (-1 + X_{p_1,p_2}) + (-1 - X_{p_1,p_2}) \\
\geq -4.
\]

\[ \delta_2 \delta_6 = 1, \ \delta_2 \delta_3 = 1, \ \delta_3 \delta_6 = 1: \text{This is impossible, since } \delta_1 \delta_2 = 1 \text{ and } \delta_1 \delta_3 = -1. \]

\[ \delta_2 \delta_6 = 1, \ \delta_2 \delta_3 = -1, \ \delta_3 \delta_6 = -1: \text{If this is the case, then } \delta_1 = \delta_2 = \delta_4 = \delta_5 = \delta_6 \text{ and different from } \delta_3; \text{ therefore using (13) and (16):} \]
\[
2 \sum_{t=2,3,4,5,6} \delta_1 \delta_4 X_{p_1,pt} + \sum_{2 \leq s < t \leq 6} \delta_s \delta_t X_{p_s,pt} \\
= 2 \left[ (1 + X_{p_1,p_2} - X_{p_2,p_4}) - X_{p_1,p_3} + X_{p_1,p_4} + X_{p_1,p_5} + X_{p_1,p_6} \right] - X_{p_2,p_3} + X_{p_2,p_4} + X_{p_2,p_5} + X_{p_2,p_6} - X_{p_3,p_4} - X_{p_3,p_5} + X_{p_3,p_6} + X_{p_4,p_5} + X_{p_4,p_6} + (1 + X_{p_4,p_6} - X_{p_4,p_5}) \\
= 3 + (-X_{p_1,p_1} + X_{p_1,p_4} - X_{p_3,p_4}) + 2(X_{p_1,p_4} + X_{p_1,p_6} + X_{p_4,p_6}) \\
+ (-X_{p_2,p_3} + X_{p_2,p_4} - X_{p_3,p_5}) + (-X_{p_1,p_3} + X_{p_1,p_5} - X_{p_3,p_5}) \\
+ (X_{p_1,p_4} - X_{p_2,p_4}) + (X_{p_1,p_5} + X_{p_2,p_5}) \\
\geq 3 + (-1) + (-1) + (-1) + (-1 + X_{p_1,p_2}) + (-1 - X_{p_1,p_2}) \\
\geq -4.
\]

**Case 4:** \[ \delta_1 \delta_4 = 1, \ \delta_1 \delta_2 = -1, \ \delta_2 \delta_4 = -1: \text{In this case, equation (9) does not hold.} \]

\[ \delta_1 \delta_5 = -1, \ \delta_1 \delta_3 = 1, \ \delta_3 \delta_5 = -1: \text{If this is the case, then } \delta_1 = \delta_3 = \delta_4 \text{ and different from } \delta_2 = \delta_5, \text{ and we consider (11); there are again four cases.} \]

\[ \delta_2 \delta_6 = -1, \ \delta_2 \delta_3 = 1, \ \delta_3 \delta_6 = -1: \text{This is impossible, since } \delta_1 \delta_2 = -1 \text{ and } \delta_1 \delta_3 = 1. \]

\[ \delta_2 \delta_6 = -1, \ \delta_2 \delta_3 = -1, \ \delta_3 \delta_6 = 1: \text{If this is the case, then } \delta_1 = \delta_3 = \delta_4 = \delta_6 \text{ and different from } \delta_2 = \delta_5, \text{ therefore using (14) and (15):} \]
\[
2 \sum_{t=2,3,4,5,6} \delta_1 \delta_4 X_{p_1,pt} + \sum_{2 \leq s < t \leq 6} \delta_s \delta_t X_{p_s,pt} \\
= 2 \left[ -X_{p_1,p_1} + X_{p_1,p_4} + X_{p_1,p_4} + (1 - X_{p_1,p_3} - X_{p_3,p_5}) + X_{p_1,p_6} \right] - X_{p_2,p_3} - X_{p_2,p_4} + X_{p_2,p_5} + X_{p_2,p_6} - X_{p_3,p_4} - X_{p_3,p_5} + X_{p_3,p_6} - X_{p_4,p_5} - X_{p_4,p_6} - X_{p_5,p_6} \\
= 3 + (-X_{p_1,p_1} + X_{p_1,p_4} - X_{p_2,p_4}) + (X_{p_1,p_4} + X_{p_1,p_6} + X_{p_4,p_6}) \\
+ (-X_{p_2,p_3} + X_{p_2,p_4} - X_{p_3,p_5}) + (X_{p_3,p_4} - X_{p_3,p_5} - X_{p_4,p_5}) \\
+ (X_{p_1,p_6} - X_{p_3,p_6}) + (-X_{p_1,p_3} - X_{p_2,p_3}) - X_{p_3,p_5} \\
\geq 3 + (-1) + (-1) + (-1) + (-1 + X_{p_1,p_2}) + (-1 - X_{p_1,p_2}) \\
\geq -3 + (X_{p_1,p_6} - X_{p_1,p_4} - X_{p_3,p_5}) \\
\geq -4.
\]

\[ \delta_2 \delta_6 = 1, \ \delta_2 \delta_3 = 1, \ \delta_3 \delta_6 = 1: \text{This is impossible, since } \delta_1 \delta_2 = -1 \text{ and } \delta_1 \delta_3 = 1. \]
\[ \delta_2 \delta_6 = 1, \ \delta_2 \delta_3 = -1, \ \delta_3 \delta_6 = -1: \] In this case, equation (11) does not hold; but then (12) must hold. Furthermore, we have \( \delta_1 = \delta_3 = \delta_4 \) and different from \( \delta_2 = \delta_5 = \delta_6 \), therefore using (14) and (16):

\[
2 \sum_{t=2,3,4,5,6} \delta_1 \delta_t X_{p_1,pt} + \sum_{2 \leq s < t \leq 6} \delta_s \delta_t X_{p_s,pt} \\
= 2 \left[ -X_{p_1,pt} - X_{p_2,pt} + X_{p_3,pt} + (1 - X_{p_1,pt} - X_{p_3,pt}) - X_{p_1,pt} \right] + X_{p_2,pt} \\
- X_{p_2,pt} + X_{p_3,pt} + (1 - X_{p_2,pt} - X_{p_3,pt}) - X_{p_2,pt} \\
+ (1 - X_{p_2,pt} - X_{p_3,pt}) + X_{p_3,pt} \\
= 3 + (-X_{p_1,pt} + X_{p_1,pt} - X_{p_2,pt}) + (-X_{p_1,pt} + X_{p_2,pt}) \\
+ (-X_{p_2,pt} + X_{p_3,pt} - X_{p_3,pt}) + (X_{p_3,pt} - X_{p_3,pt} - X_{p_4,pt}) \\
+ ((-X_{p_1,pt} - X_{p_1,pt}) + (-X_{p_1,pt} - X_{p_3,pt}) - X_{p_4,pt}) \\
\geq 3 + (-1) + (-1) + (-1) + (-1 + X_{p_3,pt}) + (-1 - X_{p_3,pt} - X_{p_4,pt}) \\
\geq -3 + (X_{p_4,pt} - X_{p_5,pt} - X_{p_4,pt}) \\
\geq -4.
\]

\[ \delta_1 \delta_5 = -1, \ \delta_1 \delta_3 = -1, \ \delta_3 \delta_5 = 1: \] If this is the case, then \( \delta_1 = \delta_4 \) and different from \( \delta_2 = \delta_3 = \delta_5 \); so we consider (11); there are again four cases.

\[ \delta_2 \delta_6 = -1, \ \delta_2 \delta_3 = 1, \ \delta_3 \delta_6 = -1: \] If this is the case, then \( \delta_1 = \delta_4 = \delta_6 \) and different from \( \delta_2 = \delta_3 = \delta_5 \), therefore using (14) and (15):

\[
2 \sum_{t=2,3,4,5,6} \delta_1 \delta_t X_{p_1,pt} + \sum_{2 \leq s < t \leq 6} \delta_s \delta_t X_{p_s,pt} \\
= 2 \left[ -X_{p_1,pt} - X_{p_1,pt} + X_{p_1,pt} + (1 - X_{p_1,pt} - X_{p_3,pt}) + X_{p_1,pt} \right] + X_{p_2,pt} \\
- X_{p_2,pt} + X_{p_3,pt} + (1 - X_{p_3,pt} - X_{p_3,pt}) - X_{p_3,pt} \\
- X_{p_3,pt} + X_{p_3,pt} + (1 - X_{p_3,pt} - X_{p_3,pt}) - X_{p_3,pt} \\
+ ((-X_{p_1,pt} + X_{p_1,pt} - X_{p_2,pt}) + (-X_{p_3,pt} + X_{p_3,pt} - X_{p_3,pt}) \\
\geq 3 + (-1) + (-2) + (-1) + (-1 + X_{p_3,pt}) + (-1 - X_{p_3,pt} - X_{p_3,pt}) \\
\geq -3 + (-X_{p_2,pt} + X_{p_3,pt} - X_{p_3,pt}) \\
\geq -4.
\]

\[ \delta_2 \delta_6 = -1, \ \delta_2 \delta_3 = -1, \ \delta_3 \delta_6 = 1: \] This is impossible, since \( \delta_1 \delta_2 = -1 \) and \( \delta_1 \delta_3 = -1. \)

\[ \delta_2 \delta_6 = 1, \ \delta_2 \delta_3 = 1, \ \delta_3 \delta_6 = 1: \] If this is the case, then \( \delta_1 = \delta_4 \) and different from \( \delta_2 = \delta_3 = \delta_5 = \delta_6 \), therefore using (14) and (16):

\[
2 \sum_{t=2,3,4,5,6} \delta_1 \delta_t X_{p_1,pt} + \sum_{2 \leq s < t \leq 6} \delta_s \delta_t X_{p_s,pt} \\
= 2 \left[ -X_{p_1,pt} - X_{p_1,pt} + X_{p_1,pt} + (1 - X_{p_1,pt} - X_{p_3,pt}) - X_{p_1,pt} \right] + X_{p_2,pt} \\
- X_{p_2,pt} + X_{p_3,pt} + (1 - X_{p_3,pt} - X_{p_3,pt}) - X_{p_3,pt} \\
- X_{p_4,pt} + X_{p_3,pt} + (1 - X_{p_3,pt} - X_{p_3,pt}) - X_{p_3,pt} \\
+ ((-X_{p_1,pt} - X_{p_1,pt} - X_{p_2,pt}) + (-X_{p_1,pt} - X_{p_3,pt} + X_{p_2,pt}) \\
\geq 3 + (-1) + (-1) + (-1) + (-1) + (-1 + X_{p_1,pt}) + (-1 - X_{p_1,pt}) \\
\geq -4.
\]

\[ \delta_2 \delta_6 = 1, \ \delta_2 \delta_3 = -1, \ \delta_3 \delta_6 = -1: \] This is impossible, since \( \delta_1 \delta_2 = -1 \) and \( \delta_1 \delta_3 = -1. \)
$\delta_1 \delta_5 = 1$, $\delta_1 \delta_3 = 1$, $\delta_3 \delta_5 = 1$: $\delta_2 \delta_6 = -1$, $\delta_2 \delta_3 = 1$, $\delta_3 \delta_6 = -1$: This is impossible, since $\delta_1 \delta_2 = -1$ and $\delta_1 \delta_3 = 1$.

$\delta_2 \delta_6 = -1$, $\delta_2 \delta_3 = -1$, $\delta_3 \delta_6 = 1$: If this is the case, then $\delta_1 = \delta_3 = \delta_4 = \delta_5 = \delta_6$ and different from $\delta_2$, therefore using (14) and (16):

$$
2 \sum_{t=2,3,4,5,6} \delta_1 \delta_t X_{p_1,p_t} + \sum_{2 \leq s < t \leq 6} \delta_s \delta_t X_{p_s,p_t} =
\begin{align*}
&= 2 \left[-X_{p_1,p_2} + (1 + X_{p_1,p_5} - X_{p_3,p_5}) + X_{p_1,p_4} + X_{p_1,p_5} - X_{p_2,p_5}\right] \\
&= 4 + 2(X_{p_1,p_4} + X_{p_1,p_6}) + (X_{p_2,p_3} - X_{p_2,p_5}) + (1 + X_{p_4,p_6} - X_{p_4,p_5})
\end{align*}
\geq
\begin{align*}
&\geq -3 + (X_{p_2,p_5} - X_{p_3,p_5}) \\
&\geq -4.
\end{align*}

$\delta_2 \delta_6 = 1$, $\delta_2 \delta_3 = 1$, $\delta_3 \delta_6 = 1$: This is impossible, since $\delta_1 \delta_2 = -1$ and $\delta_1 \delta_3 = 1$.

$\delta_2 \delta_6 = 1$, $\delta_2 \delta_3 = -1$, $\delta_3 \delta_6 = -1$: If this is the case, then $\delta_1 = \delta_3 = \delta_4 = \delta_5$ and different from $\delta_2 = \delta_6$, therefore using (14) and (16):

$$
2 \sum_{t=2,3,4,5,6} \delta_1 \delta_t X_{p_1,p_t} + \sum_{2 \leq s < t \leq 6} \delta_s \delta_t X_{p_s,p_t} =
\begin{align*}
&= 2 \left[-X_{p_1,p_2} + (1 + X_{p_1,p_5} - X_{p_3,p_5}) + X_{p_1,p_4} + X_{p_1,p_5} - X_{p_2,p_5}\right] \\
&= 3 + (1 + X_{p_1,p_4} + X_{p_1,p_5} - X_{p_2,p_5}) + (1 + X_{p_4,p_6} - X_{p_4,p_5})
\end{align*}
\geq
\begin{align*}
&\geq 3 + (1 + X_{p_4,p_6}) \\
&\geq -4.
\end{align*}

$\delta_1 \delta_5 = 1$, $\delta_1 \delta_3 = -1$, $\delta_3 \delta_5 = -1$: If this is the case, then $\delta_1 = \delta_4 = \delta_5 = \delta_6$ and different from $\delta_2 = \delta_3$.

Then both (9) and (10) do not hold, so we consider (11) and (12).

$\delta_2 \delta_6 = -1$, $\delta_2 \delta_3 = 1$, $\delta_3 \delta_6 = -1$: In this case, $\delta_1 = \delta_4 = \delta_5 = \delta_6$ and different from $\delta_2 = \delta_3$, therefore using (15) and (16):

$$
2 \sum_{t=2,3,4,5,6} \delta_1 \delta_t X_{p_1,p_t} + \sum_{2 \leq s < t \leq 6} \delta_s \delta_t X_{p_s,p_t} =
\begin{align*}
&= 2 \left[-X_{p_1,p_2} + (1 + X_{p_1,p_5} + X_{p_1,p_4} + X_{p_1,p_5} + X_{p_1,p_6}) + (1 + X_{p_2,p_6} - X_{p_3,p_6})\right] \\
&= 2 + (1 + X_{p_1,p_4} + X_{p_1,p_5}) + (1 + X_{p_4,p_6} - X_{p_4,p_5})
\end{align*}
\geq
\begin{align*}
&\geq 2 + (-1) \\
&\geq -4.
\end{align*}

$\delta_2 \delta_6 = -1$, $\delta_2 \delta_3 = -1$, $\delta_3 \delta_6 = 1$: This is impossible, since $\delta_1 \delta_2 = -1$ and $\delta_1 \delta_3 = -1$. 

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\( \delta_2 \delta_6 = 1, \delta_2 \delta_3 = 1, \delta_3 \delta_6 = 1 \): In this case, \( \delta_1 = \delta_4 = \delta_5 \) and different from \( \delta_2 = \delta_3 = \delta_6 \), therefore using (15) and (16):

\[
2 \sum_{t=2,3,4,5,6} \delta_1 \delta_t X_{p_1,p_t} + \sum_{2 \leq s \leq t \leq 6} \delta_s \delta_t X_{p_s,p_t} \\
= 2 \left[ -X_{p_1,p_2} - X_{p_1,p_3} + X_{p_1,p_4} + X_{p_1,p_5} - X_{p_1,p_6} \right] + (1 + X_{p_2,p_6} - X_{p_3,p_6}) \\
- X_{p_2,p_4} - X_{p_2,p_5} + X_{p_2,p_6} - X_{p_3,p_4} - X_{p_3,p_5} + X_{p_3,p_6} \\
+ (1 + X_{p_4,p_6} - X_{p_5,p_6}) - X_{p_4,p_4} - X_{p_5,p_5} \\
= 2 + (-X_{p_1,p_2} - X_{p_1,p_3} + X_{p_1,p_4} + X_{p_1,p_5} + X_{p_1,p_6}) \\
+ (X_{p_2,p_4} - X_{p_2,p_5} - X_{p_2,p_6}) + (-X_{p_2,p_2} + X_{p_1,p_4} - X_{p_2,p_4}) \\
+ (-X_{p_1,p_3} + X_{p_1,p_5} - X_{p_3,p_5}) + (-X_{p_1,p_3} + X_{p_1,p_4} - X_{p_3,p_4}) \\
\geq 2 + (-1) + (-1) + (-1) + (-1) + (-1) \\
\geq -4.
\]

\( \delta_2 \delta_6 = 1, \delta_2 \delta_3 = -1, \delta_3 \delta_6 = -1 \): This is impossible, since \( \delta_1 \delta_2 = -1 \) and \( \delta_1 \delta_3 = -1 \).

This concludes the proof.

By arguments akin to those in the proof of Lemma 8, we can prove:

**Lemma 9** Suppose that \( X \in \mathcal{X}_n^\Delta \), and that for the set of pairs \( \{p_1, \ldots, p_6\} \) corresponding to rows and columns of \( X \), the following hold:

\[
X_{p_1,p_4} - X_{p_1,p_2} - X_{p_2,p_4} = -1, \quad X_{p_1,p_5} - X_{p_1,p_3} - X_{p_3,p_5} = -1, \\
X_{p_2,p_6} - X_{p_2,p_3} - X_{p_3,p_6} = -1, \quad X_{p_4,p_6} - X_{p_4,p_5} - X_{p_5,p_6} = -1.
\]

Then the hexagonal inequalities

\[
2 \sum_{j=1}^{6} \delta_i \delta_j X_{p_i,p_j} + \sum_{j=1}^{6} \delta_j \delta_k X_{p_j,p_k} \geq -4
\]

hold for \( i = 2, 3, 4, 5, 6 \).

The last two lemmata yield the following theorem.

**Theorem 4** Suppose that \( X \in \mathcal{X}_n^\Delta \), and that for the set of pairs \( \{p_1, \ldots, p_6\} \) corresponding to rows and columns of \( X \), the following hold:

\[
X_{p_1,p_4} - X_{p_1,p_2} - X_{p_2,p_4} = -1, \quad X_{p_1,p_5} - X_{p_1,p_3} - X_{p_3,p_5} = -1, \\
X_{p_2,p_6} - X_{p_2,p_3} - X_{p_3,p_6} = -1, \quad X_{p_4,p_6} - X_{p_4,p_5} - X_{p_5,p_6} = -1.
\]

Then for all choices of \( \delta_k \in \{-1,1\}, k = 1, 2, 3, 4, 5, 6 \), the hexagonal inequalities

\[
2 \sum_{j=1}^{6} \delta_i \delta_j X_{p_i,p_j} + \sum_{j=1}^{6} \delta_j \delta_k X_{p_j,p_k} \geq -4, \quad i = 1, \ldots, 6
\]

hold.

Hence, for our formulation above, there are \( 192 \binom{n}{6} \) valid hexagonal inequalities in total, of which \( 192 \binom{n}{4} \) automatically hold for \( X \in \mathcal{X}_n^\Delta \).
5 Computational Results

The computational properties of the $\mathcal{X}_n$ relaxation, and in particular its ability to yield tight bounds for SRFLPs, were already studied in [3, 4]. Therefore, we focus here on the effect of adding triangle inequalities to $\mathcal{X}_n$.

The $\mathcal{X}_n$ relaxation has $O(n^3)$ linear constraints, and the $\mathcal{X}_n^\triangle$ relaxation has the same number of constraints, plus $O(n^6)$ triangle inequalities. Obviously, the triangle inequalities cannot all be included simultaneously (except perhaps for very small values of $n$). The common approach to optimizing over sets such as $\mathcal{X}_n^\triangle$ is to begin by optimizing over $\mathcal{X}_n$, then to add some violated triangle inequalities, re-optimize, and repeat until no more triangle inequalities are violated.

In this paper, we used a simple procedure to implement this strategy. After solving the current SDP relaxation, we sort the triangle inequalities in terms of their violation, and choose a cut-off value for the violations that yields the 300 to 400 most-violated inequalities. More sophisticated techniques could certainly be used, and may improve the performance of the SDP-based approach. In spite of its simplicity, this approach provided excellent results. Indeed, our results suggest that using the $\mathcal{X}_n$ relaxation augmented with a few hundred inequalities, we can routinely obtain globally optimal layouts for SRFLPs with up to 25 facilities in less than 10 hours, and in several dozen hours for SRFLPs with up to 30 facilities. In particular, allowing the computation to run for up to 52 hours, we obtained globally optimal solutions for SRFLPs with up to 30 facilities, some of which have remained unsolved since 1988.

All the computational results were obtained on a 2.0GHz Dual Opteron with 16Gb of RAM, and the SDP problems were solved using the interior-point solver CSDP (version 4.9) [6] in conjunction with the ATLAS library of routines [26].

5.1 SDP-Based Extraction Heuristic

As mentioned above in Section 2, we cannot use the well-known Goemans and Williamson randomized rounding procedure [11] because it does not ensure that the equality constraints (1) hold, and hence does not guarantee that we will obtain a valid representation of a permutation.

For this reason, we again used the problem-specific rounding scheme that was used in [4]. If $X^*$ is the optimal solution to the SDP relaxation, then each row of $X^*$ corresponds to a specific pair of $i_1 j_1$ of facilities. Therefore, for each row, if we set $R_{i_1 j_1} = +1$, then we can scan the other entries of the row and assign the value $X_{i_1 j_1, i_2 j_2}$ to the variable $R_{i_2 j_2}$, for every pair $i_2 j_2 \neq i_1 j_1$.

Using these values, we compute

$$\omega_k = \frac{1}{2} \left( n + 1 + \sum_{j \neq k} R_{kj} \right)$$

for $k = 1, \ldots, n$, and hence obtain a permutation of $[n]$ by sorting these (in decreasing or increasing order, whichever satisfies $R_{i_1 j_1} = +1$). The output of the heuristic is the best layout found by considering every row in turn$^1$.

5.2 Optimal Solutions For Five Well-Known Instances

Optimizing over $\mathcal{X}_n^\triangle$ using the strategy outlined above, we were able to obtain globally optimal solutions for the following five well-known test problems from the literature. The first three problems come from [25], while the larger two problems were first considered in [16].

$^1$We thank Robert J. Vanderbei for suggesting this extension of the heuristic in [3].
5.2.1 Optimal Solutions for Eight Instances With Clearance Requirement

The next eight instances were also introduced for the first time in [16]. They differ from the previous instances in that a clearance requirement of 0.01 unit length between each pair of consecutive facilities is required. This requirement is motivated by the context of an application in flexible manufacturing systems. Since the required clearance is always the same, it is straightforward to account for this requirement in our model by appropriately adjusting the lengths of every facility. By optimizing over $X_n$ using the strategy outlined above, we succeeded in obtaining globally optimal solutions for the eight instances. Note that for the four largest instances, the global optima are strict improvements over the best layouts previously known.

<table>
<thead>
<tr>
<th>Instance</th>
<th>Instance type</th>
<th>$n$</th>
<th>Value for $X_n$ (mm:ss)</th>
<th>CPU time (hh:mm:ss)</th>
<th>Improved CPU time (hh:mm:ss)</th>
<th>Best layout by SDP-based heuristic</th>
<th>Gap</th>
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<td>8</td>
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<tr>
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<td></td>
<td>10</td>
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<td>0:00:0.3</td>
<td>2781.5</td>
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</tr>
<tr>
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</table>

5.3 Optimal and Near-Optimal Solutions for New Large Instances

Finally, to further demonstrate the performance of the SDP-based algorithm, we generated a number of new instances of the SRFLP by starting with the connectivity data from the well-known Nugent QAP Problems with 25 and 30 facilities, and adding to them randomly generated facility lengths. The Nugent problems were originally studied in [22]. The data is readily available from QAPLIB (http://www.seas.upenn.edu/qaplib) [7].

These results suggest that by using the $X_n$ relaxation augmented with a few hundred inequalities, we can routinely obtain global optima for SRFLPs with up to 25 facilities in a few hours, and in several dozen hours for SRFLPs with up to 30 facilities.

A layout of cost 44466.5 was reported in [18], which our lower bound contradicts. We suppose that it is a typo, and that the correct figure was 44966.5, but the specific permutation was not reported by the authors.
<table>
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<tr>
<th>Instance</th>
<th>n</th>
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<th>CPU time (mm:ss)</th>
<th>Improved SDP bound from $X_n\Delta$</th>
<th>CPU time (hh:mm:ss)</th>
<th>Best layout by SDP-based heuristic</th>
<th>Gap</th>
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**References**


