Improved bounds for the symmetric rendezvous value on the line

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Abstract

A notorious open problem in the field of rendezvous search is to decide the rendezvous value of the symmetric rendezvous search problem on the line, when the initial distance apart between the two players is 2. We show that the symmetric rendezvous value is within the interval $(4.1520, 4.2574)$, which considerably improves the previous best known result $(3.9546, 4.3931)$. To achieve the improved bounds, we call upon results from absorbing markov chain theory and mathematical programming theory—particularly fractional quadratic programming and semidefinite programming. Moreover, we also establish some important properties of this problem, which may be of independent interest and useful for resolving this problem completely. Finally, we conjecture that the symmetric rendezvous value is asymptotically equal to 4.25 based on our numerical calculations.

1 Introduction

Consider two players situated on an (undirected) line who know their initial distance apart at time 0, but not the direction to the other player. Assume they move at a maximum speed of one. They obviously have no common sense of direction along the line because of the undirectness of the line. So we assume that Nature (or chance) assigns each player independently a random direction to call 'forward'. A *pure* strategy for a player is simply a continuous path that describes her position relative to her starting point, in the direction she calls 'forward'. The *rendezvous search problem on the line* involves prescribing strategies for both players to meet in least expected time, called the *rendezvous value.* We say that two players *meet*, or *rendezvous occurs*, if they occupy the same location on the line.

There are two versions of the problem depending on whether the same strategy must be employed by both players or not. In the *asymmetric rendezvous search problem*, they could use distinct mixed or pure strategies. In the *symmetric rendezvous search problem*, they must choose the same mixed strategy. Note that no pure strategy can achieve a finite meeting time for the symmetric problem because of the lack of direction for both players—for example, if they both adopt the same pure strategy and initially face in the same direction, they will move in the same direction forever and never meet. Therefore mixed strategies must be employed in the symmetric problem. We consider here only the symmetric problem.

Throughout this paper, we assume that the two players' initial distance apart is 2 unless stated otherwise. Let R^s and R^a be the rendezvous values for the symmetric and asymmetric problems, respectively. Evidently $R^s \geq R^a$ because both players could always choose to adopt the same mixed strategy. Alpern and Gal [8] show that the asymmetric rendezvous value $R^a = 3.25$, and hence completely settle the asymmetric case. On the other hand, a notorious open problem, initially posed by Alpern [1] in 1995, is to decide the symmetric rendezvous value R^s. This open question is arguably *the* most important open one in rendezvous search theory [2, 9]. The symmetric rendezvous problem itself subjectively belongs to the category of Pólya: "the simplest problem one cannot solve" [17]. Besides its theoretical significance, such kind of rendezvous search problem also has practical applications in communication synchronization, operating system design, operations research, search and rescue operations planning, military, radio, and media press. For example, the military has always had protocols for rendezvous in unfamiliar territory, and the same goes for explorers. A particular example which could lead to the rendezvous search problem considered here is the situation where two parachutists landing in an unknown field want to meet as soon as possible. For detailed discussion of rendezvous search problems and their applications, please refer to the survey paper by Alpern [2] and the book by Alpern and Gal [9].

Alpern [1] introduces the symmetric rendezvous

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search problem on the line and proposes a strategy with expected meeting time of 5. The idea is to repeat the following *moving pattern* every 3 time units until rendezvous occurs: pick a random direction and move one time unit in this direction and two time units in the opposite direction, all at speed one. Note that this kind of strategy is very special in that both players adopt the same moving pattern. Anderson and Essegaier [7] improve the upper bound to 4.5678 by repeating, every 6 time units, a *mixed movement* over four moving patterns with prespecified probabilities. Anderson and Essegaier's idea is innovative because mixed movements provide the opportunity for the two players to actually follow different moving patterns. This philosophy will be pushed further in this work to derive our bounds—both upper and lower bounds. Baston [11] further improves the upper bound to 4.4182 by repeating, every 7 time units, a mixed movement over four patterns. This improvement is a result of the new observation that accumulated information before rendezvous should be utilized in Anderson and Essegaier's strategy. Uthaisombut[1] [19] recently presented a new mixed strategy with the previous best known upper bound of 4.3931. This strategy goes further to allow the mixture of patterns with different time units, namely 6 and 7.

Note that all the previous strategies share the same spirit; namely that the same movement or mixed movement is repeated over and over again until rendezvous occurs. This kind of strategy naturally induces a Markov chain if we define its state space as half of the distance apart after each repetition. Our upper bounds will be also achieved by such Markovian type strategies, but in a more systematic way.

In terms of lower bounds, there are fewer results. Besides the obvious lower bound R^a, the previous best known lower bound of 3.9546 is recently given also by Uthaisombut [19]. The lower bound scheme of Uthaisombut is based on the simple observation that if two players are a distance d apart, then the expected extra time to meet is at least $d/2$.

Previous existing lower and upper bounds on R^s are summarized in Table 1 below.

[1]This recent work was brought to our attention after the current work was almost done.

	Upper bound	Lower bound
Alpern [1]	5.0000	-
Alpern and Gal [8]	-	3.25
Anderson and Essegaier [7]	4.5678	-
Baston [11]	4.4182	-
Uthaisombut [19]	4.3931	3.9546

Table 1: Previous upper and lower bounds for R^s

In the following, we summarize the main contributions of this work and also briefly explain the main ideas and steps that lead to them.

1. We show, in Theorem 2.1, that strategies that always move at maximum speed one and switch direction only at integer times (called grid strategies in this paper) dominate among all possible strategies. To the best knowledge of the authors, this seems to be unknown in the literature—but see a similar result by Lim et al. [15] in a different context. This greatly reduces the searching strategy space of the problem.

2. To obtain better upper bounds, we investigate the aforementioned Markovian type strategies; that is, the same mixed moving patterns are repeated every fixed time units before rendezvous occurs. For Markovian type strategies to perform well, there are two opposite driving forces involved. On the one hand, adding more moving patterns in every repetition should reduce the symmetry of the problem (or equivalently increase the asymmetry, namely, increase the chance for two players to adopt different pure strategies), and hence increase the meeting opportunity. This necessitates increasing the length of the moving patterns. On the other hand, with longer length, the expected time to meet also increases, mainly because there is at least 1/2 probability for the two players to move in the same direction whenever they adopt the same moving pattern and hence will never meet within one repetition. Therefore, the main challenge is how to balance these two forces. We manage to quantify this balance; that is, the expected time of any strategy can be expressed as the product of two inversely related expectations: one local and one global. See Section 3.1 for more details.

   (a) First, in Theorem 2.2, we show that *distance-preserving* Markovian type strategies dominate, which implies the aforementioned balancing relationship and further reduces the searching space. The essential idea behind this result is that when using a non-distance-preserving Markovian type strategy, once the distance of the two players increase to certain point it will follow a symmetric random walk and therefore the expected time to reduce such distance is infinite—a well-know result in random walk theory.

(b) Next, we reduce the performance analysis of Markovian type strategies to solving a fractional quadratic program whose value for any *feasible* solution provides an upper bound for R^s. This allows us to utilize the Mathematical Programming (MP) toolkit, particularly Quadratic Programming (QP) and Semidefinite Programming (SDP).

3. To obtain better lower bounds, we consider a related problem first introduced by Alpern and Gal [10], the *symmetric rendezvous search on a directed line*, whose only difference with the original problem is that both players are given the direction of the line and hence the rendezvous value R_U for the new problem provides a lower bound for the original one because of the new information given. In the new problem, we employ the observation that if two players have been following the same strategy (and hence in the same direction for certain time), then no information has been acquired and both players are facing exactly the same situation as started initially. Therefore the expected extra time to meet is still R_U. The new insight leads to solving a different fractional quadratic program whose *optimal* value provides a lower bound for $R^s \geq R_U$. See Section 3.2 for more details.

4. Based on the ideas above, we show that the symmetric rendezvous value is within the interval (4.1520, 4.2574).

5. Finally, based on our numerical calculations, we conjecture that the symmetric rendezvous value is equal to 4.25, which can only be achieved *asymptotically* by the procedure n-Markovian, introduced in Section 2.2. This value is only asymptotically achievable because our calculation indicates that increasing the length of the moving patterns is the dominating force among the two discussed in (2). See Section 4 for more details.

The rest of the paper is organized as follows. We first prove the two domination results in Sections 2.1 and 2.2. We then present the improved upper and lower bounds in Sections 3.1 and 3.2, respectively. Finally we give some concluding remarks in Section 4.

2 Strategies

In the symmetric rendezvous search problem on the (undirected) line, two players, I and II, are situated initially at a known distance 2 apart on a line and wish to meet in least expected time, given that they both move at a maximum speed of 1. They are told the initial distance apart, but neither the direction to the other player nor the direction along the line. So we assume that Nature (or chance) assigns each player independently a random direction to call 'forward'. A *pure* strategy for a player is simply a *Lipshitz continuous* path with maximum speed one that describes her position relative to her starting point, in the direction she calls 'forward'. Therefore the set of all pure strategies is given by

$$\mathcal{S} \;=\; \{s:\mathbb{R}_+\to\mathbb{R},\ s(0)=0,\ |s(t_1)-s(t_2)|\le|t_1-t_2|,\ \forall t_1,t_2\in\mathbb{R}_+\}.$$

A *mixed* strategy x over the pure strategy space $\mathcal{S}$ is a Borel probability measure over $\mathcal{S}$. The set of all mixed strategies over $\mathcal{S}$ will be denoted as $\mathcal{S}^*$.

Given two pure strategies $s_1, s_2 \in \mathcal{S}$, there are four equally likely meeting times, depending on the four situations in which Nature initially faces the players and the relative positions of the players, namely: towards or away from each other, same direction with player I or II in front. Let $T(s_1, s_2)$ denote the meting time of the players whenever they adopt pure strategies s_1 and s_2, respectively. Consequently, $T(s_1, s_2)$ is a (discrete) random variable whose probability distribution can be identified by considering all four equally likely possibilities above.

For computational purposes, assume that player I is placed at point 0 and player II at point 2 at time 0. Then:

1. Player I follows path $0+s_1(t)$ and Player II follows path $2+s_2(t)$. The probability is 1/4 with meeting time
$$t^{++}(s_1,s_2)=\min\{t: s_1(t)=2+s_2(t)\}.$$

2. Player I follows path $0-s_1(t)$ and Player II moves toward $2-s_2(t)$. The probability is 1/4 with meeting time
$$t^{--}(s_1,s_2)=\min\{t: 0-s_1(t)=2-s_2(t)\}.$$

3. Player I follows path $0+s_1(t)$ and Player II follows path $2-s_2(t)$. The probability is 1/4 with meeting time
$$t^{+-}(s_1,s_2)=\min\{t: 0+s_1(t)=2-s_2(t)\}.$$

4. Player I follows path $0 - s_1(t)$ and Player II follows path $2 + s_2(t)$. The probability is 1/4 with meeting time

$$t^{-+}(s_1, s_2) = \min\{t : 0 - s_1(t) = 2 + s_2(t)\}.$$

So the expected meeting time $t(s_1, s_2)$ is given by

$$t(s_1, s_2) = \mathbb{E}[T(s_1, s_2)] = \frac{1}{4} \sum_{\delta_1, \delta_2} t^{\delta_1 \delta_2}(s_1, s_2),$$

where $\delta_1, \delta_2 \in \{+, -\}$. Similarly for any two mixed strategies $x_1, x_2 \in \mathcal{S}^*$, we define their expected meeting time, or *rendezvous time*, by

$$t(x_1, x_2) = \mathbb{E}_{x_1, x_2}[t(s_1, s_2)].$$

The asymmetric and symmetric rendezvous values R^a and R^s are defined respectively as

$$\begin{aligned} R^a &= \min_{x_1, x_2} t(x_1, x_2); \\ R^s &= \min_{x} t(x, x). \end{aligned} \tag{2.1}$$

The symmetric rendezvous problem therefore is to find an optimal strategy for both players with minimum R^s, that is, to solve the optimization problem (2.1).

As a convention throughout this paper, any set $\star$ with an asterisk $(^*)$ attached—that is $\star^*$—is the set of mixed strategies over the given set. For example, $\mathcal{S}^*$ is the set of mixed strategies over $\mathcal{S}$.

2.1 Grid Strategies A strategy $s \in \mathcal{S}$ is *grid* if the speed at every instant is one and direction can only be switched at integer times. Denote $\mathcal{S}_G$ to be the set of all grid strategies, and hence $\mathcal{S}_G^*$ is the set of all mixed strategies over $\mathcal{S}_G$ according to our convention. We want to prove that we can reduce the problem to searching only mixed grid strategies $\mathcal{S}_G^*$. This result follows evidently from the result below.

THEOREM 2.1. *There is* $\widehat{\cdot} : \mathcal{S} \to \mathcal{S}_G$ *such that for every pair of pure strategies* $s_1, s_2 \in \mathcal{S}$

$$t(s_1, s_2) \geq t(\widehat{s}_1, \widehat{s}_2).$$

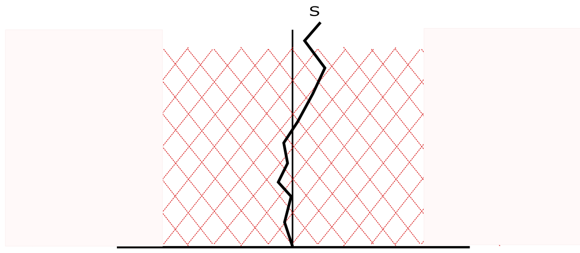

Figure 1a: Proof of Theorem 2.1

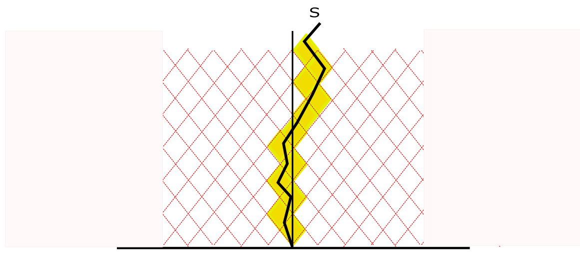

Figure 1b: Proof of Theorem 2.1

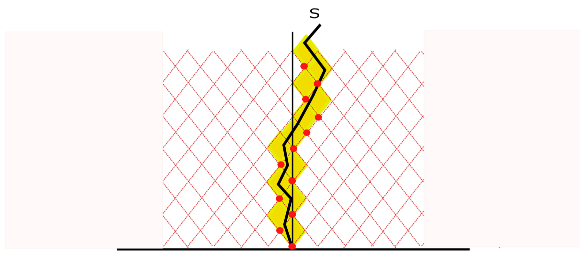

Figure 1c: Proof of Theorem 2.1

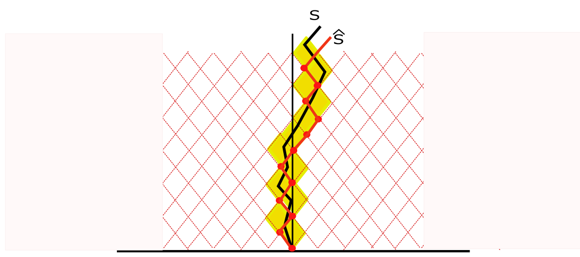

Figure 1d: Proof of Theorem 2.1

Proof. First we define $\widehat{\cdot} : \mathcal{S} \to \mathcal{S}_G$. Fix $s \in \mathcal{S}$. Consider the curve $C(t) = (s(t), t) \in \mathbb{R} \times \mathbb{R}_+$ and the "diagonal" grid with side length $\sqrt{2}$ (Figure 1a). Shadow every square that C crosses. In case the curve C goes through a corner from one square to a diagonally adjacent one above, shadow also the left one of the two squares adjacent to both squares (Figure 1b). Mark the lowest corner of each shadowed square (Figure 1c). Let $\widehat{C}$ be the piecewise linear curve obtained by joining adjacent marked corners and let $\widehat{s}$ such that $\widehat{C}(t) = (\widehat{s}(t), t)$ (Figure 1d).

Now given $s_1, s_2 \in S$ we need to show $t(s_1, s_2) \geq t(\widehat{s}_1, \widehat{s}_2)$. It suffices to show that $t^{\delta_1 \delta_2}(s_1, s_2) \geq t^{\delta_1 \delta_2}(\widehat{s}_1, \widehat{s}_2)$ for every $\delta_1, \delta_2 \in \{+, -\}$. If $t^{\delta_1 \delta_2}(s_1, s_2) = \infty$ there is nothing to prove. So let $t_0 = t^{\delta_1 \delta_2}(s_1, s_2)$,

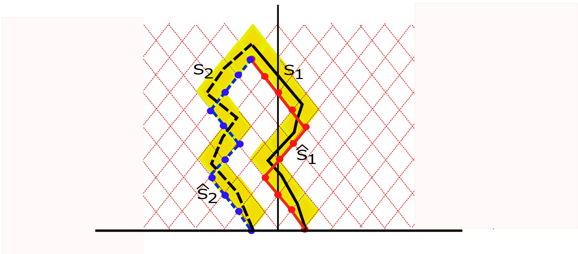

Figure 1e: Proof of Theorem 2.1

and let $x_0 := \delta_1 s_1(t_0) = \delta_2 s_2(t_0) + 2$. The paths $\delta_1 s_1$ and $\delta_2 s_2 + 2$ meet at the point (x_0, t_0). Let S_0 be the grid square containing (x_0, t_0) and let (x_1, t_1) be the lowest corner of S. By construction the two curves $(\delta_1 \widehat{s}_1, t) = (\widehat{\delta_1 s_1}, t)$ and $(\delta_2 \widehat{s}_2 + 2, t) = (\widehat{\delta_2 s_2 + 2}, t)$ meet at (x_1, t_1). Thus $\delta_1 \widehat{s}_1(t_1) = \delta_2 \widehat{s}_2(t_1) + 2$ and therefore $t(\delta_1 \widehat{s}_1, \delta_2 \widehat{s}_2) \le t_1 \le t$ (Figure 1e).

2.2 Distance-preserving strategies From now on we will only consider the grid strategy space $\mathcal{S}_G$. We introduce the notion of *string* over the alphabet set $\{-1, 1\}$ such that the first element of any string is 1, referring to a 'forward' movement of one time unit chosen by the Nature initially; and each other element 1 corresponds to the same 'forward' movement of one time unit and each -1 corresponds to the 'backward' movement of one time unit, all at speed one.

Obviously any grid strategy can be equivalently viewed as such a string of infinite length. To facilitate our analysis, we call a string of finite length n, a positive integer, an *n-generator*. For example, the 3-generator $g = \{1, -1, -1\}$ means moving 1 time unit in the 'forward' direction and 2 time units in the other, all at speed one. The set of all n-generators will be denoted as $\mathcal{G}_n$. Evidently the cardinality of $\mathcal{G}_n$ is 2^{n-1}. A mixed n-generator $x \in \mathcal{G}_n^*$ is a Borel probability measure x over the set $\mathcal{G}_n$.

Denote $\mathcal{I}_k = \{1, 2, \cdots, k\}$ as an index set for any given positive integer k. Any vector will be a column vector. A vector $x = (x_i)_{i \in \mathcal{I}_k}$ is a *probability* vector if it satisfies $x_i \ge 0$, $\forall i \in \mathcal{I}_k$ and $e^T x = 1$, where e is the vector of all ones.

Fix a positive integer n. Given a mixed n-generator $x \in \mathcal{G}_n^*$, we prescribe the following (mixed) strategy followed by both players, which will be used to derive our upper bound:

n-Markovian: Repeat the mixed n-generator x every n time units until rendezvous occurs.

Two n-generators $g_1, g_2 \in \mathcal{G}_n$ are *distance-preserving*, denoted as $g_1 \sim g_2$, if their distance apart either becomes zero or remains unchanged after one player follows g_1 and another player follows g_2 for n time units. A mixed n-generator $x \in \mathcal{S}_G^*$ is *distance-preserving* if $x_i x_j = 0$ whenever $g_i \nsim g_j$ for all $i, j \in \mathcal{I}_{2^{n-1}}$.

We shall show in this section that distance-preserving mixed generators dominate among all mixed strategies in $\mathcal{G}_n^*$ for strategy n-Markovian. Actually we shall prove a stronger result that no non-distance-preserving mixed n-generator used in n-Markovian can guarantee finite expected meeting time.

Theorem 2.2. *For any given positive integer n, the expected meeting time of strategy n-Markovian is infinite whenever the mixed n-generator used in strategy n-Markovian is non-distance-preserving.*

Proof. Let d_t be the distance of the two players after t repetitions (of n time units each) of the n-Markovian strategy. Let $x \in \mathcal{G}_n^*$ be a non-distance preserving mixed n-generator. Let $p = \Pr_{g_1, g_2 \in \mathcal{G}_n}[g_1 \nsim g_2]$, i.e. p is the probability that the distance will actually increase when starting at distance 2 and using x as the mixed generator for the n-Markovian strategy, and by assumption $p > 0$. Given $g_1, g_2 \in \mathcal{G}_n$ and $\delta_1, \delta_2 \in \{+, -\}$ assume that $d_t > 2n$ and that at time t the two players follow $g_1^{\delta_1}$ and $g_2^{\delta_2}$, respectively. As $d_t > 2n$ the two paths cannot cross and therefore

$$d_{t+1} = d_t + \delta_1 s(g_1) - \delta_2 s(g_2),$$

where for every $g \in \mathcal{G}_n$ we define $s(g) = \sum_{i=1}^n g_i$. For $i = -n, \ldots, n$, let $p_i = \Pr[d_{t+1} - d_t = 2i | d_t > 2n]$. We have $\sum_{i=-n}^n p_i = 1$ and by the symmetry of the strategy $p_{-i} = p_i$.

Now let $X_0 = 0$ and let X_t be defined for every $t = 0, 1, 2, \ldots$ by

$$X_{t+1} = X_t + i \text{ with probability } p_i,\ i = -n, \ldots, n.$$

Let T be the first t when $X_t < 0$. Conditional in $d_n > 2n$ we can couple the construction of X_t and the process n-Markovian such that $d_{t+n} > 2X_t + 2n$ for all $t = 0, 1, \ldots, T$. As $\Pr[d_n > 2n] \ge p^n$ and to get $d_t = 0$, we must first get $d_t \le 2n$ implying $X_t < 0$. We have then the expected meeting time for n-Markovian is at least $p^n(n + E[T])$. But X_t is a symmetric random walk starting at 0 and therefore $E[T] = \infty$, a well-known result of the random walk theory [13].

3 Bounding the symmetric rendezvous value R^s

3.1 Upper bound We will analyze the performance of the mixed strategy n-Markovian. Because of Theorem 2.2, we only need to focus on distance-preserving mixed generators in n-Markovian.

Let $x \in \mathcal{G}_n^*$ be a distance-preserving mixed n-generator used in n-MARKOVIAN such that $\mathcal{G}_n = \{g_1, \cdots, g_{2^{n-1}}\}$ and $x = (x_1, \cdots, x_{2^{n-1}})$.

REMARK 3.1. *We note that computational efficacy can be achieved if we reduce the cardinality of $\mathcal{G}_n$ by excluding any generator that is non-distance-preserving with itself. For example we can exclude the generator $\{1, 1, \cdots, 1\}$ because it does not preserve distance with itself. This proves to be extremely beneficial in calculating upper bounds for R^s numerically. This is implemented in our numerical calculation. However, this reduction is equivalent to setting zeros to the probabilities of those excluded generators when computational efficacy is not an issue. Therefore we will assume the cardinality of $\mathcal{G}_n$ is still 2^{n-1} for simplicity of presentation.*

Consider two n-generator $g_i, g_j \in \mathcal{G}_n$ such that one player follows g_i and another follows g_j for n time units.

- Let p_{ij} be the probability that their distance apart at time n remains unchanged, given that they have not met by then.
- Let T_{ij} be their meeting time. Let $T_{ij}^{(n)} = \min\{T_{ij}, n\}$, whose probability distribution follows from that of T_{ij}. Let $m_{ij} = \mathbb{E}[T_{ij}^{(n)}]$.

Denote matrix $P_n = (p_{ij}) \in \mathbb{R}^{2^{n-1} \times 2^{n-1}}$ and matrix $M_n = (m_{ij}) \in \mathbb{R}^{2^{n-1} \times 2^{n-1}}$. Let $T(x)$ be the expected meeting time of the mixed strategy n-MARKOVIAN, using x as the mixed n-generator. Then it is easy to verify the following:

$$T(x) = x^T M_n x + T(x) x^T P_n x,$$

where $x_i x_j = 0$ whenever $g_i \not\sim g_j$, $, i, j \in \mathcal{I}_{2^{n-1}}$. Or equivalently,

$$T(x) = \frac{x^T M_n x}{1 - x^T P_n x}. \tag{3.2}$$

REMARK 3.2. *The n-MARKOVIAN strategy can also be interpreted via absorbing Markovian chain theory. For mixed strategy n-MARKOVIAN, we associate an absorbing Markov chain with state space $\mathcal{M} = \{0, 1, 2, \cdots\}$ and transition probability matrix $\mathbb{Q} = (q_{ij})$. Each state $i \in \mathcal{M}$ corresponds to half of the distance apart after every n time units. Therefore each q_{ij} gives the conditional probability that half of the distance apart switches from i to j, for $i, j \in \mathcal{M}$. State 0 is absorbing, and the rest is transient. Theorem 2.2 basically says that the expected absorbing time starting from state 1 is infinite whenever a non-distancing-preserving mixed generator is used in n-MARKOVIAN. Therefore the formula (3.2) can also be derived based on results from absorbing Markovian chain theory where the state space consists of only states 0 and 1 when restricted to distance-preserving strategies. Note that the numerator above can be viewed as the expected "effective" meeting time, the time they spend within each state of the aforementioned Markov chain (the local factor), and the denominator is the probability that the two players will meet after time n, whose reciprocal is the expected absorbing time of the Markov chain (the global factor). Therefore the last equation is actually equal to the product of two inversely related expectations: one local and one global. This quantifies the balancing idea mentioned in the introduction.*

So the objective value of any feasible solution to the following fractional quadratic programming problem provides an upper bound for the symmetric rendezvous value R^s:

$$(3.3)\, R_n^s = \min \left\{ \frac{x^T M_n x}{1 - x^T P_n x} : x_i x_j = 0, \right. \\ \left. \forall g_i \not\sim g_j, i, j \in \mathcal{I}_{2^{n-1}}, e^T x = 1, x \geq 0 \right\}.$$

For any fixed n, the tightest upper bound is the optimal value R_n^s. However, finding R_n^s for this particular fractional quadratic program is hard when n is large. We obtain the following upper bounds of R^s for $n \leq 15$.

n	R_n^s	Achieved No. of Strategies
1	∞*	1
2	7.0000*	1
3	5.0000*	1
4	4.9441*	2
5	4.8827*	4
6	4.4634*	5
7	4.3490*	5
8	4.3209*	8
9	4.3044*	11
10	4.2866	18
11	4.2739	29
12	4.2678	38
13	4.2630	58
14	4.2595	89
15	4.2574	128
⋮	⋮	⋮
∞	**4.25?**	**∞?**

Table 2: Upper bounds for R^s

REMARK 3.3. *The Matlab code for solving (3.3) can be downloaded from our website,"http://www.unb.ca/~ddu". However we* *note that the codes therein do not directly solve (3.3). To be more computationally efficient, we reduce the search space using the idea introduced in Remark 3.1. Moreover, to obtain the bounds listed in Table 2 using nonlinear optimization solvers, one may need to choose an appropriate initial feasible solution to start with (a notorious feature of nonlinear programming). More details on the implementation are included in the self-contained explanation within the Matlab code files on our website. Finally, we only retain four digits after the decimal point for the upper bounds in Table 2. More digits can be obtained by running the Matlab code under long format environment (up to* 10^{-16} *with double precision).*

Some of the bounds given in Table 2 are actually best possible (indicated with an *) for fixed n up to computation precision. Note that, although the entries in matrices M_n and P_n are calculated exactly, we can only prove a solution is best possible up to computation precision.

FACT 3.1. *The* R^s_n *listed in Table 2 is the optimal value of the fractional quadratic program (3.3) (hence the best rendezvous value) for any fixed integer* $n \le 9$.

We prove this fact by considering each fixed $n \le 9$. It suffices to show that each value is indeed the optimal one for the fractional quadratic program (3.3). We need the following result that relates fractional quadratic program to regular quadratic program. For any $r > 0$, define

$$(3.4)\ f(r) = \min\left\{x^T\left(M_n + rP_n\right)x : x_i x_j = 0,\ \forall g_i \nsim g_j, i,j \in \mathcal{I}_{2^{n-1}}, e^T x = 1, x \ge 0\right\}.$$

LEMMA 3.1. *If* $f(r^*) = r^*$ *in (3.4), then* $R^s_n = r^*$ *in (3.3).*

So solving (3.3) reduces to searching for a fixed point of f, which involves solving a series of regular quadratic programs (3.4). Unfortunately, (3.4) in general is non-convex. To show that the solution obtained for (3.4) is indeed global, we also solve its SDP relaxation with SDP solvers SEDUMI [18] and DSDP5 [12].

$$\min\left\{(M_n + rP_n) \bullet X : X \bullet ee^T = 1, X_{ij} = 0,\ \forall g_i \nsim g_j, X_{ij} \ge 0, i,j \in \mathcal{I}_{2^{n-1}}, X \succeq 0\right\},$$

where $X \in \mathbb{R}^{2^{n-1}\times 2^{n-1}}$ and the operation $\bullet$ is the inner product of matrices by taking matrix as vectors.

If there exists a feasible solution to (3.4) that achieves the optimal value of the SDP relaxation, then it must be a global optimal solution of (3.4). In the following, we adopt this idea to (numerically up to precision 10^{-5}) prove the fact above except for the simple cases of $n = 1$, 2 and 3, whose optimality can be showed easily by first-hand analysis.

- $R^s_1 = \infty$: In this case there is just one generator $g = \{1\}$, which is non-distance-preserving and hence $R^s_1 = \infty$ based on Theorem 2.2.
- $R^s_2 = 7$: In this case, there are two generators: $g_1 = \{1, -1\}$, $g_2 = \{1, 1\}$. But g_1 is the only distance-preserving generator with $R^s_2 = 7$. This can be easily verified by substituting $M_2 = 1(1/4) + 2(3/4) = 7/4$ and $P_2 = 3/4$ into (3.3).
- $R^s_3 = 5$: In this case, there are four generators: $g_1 = \{1, 1, 1\}, g_2 = \{1, 1, -1\}, g_3 = \{1, -1, 1\}, g_4 = \{1, -1, -1\}$. But g_4 is the only distance-preserving generator with $R^s_3 = 5$. This can be easily verified by substituting $M_3 = 1(1/4) + 3(1/4) + 3(1/2) = 5/2$ and $P_3 = 1/2$ into (3.3). This is Alpern's strategy [1].

In the following, we only retain four digits after the decimal point for each value. Again, more digits can be obtained by running the Matlab code under long format environment as explained in Remark 3.3.

- $R^s_4 \approx 4.9441$: The optimal mixed 4-generator is given below.

  $g_1 = \{1,1,-1,-1\}$, $x_1 \approx 0.1118$
  $g_2 = \{1,-1,-1,1\}$, $x_2 \approx 0.8882$

- $R^s_5 \approx 4.8827$: The optimal 5-mixed generator is given below.

  $g_1 = \{1,-1,-1,-1, 1\}$, $x_1 \approx 0.3227$
  $g_2 = \{1,-1,-1, 1,-1\}$, $x_2 \approx 0.2500$
  $g_3 = \{1,-1, 1,-1,-1\}$, $x_3 \approx 0.1773$
  $g_4 = \{1, 1,-1,-1,-1\}$, $x_4 \approx 0.2500$

- $R^s_6 \approx 4.4634$: The optimal 6-mixed generator is given below.

  $g_1 = \{1,-1,-1,-1,1,1\}$, $x_1 \approx 0.2482$
  $g_2 = \{1,-1,-1,1,-1,1\}$, $x_2 \approx 0.1561$
  $g_3 = \{1,-1,-1,1,1,-1\}$, $x_3 \approx 0.1561$
  $g_4 = \{1,-1,1,-1,-1,-1\}$, $x_4 \approx 0.1867$
  $g_5 = \{1,1,-1,-1,-1,-1\}$, ,$x_5 \approx 0.2529$

  This result is a little bit better than the upper bound 4.5678 of Anderson and Essegaier [7], whose mixed 6-generator is given below.

  $g_1 = \{1,-1,-1,-1,1,1\}$, $x_1 \approx 0.4040$
  $g_2 = \{1,-1,-1,1,-1,-1\}$, $x_2 \approx 0.2120$
  $g_3 = \{1,-1,1,-1,-1,-1\}$, $x_3 \approx 0.1561$
  $g_4 = \{1,1,-1,-1,-1,-1\}$, $x_4 \approx 0.2279$

- $R^s_7 \approx 4.3489$: The optimal mixed 7-generator is given below.

$$\begin{array}{ll} g_1 = \{1,-1,-1,-1,1,1,1\}, & x_1 \approx 0.2617 \\ g_2 = \{1,-1,-1,1,-1,1,-1\}, & x_2 \approx 0.1544 \\ g_3 = \{1,-1,-1,1,1,-1,-1\}, & x_3 \approx 0.1544 \\ g_4 = \{1,-1,1,-1,-1,-1,1\}, & x_4 \approx 0.1940 \\ g_5 = \{1,1,-1,-1,-1,-1,-1\}, & x_5 \approx 0.2355 \end{array}$$

  This is better than the upper bound 4.4182 of Baston [11], whose mixed 7-generator is given below.

$$\begin{array}{ll} g_1 = \{1,-1,-1,-1,1,1,1\}, & x_1 \approx 0.2865 \\ g_2 = \{1,-1,-1,1,1,-1,-1\}, & x_2 \approx 0.2471 \\ g_3 = \{1,-1,1,-1,-1,-1,1\}, & x_3 \approx 0.2139 \\ g_4 = \{1,1,-1,-1,-1,-1,-1\}, & x_4 \approx 0.2524 \end{array}$$

  This is also better than the previous best upper bound 4.3931 of Uthaisombut[19], who mixes one 6-generator and four 7-generator's.

$$\begin{array}{l} g_1 = \{1,1,-1,-1,-1,-1,-1\}, \\ g_2 = \{1,-1,1,-1,-1,-1,1\}, \\ g_3 = \{1,-1,-1,1,-1,1,-1\}, \\ g_4 = \{1,-1,-1,1,-1,-1,1\}, \\ g_5 = \{1,-1,-1,-1,1,1\}. \end{array}$$

- $R^s_8 \approx 4.3208$: The optimal mixed 8-generator is given below.

$$\begin{array}{ll} g_1 = \{1,-1,-1,-1,1,1,1,1\}, & x_1 \approx 0.2038 \\ g_2 = \{1,-1,-1,1,-1,1,-1,1\}, & x_2 \approx 0.0881 \\ g_3 = \{1,-1,-1,1,-1,1,1,-1\}, & x_3 \approx 0.0977 \\ g_4 = \{1,-1,-1,1,1,-1,-1,-1\}, & x_4 \approx 0.1577 \\ g_5 = \{1,-1,1,-1,-1,-1,1,1\}, & x_5 \approx 0.1220 \\ g_6 = \{1,-1,1,-1,-1,1,-1,-1\}, & x_6 \approx 0.0945 \\ g_7 = \{1,1,-1,-1,-1,-1,-1,1\}, & x_7 \approx 0.1296 \\ g_8 = \{1,1,-1,-1,-1,-1,1,-1\}, & x_8 \approx 0.1066 \end{array}$$

- $R^s_9 \approx 4.3044$: The optimal mixed 9-generator is given below.

$$\begin{array}{ll} g_1 = \{1,-1,-1,-1, 1, 1,-1, 1, 1\}, & x_1 \approx 0.1150 \\ g_2 = \{1,-1,-1,-1, 1, 1, 1,-1,-1\}, & x_2 \approx 0.1485 \\ g_3 = \{1,-1,-1, 1,-1,-1, 1, 1, 1\}, & x_3 \approx 0.0831 \\ g_4 = \{1,-1,-1, 1,-1, 1,-1,-1, 1\}, & x_4 \approx 0.0613 \\ g_5 = \{1,-1,-1, 1,-1, 1,-1, 1,-1\}, & x_5 \approx 0.0577 \\ g_6 = \{1,-1,-1, 1, 1,-1,-1,-1,-1\}, & x_6 \approx 0.1160 \\ g_7 = \{1,-1, 1,-1,-1,-1, 1,-1, 1\}, & x_7 \approx 0.0507 \\ g_8 = \{1,-1, 1,-1,-1,-1, 1, 1,-1\}, & x_8 \approx 0.0672 \\ g_9 = \{1,-1, 1,-1,-1, 1,-1,-1,-1\}, & x_9 \approx 0.0634 \\ g_{10} = \{1, 1,-1,-1,-1,-1,-1, 1, 1\}, & x_{10} \approx 0.1304 \\ g_{11} = \{1, 1,-1,-1,-1,-1, 1,-1,-1\}, & x_{11} \approx 0.1067 \end{array}$$

All other cases up to $n = 15$ can be downloaded from our website "http://www.unb.ca/~ddu".

3.2 Lower bound We give a general framework for generating lower bounds. We need a new problem called the *symmetric rendezvous problem on a directed line*, first introduced by Alpern and Gal [10]. The difference between this problem and the original *symmetric rendezvous problem on a (undirected) line* is that in the new problem the players have a common sense of direction. Along a similar proof line to Theorem 2.1, we can show that grid strategies still dominates in the new problem. So we will focus only on grid strategies. But the number of grid strategies is doubled compared to the original problem and it consists of all grid strategies, denoted as $\mathcal{S}_U$, generated by strings over the alphabet set $\{1, -1\}$, not just those starting with 1. We denote R_U to be the symmetric rendezvous value of the new problem. Evidently, $R^s \geq R_U$ because more information is available (common direction) in the new problem. So any lower bound on R_U will be a valid lower bound for R^s also.

Let $x \in \mathcal{S}^*_U$ be a given mixed strategy over $\mathcal{S}_U$. Fix a positive integer n. Assume that $\mathcal{G}_n = \{g_1, g_2, \cdots, g_{2^n}\}$ is the set of n-generator's. For any n-generator $g_i \in \mathcal{G}_n$ ($i \in \mathcal{I}_{2^n}$), let $\mathcal{G}_n(g_i) \subset \mathcal{S}_G$ be the subset of strategies of $\mathcal{S}_G$ that have the same first n elements as g_i. Define further $\widehat{x}_i = \int_{\mathcal{G}_n(g_i)} dx$. Then $\widehat{x} = (\widehat{x}_1, \cdots, \widehat{x}_{2^n})^T$ is a probability vector. So $\widehat{x}(\mathcal{G}_n)$ is a mixed n-generator over $\mathcal{G}_n$.

Let $H(x)$ be the expected meeting time of the mixed strategy x when the initial distance is 2. The following scheme is an improvement over that of Uthaisombut [19]. Below we adopt some notations from [19]. For any given positive integer n, we construct the following pseudo-strategy with expected meeting time at least as good as, $H(x)$, that of x when the initial distance apart is 2.

Pseudo-strategy$_n$: Follow $\widehat{x}$ first, and, if rendezvous has not occurred yet, then

> **Case 1.** the two players move toward each other, if they have not been following the same basic grid strategy, or they have been following the same basic grid strategy and in the opposite direction up to n;
>
> **Case 2.** the two players adopt the optimal mixed strategy for the symmetric rendezvous search problem, if they have been following the same grid strategy and in the same direction up to n.

We need two simple facts before we can show that the expected meeting time of the pseudo-strategy is no more than that of x.

FACT 3.2. *In* PSEUDO-STRATEGY$_n$*, if two players are*

d distance apart after following the mixed strategy $\widehat{x}$ for n time units, then

1. *the minimum expected extra time for them to meet is at least $d/2$;*
2. *moreover, if they have been following the same grid strategy (and hence in the same direction up to n implying $d = 2$), then the minimum expected extra time for them to meet is R_U.*

Proof. The first claim is obvious [19]. We only prove the second. Suppose the two players have been following the same strategy and hence in the same direction up to n. Therefore they arrive at exactly the same situation as started initially. So the minimum expected extra time for them to meet is R_U.

We introduce more notations.

- If $i \neq j$, then let d_{ij} be the distance apart when one player follows $\widehat{s}_i$ and the another follows $\widehat{s}_j$ for n time units, assuming initial distance 2. If $i = j$, then let d_{ii} be the distance apart when both players follow $\widehat{s}_i$ in the opposite direction for n time units, assuming initial distance 2. Obviously d_{ij} is a discrete random variable whose support set is $\mathcal{D} = \{0, 2, \cdots, 2(n+1)\}$ with corresponding probabilities $p_{ij}^{(0)}, p_{ij}^{(2)}, \cdots, p_{ij}^{(2(n+1))}$.
- Let U_{ij} be their meeting time when one player follows $\widehat{s}_i$ and the another follows $\widehat{s}_j$ for n time units, assuming initial distance 2. Since both players have a common direction, Nature's only choice is to place either I or II in front. So U_{ij} is a discrete random variable with only two equi-probably supports. Let $U_{ij}^{(n)} = \min\{U_{ij}, n\}$, whose probability distribution follows from that of U_{ij}. Let $u_{ij} = \mathbb{E}[U_{ij}^{(n)}]$.

Denote the expected distance matrix $D_n = (\mathbb{E}[d_{ij}]) \in \mathbb{R}^{2^n \times 2^n}$. Denote matrix $M_n^U = (u_{ij}) \in \mathbb{R}^{2^n \times 2^n}$. Let $I_n \in \mathbb{R}^{2^n \times 2^n}$ be the identity matrix.

Lemma 3.2. *The expected meeting time of the* Pseudo-strategy$_n$ *is no more than $H(x)$, that of $x \in \mathcal{S}_G^*$, assuming initial distance 2.*

Proof. Note that we have the following formula for H:

$$\begin{aligned}
H(x) &\geq (\widehat{x})^T M_n^U \widehat{x} + \sum_{i \in \mathcal{I}_{2^n}} (R_U)(\widehat{x}_i)^2 \\
&\quad + \sum_{i,j \in \mathcal{I}_{2^n}} \left(\sum_{d \in \mathcal{D}} p_{ij}^{(d)} \frac{d}{2} \right) \widehat{x}_i \widehat{x}_j \\
&= (\widehat{x})^T M_n^U \widehat{x} + R_U (\widehat{x})^T \widehat{x} \\
&\quad + \sum_{i,j \in \mathcal{I}_{2^n}} \left(\frac{1}{2} \mathbb{E}[d_{ij}] \right) \widehat{x}_i \widehat{x}_j \\
&= (\widehat{x})^T \left(M_n^U + R_U I_n + \frac{1}{2} D_n \right) \widehat{x}.
\end{aligned}$$

The inequality above follows from Fact 3.2 and the definition of d_{ij}'s. This proves the lemma because the last quantity is exactly the expected meeting time of the Pseudo-strategy$_n$.

Let $x^* \in \mathcal{S}_G^*$ be an optimal mixed strategy for the symmetric rendezvous search problem with initial distance 2. So $R_U = H(x^*)$. Based on the lemma above, we obtain the following relationship

$$R_U = H(x^*) \geq \left(\widehat{x^*}\right)^T \left(M_n^U + R_U I_n + \frac{1}{2} D_n \right) \widehat{x^*}.$$

So,

$$R_U \geq \frac{\left(\widehat{x^*}\right)^T \left(M_n^U + \frac{1}{2} D_n \right) \widehat{x^*}}{1 - \left(\widehat{x^*}\right)^T \widehat{x^*}}.$$

Note that $\widehat{x^*}$ is obviously a feasible solution of the the following fractional quadratic program (3.5). Therefore, the optimal value of this program is a valid lower bound of R_U for any fixed n. So $R^s \geq R_U \geq r_n^s$.

(3.5)
$$r_n^s = \min \left\{ \frac{x^T (M_n^U + \frac{1}{2} D_n) x}{1 - x^T x} : e^T x = 1, x_i \geq 0, i \in \mathcal{I}_{2^n} \right\}.$$

Similarly as the upper bound, solving this fractional quadratic program is equivalent to solving a series of regular quadratic programs, which are non-convex in general. Again, to guarantee the global optimality, we also solve the SDP relaxations of the regular quadratic programs. We manage to obtain the following lower bounds for $n \leq 7$. The Matlab code for solving (3.5) can be downloaded from our website,"http://www.unb.ca/~ddu".

n	r_n^s	Uthaisombut [19]
1	3	2
2	3.2970	2.5
3	3.5869	3
4	3.8141	3.4375
5	3.9784	3.6458
6	4.0913	3.8326
7	4.1520	3.9546

Table 3: Lower bounds for R^s

We note that the lower bound scheme of Uthaisombut [19] does not distinguish the two cases in PSEUDO-STRATEGY$_n$. Instead only the idea in Case 1 of Fact 3.2 is applied. To see the advantage of adding Case 2, we also quote the lower bounds obtained in [19] (last column of Table 3).

4 Concluding remarks

First, we note that any result obtained in this work can be extended to an arbitrary initial deterministic distance d rather than just 2. For random d with bounded support, the upper bound can be extended. Moreover, we offer the following conjecture based on our numerical calculations.

Conjecture: $R^s = \lim_{n\to\infty} R_n^s = 4.25$. That is, the optimal mixed strategy is attained asymptotically by letting $n \to \infty$ in n-MARKOVIAN.

Finally, we address the computational limitations of our lower- and upper-bounding techniques. Due to computer memory limitation, we are able to find upper bounds only for $n \leq 15$ and lower bounds only for $n \leq 7$. The asymmetry here is because of two main reasons: (1) only a global optimizer is a valid lower bound while any local optimizer can serve as a valid upper bound; and (2) in the upper bound calculation, we only need to consider distance-preserving strategies, while in the lower bound calculation, we have to consider all strategies. The computational requirement becomes so extensive with n increasing, and parallel computing techniques should be adopted in order to get better numerical bounds for large n. However, we believe that proving the conjecture theoretically will be more desirable and challenging.

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