On the strength of Gomory mixed-integer cuts as group cuts

Sanjeeb Dash and Oktay Günlük

Mathematical Sciences Department,
IBM T. J. Watson Research Center,Yorktown Heights, NY 10598
`(sanjeebd@us.ibm.com, oktay@watson.ibm.com)`

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Abstract

Gomory mixed-integer (GMI) cuts generated from optimal simplex tableaus are known to be useful in solving mixed-integer programs. Further, it is well-known that GMI cuts can be derived from facets of Gomory's master cyclic group polyhedron and its mixed-integer extension studied by Gomory and Johnson. In this paper we examine why cutting planes derived from other facets of master cyclic group polyhedra (group cuts) do not seem to be as useful when used in conjunction with GMI cuts. For many practical problem instances, we numerically show that once GMI cuts from different rows of the optimal simplex tableau are added to the formulation, all other group cuts from the same tableau rows are satisfied.

Keywords: integer programming, mixed integer rounding, cyclic group polyhedra, cutting planes

1 Introduction

The Gomory mixed-integer (GMI) cut is considered to be one of the most important cutting planes for solving general mixed-integer programs. See [5], [7] for computational work in this regard. In his important work on cyclic group polyhedra, Gomory [17] derived the GMI cut for pure integer programs as a facet of the master cyclic group polyhedron. GMI cuts for mixed-integer programs can similarly be derived from the mixed-integer extension of the master cyclic group polyhedron; see Gomory and Johnson [18]. The master cyclic group polyhedron has a rich polyhedral structure and a very natural question to ask is whether or not it has other facets that would lead to useful cutting planes for mixed-integer programs. We call such cuts *group cuts*, and in this paper we present a computationally study of group cuts derived from the rows of the simplex tableau. Recently, there has been much work on facets of cyclic group polyhedra and on group cuts, see [3], [20], [11] and [12]. One goal of this recent research is to find other classes of useful group cuts.

It is well-known that for many practical problem instances GMI cuts generated from the optimal simplex tableau of the LP relaxation significantly reduce the integrality gap. More precisely, for a mixed-integer program with minimization objective define the *integrality gap closed* by a set of cuts $\mathcal{C}$ to be

$$IGC(\mathcal{C}) = \frac{(\text{LP} + \mathcal{C})\text{ bound} - \text{LP bound}}{\text{IP optimal value} - \text{LP bound}} \times 100,$$

where “LP+$\mathcal{C}$ bound” stands for the bound obtained by adding all cuts in $\mathcal{C}$ to the LP relaxation. We thus express the integrality gap closed as a percentage. For a number of MIPLIB 3.0 [6] problem instances, for example, the integrality gap closed by GMI cuts is larger than 50%. Given that GMI cuts are already quite useful for practical problem instances, it is natural to use other group cuts in conjunction with GMI cuts. Therefore, given a class $\mathcal{C}$ of group cuts different from GMI cuts, we believe

$$\Delta(\mathcal{C}) = IGC(GMI + \mathcal{C}) - IGC(GMI)$$

is a better measure of usefulness of $\mathcal{C}$ than $IGC(\mathcal{C})$.

Recently, Dash, Günlük and Goycoolea [13] studied the computational effectiveness of *two-step MIR* cuts [11], a parametric family of group cuts. They observe that for the randomly generated multiple knapsack instances described in Atamtürk[4], the integrality gap closed with two-step MIR cuts and GMI cuts is much larger than the integrality gap closed with GMI cuts alone. Yet for many MIPLIB problems, they conclude that the additional improvement over GMI cuts alone is not substantial. In other words, they observe that $\Delta(\text{two-step MIR}) \approx 0$ for a significant number of MIPLIB problems. In another recent paper, Fischetti and Saturni [16] observe a similar behavior on these problems for *scaled* GMI cuts and cuts derived from master cyclic group polyhedra via interpolation. Scaled GMI cuts are GMI cuts obtained after scaling tableau rows by positive integers. Cornuéjols, Li, and Vandenbussche [10] call them k-cuts in their recent work on this topic.

In this paper we study whether any non-GMI group cuts are useful when used in conjunction with GMI cuts. Our work is motivated by the above negative results on the usefulness of specific classes of group cuts. We numerically demonstrate that for many MIPLIB problems (21 out of 65), once the LP relaxations are augmented with GMI cuts derived from optimal simplex tableau rows (call these rows $\mathcal{R}$), the optimal solution to the augmented LP relaxation satisfies all group cuts that can be derived from $\mathcal{R}$. As a consequence, if $\mathcal{G}$ stands for all group cuts, then $\Delta(\mathcal{G}) = 0$ for these MIPLIB instances. We also show similar results with MIPLIB 2003 [2] and other test sets. Fischetti and Saturni [16] have independently suggested that for instances with many rows this might be the case.

The rest of the paper is organized as follows: In Sections 2 and 3, we describe the GMI cut and group cuts based on the rows of the simplex tableau. In Section 4, we describe how to establish the existence or otherwise of violated group cuts. In Sections 5 and 6 we present our computational approach and numerical results.

2 Gomory mixed-integer cut

Let

$$z^{IP} = \{\min ex + fv \ : \ (x, v) \in P\}$$

be a mixed-integer program where

$$P = \{(x, v) \in R^m : Ax + Cv = d, \ \ x, v \geq 0, \ x \text{ integer}\}.$$

Throughout the paper we assume that mixed-integer programs are given in this form, possibly after slack variables are added. We use P^{LP} to denote the continuous relaxation of P and assume that numerical data is rational. Let

$$Q \;=\; \Big\{ v \in R^{|J|}, x \in Z^{|I|} \;:\; \sum_{j\in J} c_j v_j + \sum_{i \in I} a_i\, x_i = b, \quad x, v \geq 0,\; x \text{ integer} \Big\}$$

where the single equation defining set Q, called the *base* equation, is obtained by taking a linear combination of the equations defining P. In other words, $a = \lambda^T A$, $c = \lambda^T C$ and $b = \lambda^T d$ for some real vector λ of appropriate dimension. As Q is a relaxation of P, valid inequalities for Q yield cutting planes for P. A well-known valid inequality for Q is the mixed integer rounding (MIR) inequality

$$\sum_{j\in J} \max\{c_j, 0\} v_j + \sum_{i\in I} \big(\, \hat{b}\, \lfloor a_i \rfloor + \min\{\hat{b}\ , \hat{a}_i\}\big) x_i \geq \hat{b}\, \lceil b \rceil, \tag{1}$$

where $\hat{b} = b - \lfloor b \rfloor$ and $\hat{a}_i = a_i - \lfloor a_i \rfloor$. As shown by Marchand and Wolsey [22], when the base equation is obtained from a row of the optimal simplex tableau associated with P, the resulting inequality (1) is equivalent to the well-known Gomory mixed integer cut:

$$\sum_{j\in J: c_j > 0} c_j v_j + \sum_{j \in J: c_j < 0} \frac{\hat{b}\, c_j}{1-\hat{b}}\, v_j + \sum_{i\in \bar{I}: \hat{a}_i \leq \hat{b}} \hat{a}_i x_i + \sum_{i\in \bar{I}: \hat{a}_i > \hat{b}} \frac{\hat{b}(1-\hat{a}_i)}{1-\hat{b}}\, x_i \geq \hat{b}, \tag{2}$$

where $\bar{I}$ denotes the set of integer variables except the one in the basis. Note that this variable does not appear in the cut and its coefficient in the base inequality (tableau row) is 1. Inequality (2) can be obtained from inequality (1) by simply substituting out the basic variable using the base equation.

Note that, as defined above, GMI cuts are generated from the optimal simplex tableau and as such they not only depend on the objective function of the associated integer program, but also on the particular optimal basis and solution chosen. Moreover, in a practical setting, these cuts actually depend on the *near*-optimal basis chosen by the linear programming (LP) solver. LP solvers typically declare the first basis that satisfies the optimality conditions with certain numerical *tolerances* as the optimal basis. We discuss the effects of this issue further in Section 5.

3 Group cuts for mixed-integer programs

As discussed earlier, the GMI cut can be derived as a facet of the master cyclic group polyhedron [17] and remaining facets of this polyhedron can also be used to derive cutting planes. In this section, we discuss how group cuts other than the GMI cut can be obtained from simplex tableau rows.

First we rewrite the set Q as follows:

$$Q \;=\; \Big\{ v \in R^{|J|}, x \in Z^{|I|} \;:\; (\sum_{i\in I} \lfloor a_i \rfloor x_i - \lfloor b \rfloor) + \sum_{j\in J} c_j v_j + \sum_{i \in I} \hat{a}_i\, x_i = \hat{b}, \quad x, v \geq 0 \Big\}$$

and assume that all $\hat{a}_i$ ($i \in I$) and $\hat{b}$ are multiples of $1/n$ for some integer n, and let $\hat{b} = r/n$, where $0 < r < n$. We call n the *scaling factor* for Q. These assumptions can be made without loss of generality provided that P^{LP} is rational.

Next, we let $I_k = \{i \in I \,:\, \hat{a}_i = k/n\}$ and define the mapping

$$\begin{aligned} w_k &= \sum_{i \in I_k} x_i, \\ v_+ &= \sum_{c_j \geq 0} c_j v_j, \quad v_- = \sum_{c_j < 0} c_j v_j, \\ z &= \sum_{i \in I} \lfloor a_i \rfloor x_i - \lfloor b \rfloor \end{aligned} \tag{3}$$

that maps each point (v, x) in Q to a point (v_+, v_-, w) in the polyhedron

$$G(n,r) = conv\{v_+, v_- \in R,\ w \in Z^{n-1} \,:\, v_+ - v_- + \sum_{i=1}^{n-1} \frac{i}{n} w_i + z = \frac{r}{n},\ \ v_+, v_-, w \geq 0, z \in Z\}.$$

(If I_k is empty for some k, then w_k is set to zero.)

$G(n,r)$ is the mixed-integer extension of the *master cyclic group polyhedron* of Gomory; we will refer to n as the *size* of $G(n,r)$. Thus the scaling factor of Q is the same as the size of the associated $G(n,r)$. Gomory and Johnson [18] showed that a valid inequality for $G(n,r)$ defines a non-trivial facet (i.e., not a non-negativity inequality) if and only if it has the form

$$n\eta_1 v_+ + n\eta_{n-1} v_- + \sum_{i=1}^{n-1} \eta_i w_i \geq 1, \tag{4}$$

where $\eta = (\eta_1, \ldots, \eta_{n-1})$ is an extreme point of the set of inequalities

$$\begin{aligned} \eta_i + \eta_j &\geq \eta_{(i+j) \bmod n} \quad \forall i, j \in \{1, \ldots, n-1\}, && (5) \\ \eta_i + \eta_j &= \eta_r \quad \forall i, j \text{ such that } r = (i+j) \bmod n, && (6) \\ \eta_j &\geq 0 \quad \forall j \in \{1, \ldots, n-1\}, && (7) \\ \eta_r &= 1. && (8) \end{aligned}$$

Clearly, valid inequalities for $G(n,r)$ yield valid inequalities for Q and, in particular, given a facet (4) of $G(n,r)$, the following is a valid inequality

$$n\eta_1 (\sum_{c_j \geq 0} c_j v_j) + n\eta_{n-1} (\sum_{c_j < 0} c_j v_j) + \sum_{i \in I} f(\hat{a}_i) x_i \geq 1, \tag{9}$$

where $f(\hat{a}_i) = \eta_k$ if $\hat{a}_i = k/n$. We call such inequalities *group cuts* for Q. It is not difficult to show that the GMI cut can be derived from an extreme point of (5)-(8).

Gomory and Johnson [18] showed that $G(n,r)$ has exponentially many facets (in n). For example, when $n = 50$ and $r = 10$, $G(n,r)$ has at least 74,000 facets, see [12]. Further, consider a point $(v', x') \in P^{LP}$ which is not in Q, and let (v', x') be mapped to (v'_+, v'_-, w') via (3). As originally described in [18], the most violated group cut for (v', x'), if there is one, can be found by solving the following *separation LP*

$$\min \Big\{ (nv'_+)\eta_1 + (nv'_-)\eta_{n-1} + \sum_i w'_i \eta_i \,:\, \eta \text{ satisfies } (5) - (8) \Big\}.$$

If the separation LP has value less than 1, then $(v'_+, v'_-, w') \notin G(n,r)$ and the corresponding group cut (9) is violated by (v', x'). On the other hand, if the optimum value of the separation LP is at least 1, then $(v'_+, v'_-, w') \in G(n,r)$ and (v', x') satisfies all group cuts.

Note that the base equation for Q does not have to come from a row of the optimal simplex tableau for this separation procedure to be valid. In general, group cuts can be generated based on any single constraint relaxation of mixed-integer programs. However, in this paper, we restrict our attention to base equations obtained from rows of the simplex tableau.

Given an optimal basis for the LP relaxation of the mixed-integer program, let x_{l_k} be the basic integer variable in the kth row of the corresponding simplex tableau. Let Q_k denote the set of non-negative integral solutions of the kth tableau row. In other words,

$$Q_k \quad = \quad \Big\{ v \in R^{|J|}, x \in Z^{|I|} \;:\; x_{l_k} + \sum_{i \in I \setminus \{l_k\}} a_{ki} x_i + \sum_{j \in J} c_{kj} v_j = b_k, \quad x, v \geq 0 \Big\}$$

As Q_k has the same form as Q, we can define a corresponding master cyclic group polyhedron $G(n_k, r_k)$, where $n_k \in Z$ is the scaling factor for Q_k – i.e., a_{ki} and b_k are multiples of $1/n_k$ – and $b_k = r_k/n_k$.

Consider the relaxation of Q_k obtained by not constraining x_{l_k} to be non-negative,

$$P_k = conv \Big\{ v \in R^{|J|}, x \in Z^{|I|} \;:\; x_{l_k} + \sum_{i \in I \setminus \{l_k\}} a_{ki} x_i + \sum_{j \in J} c_{kj} v_j = b_k, \quad x_i \geq 0, \; i \in I \setminus \{l_k\}, \; v \geq 0 \Big\}.$$

As originally shown in [17, 18] the group cuts for Q_k define P_k (in some sense P_k is a face of $G(n_k, r_k)$.) Also see [24, Section 24.2] for a short discussion of this fact. Based on this observation, note that one only has to consider the smallest scaling factor $n_k \in Z$ for Q_k. In other words, one does not obtain additional cutting planes by considering $G(tn_k, tr_k)$ for an integer $t > 1$.

Given an optimal basis to the LP relaxation, it is possible to define as many master cyclic group polyhedra as there are fractional integer variables in the solution. All facets of all these polyhedra give cutting planes for the original problem. What we call group cuts in this paper is the collection of all these cutting planes. Once again, we would like to emphasize that, as defined here, group cuts depend on the particular optimal basis used to define the group polyhedra.

4 Finding violated group cuts

As discussed in Section 1, we are interested in whether or not non-GMI group cuts are useful when they are used in conjunction with GMI cuts. Computing the actual integrality gap closed by all group cuts and comparing this value with the integrality gap closed by GMI cuts is a very challenging computational task. Instead of this we do the following: Given an optimal basis and solution to the LP relaxation of the mixed-integer program (MIP), we first strengthen the LP relaxation with all violated GMI cuts and solve it to obtain its optimal solution (v^*, x^*). We then ask the following question: *Does* (v^*, x^*) *violate any group cuts?* Note that group cuts are defined using an optimal basis to the original LP-relaxation whereas the point we are separating is an optimal solution to the strengthened LP.

The question above can also be phrased as: is it true that

$$(v^*, x^*) \in P^{LP} \cap \{\cap_k P_k\}? \tag{10}$$

As a solution to the strengthened LP, clearly $(v^*, x^*) \in P^{LP}$. Therefore, we need to check if $(v^*, x^*) \in P_k$ for all k. As discussed in Section 3 ,

$$(v^*, x^*) \in P_k \Leftrightarrow (w^*, v_+^*, v_-^*)_k \in G(n_k, r_k)$$

where $(w^*, v_+^*, v_-^*)_k$ is the image of (v^*, x^*) via the mapping (3), with n replaced by n_k. If the answer is affirmative for all k, (i.e., (v^*, x^*) satisfies all group cuts) then we can conclude that non-GMI group cuts are not useful when GMI cuts are present. Note that even when (10) does not hold, the additional improvement in closing the integrality gap due to non-GMI group cuts can still be zero.

As discussed in the previous section, it is possible to answer the question in (10) by solving LPs (one for each tableau row). Recall that there exists a violated group cut if and only if the separation LP has value less than 1. In practice, however, this can be a difficult task as the size of the LPs can be extremely large: for example, if $n_k = 2000$, (3) has about 2,000 variables and 2,000,000 constraints. In this section we develop techniques to determine if (10) is true without solving the separation LP.

Establishing the existence of a violated group cut can sometimes also be done without solving the separation LP. We use a separation heuristic [13] for finding violated two-step MIR cuts, a particular class of group cuts [11]. As we discuss later, for most of the problems we consider, if there exists a violated group cut, we are able to establish this by finding a violated two-step MIR cut.

4.1 Solving the separation LP

Given the optimal solution (v^*, x^*) to the strengthened LP, let $\bar{I} \subseteq I$ denote the set of indices of non-zero integral variables, that is, $\bar{I} = \{i \in I \ : \ x_i^* \neq 0\}$. Furthermore, let $\bar{n}_k$ be the smallest integer such that b_k and a_{ki} are multiples of $1/\bar{n}_k$ for all $i \in \bar{I}$. Clearly, $n_k \geq \bar{n}_k$ as b_k and all a_{ki} for $i \in I$ are multiples of $1/n_k$. We next show that to find the most violated group cut, it suffices to work with a group polyhedron of size $\bar{n}_k$ instead of the original $G(n_k, r_k)$. This, in turn, implies that one can solve a smaller separation LP to obtain the cut.

Proposition 1 *The most violated group cut for* (v^*, x^*) *can be found by solving the separation problem over* $G(\bar{n}_k, \bar{r}_k)$ *where* $\hat{b} = \bar{r}_k/\bar{n}_k$.

Proof. Let

$$\bar{P}_k = conv\Big\{v \in R^{|J|}, x \in Z^{|\bar{I}|} \ : \ x_{l_k} + \sum_{i \in \bar{I} \setminus \{l_k\}} a_{ki} x_i + \sum_{j \in J} c_{kj} v_j = b_k, \ \ x_i \geq 0, \ i \in \bar{I} \setminus \{l_k\}, \ v \geq 0\Big\}$$

be the set of points obtained from P_k by restricting x_i to be zero for $i \in I \setminus \bar{I}$. Furthermore, let $(\bar{v}, \bar{x}) \in R_+^{J|+|\bar{I}|}$ be obtained from (v^*, x^*) by deleting zero entries for integer components.

As $P_k \subset R_+^{|J|+|\bar{I}|}$ we have

$$(v^*, x^*) \in P_k \Leftrightarrow (\bar{v}, \bar{x}) \in \bar{P}_k.$$

As discussed earlier, membership of $(\bar{v}, \bar{x})$ in $\bar{P}_k$ can be verified (or falsified) by checking whether the image of $(\bar{v}, \bar{x})$ obtained via the mapping (3) is in $G(\bar{n}_k, \bar{r}_k)$, or not. If the image of $(\bar{v}, \bar{x})$ is not in $G(\bar{n}_k, \bar{r}_k)$, then a most violated inequality for $(\bar{v}, \bar{x})$ can be obtained from the separation LP. From this valid inequality (defined in the space of $\bar{P}_k$) it is easy to obtain a violated valid inequality in the original space P_k by using Gomory's interpolation procedure. (it is also possible to see this last step as a lifting step using the sub-additive function given by the separation LP.) ■

Therefore, $\bar{n}_k$ determines the size of the smallest separation LP that needs to be solved for exact separation. Note that as $\bar{I} \subseteq I$, we have $n_k \geq \bar{n}_k$ and in practice the difference can very be significant. If the number $\bar{n}_k$ is still too large for practical purposes, one can also solve a relaxation of the separation LP.

4.2 Relaxing the separation LP

Note that for every pair of indices $1 \leq i, j \leq n-1$, the separation LP has a sub-additivity constraint of the form

$$\eta_i + \eta_j \ \geq \ \eta_{(i+j) \bmod n}$$

and these constraints dominate the size of the LP.

We define the *relaxed separation LP* to be a relaxation of the separation LP obtained by deleting the sub-additivity constraints where all three η variables in the constraint have zero objective coefficient. More precisely, a sub-additivity constraint appears in the relaxed separation LP if only if at least one of the indices of the associated variables belongs to the set $T = \{1, r, n-1\} \cup \{i : w_i^* \neq 0\}$. Notice that the relaxed separation LP has the same number of variables, but fewer constraints than the exact separation LP. In practice the relaxed separation LP has significantly fewer constraints. Clearly, if the optimal value of the relaxed separation LP is at least 1, the optimal value of the exact separation LP is at least 1 and therefore (v^*, x^*) satisfies all group cuts.

4.3 Lower bounds for the separation LP

We next develop lower bounds for the optimal value of the separation LP that help prove that no violated group cuts exist without actually solving an LP.

Let (v^*, x^*) be an optimal solution of the strengthened LP. Assume we are dealing with the lth tableau row, and the sets Q_l and P_l associated with this tableau row. Further, let $n = n_l$ and $r = r_l$ for convenience, where n_l is the scaling factor associated with Q_l. Let (w^*, v_+^*, v_-^*) be the image of (v^*, x^*) via the mapping (3). Note that (w^*, v_+^*, v_-^*) will vary with the tableau rows.

We next show that

$$w_r^* + \frac{1}{r}(nv_+^* + w_1^*) + \frac{1}{n-r}(nv_-^* + w_{n-1}^*) \tag{11}$$

is a lower bound on the objective function value of the separation LP corresponding to a tableau row.

Proposition 2 *If (11) has value 1 or more, then no facet of $G(n, r)$ is violated by (w^*, v_+^*, v_-^*).*

Proof. From (5), it is easy to infer that for any facet η and any index i between 1 and $n-1$

$$i\eta_1 \geq \eta_i \quad \text{and} \quad i\eta_{n-1} \geq \eta_{n-i}. \tag{12}$$

Using $\eta_r = 1$, this implies that

$$\eta_1 \geq \frac{1}{r} \quad \text{and} \quad \eta_{n-1} \geq \frac{1}{n-r}. \tag{13}$$

Therefore (11) is less than or equal to $w_r^* + \eta_1(nv_+^* + w_1^*) + \eta_{n-1}(nv_-^* + w_{n-1}^*)$. which in turn is less than or equal to the objective in (3) evaluated at the facet η. For the MIR facet, note that $\eta_i = i/r$ if $i < r$ and $\eta_i = (n-i)/(n-r)$ if $i > r$, and the inequalities in (13) hold with equality. ∎

We next develop a lower bound using the fact that (v^*, x^*) is not an arbitrary point. As an optimal solution of the strengthened LP, (v^*, x^*) satisfies the GMI cut for each tableau row.

Proposition 3 *Assume (w^*, v_+^*, v_-^*) satisfies the MIR facet inequality for $G(n,r)$. Then no facet of $G(n,r)$ is violated by (w^*, v_+^*, v_-^*) if the following two conditions hold:*

$$(i) \quad nv_+^* + w_1^* \geq \sum_{1<i<r} (r-i)w_i^*, \tag{14}$$

$$(ii) \quad nv_-^* + w_{n-1}^* \geq \sum_{r<i<n-1} (i-r)w_i^*. \tag{15}$$

Proof. Let η' stand for the MIR facet, and η'' for any other facet. We know that

$$\sum_i w_i^*\eta_i' + (nv_+^*)\eta_1' + (nv_-^*)\eta_{n-1}' \geq 1. \tag{16}$$

We will show that the conditions in (14) and (15) imply that if we replace η' by η'', the left hand side of the expression above can only increase, and hence no facet is violated. Recall from the proof of Proposition 2 that $\eta_1'' \geq \eta_1' = 1/r$. Let $\eta_1'' = \eta_1' + \epsilon$ for some $\epsilon \geq 0$. Let k be an index between 2 and $r-1$. We know that

$$\begin{array}{rcl}
1 - \eta_k'' & = & \eta_{r-k}'' \quad \text{(by (6) and (8))} \\
& \leq & (r-k)\eta_1'' \quad \text{(by (12))} \\
& = & (r-k)(\eta_1' + \epsilon) = (r-k)\eta_1' + \epsilon(r-k) \\
& = & \eta_{r-k}' + \epsilon(r-k) \quad (\text{as } \eta_i' = i/r \text{ for } i = 1, \ldots, r) \\
& = & 1 - \eta_k' + \epsilon(r-k) \quad \text{(by (6) and (8))} \\
& \Rightarrow & \eta_k'' \;\geq\; \eta_k' - \epsilon(r-k).
\end{array}$$

Denote $nv_+^* + w_1^*$ by w_1^{**} and $nv_-^* + w_{n-1}^*$ by w_{n-1}^{**}. The above inequalities imply that

$$\begin{array}{rcl}
w_1^{**}\eta_1'' + \displaystyle\sum_{1<i<r} w_i^*\eta_i'' & \geq & w_1^{**}(\eta_1' + \epsilon) + \displaystyle\sum_{1<i<r} w_i^*(\eta_i' - \epsilon(r-i)) \\
& = & w_1^{**}\eta_1' + \displaystyle\sum_{1<i<r} w_i^*\eta_i' + \epsilon(w_1^{**} - \displaystyle\sum_{1<i<r}(r-i)w_i^*) \\
& \geq & w_1^{**}\eta_1' + \displaystyle\sum_{1<i<r} w_i^*\eta_i'.
\end{array} \tag{17}$$

The condition in (14) says that $w_1^{**} \geq \sum_{1<i<r}(r-i)w_i^*$; this and the fact that $\epsilon \geq 0$ imply the last inequality above. Similarly, one can show that the condition in (15) implies that

$$w_{n-1}^{**}\eta_{n-1}'' + \sum_{r<i<n-1} w_i^*\eta_i'' \;\geq\; w_{n-1}^{**}\eta_{n-1}' + \sum_{r<i<n-1} w_i^*\eta_i'. \tag{18}$$

Further, $w_r^*\eta'' = w_r^*\eta_i'$, because of (8). Adding the inequalities (17) and (18) and the equation $w_r^*\eta'' = w_r^*\eta_i'$, we see that replacing η' by η'' in (16) causes the left hand side of (16) to increase. ■

Depending on the data, the number $n = n_l$ can be very large and it might not be practically possible or useful to compute n, the associated right hand side r, and the values of w_i^*. Even in this case, it is still possible to derive lower bounds on the value of the separation LP. The next two propositions give weaker bounds on the separation LP than Propositions 2 and 3, but they do not use n_l.

First we show that

$$\sum_{\hat{a}_i=\hat{b}} x_i^* + \frac{v_+^*}{\hat{b}} + \frac{v_-^*}{1-\hat{b}} \tag{19}$$

gives a lower bound on the optimal value of the separation LP. Note that without knowing the n and r in $G(n,r)$, we can still compute (19).

Proposition 4 *If (19) has value 1 or more, then (v^*, x^*) satisfies all group cuts for Q_l, that is, $(v^*, x^*) \in P_l$.*

Proof. By definition, $\hat{b} = r/n$, and thus

$$\frac{n}{r} = \frac{1}{\hat{b}} \quad \text{and} \quad \frac{n}{1-r} = \frac{1}{1-\hat{b}}.$$

Further,

$$w_r^* = \sum_{\hat{a}_i=\hat{b}} x_i^*$$

by definition. This and the non-negativity of w^* imply that (19) is less than or equal to (11), which in turn is a lower bound on the value of the separation LP. ■

Proposition 5 *Assume (v^*, x^*) satisfies the MIR inequality for Q_l. Then (v^*, x^*) satisfies all group cuts for Q_l, if*

$$v_+^* \ge \sum_{\hat{a}_i<\hat{b}} (\hat{b}-\hat{a}_i)x_i^* \text{ and } v_-^* \ge \sum_{\hat{b}<\hat{a}_i} (\hat{a}_i-\hat{b})x_i^*.$$

Proof. Divide the inequalities in (14) and (15) by n. If $\hat{a}_i = k/n$, then $(\hat{b}-\hat{a}_i)x_i^* = w_k^*(r-k)/n$. Also, $w_1^*, w_{n-1}^* \ge 0$ (recall we do not know their exact values). Therefore, if the inequalities above are satisfied, then so are the inequalities in (14) and (15). ■

5 Computational approach

Given a mixed-integer program (MIP), the first step of our procedure is to solve its LP relaxation and redefine the variables by complementing the ones at their upper bounds and shifting the ones at their lower bounds. This transformation step gives a solution where all non-basic variables are at value zero. Based on the rows of the optimal (transformed) tableau, we next construct the sets P_k, one for each integer variable with fractional value. We consider a value v fractional if

$\min\{v - \lfloor v \rfloor, \lceil v \rceil - v\} > 10^{-4}$. Though some problem instances have inequality constraints in their formulation, the simplex tableau implicitly requires slack variables to convert these to equality form. Therefore, the defining equations for sets P_k include (continuous) slack variables.

Next, we add all GMI cuts to the relaxation to obtain the strengthened LP. Let (v^*, x^*) denote an optimal solution to the strengthened LP. If the heuristic procedure described in [13] can produce a violated two-step MIR cut then we conclude that there are indeed violated group cuts and we stop.

If the heuristic separation procedure fails, we then apply the following procedure for each P_k: First we use Proposition 1 and only consider the non-zero integer variables to compute the group polyhedron size which we denote by $\bar{n}_k$. If $\bar{n}_k > 20,000$ we do not compute it. Depending on $\bar{n}_k$, we then do the following:

1. If $\bar{n}_k \leq 20,000$, we apply Propositions 2 and 3 to prove that there are no violated group cuts. If this test is inconclusive, depending on the value of $\bar{n}_k$ we perform one of the following:
   (a) If $\bar{n}_k \leq 400$, we explicitly solve the separation LP. Based on the optimal value of the separation LP, we establish whether or not there exists a violated group cut.
   (b) If $\bar{n}_k > 400$, we instead solve the relaxation described in Section 4.2 and if the optimal value is at least 1, we conclude that there are no violated group cuts. On the other hand, if the optimal value is less than 1, we declare that our procedure has failed.
2. If $\bar{n}_k > 20,000$, we apply Propositions 4 and 5 to prove that there are no violated group cuts. If this test is inconclusive, we declare that our procedure has failed.

If for each P_k associated with the optimal simplex tableau for the linear relaxation of an MIP, we can establish that there are no violated group cuts, we conclude that $(v^*, x^*) \in P^{LP} \cap \{\cap_k P_k\}$. If on the other hand, we can find a violated group cut (using the heuristic separation of two-step MIR cuts or through the separation LP) we conclude that $(v^*, x^*) \notin P^{LP} \cap \{\cap_k P_k\}$. Finally, if we cannot find any group cuts and Steps (1b) or (2) are inconclusive (fail) for some P_k, we conclude that we can not answer the membership question. In our numerical experiments, this does not happen very often.

It is important to clarify some limitations of this approach. Note that the sets P_k and therefore the group cuts are defined using an optimal solution and an associated optimal basis of the LP relaxation. If the LP relaxation of the MIP has multiple optimal solutions, then the collection of group cuts based on other solutions could be different. Therefore our conclusion regarding whether or not (v^*, x^*) satisfies all group cuts depends on the particular solution used to construct the sets P_k. In addition, the point (v^*, x^*) could also be one of the many optimal solutions to the strengthened LP. If there is more than one optimal solution to the strengthened LP, it could very well be the case that other optimal solutions would be classified differently. This second limitation is not a major one in the following sense. Suppose we can show that some optimal solution of the strengthened LP satisfies $P^{LP} \cap \{\cap_k P_k\}$. For any other optimal solution of the strengthened LP, the violated group cuts – if any – are not important in the sense that their addition does not change the bound from the strengthened LP.

In addition, our classification depends to some extent on the level of precision used in solving LPs; all practical LP solvers only return approximately optimal solutions. Thus it is possible that a different near-optimal simplex tableau than the one we work with would yield different results.

Finally, our procedure involves some numerical approximations when determining the group polyhedron $G(\bar{n}_k, \bar{r}_k)$ associated with a given set P_k. Consider the kth tableau row, and the associated set P_k. To check if a_{ki} is a multiple of some integer, say n, we check if $|a_{ki} - t_i/n| < 10^{-14}$ for $t_i = \lfloor na_{ki} \rceil$. Equivalently, we check if $|na_{ki} - \lfloor na_{ki} \rceil| < 10^{-14}n$. For $n = 1, \ldots, 20,000$ we test if a_{ki} $(i \in I_k \setminus \{l_k\})$ and b_k are all multiples of $1/n$. We denote the smallest value of $n \leq 20,000$ for which this is true as $\bar{n}_k$. Thus, our computed $\bar{n}_k$s are approximate, though accurate to a fairly high precision. The coefficients of tableau rows would typically be represented as doubles, and on most processors, the default precision of doubles is only 10^{-16}.

6 Computational experiments

We apply the procedure described in Section 5 to the following data sets:

1. MIPLIB 3.0 [6] available at [1],
2. MIPLIB 2003 [2], available at [1],
3. MILPLib collected by Hans Mittelmann available at [23], and,
4. A set of instances collected by Fischetti and Lodi [15, 14], available at [21], which we refer to as "Unibo instances".

We use CLP, the COIN LP solver [8], in our experiments.

To avoid repetition, we only report on the 27 problems in MIPLIB 2003 which are not part of MIPLIB 3.0 and on the 30 Unibo instances which are not part of MIPLIB 2003. There are also a number of instances in MILPLib, listed in Section 6.3 , that are obtained from MIPLIB instances by relaxing the integrality requirement of some of the variables. In Table 1, we summarize our computational results where we retain all MILPLib instances. For each problem set, we report the number of instances for which we can verify that

$$(v^*, x^*) \in P^{LP} \cap \{\cap_k P_k\} \tag{20}$$

together with the number of instances for which this is not true and the instances on which our method fails to verify either property. We omit from this table a few instances for which we failed to solve the strengthened LP due to large memory requirements.

Problem set	(20) is True	(20) is False	Fail
MIPLIB 3.0	21	38	6
MIPLIB 2003	8	16	2
MILPLib	22	14	3
Unibo	4	18	4
All 156 instances	55	86	15

Table 1: Summary of computational tests

As seen in Table 1, we are able to classify nearly 90% of the problems instances and from among those, close to 40% of them satisfy (20). We note that these observations essentially remain the same even after removing the MILPLib instances that are relaxations of MIPLIB instances.

6.1 Miplib 3.0 Instances

For 38 of the 65 problems in MIPLIB 3.0, listed in Table 3, we can find violated group cuts, for the 21 problems in Table 2 we can assert that no such cuts exist, and for the remaining 6 problems – in Table 4 – we can do neither.

In Tables 2 – 4, the first two columns give the problem name, and whether the problem has continuous variables ('y') or not ('n'). The third column gives the percentage integrality gap closed by GMI cuts. The fourth column gives the number of "interesting" tableau rows, that is the tableau rows from which we generate GMI cuts, and which we use to test for other violated group cuts. The fifth and sixth columns give, the number of interesting tableau rows for which $n_k \leq 20,000$ and $n_k \leq 200$, respectively. These numbers indicate whether the tableau coefficients are complicated or easy to handle from the perspective of the separation LPs. For Tables 2, and 3, the last column gives the number of tableau rows for which we solve the corresponding separation LPs (either the exact separation LP or the relaxation in Section 4.2). For Table 4 the last column stands for the number of tableau rows we cannot analyze in the sense that we neither find violated group cuts from those rows nor prove that none exist.

Instance	cont var	GMI%	# tab. rows	$n_k \leq 20,000$	$n_k \leq 200$	sep-LP
air03	n	100.00	35	-	-	
bell3a	y	45.10	32	30	6	13
blend2	y	15.98	6	0	0	
dsbmip	y	-	61	24	12	
egout	y	55.63	40	40	2	
fixnet6	y	12.88	60	60	29	
flugpl	y	11.74	10	6	3	
khb05250	y	74.91	19	19	0	
misc06	y	28.47	10	6	3	
misc07	y	0.00	18	18	18	2
mod010	n	100.00	16	-	-	
mod011	y	31.15	16	0	0	
noswot	y	-	15	0	0	
nw04	n	62.27	6	6	6	
p0201	n	26.71	40	40	40	32
pp08a	y	54.42	53	53	53	
qiu	y	2.53	36	0	0	
rentacar	y	28.07	32	8	0	
set1ch	y	39.16	138	138	58	
vpm1	y	22.91	14	14	14	
vpm2	y	10.93	32	31	21	

Table 2: MIPLIB instances where point is in convex hull

In Table 2 problems **air03** and **mod010** are unusual in the sense that GMI cuts alone close 100% of the integrality gap and produce an integral optimal solution. Further, **dsbmip** and **noswot** have no integrality gap, and yet the point (v^*, x^*) is fractional, and we use the techniques developed earlier to show that no group cuts are violated. From the remaining 17 problems, for only 3 problems – **bell3a**, **misc07**, **p0201** – we solve separation LPs, and for the rest, the bounds in Propositions 1-4 are sufficient to establish that no violated group cuts exist. Note that for 10 of the 21 problems, all tableau rows are easy to handle in the sense that $n_k \leq 20,000$; thus these rows do not have too ugly coefficients.

Instance	cont var	GMI%	# tab. rows	$n_k \leq 20,000$	$n_k \leq 200$	sep-LP
10teams	y	57.14	163	2	2	
air04	n	6.95	290	0	0	
air05	n	4.64	224	0	0	
arki001	y	29.26	69	10	9	
bell5	y	14.53	25	17	10	
cap6000	n	41.65	2	0	0	
dcmulti	y	47.65	49	39	9	
enigma	n	-	6	0	0	
fast0507	n	2.00	304	9	9	
fiber	y	63.09	45	45	44	
gen	y	60.69	42	30	18	
gesa2	y	28.53	58	17	0	
gesa3	y	47.53	85	25	0	^{*1}
gesa3_o	y	50.54	106	32	0	^{*1}
gt2	n	69.71	11	11	0	
harp2	n	24.07	30	0	0	
l152lav	n	10.88	55	53	53	
lseu	n	41.59	10	10	0	
markshare1	y	0.00	6	0	0	
markshare2	y	0.00	7	0	0	
mas74	y	6.67	12	0	0	
mas76	y	6.42	11	0	0	
misc03	y	87.21	20	20	8	2
mitre	n	7.24	762	683	124	
mkc	y	6.83	72	52	0	
mod008	n	20.11	5	1	0	
p0033	n	56.82	6	6	2	
p0282	n	3.70	26	26	21	
p0548	n	39.20	47	44	3	
p2756	n	0.54	27	27	1	
pk1	y	0.00	15	0	0	
qnet1	y	11.91	50	0	0	
qnet1_o	y	42.99	11	3	0	
rgn	y	3.15	14	0	0	
seymour	n	7.69	616	116	116	
stein27	n	0.00	21	21	21	
stein45	n	0.00	35	35	22	
swath	y	8.42	45	45	45	

Table 3: MIPLIB instances where point is not in convex hull

In Table 3, for 35 of these 38 problems, the heuristic two-step MIR separation procedure finds violated cuts. Note that for **enigma**, there is no integrality gap, and yet the resulting point after adding GMI cuts does not satisfy all group cuts. For only **misc03**, **gesa3** and **gesa3_o**, we do not find violated two-step MIR cuts, but verify the existence of violated group cuts by solving separation LPs (for 2,1,1 tableau rows, respectively). In both **gesa3** and **gesa3_o**, however, the the separation LP has a value of 0.9998, which in turn means that the violation of the maximally violated group cut written in the form (9) is 0.0002. Thus, for these two problem instances (v^*, x^*) approximately satisfies (20). We have marked these two instances with a '*'. Observe that out of the 21 problems that satisfy (20), only 4 of them have continuous variables. On the other hand, the problems that do not satisfy this condition are divided more evenly between the pure and mixed-integer classes. One point to note is that the effect of adding the violated two-step MIR cuts varies a lot for the problems in Table 3; see [13] for more details on Δ(two-step MIR).

Instance	cont var	GMI%	# tab. rows	$n_k \leq 20,000$	$n_k \leq 200$	fail
dano3mip	y	0.11	124	2	0	*15
danoint	y	1.73	52	0	0	2
gesa2_o	y	31.03	73	25	0	*3
modglob	y	17.28	30	0	0	2
pp08aCUTS	y	33.79	46	24	17	6
rout	y	0.81	36	10	9	7

Table 4: Unclassified MIPLIB instances

We are unable to classify the problems in Table 4 partly because the tableau rows are complicated, in the sense that we cannot compute the scaling factors n_k for them. However, as shown in the column titled "fail", only a few tableau rows have the potential to yield violated group cuts, e.g., 2 out of 52 for **danoint**. For the rest of the tableau rows, no violated group cuts exist. Further, for the 15 tableau rows indicated in the "fail" column in the case of **dano3mip**, the value of the separation LP is at least 0.9975. That is, if there are any violated group cuts, then the maximum violation is at most 0.0025. Similarly, for **gesa2_o**, the most violated group cut, if there is one, has a violation of at most 0.005. We have marked these two instances with a '*'.

6.2 Miplib 2003 Instances

In Table 5, we present results on the 27 problems in MIPLIB2003 which are not contained in MIPLIB 3.0. Here we omit the column for integrality gap closed by GMI cuts, as the integral optimal solutions are not known for many of these problems. The remaining columns have the same meaning as the columns in the earlier tables. The three parts of Table 5, separated by horizontal lines, correspond to Tables 2 – 4, respectively. In this table, we omitted the instance **stp3d** as we ran out of memory (4 GB) before we could solve the strengthened LP (after adding GMI cuts). Note that for **manna81** GMI cuts alone close 100% of the integrality gap and the point (v^*, x^*) is an integral optimal solution.

6.3 MILPLib Instances

In Table 6, we present results on 39 instances in MILPLib. We omitted 5 instances from this test set where we either ran out of memory (4 GB) or ran out of time (4 hours) before we could

Instance	cont var	# tab. rows	$n_k \leq 20,000$	$n_k \leq 200$	sep-LP (fail)
a1c1s1	y	173	173	89	
glass4	y	72	72	68	
manna81	n	870	-	-	
net12	y	470	468	392	19
timtab1	y	130	130	130	
timtab2	y	236	236	236	
tr12-30	y	348	348	36	
van	y	192	64	64	
aflow30a	y	31	31	30	
aflow40b	y	38	25	17	8
atlanta-ip	y	2207	6	6	
ds	n	519	0	0	
liu	y	536	536	8	
momentum1	y	300	98	98	
momentum2	y	609	324	323	
msc98-ip	y	2930	907	760	
mzzv11	n	801	417	191	
mzzv42z	n	810	383	372	
nsrand-ipx	y	65	65	65	
opt1217	y	29	29	0	
protfold	n	558	0	0	
roll3000	y	208	108	47	
sp97ar	n	196	8	8	
t1717	n	511	0	0	
momentum3	y	1249	14	12	(187)
rd-rplusc-21	y	80	4	4	(8)

Table 5: MIPLIB 2003 instances

solve the strengthened LP. These instances are **neos4**, **neos19** and the three versions of the **30:70:4:5:0** instances. The format of this table is the same as Table 5, except that we add a column for percentage integrality gap closed by GMI cuts. Note that for these instances we solve the separation LP only once (for **neos18**, and for only one of its tableau rows).

Instance	cont var	GMI	# tab. rows	$n_k \leq 20,000$	$n_k \leq 200$	sep-LP (fail)
bc1	y	0.11	7	0	0	
bienst1	y	11.95	26	20	4	
bienst2	y	9.77	32	26	5	
dano3_3	y	6.90	12	0	0	
dano3_4	y	8.15	18	0	0	
dano3_5	y	4.34	25	1	0	
neos2	y	4.25	24	24	18	
neos3	y	4.36	31	31	27	
neos5	y		35	35	35	
neos7	y	7.35	138	46	11	
neos9	y	5.56	77	77	77	
neos11	y	0.0	315	0	0	
neos13	y	36.05	367	0	0	
neos14	y	58.57	136	136	19	
neos15	y	51.66	160	160	22	
neos16	n		173	173	173	
neos17	y	4.42	171	171	0	
neos18	n	11.11	775	775	775	1
neos22	y	58.28	454	454	454	
neos23	y		60	60	36	
neos648910	y		296	296	296	
neos671048	y		17	17	0	
markshare1-1	y	0.0	6	0	0	
markshare2-1	y	0.0	6	0	0	
mkc1	y	0.0	38	27	1	
neos1	n	0.0	200	0	0	
neos8	n	22.0	174	84	0	
neos10	n	4.93	184	25	6	
neos20	y	0.0	436	321	62	
neos21	y		165	154	126	
nug08	n	5.78	500	0	0	
qap10	n	12.56	1223	0	0	
ran14x18_1	y	6.78	18	18	18	
swath1	y	0.11	13	13	13	
swath2	y		14	14	14	
swath3	y		16	16	16	
neos6	y	100.00	122	0	0	(122)
neos12	y	0.00	1103	40	0	(194)
seymour1	y	11.59	158	96	96	(24)

Table 6: MILPLib instances

We would like to emphasize that some of the instances in this data set are very similar to one another and to some instances in MIPLIB 3.0. For example

1. **dano3_3**, **dano3_4** and **dano3_5** are obtained from **dano3mip** in MIPLIB 3.0 by removing integrality requirements of some of the variables;
2. similarly **markshare1-1** and **markshare2-1** are obtained from **markshare1** and **markshare2**.

3. **swath1**, **swath2** and **swath3** are obtained from **swath**;
4. **mkc1** is obtained from **mkc**.
5. **seymour1** is obtained from **seymour**.

In addition **bienst1** is a relaxation of **bienst2** and **neos14** is a relaxation of **neos15**. If we leave out repeated instances, for 17 of the remaining instances we can verify that there are no violated group cuts after adding GMI cuts; for 8 instances violated group cuts exist and for 2 instances we cannot verify either claim.

6.4 Unibo Instances

In Table 7, we present results on the 26 problems in the Fischetti and Lodi test set. We omitted 3 instances where we ran out of memory (4 GB) before we could solve the strengthened LP. These instances are **NSR8K**, **core4284** and **core4872**. The format of this table is the same as Table 5. Note that in these instances we do not solve the separation LP at all.

Instance	cont var	# tab. rows	$n_k \leq 20,000$	$n_k \leq 200$	sep-LP (fail)
A2C1S1	y	174	174	115	
B1C1S1	y	266	266	159	
B2C1S1	y	262	262	177	
dg012142	y	398	26	1	
CMS750-4	y	1935	1900	1746	
UMTS	y	367	4	0	
berlin-5-8-0	y	242	217	140	
blp-ar98	y	148	141	90	
blp-ic97	y	68	68	50	
blp-ic98	y	56	43	27	
blp-ir98	y	37	37	23	
core2536-691	y	762	8	8	
core2586-950	y	1571	0	0	
dc1c	y	635	36	36	
dolom1	y	593	0	0	
rail507	y	320	0	0	
railway-8-1-0	y	333	327	249	
sp97ic	n	82	8	8	
sp98ar	n	154	28	27	
sp98ic	n	77	0	0	
trento1	y	478	0	0	
usAbbrv	y	703	682	603	
bg512142	y	204	5	0	(7)
biella1	y	579	11	11	(135)
dc1l	y	686	1	1	(331)
siena1	y	845	0	0	(839)

Table 7: Unibo instances

7 Concluding remarks

Our main conclusion is that for a significant number of problem instances (35% in our tests) GMI cuts are not only the most important group cuts generated from individual tableau rows, they actually are the only relevant ones. Fischetti and Saturni have independently suggested this possibility [16]. Part of our observation on the strength of GMI cuts alone can be attributed to the fact that GMI cuts have the smallest possible cut coefficient for continuous variables among all group cuts. Note that most of the pure integer instances in our test set have inequality constraints in their formulation that introduce continuous variables (slacks) in the simplex tableau rows. However, we observed that pure integer problems are more likely to have violated group cuts after GMI cuts are added. More precisely, more than 80% of the pure integer instances fall in this category, whereas this number is less than 60% for instances with continuous variables.

The observations above leave open the possibility that non-GMI group cuts from individual tableau rows could still be useful for many problems (65% in our test set). Our tests strengthen the evidence [12] that two-step MIR cuts are quite important from among the non-GMI group cuts. In the instances in our test set where a violated non-GMI group cut might exist, it is usually possible to find a violated two-step MIR cut (82% of these instances).

Finally, we would like to emphasize the effectiveness of the simple tests described in Section 4. For 75% of the instances where we could not find violated two-step MIR cuts, we could verify that violated group cuts do not exist using these simple tests instead of solving separation LPs.

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