Static-arbitrage bounds on the prices of basket options via linear programming

Javier Peña*[1], Juan C. Vera†[2], and Luis F. Zuluaga‡[3]

[1]Tepper School of Business, Carnegie Mellon University
[2]Mellon School of Sciences, Carnegie Mellon University
[3]Faculty of Business Administration, University of New Brunswick

July 2006

Abstract

We show that the problem of computing sharp upper and lower static-arbitrage bounds on the price of a European basket option, given the prices of other similar options, can be cast as a linear program (LP). The LP formulations readily yield super-replicating (sub-replicating) strategies for the upper (lower) bound problem. The dual counterparts of the LP formulations in turn yield underlying asset price distributions that replicate the given option prices, and the bound on the new basket option's price. In the special case when the given option prices are those of vanilla options on the underlying assets, we show that the LP formulations admit further simplifications. In particular, for the upper bound problem we derive closed-form formulas for the basket's price bound, and for the corresponding super-replicating strategy. In addition, our LP approach admits efficient modeling of additional features such as basket options with negative weights, bid/ask spreads, transaction costs, and diversification constraints. We provide numerical experiments to illustrate some of our results.

1 Introduction

We address the problem of computing sharp bounds on the price of a European basket option, given the only assumption of absence of arbitrage, and information on the prices of other European basket options on the same underlying assets and with the same maturity. Bounds of this type are called *static-arbitrage bounds.* Such kinds of bounds can be seen as *robust* bounds that any reasonable pricing model must satisfy [4, 10]. They provide a mechanism for checking consistency of prices, as well as an initial price estimate for options regardless of any model specifics.

The computation of sharp upper (lower) static-arbitrage bounds can be formulated in terms of finding the least (most) expensive portfolio of the basket options with known prices and cash whose combined payoff super-replicates (sub-replicates) the payoff of the new basket option of interest (see, e.g. [4]). Under reasonably mild assumptions (see, e.g., Proposition 1 in Section 2),

*jfp@andrew.cmu.edu
†jvera@andrew.cmu.edu
‡lzuluaga@unb.ca

an equivalent dual formulation is to find an underlying asset price distribution that maximizes (for the upper bound) or minimizes (for the lower bound) the discounted expected payoff of the new option, and *replicates* the given option prices, i.e., the discounted expected payoffs of the given options match the observed prices. This *semiparametric* approach can be seen as an alternative to *parametric* techniques (such as Monte Carlo simulations) that determine the price of an option as the discounted expected option's payoff under an appropriate risk-neutral measure. Semiparametric techniques are especially useful in incomplete market conditions, or when the number of underlying assets makes the computation of parametric prices numerically challenging.

The problem of computing static-arbitrage bounds has received a fair amount of attention in recent years. Of particular relevance to our work are the recent articles by d'Aspremont and El Ghaoui [3], and by Hobson, Laurence, and Wang [4, 5]. D'Aspremont and El Ghaoui [3] address the same general problem that we study in this paper. They derive linear programming relaxations for static-arbitrage bounds based on an integral transformation of the options' price functions. They also provide linear programming, or closed-form expressions for the bounds in some special cases when only one or two vanilla options prices per asset are given. On the other hand, Hobson, Laurence and Wang [4, 5] derive more general results for the upper bound when vanilla options prices on every asset are given for a continuum of strikes. Their approach relies on a Lagrangian programming formulation and the fact that the continuum of options determines the full marginal distributions of each of the assets.

We undertake a different approach to the generic static-arbitrage bound problem. We focus on deriving polyhedral representations for the super/sub-replication constraints. A main feature of our approach is that it relies entirely on linear programming (LP) techniques. Another main feature is that because we directly work with the super/sub replicating constraints, we can readily consider some important features in the static bounds' computations that have not been considered before, such as basket options with negative weights, bid/ask spreads, limits on the sizes of long/short positions, and transaction costs. We formalize the intuitively clear fact that the super/sub-replication constraints can be cast purely in terms of linear inequalities, which automatically yields LP formulations for these bounds (Propositions 2 and 5). Our initial LP formulations treat the upper and lower bounds in a symmetric manner. We subsequently show that the LP formulations for the upper and lower bounds can be further simplified (Propositions 6 and 7). The simplifications stress a fundamental difference between the upper and lower bound computation. The different nature of the upper and lower bound computation has been recognized previously, as it was apparent that the computation of the upper bounds was much easier in certain cases [3, 4, 5]. However, no theoretical explanation had been previously provided. Our results highlight the different character of the super and sub-replication constraints.

Although the linear programming formulations for general arbitrage bounds may lead to large linear programs, we show that these linear programs are of a manageable size if either the number of assets, or the number of known basket prices is small. Furthermore, we show that in the special case when only forwards and vanilla option prices are given, the upper bound can be simplified substantially. This allow us to derive a closed-form expression for the basket option's price bound (Theorem 8). Although this problem had been previously studied in [4], our derivation is fundamentally different and the closed-form solution is new. The results in [4] require the solution of a certain Lagrangian problem, whereas our closed-form expression lends itself to a simple binary search to find the maximum value of a one-variable piece-wise concave function. In contrast with previous work [3, 4], our results apply to general basket options without any sign restrictions on the basket's weights. We also derive a closed-form expression for the corresponding super-replicating strategy (Theorem 10). Related but weaker results can also be derived for corresponding lower bound problem. For ease of exposition, such results are presented in a separate article [7].

The paper is organized as follows. Section 2 presents the basic notation and general linear programming formulations for both upper and lower static-arbitrage bounds. We also discuss further simplifications that are possible in each case. In Section 3 we specialize the linear programming formulations for the upper bound problem when prices of forward and vanilla options are given. In Section 4 we provide numerical experiments to illustrate some of our results.

2 Static-arbitrage bounds via linear programming

Consider the problem of computing sharp upper and lower *static-arbitrage* bounds on the price of a European basket option, given information on the prices of other similar options, without making any assumptions other than the absence of arbitrage. Finding the upper static-arbitrage bound for the price of a basket option can be formulated as the following optimization problem (see, e.g., [3]):

$$\begin{array}{lll} \sup\limits_{\pi} & \mathbb{E}_\pi[(w^0 \cdot S - K_0)^+] & \\ \text{s.t.} & \mathbb{E}_\pi[1] = 1 & \\ & \mathbb{E}_\pi[(w^j \cdot S - K_j)^+] = p_j, \quad j = 1, ..., r & \\ & \pi \quad \text{a distribution in } \mathbf{R}^n_+. & \end{array} \tag{U}$$

Above, the multivariate random variable S represents the prices of the n underlying assets (at maturity) in the basket. The given vectors $w^j \in \mathbf{R}^n$, and constants $K_j \in \mathbf{R}$, $j = 0, 1, \dots, r$, (respectively) represent the weights of the underlying assets and the strike price of the basket options involved in the problem; that is, $(w^j \cdot S - K_j)^+ := \max\{0, \sum_{i=1}^n w_i^j S_i - K_j\}$. Problem (U) maximizes the expected payoff of a basket option over all underlying asset price distributions π in $\mathbf{R}^n_+$ that *replicate* the given basket option's prices p_j, $j = 1, \dots, r$.

Following [3], we implicitly assume that all the options have the same maturity, and that without loss of generality, the risk-free interest rate is zero; or equivalently, we compare the prices in the forward market (at maturity).

The lower static-arbitrage bound on the price of a basket option corresponding to (U), can be obtained by changing sup to inf in (U). As shown in Section 2.2, the results in this section extend in a straightforward fashion to the corresponding lower static-arbitrage bound problem.

Problem (U) has the following associated dual (see, [6]):

$$\begin{array}{lll} \inf\limits_{z,y} & z + \sum\limits_{j=1}^r p_j y_j & \\ \text{s.t.} & z + \sum\limits_{j=1}^r y_j (w^j \cdot s - K_j)^+ \geq (w^0 \cdot s - K_0)^+ \quad \text{for all } s \in \mathbf{R}^n_+ & \\ & y \in \mathbf{R}^r, \quad z \in \mathbf{R}. & \end{array} \tag{DU}$$

In this context the dual problem has a natural interpretation: it aims to find the cheapest portfolio of positions in cash and in the basket options $(w^j \cdot S - K_j)^+$, $j = 1, \dots, r$ that super-replicates the payoff of the basket option $(w^0 \cdot S - K_0)^+$. It is easy to see that weak duality holds between (U) and (DU). Furthermore, under reasonably mild assumptions, strong duality holds as well. For instance, Proposition 1 states two generic conditions that ensure strong duality in our context. Proposition 1 follows from general convex duality results [8, 9], as it is discussed in [11, Sec. 3].

Proposition 1 *The optimal values of* (U) *and* (DU) *coincide if at least one of the following two conditions holds.*

(i) *Strong primal feasibility:*

$$\begin{bmatrix} 1 \\ p \end{bmatrix} \in \operatorname{int}\left(\left\{\begin{bmatrix} \mathbb{E}_\pi[1] \\ \left(\mathbb{E}_\pi[(w^j \cdot S - K_j)^+)\right)_{j=1,\dots,r} \end{bmatrix} : \pi \quad \text{is a distribution in } \mathbf{R}^n_+\right\}\right).$$

In particular, strong duality holds provided the prices p are arbitrage-free and remain arbitrage-free after slight perturbations.

(ii) *Strong dual feasibility: There exists $(\hat{z}, \hat{y}) \in \mathbf{R}^{r+1}$ such that*

$$(\hat{z}, \hat{y}) \in \operatorname{int}\left(\left\{(z, y) \in \mathbf{R}^{r+1} : z + \sum_{j=1}^{r} y_j (w^j \cdot s - K_j)^+ \geq (w^0 \cdot s - K_0)^+ \text{ for all } s \in \mathbf{R}^n_+\right\}\right).$$

In particular, strong duality holds provided that for each asset at least one vanilla option price is known.

We note that the problems (U) and (DU) are semi-infinite linear programs. However, as Proposition 2 below shows, (DU) can be recast as a linear program. For ease of exposition, we shall use the following notational conventions. Let W denote the $r \times n$ matrix whose j-th row is the vector w^j for $j = 1, \dots r$ and let $\overline{W}$ the $(r+1) \times n$ matrix whose j-th row is the vector w^j for $j = 0, 1, \dots r$. Also, let K denote the vector $\begin{bmatrix} K_1, \dots, K_r \end{bmatrix}^{\mathrm{T}}$ and $\overline{K} = \begin{bmatrix} K_0, K_1, \dots, K_r \end{bmatrix}^{\mathrm{T}}$. Given $v \in \mathbf{R}^I$ for some finite index set I, and $J \subseteq I$, let $v_J \in \mathbf{R}^J$ denote the vector formed by the entries v_j with $j \in J$. Likewise, if the rows of M are indexed by I and $J \subseteq I$, let M_J denote the matrix formed by the rows of M indexed by J. Also, we shall write J^c as a shorthand for $I \setminus J$. The larger set I will typically be of the form $\{0, 1, \dots, r\}$ or $\{1, \dots, r\}$, for some positive integer r.

Given $J \subseteq \{0, 1, \dots, r\}$, define

$$P_J = P_J(\overline{W}, \overline{K}) := \left\{ s : \begin{array}{l} \overline{W}_J \, s \geq \overline{K}_J \\ \overline{W}_{J^c} \, s \leq \overline{K}_{J^c} \\ s \geq 0 \end{array} \right\}, \tag{1}$$

and let $\overline{\mathcal{J}} = \{J \subseteq \{0, \dots, r\} : P_J \neq \emptyset\}$.

2.1 General upper bound problem

Proposition 2 **(i)** *The dual problem* (DU) *can be rewritten as the following linear program*

$$\begin{array}{lll} \min & z + p \cdot y & \\ \text{s.t.} & \qquad \begin{bmatrix} -1 \\ y \end{bmatrix}_J \cdot \overline{W}_J \geq \gamma^J \cdot \overline{W}_J - \beta^J \cdot \overline{W}_{J^c} & J \in \overline{\mathcal{J}} \\ & -z + \begin{bmatrix} -1 \\ y \end{bmatrix}_J \cdot \overline{K}_J \leq \gamma^J \cdot \overline{K}_J - \beta^J \cdot \overline{K}_{J^c} & J \in \overline{\mathcal{J}} \\ & y \in \mathbf{R}^r, \quad z \in \mathbf{R} & \\ & \gamma^J \in \mathbf{R}^J_+, \quad \beta^J \in \mathbf{R}^{J^c}_+, & J \in \overline{\mathcal{J}}. \end{array} \tag{LDU}$$

(ii) *Let* (LU) *denote the linear programming dual of* (LDU). *Then* (LU) *can be modified to produce a sequence of linear programs* (LU_k) *that yield a sequence of atomic underlying asset price distributions feasible for* (U) *and whose (objective) value for* (U) *converges to the optimal value of* (U). *Moreover if the optimal value of* (U) *is attained then this sequence is eventually constant.*

Notice that (LDU) has $(r+1)(|\overline{\mathcal{J}}|+1)$ variables and $2n|\overline{\mathcal{J}}|$ constraints. We now show that if either r or n is small, then problem (LDU) is of manageable size. Let $L(n,r)$ be the maximum number of non-empty regions in which r hyperplanes divide $\mathbf{R}^n$. Notice that $L(n,0) = 1$ and $L(n,r+1) \leq L(n,r) + L(n-1,r)$. By induction, it follows that $L(n,r) \leq \frac{1}{2}\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{r+2}{n-2k}$. Since $|\overline{\mathcal{J}}| \leq L(n,r)$, it follows that $|\overline{\mathcal{J}}| \leq n\binom{r+2}{n}$ if $n \leq r/2$ and $|\overline{\mathcal{J}}| \leq 2^r$ for all r. Therefore if either r or n is small, then the problem (LDU) is of manageable size. Notice that the set $\overline{\mathcal{J}}$, can be generated in $r+1$ main steps as follows: For $\ell = 0, \dots, r$ define

$$\overline{\mathcal{J}}_\ell = \left\{ J \subseteq \{0, \dots, \ell\} : P_J(\overline{W}_{\{0,\dots,\ell\}}, \overline{K}_{\{0,\dots,\ell\}}) \neq \emptyset \right\}.$$

Then $\overline{\mathcal{J}} = \overline{\mathcal{J}}_r$. Notice that given an (efficient) enumeration of $\overline{\mathcal{J}}_\ell$ it is possible to construct an (efficient) enumeration of $\overline{\mathcal{J}}_{\ell+1}$ by checking for each $J \in \overline{\mathcal{J}}_\ell$ whether $J, J \cup \{\ell+1\} \in \overline{\mathcal{J}}_{\ell+1}$. The latter corresponds to checking two linear feasibility problem with $\ell+1$ constraints and n variables.

The proof of Proposition 2 is based on the following lemma.

Lemma 3 *Let* $\overline{W} \in \mathbf{R}^{(r+1)\times n}$, $\overline{K} \in \mathbf{R}^{r+1}$, $J \subseteq \{0,1,\dots,r\}$ *and let* P_J *be as in* (1). *If* $P_J \neq \emptyset$, *then for* $y \in \mathbf{R}^{r+1}$, $a \in \mathbf{R}^n$ *and* $b \in \mathbf{R}$ *we have*

$$y \cdot (\overline{W}s - \overline{K})^+ \geq a \cdot s - b \text{ for all } s \in P_J \tag{2}$$

if and only if there exist $\gamma^J \in \mathbf{R}^J_+$, $\beta^J \in \mathbf{R}^{J^c}_+$ *such that*

$$\begin{aligned} y_J \cdot \overline{W}_J - a &\geq \gamma^J \cdot \overline{W}_J - \beta^J \cdot \overline{W}_{J^c} \\ y_J \cdot \overline{K}_J - b &\leq \gamma^J \cdot \overline{K}_J - \beta^J \cdot \overline{K}_{J^c} \end{aligned} \tag{3}$$

Proof. For all $s \in P_J = \{s : \overline{W}_J s \geq \overline{K}_J,\ \overline{W}_{J^c} s \leq \overline{K}_{J^c},\ s \geq 0\}$ we have

$$y \cdot (\overline{W}s - \overline{K})^+ - a \cdot s + b = y_J \cdot (\overline{W}_J s - \overline{K}_J) - a \cdot s + b = (y_J \cdot \overline{W}_J - a) \cdot s - (y_J \cdot \overline{K}_J - b).$$

So if we consider the linear program

$$\begin{array}{ll} \min\limits_{s} & (y_J \cdot \overline{W}_J - a) \cdot s \\ \text{s.t.} & \overline{W}_J\, s \geq \overline{K}_J \\ & -\overline{W}_{J^c}\, s \geq -\overline{K}_{J^c} \\ & s \geq 0, \end{array} \tag{4}$$

it follows that (2) holds if and only if the optimal value of the linear program (4) is at least $y_J \cdot \overline{K}_J - b$. By linear programming duality, the latter holds if and only if there exist $\gamma^J \in \mathbf{R}^J_+,\ \beta^J \in \mathbf{R}^{J^c}_+$ such that (3) holds. □

Proof of Proposition 2. Observe that the problem (DU) can be rewritten as

$$\begin{array}{ll} \inf\limits_{y} & z + p \cdot y \\ \text{s.t.} & \begin{bmatrix} -1 \\ y \end{bmatrix} \cdot (\overline{W}s - \overline{K})^+ \geq -z \quad \forall s \in P_J, \quad J \in \overline{\mathcal{J}}. \end{array}$$

Then by applying Lemma 3 it follows that (DU) can be rewritten as (LDU), proving **(i)**.

Notice that the linear programming dual of (LDU) is

$$\begin{array}{lll} \max & \sum\limits_{J \in \mathcal{J}: 0 \in J} u^J \cdot w^0 - t^J K_0 & \\ \text{s.t.} & \sum\limits_{J \in \mathcal{J}} t^J = 1 & \\ & \sum\limits_{J \in \mathcal{J}: j \in J} u^J \cdot w^j - t^J K_j = p_j & j = 1, \ldots, r \\ & \overline{W}_J u^J \geq t^J \cdot \overline{K}_J & J \in \overline{\mathcal{J}} \\ & \overline{W}_{J^c} u^J \leq t^J \cdot \overline{K}_{J^c} & J \in \overline{\mathcal{J}} \\ & & \\ & u^J \in \mathbf{R}^n_+, \quad t^J \in \mathbf{R}_+ & J \in \overline{\mathcal{J}} \end{array} \tag{LU}$$

For a given $k > 0$ let (LU_k) be the linear program obtained adding the constrains

$$u^J_i \leq t^J k \qquad i = 1, \ldots, n \quad J \in \overline{\mathcal{J}}$$

to the program (LU). To prove **(ii)** it is enough to prove the following claim.

Claim 4 *The sequence* $\nu_k =$ optimal value of (LU_k), $k = 1, 2, \ldots,$ *converges to* $\nu_U =$ optimal value of (U). *Also, the optimal solution of* (LU_k) *yields a* (U)*-feasible atomic underlying asset price distribution whose* (U)*-value is equal to* v_k. *Moreover if the optimal value of* (U) *is attained then there is* M *such that* $\nu_k = \nu_U$ *for all* $k > M$.

Proof. First notice that ν_k is increasing. Moreover, given k and a (LU$_k$)-feasible solution $(t, u) = (t^J : J \in \overline{\mathcal{J}}, u^J : J \in \overline{\mathcal{J}})$, let π be the atomic underlying asset price distribution concentrated on the atoms

$$\left\{ \frac{u^J}{t^J} : J \in \overline{\mathcal{J}},\, t^J > 0 \right\},$$

defined by $\pi\left(\frac{u^J}{t^J}\right) = t^J$. Then the atomic asset price distribution π is a feasible solution for (U) with (U)-value equal to the (LU)-value of (t, u). Therefore $\nu_k \leq \nu_U$ for all k. Next, given $\epsilon > 0$ let π be a (U)-feasible measure with (U)-value greater than $\nu_U - \epsilon$. Define

$$t_\pi^J = \pi(P_J) \quad \text{and} \quad u_\pi^J = t_\pi^J \int_{P_J} s d\pi(s)$$

for every $J \in \overline{\mathcal{J}}$. Notice that $t_\pi^J = 0$ implies $u_\pi^J = 0$ and therefore there exists $M > 0$ such that for every $k > M$,

$$(t_\pi, u_\pi) = (t_\pi^J : J \in \mathcal{J}, u_\pi^J : J \in \overline{\mathcal{J}}),$$

is (LU$_k$)-feasible and

$$\nu_k \geq \sum_{J \in \overline{\mathcal{J}} : 0 \in J} u_\pi^J \cdot w^0 - t_\pi^J K_0 = \mathbb{E}_\pi \left[w^0 \cdot S - K_0 \right] \geq \nu_U - \epsilon.$$

Notice that if the optimal value of (U) is attained then we can take $\epsilon = 0$ and therefore $\nu_k = \nu_U$ for all $k > M$. □

2.2 General lower bound problem

We can apply the same analysis to the static-arbitrage lower bound on the price of the call $(w^0 \cdot S - K_0)^+$ given the prices of the calls $(w^j \cdot S - K^j)^+$, $j = 1, \ldots, r$. In this case the primal problem is

$$\begin{array}{lll} \inf\limits_{\pi} & \mathbb{E}_\pi[(w^0 \cdot S - K_0)^+] & \\ \text{s.t.} & \mathbb{E}_\pi[1] = 1 & \\ & \mathbb{E}_\pi[(w^j \cdot S - K_j)^+] = p_j, \quad j = 1, ..., r & \\ & \pi \quad \text{a distribution in } \mathbf{R}_+^n. & \end{array} \tag{L}$$

The dual of (L) is

$$\begin{array}{ll} \sup\limits_{y,z} & z + p \cdot y \\ \text{s.t.} & z + y \cdot (Ws - K)^+ \leq (w^0 \cdot s - K_0)^+ \quad \text{for all } s \in \mathbf{R}_+^n \end{array} \tag{DL}$$

Now the dual problem (DL) aims to find the most expensive portfolio of cash and options $(w^j \cdot S - K^j)^+$, $j = 1, \ldots, r$ that sub-replicates the option $(w^0 \cdot S - K_0)^+$.

Proceeding exactly as in Section 2.1, we obtain the following result.

Proposition 5 **(i)** *The dual problem* (DL) *can be rewritten as the following linear program*

$$\begin{aligned}
\max \quad & z + p \cdot y \\
\text{s.t.} \quad & \begin{bmatrix} -1 \\ y \end{bmatrix}_J \cdot \overline{W}_J \le -\gamma^J \cdot \overline{W}_J + \beta^J \cdot \overline{W}_{J^c} && J \in \overline{\mathcal{J}} \\
& -z + \begin{bmatrix} -1 \\ y \end{bmatrix}_J \cdot K_J \ge -\gamma^J \cdot K_J + \beta^J \cdot K_{J^c} && J \in \overline{\mathcal{J}} \\
& y \in \mathbf{R}^r, \quad z \in \mathbf{R} \\
& \gamma^J \in \mathbf{R}^J_+, \quad \beta^J \in \mathbf{R}^{J^c}_+, && J \in \overline{\mathcal{J}}.
\end{aligned} \tag{LDL}$$

(ii) *Let* (LL) *denote the linear programming dual of* (LDL)*. Then* (LL) *can be modified to produce a sequence of programs* (LL_k) *that yield a sequence of atomic underlying asset price distributions whose value for* (L) *converges to the optimal value of* (L)*. Moreover if the optimal value of* (L) *is attained then this sequence is eventually constant.*

2.3 Upper versus lower bounds

Sections 2.1 and 2.2 provide similar linear programming formulations for the computation of upper and lower bounds. We next show that in each case the linear programs can be simplified a bit further. Such simplifications illustrate the different character of the upper and lower bounds. In Section 3 we use the simplification for the upper bound to obtain specialized results in the case when only call prices on single assets are available.

Let $\mathcal{J}$ be defined in terms of (W, K) in the same way as $\overline{\mathcal{J}}$ is defined in terms of $(\overline{W}, \overline{K})$. In other words, for $J \subseteq \{1, \ldots, n\}$ let

$$P_J = P_J(W, K) := \left\{ s : \begin{array}{c} W_J\, s \ge K_J \\ W_{J^c}\, s \le K_{J^c} \\ s \ge 0 \end{array} \right\}$$

And let $\mathcal{J} = \{J \subseteq \{1, \ldots, r\} : P_J \ne \emptyset\}$.

Proposition 6 *The constraints of the linear program* (LDU) *can also be written as*

$$\begin{aligned}
& -w^0 + y_J \cdot W_J \ge \gamma^J \cdot W_J - \beta^J \cdot W_{J^c}, && J \in \mathcal{J} \\
& -z - K_0 + y_J \cdot K_J \le \gamma^J \cdot K_J - \beta^J \cdot K_{J^c}, && J \in \mathcal{J} \\
& \tilde{y}_J \cdot W_J \ge \tilde{\gamma}^J \cdot W_J - \tilde{\beta}^J \cdot W_{J^c}, && J \in \mathcal{J} \\
& -K_0 + \tilde{y}_J \cdot K_J \le \tilde{\gamma}^J \cdot K_J - \tilde{\beta}^J \cdot K_{J^c}, && J \in \mathcal{J} \\
& y, \tilde{y} \in \mathbf{R}^r, \quad z, \tilde{z} \in \mathbf{R} \\
& \gamma^J, \tilde{\gamma}^J \in \mathbf{R}^J_+, \quad \beta^J, \tilde{\beta}^J \in \mathbf{R}^{J^c}_+, && J \in \mathcal{J}.
\end{aligned} \tag{5}$$

Proof. We modify the proof of Proposition 2: Observe that the constraints of the problem (DU) can also be rewritten as

$$\begin{aligned}
& z + y \cdot (Ws - K)^+ \ge 0 \quad \forall s \in P_J, && J \in \mathcal{J} \\
& z + y \cdot (Ws - K)^+ - w^0 \cdot s + K_0 \ge 0 \quad \forall\, s \in P_J, && J \in \mathcal{J}.
\end{aligned}$$

The statement above then follows from Lemma 3. □

Proposition 7 *The constraints of the linear program* (LDL) *can also be written as*

$$\begin{array}{ll} -\zeta^J w^0 + y_J \cdot W_J \leq -\gamma^J \cdot W_J + \beta^J \cdot W_{J^c} & J \in \mathcal{J} \\ -z - \zeta^J K_0 + y_J \cdot K_J \geq -\gamma^J \cdot K_J + \beta^J \cdot K_{J^c} & J \in \mathcal{J} \\ \zeta^J \leq 1, & J \in \mathcal{J} \\ y \in \mathbf{R}^r, \quad z \in \mathbf{R} & \\ \zeta^J \in \mathbf{R}_+, \quad \gamma^J \in \mathbf{R}^J_+, \quad \beta^J \in \mathbf{R}^{J^c}_+, & J \in \mathcal{J}. \end{array} \tag{6}$$

Proof. Observe that the constraints of the problem (DL) can be rewritten as

$$z + y \cdot (Ws - K)^+ \leq (w^0 \cdot s - K_0)^+ \quad \text{for all } \ s \in P_J, \quad J \in \mathcal{J}.$$

Thus we need to show that for a given $J \in \mathcal{J}$

$$z + y \cdot (Ws - K)^+ \leq (w^0 \cdot s - K_0)^+ \text{ for all } s \in P_J \tag{7}$$

if and only if there exists $y \in \mathbf{R}^r$, $z \in \mathbf{R}, \zeta^J \in \mathbf{R}_+, \gamma^J \in \mathbf{R}^J_+, \beta^J \in \mathbf{R}^{J^c}_+$ such that

$$\begin{array}{l} -\zeta^J w^0 + y_J \cdot W_J \leq -\gamma^J \cdot W_J + \beta^J \cdot W_{J^c} \\ -z - \zeta^J K_0 + y_J \cdot K_J \geq -\gamma^J \cdot K_J + \beta^J \cdot K_{J^c} \\ \zeta^J \leq 1. \end{array} \tag{8}$$

If $P_1 := P_J \cap \{s : w^0 \cdot s - K_0 \leq 0\} = \emptyset$ then (7) holds if and only if $y(Ws - K)^+ \leq 0$ for all $s \in P_J$ and the lemma follows from Lemma 3 with $\zeta^J = 0$. Likewise, if $P_2 := P_J \cap \{s : w^0 \cdot s - K_0 \geq 0\} = \emptyset$, then the lemma follows from Lemma 3 with $\zeta^J = 1$. If both P_1 and P_2 are nonempty then (7) holds if and only if both

$$z + y \cdot (Ws - K)^+ \leq 0 \text{ for all } s \in P_1$$

and

$$z + y \cdot (Ws - K)^+ \leq (w^0 \cdot s - K_0) \text{ for all } s \in P_2,$$

which, by Lemma 3, hold if and only if there exist $\zeta^J \in \mathbf{R}_+, \gamma^J \in \mathbf{R}^J_+, \beta^J \in \mathbf{R}^{J^c}_+$ and $\tilde{\zeta}^J \in \mathbf{R}_+, \tilde{\gamma}^J \in \mathbf{R}^J_+, \tilde{\beta}^J \in \mathbf{R}^{J^c}_+$ such that

$$\begin{array}{l} -\zeta^J w^0 + y_J \cdot W_J \leq -\gamma^J \cdot W_J + \beta^J \cdot W_{J^c} \\ -z - \zeta^J K_0 + y_J \cdot K_J \geq -\gamma^J \cdot K_J + \beta^J \cdot K_{J^c} \end{array} \tag{9}$$

and

$$\begin{array}{l} (\tilde{\zeta}^J - 1) w^0 + y_J \cdot W_J \leq -\tilde{\gamma}^J \cdot W_J + \tilde{\beta}^J \cdot W_{J^c} \\ -\tilde{z} + (\tilde{\zeta}^J - 1) K_0 + y_J \cdot K_J \geq -\tilde{\gamma}^J \cdot K_J + \tilde{\beta}^J \cdot K_{J^c} \end{array} \tag{10}$$

To finish, observe that if both (9) and (10) hold, then a suitable convex combination of them yields (8). Conversely, if (8) holds, then (9) holds for the same choice of variables, and (10) holds as well for $\tilde{\zeta}^J = 1 - \zeta^J$ and $\tilde{\gamma}^J = \gamma^J, \tilde{\beta}^J = \beta^J$. □

3 Upper bound given calls on single assets

We now consider the computation of the upper bound on the price of the basket $(w^0 \cdot S - K_0)^+$ in the special case when prices of m calls $p_i^j = \mathbb{E}_\pi[(S_i - K_i^j)^+]$, $j = 1, \dots, m$ and a forward $p_i^0 = \mathbb{E}_\pi[S_i]$ for each asset $i = 1, \dots, n$ are known. Notice that the assumption on the number of options per asset can be made without loss of generality as long as only vanilla options are given. If one of the assets has fewer than m options, we can artificially increase the number of known options to m by repeating one of the options.

In this case static upper bound problem (U) is

$$\begin{array}{lll} \sup\limits_{\pi} & \mathbb{E}_\pi[(w^0 \cdot S - K_0)^+] & \\ \text{s.t.} & \mathbb{E}_\pi[1] = 1 & \\ & \mathbb{E}_\pi[S] = p^0 & \\ & \mathbb{E}_\pi[(S - K^j)^+] = p^j, & j = 1, \dots, m \\ & \pi \quad \text{a distribution in } \mathbf{R}_+^n. & \end{array} \tag{11}$$

Without loss of generality assume

$$\vec{0} \leq K^1 \leq \cdots \leq K^m \in \mathbf{R}^n.$$

It is convenient to put $K^0 := \vec{0}$ so that the dual of (11) can be written as

$$\begin{array}{lll} \inf\limits_{z,y} & z + \sum\limits_{j=0}^{m} p^j \cdot y^j & \\ \text{s.t.} & z + \sum\limits_{j=0}^{m} y^j \cdot (s - K^j)^+ \geq (w^0 \cdot s - K_0)^+ & \text{for all } s \in \mathbf{R}_+^n \\ & y^j \in \mathbf{R}^n, \ j = 0, \dots, m & \\ & z \in \mathbf{R}. & \end{array} \tag{12}$$

By Proposition 1, the optimal values of (11) and (12) are the same. In this section we show that the linear programming formulation of (12) derived in Section 2.1 can be greatly simplified (see Theorem 11 below). This simplification in turn yields a closed-form formula (14) for the optimal value of (11), (12) provided that the prices of the given vanilla options are arbitrage-free. Furthermore, the simplification given by Theorem 11 enables the efficient modeling of additional features such as bid-ask spreads, transaction costs, and diversification constraints (see Section 3.3).

The closed-form formula (14) generalizes the closed-form formula derived by d'Aspremont and El Ghaoui [3, Prop.4]. The latter corresponds to (14) for the special case $m = 1$. The optimal super-replicating strategy that solves (12), given by Theorem 10 below, is a kind of "dual counterpart" of (14). For ease of exposition, we first present the results for the case $w^0 \geq 0$. In Section 3.2 we show how these results extend to general w^0.

It is well-known [1] that a set of vanilla options $(S_i - K_i^j)^+, \ j = 0, 1, \dots, m$ in asset i is arbitrage-free if and only if the following convexity condition holds:

$$0 \leq \frac{p_i^{m-1} - p_i^m}{K_i^m - K_i^{m-1}} \leq \frac{p_i^{m-2} - p_i^{m-1}}{K_i^{m-1} - K_i^{m-2}} \leq \cdots \leq \frac{p_i^1 - p_i^2}{K_i^2 - K_i^1} \leq \frac{p_i^0 - p_i^1}{K_i^1} \leq 1. \tag{13}$$

3.1 Special case: w^0 non-negative

Theorem 8 *If $w^0 \geq 0$ and the arbitrage-free condition* (13) *holds, then the optimal value of* (11) *is*

$$\max_{\tau\in[0,1]} (w^0 \cdot \nu(\tau) - \tau K_0) \tag{14}$$

where

$$\nu(\tau)_i := \min_{j=0,\ldots,m} \left\{ p_i^j + \tau K_i^j \right\},\ i = 1,\ldots,n.$$

Remark 9 Observe that because the components of $\nu(\tau)$ are piece-wise linear, the value (14) can be found by evaluating the points where the components of $\nu(\tau)$ change slopes, that is, the $mn+2$ points:

$$\tau = 0, 1,\ \text{and}\ \tau = \frac{p_i^{j-1} - p_i^j}{K_i^j - K_i^{j-1}},\ i = 1,\ldots,n,\ j = 1,\ldots,m. \tag{15}$$

Furthermore, because each $\nu(\tau)_i$ is concave, it follows that the function $w^0 \cdot \nu(\tau) - \tau K_0$ is concave as well. Thus once the $mn+2$ breakpoints in (15) are sorted, the one-dimensional optimization problem (14) can be solved via a binary search by evaluating $w^0 \cdot \nu(\tau) - \tau K_0$ at $2\log(nm+2)$ points.

We next present an optimal super-replication trading strategy that solves (12), and whose value is consequently (14). To this end, we first introduce some convenient notation: For a given $\tau \in [0,1)$ and each $i = 1,\ldots,n$ let $j_i[\tau] \in \{1,\ldots,m\}$ be such that

$$\nu(t)_i = p_i^{j_i[\tau]} + tK_i^{j_i[\tau]} \text{ for } t \downarrow \tau.$$

Likewise, for $\tau \in (0,1]$ and each $i = 1,\ldots,n$ let $j_i'[\tau] \in \{1,\ldots,m\}$ be such that

$$\nu(t)_i = p_i^{j_i'[\tau]} + tK_i^{j_i'[\tau]} \text{ for } t \uparrow \tau.$$

Notice that for $\tau \in (0,1)$ we have $j_i'[\tau] = j_i[\tau] + 1$ if τ is one of the relevant breakpoints (15) for $\nu(\tau)_i$. Otherwise $j_i'[\tau] = j_i[\tau]$.

Theorem 10 *Assume $w^0 \geq 0$ and the arbitrage-free condition* (13) *holds. Let $\overline{\tau}$ be the maximizer of* (14)*.*

(a) *If $\overline{\tau} \in (0,1)$, then an optimal solution to* (12) *is*

$$z = 0, \quad y_i^{j_i[\overline{\tau}]} = \lambda w_i^0,\ y_i^{j_i'[\overline{\tau}]} = (1-\lambda) w_i^0, \quad \textit{and} \quad y_i^j = 0 \ \text{ for all other } i, j, \tag{16}$$

where $\lambda \in [0,1]$ satisfies

$$\sum_{i=1}^n w_i^0 (\lambda K_i^{j_i[\overline{\tau}]} + (1-\lambda) K_i^{j_i'[\overline{\tau}]}) = K_0. \tag{17}$$

(b) *If $\overline{\tau} = 0$ then an optimal solution to* (12) *is*

$$z = 0, \quad y_i^{j_i[\overline{\tau}]} = w_i^0, \quad \textit{and} \quad y_i^j = 0 \textit{ for all other } i, j.$$

(c) *If $\overline{\tau} = 1$ then an optimal solution to* (12) *is*

$$z = \sum_{i=1}^{n} w_i^0 K_i^{j_i'[\overline{\tau}]} - K_0, \;\; y_i^{j_i'[\overline{\tau}]} = w_i^0, \quad \text{and } y_i^j = 0 \text{ for all other } i, j.$$

For ease of exposition, we present the two key components of Theorems 8 and 10, namely Theorem 11 and Lemma 12. We defer the proofs of these two components to Section 3.4.

In what follows, we use the following convenient notation: Given two vectors $u, v \in \mathbf{R}^n$ let $x \circ y \in \mathbf{R}^n$ denote the component-wise product of x and y, i.e., $(x \circ y)_i = x_i y_i, \; i = 1, \ldots, n$. Let $\text{cone}(K) \subseteq \mathbf{R}_+^{2n(m+1)+1}$ denote the cone of vectors $(v, T, \tau) := (v^0, \ldots, v^m, T^0, \ldots, T^m, \tau), \; v^i \in \mathbf{R}_+^n, \; T^i \in \mathbf{R}_+^n, \; \tau \in \mathbf{R}_+$ that satisfy the following conditions

$$\begin{array}{ll} T^i \circ K^i \le v^i, & i = 0, \ldots, m, \\ v^i \le T^i \circ K^{i+1}, & i = 0, \ldots, m-1, \\ \displaystyle\sum_{i=0}^{m} T^i - \tau e = 0. & \end{array}$$

Theorem 11 *Assume $\vec{0} = K^0 \le K^1 \le \cdots \le K^m \in \mathbf{R}^m$ are given. Then $z \in \mathbf{R}$, $y^i \in \mathbf{R}^n$ satisfy*

$$z + \sum_{j=0}^{m} y^j \cdot (s - K^j)^+ \ge (w^0 \cdot s - K_0)^+ \text{ for all } s \in \mathbf{R}_+^n \tag{18}$$

if and only if

$$\begin{array}{l} \left(\left(\sum_{j=0}^{i} y^j - w_0 \right)_{i=0,\ldots,m}, \left(-\sum_{j=0}^{i} K^j \circ y^j \right)_{i=0,\ldots,m}, z + K_0 \right) \in \text{cone}(K)^* \\ \left(\left(\sum_{j=0}^{i} y^j \right)_{i=0,\ldots,m}, \left(-\sum_{j=0}^{i} K^j \circ y^j \right)_{i=0,\ldots,m}, z \right) \in \text{cone}(K)^*. \end{array} \tag{19}$$

In particular, the optimal super-replication problem (12) *can be rewritten as*

$$\begin{array}{ll} \displaystyle\min_{z,y} & z + \displaystyle\sum_{j=0}^{m} p^j \cdot y^j \\ \text{s.t.} & \left(\left(\sum_{j=0}^{i} y^j - w_0 \right)_{i=0,\ldots,m}, \left(-\sum_{j=0}^{i} K^j \circ y^j \right)_{i=0,\ldots,m}, z + K_0 \right) \in \text{cone}(K)^* \\ & \left(\left(\sum_{j=0}^{i} y^j \right)_{i=0,\ldots,m}, \left(-\sum_{j=0}^{i} K^j \circ y^j \right)_{i=0,\ldots,m}, z \right) \in \text{cone}(K)^* \\ & z \in \mathbf{R}, \; y^i \in \mathbf{R}^n, \; i = 0, \ldots, m, \end{array} \tag{20}$$

and its linear programming dual computes the optimal value of (11)*:*

$$\begin{array}{ll} \displaystyle\max_{v,T,\tau,\tilde{v},\tilde{T},\tilde{\tau}} & w^0 \cdot \displaystyle\sum_{i=0}^{m} v^i - K_0 \tau \\ \text{s.t.} & \tau + \tilde{\tau} = 1 \\ & \displaystyle\sum_{i=j}^{m} \left(v^i + \tilde{v}^i - (T^i + \tilde{T}^i) \circ K^j \right) = p^j, \quad j = 0, \ldots, m \\ & (v, T, \tau), (\tilde{v}, \tilde{T}, \tilde{\tau}) \in \text{cone}(K). \end{array} \tag{21}$$

Proof. See Section 3.4. □

The format of the linear program (21) suggests to a one-dimensional parametrization in term of τ: Once τ is fixed, the problem (21) decomposes into n one-dimensional problems of similar structure. It turns out that each of these problems has a closed-form solution, as stated in the following lemma. For notational convenience, the statement of Lemma 12 uses the objects $p, K, \mathrm{cone}(K)$ as above but in the special one-dimensional case.

Lemma 12 *Assume $n = 1$, $\tau \in [0,1]$, and the strikes K^j and prices p^j satisfy the arbitrage-free condition* (13)*. Then the optimal value of the linear program*

$$\begin{aligned} \max_{T,v,\tilde{T},\tilde{v}} \quad & \sum_{i=0}^{m} v^i \\ \text{s.t.} \quad & \sum_{i=j}^{m} \left(v^i + \tilde{v}^i - (T^i + \tilde{T}^i) K^j \right) = p^j, \quad j = 0, \ldots, m \\ & (v, T, \tau), (\tilde{v}, \tilde{T}, 1-\tau) \in \mathrm{cone}(K) \end{aligned} \tag{22}$$

is

$$\min_{j=0,\ldots,m} \left\{ p^j + \tau K^j \right\}.$$

Proof. See Section 3.4. □

Proof of Theorem 8. From Lemma 12, the fact that $w^0 \geq 0$, and the structure of the constraints in (21), it follows that the optimal value of (21) is (14). The statement of Theorem 8 then follows from Theorem 11. □

Proof of Theorem 10.

(a) From the construction of $j_i[\overline{\tau}]$ and $j_i'[\overline{\tau}]$ it follows that

$$w^0 \cdot \nu(\tau) - \tau K_0 = \sum_{i=1}^{n} w_i^0 p_i^{j_i[\overline{\tau}]} + \tau \left(\sum_{i=1}^{n} w_i^0 K_i^{j_i[\overline{\tau}]} - K_0 \right) \quad \text{for } \tau \downarrow \overline{\tau},$$

and

$$w^0 \cdot \nu(\tau) - \tau K_0 = \sum_{i=1}^{n} w_i^0 p_i^{j_i'[\overline{\tau}]} + \tau \left(\sum_{i=1}^{n} w_i^0 K_i^{j_i'[\overline{\tau}]} - K_0 \right) \quad \text{for } \tau \uparrow \overline{\tau}.$$

The optimality of $\overline{\tau}$ implies that the right derivative of $w^0 \cdot \nu(\tau) - \tau K_0$ at $\overline{\tau}$ is non-positive, i.e., $\sum_{i=1}^{n} w_i^0 K_i^{j_i[\overline{\tau}]} - K_0 \leq 0$; it also implies that the right derivative of $w^0 \cdot \nu(\tau) - \tau K_0$ at $\overline{\tau}$ is non-negative, i.e., $\sum_{i=1}^{n} w_i^0 K_i^{j_i'[\overline{\tau}]} - K_0 \geq 0$. Thus there is some $\lambda \in [0,1]$ such that (17) holds. In particular,

$$w^0 \cdot \nu(\overline{\tau}) - \overline{\tau} K_0 = \sum_{i=1}^{n} \left(\lambda w_i^0 p_i^{j_i[\overline{\tau}]} + (1-\lambda) w_i^0 p_i^{j_i'[\overline{\tau}]} \right).$$

Thus the point z, y_i^j given by (16) has objective value equal to (14). Therefore to finish it suffices to show that it is feasible for (12). Indeed, since $w^0 \geq 0$ and $\lambda \in [0,1]$,

$$\sum_{i=1}^{n} \left(\lambda w_i^0 (s_i - K_i^{j_i[\overline{\tau}]})^+ + (1-\lambda) w_i^0 (s_i - K_i^{j_i'[\overline{\tau}]})^+ \right) \geq 0,$$

and

$$\begin{aligned}
&\sum_{i=1}^{n} \left(\lambda w_i^0 (s_i - K_i^{j_i[\overline{\tau}]})^+ + (1-\lambda) w_i^0 (s_i - K_i^{j_i'[\overline{\tau}]})^+ \right) \\
&\qquad \geq \sum_{i=1}^{n} \left(\lambda w_i^0 (s_i - K_i^{j_i[\overline{\tau}]}) + (1-\lambda) w_i^0 (s_i - K_i^{j_i'[\overline{\tau}]}) \right) \\
&\qquad = \sum_{i=1}^{n} w_i^0 s_i - \sum_{i=1}^{n} \left(\lambda w_i^0 K_i^{j_i[\overline{\tau}]} + (1-\lambda) w_i^0 K_i^{j_i'[\overline{\tau}]} \right) = w^0 \cdot s - K_0.
\end{aligned}$$

(The last step follows from (17).)

(b,c) These follow via similar (but simpler) arguments to that in (a).

□

3.2 General case: unrestricted w^0

At the expense of slightly more complicated notation, Theorems 8 and 10 extend to the general case when w^0 is unrestricted. First, extend the definition of $\nu(\tau)_i$ as follows. Define

$$I^+ := \left\{ i \in \{1, \ldots, n\} : w_i^0 \geq 0 \right\}, \quad I^- := \left\{ i \in \{1, \ldots, n\} : w_i^0 < 0 \right\},$$

and let

$$\nu(\tau)_i = \begin{cases} \min\limits_{j=0,\ldots,m} \left\{ K_i^j + \tau K_i^j \right\} & \text{if } i \in I^+ \\ p_i^0 - \min\limits_{j=0,\ldots,m} \left\{ p_i^j + (1-\tau) K_i^j \right\} & \text{if } i \in I^-. \end{cases}$$

Theorem 13 *If the arbitrage-free condition* (13) *holds, then the optimal value of* (11) *is*

$$\max_{\tau \in [0,1]} (w^0 \cdot \nu(\tau) - \tau K_0). \tag{23}$$

Theorem 13 follows from Theorem 11 and the following counterpart of Lemma 12.

Lemma 14 *Assume $n = 1$, $\tau \in [0,1]$, and the strikes K^j and prices p^j satisfy the arbitrage-free condition* (13)*. Then the optimal value of the linear program*

$$\begin{array}{ll} \min\limits_{T,v,\tilde{T},\tilde{v}} & \displaystyle\sum_{i=0}^{m} v^i \\ \text{s.t.} & \displaystyle\sum_{i=j}^{m} \left(v^i + \tilde{v}^i - (T^i + \tilde{T}^i) K^j \right) = p^j, \quad j = 0, \ldots, m \\ & (v, T, \tau), (\tilde{v}, \tilde{T}, 1-\tau) \in \operatorname{cone}(K) \end{array} \tag{24}$$

is

$$p^0 - \min_{j=0,\ldots,m} \left\{ p^j + (1-\tau) K^j \right\}. \tag{25}$$

Proof. From the first constraint in (24) for $j = 0$ and the last constraint, we get

$$\sum_{i=0}^{m} v^i = p^0 - \sum_{i=0}^{m} \tilde{v}^i. \tag{26}$$

Hence the optimal solution of (24) can be obtained by replacing the objective with $\max_{T,v,\tilde{T},\tilde{v}} \sum_{i=0}^{m} \tilde{v}^i$ subject to the same constraints. The latter problem is identical to (22) with the roles of (v, T, τ) and $(\tilde{v}, \tilde{T}, 1-\tau)$ reversed. Therefore, (25) follows from Lemma 12 and (26). □

Remark 15 Because the components of $\nu(\tau)$ are piecewise linear, the value (23) can be found by evaluating the points where the components of $\nu(\tau)$ change slopes, that is, the $mn + 2$ points:

$$\tau = 0, 1, \;\; \tau = \frac{p_i^{j-1} - p_i^j}{K_i^j - K_i^{j-1}}, \; i \in I^+, \;\; \text{and} \;\; \tau = 1 - \frac{p_i^{j-1} - p_i^j}{K_i^j - K_i^{j-1}}, \; i \in I^-, \; j = 1, \ldots, m. \tag{27}$$

Furthermore, because each $w_i^0 \nu(\tau)_i$ is concave, it follows that the function $w^0 \cdot \nu(\tau) - \tau K_0$ is concave as well. Thus once the $mn + 2$ breakpoints in (27) are sorted, the one-dimensional optimization problem (23) can be solved via a binary search by evaluating $w^0 \cdot \nu(\tau) - \tau K_0$ at $2 \log(nm + 2)$ points.

We next describe the optimal super-replicating strategy that solves (12) and yields the optimal bound (23) in this general case. Once again, we introduce some convenient notation: Let $\tau \in [0, 1]$ be given. For each $i \in I^+$ let $j_i[\tau]$ and $j'_i[\tau]$ be such that

$$\nu(t)_i = p_i^{j_i[\tau]} + t K_i^{j_i[\tau]} \text{ for } t \downarrow \tau, \;\; \text{and } \nu(t)_i = p_i^{j'_i[\tau]} + \tau K_i^{j'_i[\tau]} \text{ for } t \uparrow \tau.$$

And for each $i \in I^-$ let $j_i[\tau]$ and $j'_i[\tau]$ be such that

$$\nu(t)_i = p_i^0 - (p_i^{j_i[\tau]} + (1-t) K_i^{j_i[\tau]}) \text{ for } t \uparrow \tau, \;\; \text{and } \nu(t)_i = p_i^0 - (p_i^{j'_i[\tau]} + (1-t) K_i^{j'_i[\tau]}) \text{ for } t \downarrow \tau.$$

Notice that $j'_i[\tau] = j_i[\tau] + 1$ if τ is one of the relevant breakpoints (27) for $\nu(\tau)_i$, and otherwise $j'_i[\tau] = j_i[\tau]$.

Theorem 16 *Assume the arbitrage-free condition* (13) *holds. Let* $\overline{\tau}$ *be the maximizer of* (23).

(a) *If* $\overline{\tau} \in (0, 1)$ *then an optimal solution to* (12) *is*

$$\begin{array}{ll} z = \sum_{i \in I^-} w_i^0 \left(-\lambda K_i^{j'_i[\overline{\tau}]} - (1-\lambda) K_i^{j_i[\overline{\tau}]} \right) & \\ y_i^{j_i[\overline{\tau}]} = \lambda w_i^0, \; y_i^{j'_i[\overline{\tau}]} = (1-\lambda) w_i^0, & \textit{for } \; i \in I^+ \\ y_i^{j'_i[\overline{\tau}]} = -\lambda w_i^0, \; y_i^{j_i[\overline{\tau}]} = -(1-\lambda) w_i^0, & \textit{for } \; i \in I^- \\ y_i^0 = w_i^0 & \textit{for } \; i \in I^- \\ y_i^j = 0 & \textit{for all other } i, j, \end{array} \tag{28}$$

where $\lambda \in [0, 1]$ *satisfies*

$$\sum_{i \in I^+} w_i^0 \left(\lambda K_i^{j_i[\overline{\tau}]} + (1-\lambda) K_i^{j'_i[\overline{\tau}]} \right) + \sum_{i \in I^-} w_i^0 \left(\lambda K_i^{j'_i[\overline{\tau}]} + (1-\lambda) K_i^{j_i[\overline{\tau}]} \right) = K_0. \tag{29}$$

(b) *If* $\overline{\tau} = 0$ *then an optimal solution to* (12) *is*

$$\begin{aligned}
&z = -\sum_{i\in I^-} w_i^0 K_i^{j_i'[\overline{\tau}]} && \\
&y_i^{j_i[\overline{\tau}]} = w_i^0 && \text{for } i \in I^+ \\
&y_i^{j_i'[\overline{\tau}]} = -w_i^0 && \text{for } i \in I^- \\
&y_i^0 = w_i^0 && \text{for } i \in I^- \\
&y_i^j = 0 && \text{for all other } i, j.
\end{aligned}$$

(c) *If* $\overline{\tau} = 1$ *then an optimal solution to* (12) *is*

$$\begin{aligned}
&z = -\sum_{i\in I^+} w_i^0 K_i^{j_i'[\overline{\tau}]} - K_0 && \\
&y_i^{j_i'[\overline{\tau}]} = w_i^0 && \text{for } i \in I^+ \\
&y_i^{j_i[\overline{\tau}]} = -w_i^0 && \text{for } i \in I^- \\
&y_i^0 = w_i^0 && \text{for } i \in I^- \\
&y_i^j = 0 && \text{for all other } i, j.
\end{aligned}$$

Proof.

(a) From the construction of $j_i[\overline{\tau}]$ and $j_i'[\overline{\tau}]$ it follows that

$$\begin{aligned}
w^0 \cdot \nu(\tau) - \tau K_0 = &\sum_{i\in I^+} w_i^0 p_i^{j_i[\overline{\tau}]} - \sum_{i\in I^-} w_i^0 p_i^{j_i'[\overline{\tau}]} + \sum_{i\in I^-} w_i^0 (p_i^0 - K_i^{j_i'[\overline{\tau}]}) \\
&+ \tau \left(\sum_{i\in I^+} w_i^0 K_i^{j_i[\overline{\tau}]} + \sum_{i\in I^-} w_i^0 K_i^{j_i'[\overline{\tau}]} - K_0 \right) \quad \text{for } \tau \downarrow \overline{\tau},
\end{aligned}$$

and

$$\begin{aligned}
w^0 \cdot \nu(\tau) - \tau K_0 = &\sum_{i\in I^+} w_i^0 p_i^{j_i'[\overline{\tau}]} - \sum_{i\in I^-} w_i^0 p_i^{j_i[\overline{\tau}]} + \sum_{i\in I^-} w_i^0 (p_i^0 - K_i^{j_i[\overline{\tau}]}) \\
&+ \tau \left(\sum_{i\in I^+} w_i^0 K_i^{j_i'[\overline{\tau}]} + \sum_{i\in I^-} w_i^0 K_i^{j_i[\overline{\tau}]} - K_0 \right) \quad \text{for } \tau \uparrow \overline{\tau},
\end{aligned}$$

The optimality of $\overline{\tau}$ implies $\sum_{i\in I^+} w_i^0 K_i^{j_i[\overline{\tau}]} + \sum_{i\in I^-} w_i^0 K_i^{j_i'[\overline{\tau}]} - K_0 \leq 0$ and $\sum_{i\in I^+} w_i^0 K_i^{j_i'[\overline{\tau}]} + \sum_{i\in I^-} w_i^0 K_i^{j_i[\overline{\tau}]} - K_0 \geq 0$. Thus there is some $\lambda \in [0, 1]$ such that (29) holds. In particular,

$$\begin{aligned}
w^0 \cdot \nu(\overline{\tau}) - \overline{\tau} K_0 = &\sum_{i\in I^+} \left(\lambda w_i^0 p_i^{j_i[\overline{\tau}]} + (1-\lambda) w_i^0 p_i^{j_i'[\overline{\tau}]} \right) - \sum_{i\in I^-} \left(\lambda w_i^0 p_i^{j_i'[\overline{\tau}]} + (1-\lambda) w_i^0 p_i^{j_i[\overline{\tau}]} \right) \\
&+ \sum_{i\in I^-} w_i^0 \left(p_i^0 - \lambda K_i^{j_i'[\overline{\tau}]} - (1-\lambda) K_i^{j_i[\overline{\tau}]} \right).
\end{aligned}$$

Thus the point z, y_i^j given by (28) has objective value equal to (23). Therefore to finish it suffices to show that it is feasible for (12). Indeed, since $w^0_{I^+}, -w^0_{I^-} \geq 0$ and $\lambda \in [0,1]$,

$$\begin{aligned}
&\sum_{i\in I^+} w_i^0 \left(\lambda(s_i - K_i^{j_i[\overline{\tau}]})^+ + (1-\lambda)(s_i - K_i^{j'_i[\overline{\tau}]})^+\right) \\
&\qquad - \sum_{i\in I^-} w_i^0 \left(\lambda(s_i - K_i^{j'_i[\overline{\tau}]})^+ + (1-\lambda)(s_i - K_i^{j_i[\overline{\tau}]})^+\right) \\
&\qquad + \sum_{i\in I^-} w_i^0 s_i - \sum_{i\in I^-} w_i^0 \left(\lambda K_i^{j'_i[\overline{\tau}]} + (1-\lambda) K_i^{j_i[\overline{\tau}]}\right) \\
&\qquad \geq - \sum_{i\in I^-} w_i^0 \left(\lambda(s_i - K_i^{j'_i[\overline{\tau}]}) + (1-\lambda)(s_i - K_i^{j_i[\overline{\tau}]})\right) \\
&\qquad\qquad + \sum_{i\in I^-} w_i^0 s_i - \sum_{i\in I^-} w_i^0 \left(\lambda K_i^{j'_i[\overline{\tau}]} + (1-\lambda) K_i^{j_i[\overline{\tau}]}\right) = 0,
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{i\in I^+} w_i^0 \left(\lambda(s_i - K_i^{j_i[\overline{\tau}]})^+ + (1-\lambda)(s_i - K_i^{j'_i[\overline{\tau}]})^+\right) \\
&\qquad - \sum_{i\in I^-} w_i^0 \left(\lambda(s_i - K_i^{j'_i[\overline{\tau}]})^+ + (1-\lambda)(s_i - K_i^{j_i[\overline{\tau}]})^+\right) \\
&\qquad + \sum_{i\in I^-} w_i^0 s_i - \sum_{i\in I^-} w_i^0 \left(\lambda K_i^{j'_i[\overline{\tau}]} + (1-\lambda) K_i^{j_i[\overline{\tau}]}\right) \\
&\qquad \geq \sum_{i\in I^+} w_i^0 \left(\lambda(s_i - K_i^{j_i[\overline{\tau}]}) + (1-\lambda)(s_i - K_i^{j'_i[\overline{\tau}]})\right) \\
&\qquad\qquad + \sum_{i\in I^-} w_i^0 s_i - \sum_{i\in I^-} w_i^0 \left(\lambda K_i^{j'_i[\overline{\tau}]} + (1-\lambda) K_i^{j_i[\overline{\tau}]}\right) = w^0 \cdot s - K_0.
\end{aligned}$$

(The last step follows from (29).)

(b,c) These follow via similar (but simpler) arguments to that in (a). □

3.3 Bid-ask spreads and other additional features

The main statement of Theorem 11 is the reformulation of the super-replication constraint (18) in the drastically more efficient and concise form (19). This also yields efficient reformulations of the optimal super-replication problem when additional features are incorporated in the problem. For instance, as we show below, the more realistic problem that takes into account bid-ask spread in the prices of the known options also lends itself to an efficient linear programming formulation. More specifically, assume the vector of ask (buy) and bid (sell) prices of the options $(S - K^j)^+$ are $p_+^j \geq p_-^j$ respectively. In this case the optimal super-replication problem becomes

$$\begin{array}{lll}
\inf\limits_{z,y,y_+,y_-} & z+\sum\limits_{j=0}^{m}\left(p_+^j\cdot y_+^j - p_-^j\cdot y_-^j\right) & \\
\text{s.t.} & z+\sum\limits_{j=0}^{m} y^j\cdot(s-K^j)^+ \geq (w^0\cdot s-K_0)^+ \quad \text{for all } s\in\mathbf{R}_+^n & \\
& y^j = y_+^j - y_-^j,\ j=0,\ldots,m & \\
& y^j\in\mathbf{R}^n,\ y_+^j,y_-^j\in\mathbf{R}_+^n,\ j=0,\ldots,m & \\
& z\in\mathbf{R},. &
\end{array} \tag{30}$$

The second part of Theorem 11 generalizes as follows.

Theorem 17 *The optimal super-replication problem* (30) *can be rewritten as*

$$\begin{array}{ll}
\min\limits_{z,y,y_+,y_-} & z+\sum\limits_{j=0}^{m}\left(p_+^j\cdot y_+^j - p_-^j\cdot y_-^j\right) \\
\text{s.t.} & \left(\left(\sum_{j=0}^{i} y^j - w_0\right)_{i=0,\ldots,m}, \left(-\sum_{j=0}^{i} K^j\circ y^j\right)_{i=0,\ldots,m}, z+K_0\right)\in\operatorname{cone}(K)^* \\
& \left(\left(\sum_{j=0}^{i} y^j\right)_{i=0,\ldots,m}, \left(-\sum_{j=0}^{i} K^j\circ y^j\right)_{i=0,\ldots,m}, z\right)\in\operatorname{cone}(K)^* \\
& y^j = y_+^j - y_-^j,\ j=0,\ldots,m \\
& y^j\in\mathbf{R}^n,\ y_+^j,y_-^j\in\mathbf{R}_+^n,\ j=0,\ldots,m \\
& z\in\mathbf{R},.
\end{array} \tag{31}$$

Proof. This follows from the equivalence between (18) and (19) in Theorem 11. □

Other features, such as rebalancing transaction costs, as well as linear constraints on the composition of the super-replication portfolio can be incorporated in a similar fashion. These features readily appear in real pricing problems. Thus, our results greatly broaden the range of applicability of the static-arbitrage approach to practical pricing problems. In Section 4, we provide experimental results that illustrate the incorporation of these additional features into the static-arbitrage pricing problem.

3.4 Proofs of Theorem 11 and Lemma 12

The main statement of Theorem 11 is a reformulation of the optimal super-replicating problem (12). The crux for the latter is Theorem 18, which is of independent interest. We shall rely on the following notation. Define the set of partitions $\mathcal{P}(n,m)$ of $\{1,\ldots,n\}$ as follows:

$$\mathcal{P}(n,m) := \left\{ \left(J^0,J^1,\ldots,J^m\right) : \bigcup_{i=0}^{m} J^i = \{1,\ldots,n\}, J^i\cap J^j=\emptyset \text{ for } i\neq j \right\}.$$

Given $J\in\mathcal{P}(n,m)$, define

$$P_J := \left\{ s : K^i_{J^i} \leq s_{J^i} \leq K^{i+1}_{J^i} \text{ for } i=0,1,\ldots,m-1, \ \text{ and } s_{J^m}\geq K^m_{J^m} \right\}.$$

Theorem 18 *Assume $b \in \mathbf{R}^n$ and $c \in \mathbf{R}$ are given. Then $y^j \in \mathbf{R}^n$, $j = 0, \ldots, m$ satisfy*

$$\sum_{j=0}^{m} y^j \cdot (s - K^j)^+ \geq b \cdot s - c \ \ \textit{for all } s \in \mathbf{R}^n_+ \tag{32}$$

if and only if

$$\left(\left(\textstyle\sum_{j=0}^{i} y^j - b \right)_{i=0,\ldots,m}, \left(- \textstyle\sum_{j=0}^{i} K^j \circ y^j \right)_{i=0,\ldots,m}, c \right) \in \operatorname{cone}(K)^*. \tag{33}$$

Proof. The proof consists of the following two main steps.
Step 1: (32) holds if and only if there exist $\gamma^i, \beta^i \in \mathbf{R}^n_+$, $i = 0, \ldots, m-1$ and $\gamma^m \in \mathbf{R}^n_+$ such that

$$\begin{aligned}
&-b + \sum_{j=0}^{i} y^j = \gamma^i - \beta^i, && i = 0, \ldots, m-1 \\
&-b + \sum_{j=0}^{m} y^j = \gamma^m \\
&\sum_{i=0}^{m} \left(\sum_{j=0}^{i} y^j_{J^i} \cdot K^j_{J^i} \right) - c \leq \sum_{i=0}^{m-1} \left(K^i_{J^i} \cdot \gamma^i_{J^i} - K^{i+1}_{J^i} \cdot \beta^i_{J^i} \right) + K^m_{J^m} \cdot \gamma^m_{J^m}, && J \in \mathcal{P}(n, m).
\end{aligned} \tag{34}$$

Step 2: (33) holds if and only if (34) holds.

Details of step 1: Proceeding as in Lemma 3 in Section 2.1, it follows that (32) holds if and only if for each $J \in \mathcal{P}(n, m)$ there exist $\gamma^{i,J}, \beta^{i,J} \in \mathbf{R}^{J^i}_+$, $i = 0, \ldots, m-1$, $\gamma^{m,J} \in \mathbf{R}^{J^m}_+$ such that

$$\begin{aligned}
&-b_{J^i} + \sum_{j=0}^{i} y^j_{J^i} = \gamma^{i,J} - \beta^{i,J}, && i = 0, \ldots, m-1 \\
&-b_{J^m} + \sum_{j=0}^{m} y^j_{J^m} = \gamma^{m,J} \\
&\sum_{i=0}^{m} \left(\sum_{j=0}^{i} y^j_{J^i} \cdot K^j_{J^i} \right) - c \leq \sum_{i=0}^{m-1} \left(K^i_{J^i} \cdot \gamma^{i,J} - K^{i+1}_{J^i} \cdot \beta^{i,J} \right) + K^m_{J^m} \gamma^{m,J} && J \in \mathcal{P}(n, m).
\end{aligned} \tag{35}$$

Notice that (34) implies (35), and henceforth (32). Thus to prove the equivalence between (32) and (34), it suffices to show that if (35) holds, then there exist $\gamma^i, \beta^i \in \mathbf{R}^n_+$, $i = 0, \ldots, m-1$ and $\gamma^m \in \mathbf{R}^n_+$ such that (34) holds. Assume (35) holds. Define $\gamma^m \in \mathbf{R}^n_+$ and $\gamma^i, \beta^i \in \mathbf{R}^n_+$, $i = 0, \ldots, m-1$ as follows. Let $\overline{J} = (\emptyset, \ldots, \emptyset, \{1, \ldots, n\})$ and put

$$\gamma^m := \gamma^{m, \overline{J}}.$$

Then the second equation in (34) holds. Furthermore, $\gamma^{m,J} = \gamma^m_{J^m}$ for all $J \in \mathcal{P}(n, m)$.

Next, fix $i \in \{0, \ldots, m-1\}$. For each $\ell \in \{1, \ldots, n\}$ define the partition $J[i, \ell]$ by

$$J[i, \ell] := \underset{\{J \in \mathcal{P}(n,r) : l \in J^i\}}{\operatorname{argmax}} \left(K^i_\ell \gamma^{i,J}_\ell - K^{i+1}_\ell \beta^{i,J}_\ell \right).$$

Let $\gamma^i, \beta^i \in \mathbf{R}^n_+$ be defined by $\gamma^i_\ell = \gamma^{i,J[i,\ell]}_\ell$ and $\beta^i_\ell = \beta^{i,J[i,\ell]}_\ell$, $\ell \in \{1, \dots, n\}$. From the first identity in (35), applied to $J = J[i, \ell]$, we get

$$-b_{J[i,\ell]^i} + \sum_{j=0}^{i} y^j_{J[i,l]^i} = \gamma^{i,J[i,l]} - \beta^{i,J[i,l]}.$$

In particular,

$$-b_\ell + \sum_{j=0}^{i} y^j_\ell = \gamma^{i,J[i,\ell]}_\ell - \beta^{i,J[i,\ell]}_\ell = \gamma^i_\ell - \beta^i_\ell.$$

This holds for $i \in \{0, \dots, m-1\}$ and $\ell \in \{1, \dots, n\}$ thus the first equation in (34) follows. It only remains to prove the last inequality in (34). To that end, fix $J \in \mathcal{P}(n, m)$. For $i = 0, \dots, m-1$ and $\ell \in J^i$, the construction of $J[i, \ell]$ implies that

$$K^i_\ell \gamma^{i,J}_\ell - K^{i+1}_\ell \beta^{i,J}_\ell \le K^i_\ell \gamma^{i,J[i,\ell]}_\ell - K^{i+1}_\ell \beta^{i,J[i,\ell]}_\ell = K^i_\ell \gamma^i_\ell - K^{i+1}_\ell \beta^i_\ell.$$

Thus

$$K^i_{J^i} \cdot \gamma^{i,J} - K^{i+1}_{J^i} \cdot \beta^{i,J} \le K^i_{J^i} \cdot \gamma^i_{J_i} - K^{i+1}_{J^i} \cdot \beta^i_{J_i}.$$

Hence from the last inequality in (35) and the fact that $\gamma^{m,J} = \gamma^m_{J^m}$ we get

$$\begin{aligned} \sum_{i=0}^{m} \left(\sum_{j=0}^{i} y^j_{J^i} \cdot K^j_{J^i} \right) - c \quad &\le \quad \sum_{i=0}^{m-1} \left(K^i_{J^i} \cdot \gamma^{i,J} - K^{i+1}_{J^i} \cdot \beta^{i,J} \right) + K^m_{J^m} \gamma^{m,J} \\ &\le \quad \sum_{i=0}^{m-1} \left(K^i_{J^i} \cdot \gamma^i_{J_i} - K^{i+1}_{J^i} \cdot \beta^i_{J_i} \right) + K^m_{J^m} \gamma^m_{J^m}. \end{aligned}$$

We thus get (34). This completes the equivalence between (32) and (34).

Details of step 2: Note that the dual $\mathrm{cone}(K)^* \subseteq \mathbf{R}^{2n(m+1)+1}$ of $\mathrm{cone}(K)$ is the set of vectors $(u, \Theta, t) := (u^0, \dots, u^m, \Theta^0, \dots, \Theta^m, t)$ that satisfy the following conditions for some $\gamma^m \in \mathbf{R}^n_+$, $\gamma^i, \beta^i \in \mathbf{R}^n_+$, $i = 0, \dots, m-1$, $\xi \in \mathbf{R}^n$

$$\begin{array}{ll} u^i = \gamma^i - \beta^i, & i = 0, \dots, m-1 \\ u^m = \gamma^m & \\ \Theta^i \ge -\xi - K^i \circ \gamma^i + K^{i+1} \circ \beta^i, & i = 0, \dots, m-1 \\ \Theta^m \ge -\xi - K^m \circ \gamma^m & \\ t \ge e^{\mathrm{T}} \xi. & \end{array} \tag{36}$$

Therefore (33) holds if and only if for some $\gamma^m \in \mathbf{R}^n_+$, $\gamma^i, \beta^i \in \mathbf{R}^n_+$, $i = 0, \dots, m-1$, and $\xi \in \mathbf{R}^n$

$$\begin{array}{ll} \displaystyle\sum_{j=0}^{m} y^j - b = \gamma^m & \\ \displaystyle\sum_{j=0}^{i} y^j - b = \gamma^i - \beta^i & i = 0, \dots, m-1 \\ \displaystyle\sum_{j=0}^{i} K^j \circ y^j \le \xi + K^i \circ \gamma^i - K^{i+1} \circ \beta^i & i = 0, \dots, m-1 \\ \displaystyle\sum_{j=0}^{m} K^j \circ y^j \le \xi + K^m \circ \gamma^m & \\ -c \le -e^{\mathrm{T}} \xi. & \end{array} \tag{37}$$

We next show the equivalence between (37) and (34). Let $J \in \mathcal{P}(n,m)$ be given. From (37) we have

$$\sum_{j=0}^{i} K^j_{J^i} \circ y^j_{J^i} \le \xi_{J^i} + K^i_{J^i} \circ \gamma^i_{J^i} - K^{i+1}_{J^i} \circ \beta^i_{J^i},\ i = 0, \dots, m-1.$$

and

$$\sum_{j=0}^{m} K^j_{J^m} \circ y^j_{J^m} \le \xi_{J^m} + K^m_{J^m} \circ \gamma^m_{J^m},$$

Adding all of these inequalities and rearranging terms, we get

$$\sum_{i=0}^{m} \left(\sum_{j=0}^{i} y^j_{J^i} \cdot K^j_{J^i} \right) - e^{\mathrm{T}}\xi \le \sum_{i=0}^{m-1} \left(K^i_{J^i} \cdot \gamma^i_{J^i} - K^{i+1}_{J^i} \cdot \beta^i_{J^i} \right) + K^m_{J^m} \cdot \gamma^m_{J^m}.$$

Since $-c \le -e^{\mathrm{T}}\xi$, we get (34).

Now assume that (34) holds. For $\ell = 1, \dots, n$ let

$$\xi_\ell := \max \left\{ \sum_{j=0}^{m} K^j_\ell y^j_\ell - K^m_\ell \gamma^m_\ell, \sum_{j=0}^{i} K^j_\ell y^j_\ell - K^i_\ell \gamma^i_\ell + K^{i+1}_\ell \beta^i_\ell : i = 0, \dots, m-1 \right\}.$$

This choice of ξ ensures that the first four constraints in (37) hold. Let $\overline{J} \in \mathcal{P}(n,m)$ be such that

$$\xi_{\overline{J}^m} = \sum_{j=0}^{m} K^j_{\overline{J}^m} \circ y^j_{\overline{J}^m} - K^m_{\overline{J}^m} \circ \gamma^m_{\overline{J}^m},$$

and

$$\xi_{\overline{J}^i} = \sum_{j=0}^{i} K^j_{\overline{J}^i} \circ y^j_{\overline{J}^i} - K^i_{\overline{J}^i} \gamma^i_{\overline{J}^i} + K^{i+1}_{\overline{J}^i} \beta^i_{\overline{J}^i} \quad \text{for } i = 0, \dots, m-1.$$

Furthermore, from (34) (applied to the partition $\overline{J}$) we have

$$\begin{aligned}
-c &\le -\sum_{i=0}^{m} \left(\sum_{j=0}^{i} y^j_{\overline{J}^i} \cdot K^j_{\overline{J}^i} \right) + \sum_{i=0}^{m-1} \left(K^i_{\overline{J}^i} \cdot \gamma^i_{\overline{J}^i} - K^{i+1}_{\overline{J}^i} \cdot \beta^i_{\overline{J}^i} \right) + K^m_{\overline{J}^m} \cdot \gamma^m_{\overline{J}^m} \\
&= -\sum_{i=0}^{m-1} \left(\sum_{j=0}^{i} y^j_{\overline{J}^i} \cdot K^j_{\overline{J}^i} - K^i_{\overline{J}^i} \cdot \gamma^i_{\overline{J}^i} + K^{i+1}_{\overline{J}^i} \cdot \beta^i_{\overline{J}^i} \right) - \sum_{j=0}^{m} y^j_{\overline{J}^m} \cdot K^j_{\overline{J}^m} + K^m_{\overline{J}^m} \cdot \gamma^m_{\overline{J}^m} \\
&= -\sum_{i=0}^{m} \sum_{\ell \in \overline{J}^i} \xi_\ell \\
&= -e^{\mathrm{T}}\xi.
\end{aligned}$$

Hence the last constraint in (37) holds as well. This completes the equivalence between (33) and (34). □

Proof of Theorem 11. The equivalence between (18) and (19) follows from Theorem 18. The latter in turn yields the equivalence between (12) and (20). Finally, by linear programming duality and Proposition 1 it follows that (21) yields the optimal value of (11). □

Proof of Lemma 12. First, note if $(T, v, \tilde{T}, \tilde{v})$ is feasible for (22) then for each $j = 0, \ldots, m$ we have

$$\sum_{i=j}^{m} v^i = p^j + \sum_{i=j}^{m}((T^i + \tilde{T}^i)K^j - \tilde{v}^i) \leq p^j + \sum_{i=j}^{m} T^i K^j,$$

and

$$\sum_{i=0}^{j-1} v^i \leq \sum_{i=0}^{j-1} T^i K^{i+1} \leq \sum_{i=0}^{j-1} T^i K^j.$$

Thus for each $j = 0, \ldots, m$

$$\sum_{i=0}^{m} v^i \leq p^j + \sum_{i=0}^{m} T^i K^j = p^j + \tau K^j.$$

Therefore the optimal value of (22) is at most $\min\limits_{j=0,\ldots,m} \left\{p^j + \tau K^j\right\}$.

To complete the proof, we next construct a feasible solution to (22) whose objective for (22) attains this value. Put

$$\sigma^0 := 1;\ \sigma^i := \frac{p^i - p^{i+1}}{K^{i+1} - K^i},\ i = 1, \ldots, m-1;\ \sigma^m := 0,$$

and construct the point $(T, v, \tilde{T}, \tilde{v})$ as follows

$$\begin{array}{l} T^m = 0, \\ T^i = \min(\tau, \sigma^i) - \min(\tau, \sigma^i), \\ v^m = p^m, \\ v^i = T^i K^{i+1},\ i = 1, \ldots, m-1, \\ v^0 = \min\left\{p^0 - p^1 - \sigma^1 K^1, T^0 K^1\right\}, \end{array}$$

and

$$\begin{array}{l} \tilde{T}^i = \sigma^i - \sigma^{i+1} - T^i,\ i = 0, \ldots, m-1, \\ \tilde{T}^m = 0, \\ \tilde{v}^0 = p^0 - p^1 - \sigma^1 K^1 - v^0, \\ \tilde{v}^i = \tilde{T}^i K^{i+1},\ i = 1, \ldots, m-1 \\ \tilde{v}^m = 0. \end{array}$$

By construction $\sum_{j=i}^m T^i = \min(\tau, \sigma^i),\ i = 0, \ldots, m$. Hence $(v, T, \tau) \in \text{cone}(K)$. Likewise, $(\tilde{v}, \tilde{T}, 1-\tau) \in \text{cone}(K)$. The rest of the proof relies on the following identity, which is clearly valid for $j = 0, 1, \ldots, m-1$:

$$\begin{array}{rcl} \displaystyle\sum_{i=j}^{m-1}(\sigma^i - \sigma^{i+1})K^{i+1} + p^m & = & \sigma^j K^{j+1} + \displaystyle\sum_{i=j+1}^{m-1} \sigma^i(K^{i+1} - K^i) + p^m \\ & = & \sigma^j K^{j+1} + p^{j+1} - p^m + p^m \\ & = & \sigma^j K^{j+1} + p^{j+1} \\ & = & \sigma^j K^j + p^j. \end{array} \tag{38}$$

Notice that the identity between the first and last quantities also holds for $j = m$.

From (38), for $j \geq 1$ we get

$$\begin{aligned}
\sum_{i=j}^{m} \left(v^i + \tilde{v}^i - (T^i + \tilde{T}^i) K^j \right) &= \sum_{i=j}^{m-1} (T^i + \tilde{T}^i)(K^{i+1} - K^j) + p^m \\
&= \sum_{i=j}^{m-1} (\sigma^i - \sigma^{i+1})(K^{i+1} - K^j) + p^m \\
&= \sum_{i=j}^{m-1} (\sigma^i - \sigma^{i+1}) K^{i+1} + p^m - \sigma^j K^j \\
&= \sigma^j K^j + p^j - \sigma^j K^j \\
&= p^j.
\end{aligned}$$

From (38) we also get

$$\begin{aligned}
\sum_{i=0}^{m} \left(v^i + \tilde{v}^i - (T^i + \tilde{T}^i) K^0 \right) &= \sum_{i=0}^{m} \left(v^i + \tilde{v}^i \right) \\
&= v^0 + \tilde{v}^0 + \sum_{i=1}^{m-1} (\sigma^i - \sigma^{i+1}) K^{i+1} + p^m \\
&= p^0 - p^1 - \sigma^1 K^1 + \sigma^1 K^1 + p^1 \\
&= p^0.
\end{aligned}$$

Hence $(T, v, \tilde{T}, \tilde{v})$ is indeed a feasible solution for (22). To finish, we next show that the objective value of this point is $\min_{j=0,\dots,m} \left\{ p^j + \tau K^j \right\}$. To that end, let $j^* := \operatorname{argmin}_{j=0,\dots,m} \left\{ p^j + \tau K^j \right\}$. If $1 \leq j^* \leq m$ then $\sigma^{j^*} \leq \tau \leq \sigma^{j^*-1}$, and consequently $T^i = 0,\ i < j^* - 1,\ T^{j^*-1} = \tau - \sigma^{j^*}$, and $T^i = \sigma^i - \sigma^{i+1},\ j^* \leq i \leq m-1$. Thus from (38) we get

$$\begin{aligned}
\sum_{i=0}^{m} v^i &= (\tau - \sigma^{j^*}) K^{j^*} + \sum_{i=j^*}^{m-1} (\sigma^i - \sigma^{i+1}) K^{i+1} + p^m \\
&= \tau K^{j^*} - \sigma^{j^*} K^{j^*} + \sigma^{j^*} K^{j^*} + p^{j^*} \\
&= \tau K^{j^*} + p^{j^*}.
\end{aligned}$$

If $j^* = 0$ then $\tau > \frac{p^0 - p^1}{K^1} \geq \sigma^1$ and consequently $T^0 = \tau - \sigma^1,\ v^0 = p^0 - p^1 - \sigma^1 K^1$, and $T^i = \sigma^i - \sigma^{i+1},\ i = 1, \dots, m-1$. Thus, from (38) we get

$$\begin{aligned}
\sum_{i=0}^{m} v^i &= p^0 - p^1 - \sigma^1 K^1 + \sum_{i=1}^{m-1} (\sigma^i - \sigma^{i+1}) K^{i+1} + p^m \\
&= p^0 - p^1 - \sigma^1 K^1 + \sigma^1 K^2 + p^2 - p^m + p^m \\
&= p^0.
\end{aligned}$$

In either case $\sum_{i=0}^{m} v^i = p^{j^*} + \tau K^{j^*} = \min_{j=0,\dots,m} \left\{ p^j + \tau K^j \right\}$. □

4 Some numerical results

We next present computational experiments that illustrate some of our results. The experiments focus on the new features that can be incorporated as a result of our linear programming approach. In particular, we present a numerical example for an *exchange* option, to illustrate how our approach allow us to consider basket options with negative weights. We also discuss an example that takes into account the presence of bid/ask spreads in option prices. Finally, we discuss the possibility of adding diversification constraints to the super-replication strategy problem. Although these features are prevalent in real pricing problems, they were beyond the scope of previous approaches to static-arbitrage bounds.

Related numerical results are presented in [3, 4], where the authors provide extensive numerical experiments comparing static-arbitrage pricing techniques and parametric pricing techniques (such as Monte Carlo simulations) for basket options.

4.1 Basket options with negative weights

Consider the problem of finding static-arbitrage bounds for a European *exchange* option, given the prices of vanilla options on the two assets involved. This corresponds to (11) with $n = 2$, $w^0 = (1, -1)$, and $K_0 = 0$. We will consider the case in which information about the forward prices of the two assets, and the following $m = 5$ call options is given:

$$K^1 = \begin{bmatrix} 0.85 \\ 0.85 \end{bmatrix},\ K^2 = \begin{bmatrix} 0.90 \\ 0.90 \end{bmatrix},\ K^3 = \begin{bmatrix} 0.95 \\ 0.95 \end{bmatrix},\ K^4 = \begin{bmatrix} 1.00 \\ 1.00 \end{bmatrix},\ K^5 = \begin{bmatrix} 1.05 \\ 1.05 \end{bmatrix}. \quad (39)$$

In what follows, we compare the static-arbitrage bound on the exchange option described above with the semiparametric bounds for the exchange option computed in [11] and [2]. The latter bounds are computed using a more sophisticated *semidefinite programming* approach. For our problem set up, assume the stock prices (S_1, S_2) follow a geometric Brownian motion. Hence we can sample the values of p^j, $j = 0, \dots, 5$ in (11) corresponding to the strikes in (39), from the correlated multivariate lognormal distribution model for asset prices under the risk neutral probability measure (see, e.g., eq. (15) in [2]). In particular we use a riskless interest rate $r = 0$, option maturity $T = 1$, current prices $S_1(0) = 0.95$, $S_2(0) = 0.90$, volatilities $\delta_1 = 0.2$, $\delta_2 = 0.22$. Thus, we obtain the following given option prices for the problem

$$p^0 = \begin{bmatrix} 0.9500 \\ 0.9000 \end{bmatrix},\ p^1 = \begin{bmatrix} 0.1324 \\ 0.1042 \end{bmatrix},\ p^2 = \begin{bmatrix} 0.1013 \\ 0.0788 \end{bmatrix},$$

$$p^3 = \begin{bmatrix} 0.0757 \\ 0.0584 \end{bmatrix},\ p^4 = \begin{bmatrix} 0.0552 \\ 0.0425 \end{bmatrix},\ p^5 = \begin{bmatrix} 0.0394 \\ 0.0304 \end{bmatrix}.$$

The arbitrage condition (13) follows from computing the prices using the lognormal model. Thus, using Theorem 13, we obtain an upper static-arbitrage bound on the exchange option of 0.1802. To compare this result with the bounds provided in [11, Example 5.1 and Table 1], we construct Table 1.

In Table 1, the first column corresponds to the correlation of the Wiener processes involved in the lognormal distribution model of the asset prices. The second column corresponds to the *exact* price obtained under the lognormal distribution model assumption (see [2]). The third column corresponds to the tight upper bound on the payoff of a European exchange option on a pair of stocks whose prices at maturity have known means, variances and covariance; that is, a *second-order semiparametric* upper bound. The static-arbitrage bound does not change

Table 1: Static-arbitrage vs. second-order semiparametric bounds for a exchange option

Correlation	Exact price	Tight upper semiparametric bound [11]	Tight upper static-arbitrage bound (11)
-1.0	0.1801	0.2206	0.1802
-0.5	0.1600	0.1958	0.1802
0.0	0.1361	0.1660	0.1802
0.5	0.1051	0.1268	0.1802
1.0	0.0500	0.0504	0.1802

for different values of the correlation between the assets because the vanilla options do not provide any correlation information. Table 1 shows that for negative values of the correlation between the assets in the exchange option, the upper static-arbitrage bound is almost as good or better than the second-order semiparametric upper bound. Considering that in order to compute the second-order semiparametric upper bound it is necessary to use *semidefinite programming*, a much more involved and computationally costly optimization technique than linear programming, the fact that good static-arbitrage bounds can be computed with a closed-form formula is somewhat remarkable.

4.2 Bid-ask prices

We next provide some experimental results that illustrate how bid-ask spreads on financial instrument prices can be incorporated in the static-arbitrage pricing problem.

In the formulation of the upper static-arbitrage bound (12), it is assumed that the options can be bought and sold at the same price. In practice, the price at which an investor buys the option, i.e., the *ask* price, is higher that the price at which the investor can sell the option, i.e., the *bid* price. This gives rise to the so-called bid-ask spread as can be observed in Table 2, which lists the prices of vanilla options the on stocks in the DJX index as traded on May 17th, 2004 on the June contracts with maturity on June 18th, 2004. This dataset is similar to that of [4, Section 6.2, Table 2]. However, we have only included traded contracts (with volume greater than zero), for liquidity considerations. With the data in Table 2, we can use the LP formulation (31) in Section 3.3 to compute the cheapest super-replicating strategy for the DJX basket call option with strike price 80.00 taking into account the bid-ask spread. We obtain the super-replication strategy given in Table 3, which yields an upper bound of 19.8872. From market data, the best bid price for this option was 18.7, and the best ask price was 19.5. Table 3 provides the long (buy) positions on the call options with position different from zero in the super-replicating portfolio. In this particular experiment, the super-replicating portfolio does not contain any short (sell) positions.

Using bid and ask prices in the computation of the super-replicating strategy gives a more practical value to the static-arbitrage pricing approach. In particular, this resolves a major limitation in previous approaches [3, 4] that used mid-market prices (i.e., the average of the bid and ask prices) as the "nominal" option prices. Such approximation systematically underestimates the actual buying prices and overestimates the actual selling prices. It is then not surprising that the market data used in [3, 4] requires a fair amount of "cleaning" to rule out apparent arbitrage opportunities created by these estimates (see [4, Section 6.2]). By contrast, the model herein that takes into account bid-ask spreads does not suffer from this limitation.

We note that although the super-replicating strategy in Table 3 contains only long positions, this does not mean that the bid-ask DJX option price upper bound of 19.8872 could be found by simply using the ask (buy) prices as the option prices in the *original* LP formulation of the problem (20). If this naive approach were attempted, the LP (20) would diverge, since the ask prices alone do not satisfy the arbitrage-free condition (13).

4.3 Diversifying the super-replicating strategy

Consider an investor looking at the strategy in Table 3, who wishes to create a super-replicating strategy that contains more positions in higher strike options. One possible way to obtain such super-replicating strategy is by simply adding the following *diversifying* linear constraints to the LP formulation (31):

$$e \cdot y^j \geq 0.05, j = 0, \ldots, m. \tag{40}$$

Above e represents the vector of all-ones. The solution to this *diversified* super-replicating strategy gives a portfolio whose cost is 19.9022, just 0.08% more expensive than the cheapest super-replicating strategy computed in Table 3. As Table 4 shows, such a strategy has the desired investor's property.

References

[1] D. Bertsimas and I. Popescu. On the relation between option and stock prices: a convex optimization approach. *Oper. Res.50 (2002) 358–374.*

[2] P. P. Boyle and X. S. Lin, Bounds on contingent claims based on several assets, J. Fin. Econ., 46(3):383–400, 1997.

[3] A. d'Aspremont and L. El Ghaoui. Static arbitrage bounds on basket option prices. *To Appear in Math. Program..*

[4] D. Hobson, P. Laurence, and TH. Wang. Static arbitrage upper bounds for the prices of basket options *To Appear in Quantitative Finance.*

[5] D. Hobson, P. Laurence, and TH. Wang. Static arbitrage optimal sub-replicating strategies for basket options *To Appear in Insurance Mathematics and Economics.*

[6] S. Karlin and W. Studden. *Tchebycheff Systems With Applications in Analysis and Statistics. John Wiley and Sons, New York.*

[7] J. Peña, J. Vera, and L. Zuluaga, "Static-arbitrage lower bounds for basket options via linear programming," In preparation.

[8] J. Renegar, *A Mathematical View of Interior-Point Methods for Convex Optimization,* SIAM 2001.

[9] T. Rockafellar, *Convex Analysis,* Princeton University Press, 1970.

[10] V. H. de la Peña, R. Ibragimov, and S. Jordan. Option bounds J. Appl. Probab. 41A (2004), 145–156.

[11] L. F. Zuluaga, and J. Peña. A Conic Programming approach to Generalized Tchebycheff Inequalities Math. OR. 30(2), pp. 369–388, 2005.

Table 2: CBOE data from May 17th, 2004 on June 2004 contracts expiring June 18. The table gives prices of call options traded (volume greater than zero) on May 17th for the 30 stocks underlying the DJX index basket option. For every stock, the first row corresponds to the different strike prices, and the second and third rows correspond to the respective ask and bid prices. The entry of 0.00 for each stock gives the close price of the stock, which can be considered as the forward option (call option with strike price zero) price.

	0.00	20.00	22.50	25.00	27.50				
MSFT	25.79	5.70	3.20	1.15	0.20				
	25.42	5.50	3.10	1.05	0.15				
	0.00	25.00	27.50	30.00	32.50	35.00			
AA	29.70	4.10	2.20	0.95	0.30	0.15			
	28.60	3.90	2.05	0.85	0.20	0.05			
	0.00	65.00	70.00	75.00					
AIG	70.15	5.40	2.10	0.45					
	69.22	5.30	2.00	0.40					
	0.00	47.50	50.00						
AXP	49.30	2.20	0.80						
	48.20	2.05	0.70						
	0.00	40.00	42.50	45.00					
BA	43.61	3.10	1.35	0.40					
	42.49	2.90	1.25	0.30					
	0.00	35.00	37.50						
VZ	36.74	1.50	0.40						
	35.68	1.40	0.30						
	0.00	60.00	70.00	75.00	80.00	85.00			
CAT	74.45	13.80	4.90	1.95	0.60	0.20			
	72.70	13.60	4.80	1.90	0.50	0.10			
	0.00	40.00	42.50	45.00					
DD	41.48	2.00	0.70	0.15					
	41.01	1.80	0.55	0.10					
	0.00	20.00	22.50	25.00	27.50				
DIS	22.99	3.00	1.05	0.20	0.10				
	22.69	2.95	0.90	0.15	0.00				
	0.00	25.00	27.50	30.00	32.50	35.00	37.50		
GE	30.06	5.10	2.70	0.85	0.15	0.05	0.05		
	29.68	4.90	2.60	0.75	0.10	0.00	0.00		
	0.00	47.50	50.00	55.00	60.00	65.00	70.00		
WMT	55.25	7.40	5.10	1.45	0.15	0.05	0.05		
	54.14	7.20	4.90	1.30	0.10	0.00	0.00		
	0.00	35.00	40.00	42.50	45.00	47.50	50.00	55.00	
GM	43.90	8.70	4.20	2.30	1.05	0.40	0.15	0.05	
	42.88	8.60	4.00	2.20	0.95	0.30	0.10	0.00	
	0.00	30.00	32.50	35.00	37.50	40.00			
HD	33.75	3.80	1.85	0.60	0.15	0.05			
	33.07	3.60	1.70	0.55	0.10	0.00			
	0.00	30.00	32.50	35.00	37.50	40.00			
HON	33.43	2.85	1.15	0.30	0.10	0.05			
	32.44	2.70	1.00	0.20	0.00	0.00			
	0.00	15.00	17.50	20.00	22.50				
HPQ	19.70	4.60	2.30	0.70	0.15				
	19.21	4.50	2.20	0.65	0.10				
	0.00	80.00	85.00	90.00	95.00	100.00			
IBM	86.03	6.30	2.65	0.70	0.20	0.05			
	85.15	6.10	2.50	0.65	0.15	0.00			
	0.00	27.50	30.00	32.50	35.00	37.50	40.00	42.50	45.00
JPM	35.47	8.00	5.60	3.30	1.45	0.45	0.10	0.10	0.05
	34.75	7.80	5.40	3.10	1.35	0.35	0.05	0.00	0.00
	0.00	47.50	50.00	55.00					
KO	50.12	2.70	1.00	0.05					
	49.51	2.55	0.85	0.00					
	0.00	40.00	42.50	45.00					
XOM	43.54	3.40	1.45	0.40					
	43.01	3.20	1.35	0.30					
	0.00	20.00	22.50	25.00	27.50	30.00			
INTC	27.30	6.90	4.50	2.30	0.80	0.20			
	26.44	6.80	4.30	2.25	0.70	0.10			
	0.00	50.00	55.00						
JNJ	55.10	5.00	1.10						
	54.13	4.80	1.05						
	0.00	80.00	85.00	90.00					
UTX	82.80	3.60	1.20	0.30					
	81.50	3.40	1.10	0.20					
	0.00	80.00	85.00	90.00					
MMM	83.89	4.20	1.30	0.25					
	82.75	4.00	1.15	0.15					
	0.00	45.00	47.50	50.00	55.00	60.00			
MO	50.00	4.90	2.80	1.20	0.15	0.10			
	48.50	4.70	2.65	1.15	0.10	0.00			
	0.00	45.00	47.50	50.00					
MRK	46.89	2.15	0.70	0.15					
	46.00	1.95	0.60	0.10					
	0.00	30.00	35.00	37.50	40.00	42.50			
PFE	35.91	5.70	1.30	0.30	0.10	0.05			
	35.00	5.50	1.20	0.25	0.05	0.00			
	0.00	90.00	95.00	100.00	105.00	110.00	115.00		
PG	107.15	16.50	11.70	7.10	3.30	1.00	0.25		
	105.81	16.30	11.40	6.90	3.10	0.90	0.20		
	0.00	25.00							
SBC	24.49	0.40							
	24.11	0.35							
	0.00	20.00	25.00	27.50	30.00				
MCD	26.05	5.90	1.40	0.35	0.05				
	25.50	5.80	1.30	0.25	0.05				
	0.00	30.00	35.00	40.00	42.50	45.00	47.50	50.00	55.00
C	45.30	15.00	10.00	5.20	3.00	1.30	0.40	0.15	0.05
	44.83	14.80	9.80	5.10	2.90	1.25	0.35	0.05	0.00

Table 3: Strike 80.00 DJX basket super-replicating strategy from LP formulation (eq. (31)). Option price upper bound = 19.8872. For every asset, the first row gives the strikes of the asset's call options with a position greater than zero in the super-replicating strategy. The second row gives the corresponding long position. In this particular experiment, the super-replicating portfolio does not contain any short positions.

	22.50		47.50		50.00	
MSFT	0.071	WMT	0.071	JNJ	0.071	
	25.00		35.00		0.00	80.00
AA	0.071	GM	0.071	UTX	0.054	0.017
	65.00		30.00		80.00	
AIG	0.071	HD	0.071	MMM	0.071	
	47.50		30.00		45.00	
AXP	0.071	HON	0.071	MO	0.071	
	40.00		15.00		45.00	
BA	0.071	HPQ	0.071	MRK	0.071	
	35.00		80.00		30.00	
VZ	0.071	IBM	0.071	PFE	0.071	
	60.00		27.50		90.00	
CAT	0.071	JPM	0.071	PG	0.071	
	0.00		47.50		0.00	
DD	0.071	KO	0.071	SBC	0.071	
	20.00		40.00		20.00	
DIS	0.071	XOM	0.071	MCD	0.071	
	25.00		20.00		35.00	
GE	0.071	INTC	0.071	C	0.071	

Table 4: Strike 80.00 DJX basket super-replicating strategy from LP formulation (eq. (31)) plus diversification constraints (eq. (40)). Option price upper bound = 19.9022. For every asset, the first row gives the strikes of the asset's call options with a position greater than zero in the super-replicating strategy. The second row gives the corresponding long position. In this particular experiment, the super-replicating portfolio does not contain any short positions.

	22.50		
MSFT	0.071		
	25.00		
AA	0.071		
	65.00		
AIG	0.071		
	47.50		
AXP	0.071		
	40.00		
BA	0.071		
	35.00		
VZ	0.071		
	60.00		
CAT	0.071		
	0.00		
DD	0.071		
	20.00		
DIS	0.071		
	25.00	35.00	37.50
GE	0.071	0.011	0.025

	47.50	65.00	70.00
WMT	0.071	0.010	0.025
	35.00	55.00	
GM	0.071	0.050	
	30.00	40.00	
HD	0.071	0.007	
	30.00	40.00	
HON	0.071	0.007	
	15.00		
HPQ	0.071		
	80.00	100.00	
IBM	0.071	0.007	
	27.50	45.00	
JPM	0.071	0.025	
	47.50	55.00	
KO	0.071	0.050	
	40.00		
XOM	0.071		
	20.00		
INTC	0.071		

	50.00	
JNJ	0.071	
	0.00	80.00
UTX	0.054	0.017
	80.00	
MMM	0.071	
	45.00	
MO	0.071	
	45.00	
MRK	0.071	
	30.00	42.50
PFE	0.071	0.007
	90.00	
PG	0.071	
	0.00	
SBC	0.071	
	20.00	30.00
MCD	0.071	0.050
	35.00	55.00
C	0.071	0.025