A New Cone Programming Approach for Robust Portfolio Selection *

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Abstract

The robust portfolio selection problems have recently been studied by several researchers (e.g., see [15, 14, 17, 25]). In their work, the "separable" uncertainty sets of the problem parameters (e.g., mean and covariance of the random returns) were considered. These uncertainty sets share two common drawbacks: i) the actual confidence level of the uncertainty set is unknown, and it can be much higher than the desired one; and ii) the uncertainty set is fully or partially box-type. The consequence of these drawbacks is that the resulting robust portfolio can be too conservative and moreover, it is usually highly non-diversified as observed in computational experiments. To combat these drawbacks, we consider a factor model for the random asset returns. For this model, we introduce a "joint" ellipsoidal uncertainty set for the model parameters and show that it can be constructed as a confidence region associated with a statistical procedure applied to estimate the model parameters for any desired confidence level. We further show that the robust maximum risk-adjusted return problem with this uncertainty set can be reformulated and solved as a cone programming problem. Some computational experiments are performed to compare the performances of the robust portfolios corresponding to our "joint" uncertainty set and Goldfarb and Iyengar's "separable" uncertainty set [15]. We observe that our robust portfolio has much better performance than Goldfarb and Iyengar's in terms of wealth growth rate and transaction cost, and moreover, our robust portfolio is fairly diversified, but Goldfarb and Iyengar's is highly non-diversified.

Key words: Robust optimization, mean-variance portfolio selection, maximum risk-adjusted return portfolio selection, cone programming, linear regression.

MSC 2000 subject classification. Primary: 91B28; secondary: 90C20, 90C22.

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1 Introduction

Portfolio selection problem is concerned with determining a portfolio such that the "return" and "risk" of the portfolio have a favorable trade-off. The first mathematical model for portfolio selection problem was developed by Markowitz [20] five decades ago, in which an optimal or efficient portfolio can be identified by solving a convex quadratic program. In his model, the "return" and "risk" of a portfolio are measured by the mean and variance of the random portfolio return, respectively. Thus, the Markowitz portfolio model is also widely referred to as the mean-variance model.

Despite the theoretical elegance and importance of the mean-variance model, it continues to encounter skepticism among the investment practitioners. One of the main reasons is that the optimal portfolios determined by the mean-variance model are often sensitive to perturbations in the parameters of the problem (e.g., expected returns and the covariance matrix), and thus lead to large turnover ratios with periodic adjustments of the problem parameters; see for example Michaud [21]. Various aspects of this phenomenon have also been extensively studied in the literature, for example, see [10, 8, 9].

As a recently emerging modeling tool, robust optimization can incorporate the perturbations in the parameters of the problems into the decision making process. Generally speaking, robust optimization aims to find solutions to given optimization problems with uncertain problem parameters that will achieve good objective values for all or most of realizations of the uncertain problem parameters. For details, see [3, 4, 5, 11, 13]. Recently, robust optimization has been applied to model portfolio selection problems in order to alleviate the sensitivity of optimal portfolios to statistical errors in the estimates of problem parameters. In particular, Goldfarb and Iyengar [15] introduced a factor model for the random portfolio returns and proposed some statistical procedures for constructing uncertainty sets for the model parameters. For these uncertainty sets, they showed that the robust portfolio selection problems can be reformulated as second-order cone programs. Subsequently, Erdoğan et al.[14] extended this method to robust index tracking and active portfolio management problems. Alternatively, Tütüncü and Koenig [25] (see also Halldórsson and Tütüncü [17]) considered a box-type uncertainty structure for the mean and covariance matrix of the assets returns. For this uncertainty structure, they showed that the robust portfolio selection problems can be formulated and solved as smooth saddle-point problems that involve semidefinite constraints. In addition, for finite uncertainty sets, Ben-Tal et al. [2] studied the robust formulations of multi-stage portfolio selection problems. Also, El Ghaoui et al. [12] considered the robust value-at-risk (VaR) problems given the partial information on the distribution of the returns, and they showed that these problems can be cast as semidefinite programs. Recently, Zhu and Fukushima [27] showed that the robust conditional value-at-risk (CVaR) problems can be reformulated as linear programs or second-order cone programs for some simple uncertainty structures of the distributions of the returns.

The structure of uncertainty set is an important ingredient in formulating and solving robust portfolio selection problems. The "separable" uncertainty sets have been commonly considered in the literatures. For example, Tütüncü and Koenig [25] (see also Halldórsson and Tütüncü [17]) proposed the box-type uncertainty sets $\mathcal{S}_m = \{\mu : \mu^L \leq \mu \leq \mu^U\}$ and $\mathcal{S}_v = \{\Sigma : \Sigma \succeq 0, \Sigma^L \leq \Sigma \leq \Sigma^U\}$ for the mean return vector μ and the covariance matrix Σ of the asset returns r, respectively. Here $A \succeq 0$ (resp. $\succ 0$) denotes that the matrix A is symmetric and positive semidefinite (resp. definite). In addition, Goldfarb and Iyengar [15] introduced a factor model for the random asset return vector r in the form of

$$r = \mu + V^T f + \epsilon,$$

where μ is the mean return vector, f is the random factor return vector that drives the market, V is the factor loading matrix and ϵ is the residual return vector (see Section 2 for details). They proposed specific uncertainty sets $\mathcal{S}_m$ and $\mathcal{S}_v$ for μ and V, respectively; in particular, $\mathcal{S}_m$ is a box (see (2)) and $\mathcal{S}_v$ is a Cartesian product of a bunch of ellipsoids (see (3)). It shall be stressed that these "separable" uncertainty sets share two common drawbacks: i) viewed as a joint uncertainty set, the actual confidence level of $\mathcal{S}_m \times \mathcal{S}_v$ is unknown even though $\mathcal{S}_m$ and/or $\mathcal{S}_v$ may have known confidence levels individually; and ii) $\mathcal{S}_m \times \mathcal{S}_v$ is fully or partially box-type. The consequence of the first drawback is that the resulting robust portfolio can be too conservative since the actual confidence level of $\mathcal{S}_m \times \mathcal{S}_v$ can be much higher than the desired one; and the consequence of the second drawback is that the resulting robust portfolio is highly non-diversified as observed in computational experiments. The details of these drawbacks and the associated consequences are addressed in Sections 2 and 5.

In this paper, we consider the same factor model as introduced in [15]. To combat the aforementioned drawbacks of the "separable" uncertainty sets, we propose a "joint" ellipsoidal uncertainty set for (μ, V), and show that it can be constructed as a confidence region associated with a statistical procedure applied to estimate (μ, V) for any desired confidence level. We further show that the robust maximum risk-adjusted return (RAR) portfolio selection problem with this uncertainty set can be reformulated and solved as a cone programming problem. Some computational experiments are performed to compare the performances of the robust portfolios corresponding to our "joint" uncertainty set and Goldfarb and Iyengar's "separable" uncertainty set [15]. From the computational results, we observe that our robust portfolio has much better performance than Goldfarb and Iyengar's in terms of wealth growth rate and transaction cost, and moreover, our robust portfolio is fairly diversified, but Goldfarb and Iyengar's is highly non-diversified.

The organization of this paper is as follows. In Section 2, we describe the factor model introduced by Goldfarb and Iyengar [15], and review the statistical procedure proposed in [15] for constructing a "separable" uncertainty set of the model parameters. Some drawbacks of this uncertainty set are also addressed in this section. In Section 3, we introduce a "joint" uncertainty set for the model parameters, and propose a statistical procedure for constructing such an uncertainty set for any desired confidence level. Several robust portfolio selection problems with this uncertainty set are also discussed. In Section 4, we show that the robust maximum RAR portfolio selection problem with our "joint" uncertainty can be reformulated as a cone programming problem. In Section 5, we present some preliminary computational results for the robust maximum RAR portfolio selection problems with our "joint" uncertainty set and Goldfarb and Iyengar's "separable" uncertainty set, respectively. Finally, we present some concluding remarks in Section 6.

2 Factor model and separable uncertainty sets

In this section, we first describe the factor model for asset returns introduced by Goldfarb and Iyengar [15]. Then we review the statistical procedure proposed in [15] for constructing a "separable" uncertainty set for the model parameters. Some drawbacks of this uncertainty set are addressed in the end of this section.

We now describe the factor model studied in [15] (see Section 2 of [15] for more details). Let us consider a discrete-time market with n traded assets. The vector of asset returns over a single market period is denoted by $r \in \Re^n$. The returns on the assets in different market periods are assumed to be independent. The single period return r is assumed to be a random vector given by

$$r = \mu + V^T f + \epsilon, \tag{1}$$

where $\mu \in \Re^n$ is the vector of mean returns, $f \sim \mathcal{N}(0, F) \in \Re^m$ denotes the returns of the factors driving the market, $V \in \Re^{m\times n}$ denotes the factor loading matrix of the n assets, and $\epsilon \sim \mathcal{N}(0, D) \in \Re^n$ is the vector of residual returns. Here $x \sim \mathcal{N}(\mu, \Sigma)$ denotes that x is a multivariate normal random variable with mean vector μ and covariance matrix Σ. In addition, we assume that the covariance matrix $D = \text{diag}(d) \succeq 0$, where $\text{diag}(d)$ denotes a diagonal matrix with the vector d along the diagonal, and the vector of residual returns ϵ is independent of the vector of factor returns f. Thus, the vector of assets return $r \sim \mathcal{N}(\mu, V^T F V + D)$.

Goldfarb and Iyengar [15] also developed a robust counterpart for the aforementioned factor model. In [15], even though some uncertainty structures were proposed for the parameters F and D, they are usually assumed to be known and fixed in real computations, and they can be obtained by some standard statistical approaches (see Section 7 of [15]). Thus, for the convenience of presentation, we only assume the uncertainty structures for the parameters μ and V in this paper. However, it shall be mentioned that the results of this paper can be extended to the case where F and D have the same uncertainty structures as described in [15].

We now describe the "separable" uncertainty structure for the model parameters μ and V that was proposed in [15] (see Section 2 of [15] for details). The mean returns vector μ is assumed to lie in the uncertainty set $\mathcal{S}_m$ given by

$$\mathcal{S}_m = \{\mu : \ \mu = \mu_0 + \xi, \ |\xi_i| \le \gamma_i, i = 1, \dots, n\}, \tag{2}$$

i.e., each component of μ is assumed to lie within a certain interval. The columns of the matrix V, i.e., the factor loadings of the individual assets, are also assumed to be known approximately. In particular, V is assumed to belong to the elliptical uncertainty set $\mathcal{S}_v$ given by

$$\mathcal{S}_v = \{V : \; V = V_0 + W, \; \|W_i\|_G \leq \rho_i, i = 1, \ldots, n\}, \tag{3}$$

where W_i is the ith column of W and $\|w\|_G = \sqrt{w^T G w}$ denotes the elliptic norm of w with respect to a symmetric, positive definite matrix G.

In [15], two similar statistical procedures were proposed for constructing the above uncertainty sets $\mathcal{S}_m \times \mathcal{S}_v$ for (μ, V). As observed in computational experiments, the behavior of these uncertainty sets is almost identical. For the sake of brevity, we only describe the first statistical procedure proposed in [15].

Indeed, suppose the market data consists of asset returns $\{r^t : t = 1, \ldots, p\}$ and the corresponding factor returns $\{f^t : t = 1, \ldots, p\}$ for p trading periods. Then the linear model (1) implies that

$$r_i^t = \mu_i + \sum_{j=1}^{m} V_{ji} f_j^t + \epsilon_i^t, \; i = 1, \ldots, n, \; t = 1, \ldots, p.$$

As in the typical linear regression analysis, it is assumed that $\{\epsilon_i^t : \; i = 1, \ldots, n, \; t = 1, \ldots, p\}$ are all independent normal random variables and $\epsilon_i^t \sim \mathcal{N}(0, \sigma_i^2)$ for all $t = 1, \ldots, p$. Now, let $B = (f^1, f^2, \ldots, f^p) \in \Re^{m \times p}$ be the matrix of factor returns, and let $e \in \Re^p$ be an all-one vector. Collecting together terms corresponding to a particular asset i over all periods $t = 1, \ldots, p$, we get the following linear model for the returns $\{r_i^t : \; t = 1, \ldots, p\}$,

$$y_i = A x_i + \epsilon_i,$$

where

$$y_i = (r_i^1, r_i^2, \ldots, r_i^p)^T, \quad A = (e \; B^T), \quad x_i = (\mu_i, V_{1i}, V_{2i}, \ldots, V_{mi})^T, \tag{4}$$

and $\epsilon_i = (\epsilon_i^1, \ldots, \epsilon_i^p)^T$ is the vector of residual returns corresponding to asset i. The least-squares estimate $\bar{x}_i$ of the true parameter x_i is given by the solution of the normal equation

$$A^T A \bar{x}_i = A^T y_i.$$

In practice, $p \gg m$ and the columns of A are linearly independent. Thus, it is assumed throughout the paper that A has full column rank. Then we have

$$\bar{x}_i = (A^T A)^{-1} A^T y_i. \tag{5}$$

Substituting $y_i = A x_i + \epsilon_i$, we get

$$\bar{x}_i - x_i = (A^T A)^{-1} A^T \epsilon_i \sim \mathcal{N}(0, \Sigma), \tag{6}$$

where $\Sigma = \sigma_i^2 (A^T A)^{-1}$. Hence,

$$\mathscr{X} = \frac{1}{\sigma_i^2} (\bar{x}_i - x_i)^T A^T A (\bar{x}_i - x_i) \sim \chi^2_{m+1}, \tag{7}$$

i.e., $\mathscr{X}$ is a χ^2 random variable with $m+1$ degrees of freedom. Let s_i^2 be the unbiased estimate of σ_i^2 given by

$$s_i^2 = \frac{\|y_i - A\bar{x}_i\|^2}{p-m-1}. \tag{8}$$

Replacing σ_i^2 in (7) by $(m+1)s_i^2$ and following a standard result in regression theory (e.g., see Anderson [1] and Greene [16]), we see that the resulting random variables

$$\mathscr{Y}^i = \frac{1}{(m+1)s_i^2}(\bar{x}_i - x_i)^T A^T A(\bar{x}_i - x_i),\ i = 1, \ldots, n, \tag{9}$$

are distributed according to the F-distribution with $m+1$ degrees of freedom in the numerator and $p-m-1$ degrees of freedom in the denominator.

Let $0 < \tilde{\omega} < 1$, $\mathscr{F}_J$ denote the cumulative distribution function with J degrees of freedom in the numerator and $p-m-1$ degrees of freedom in the denominator and let $c_J(\tilde{\omega})$ be its $\tilde{\omega}$-critical value, i.e., $\mathscr{F}_J(c_J(\tilde{\omega})) = \tilde{\omega}$. Then we have

$$\mathrm{P}\left((\bar{x}_i - x_i)^T A^T A(\bar{x}_i - x_i) \le (m+1)c_{m+1}(\tilde{\omega})s_i^2\right) = \tilde{\omega}. \tag{10}$$

We now define

$$\mathcal{S}_i(\tilde{\omega}) = \left\{x_i :\ (\bar{x}_i - x_i)^T A^T A(\bar{x}_i - x_i) \le (m+1)c_{m+1}(\tilde{\omega})s_i^2\right\}. \tag{11}$$

Then, (10) implies that $S_i(\tilde{\omega})$ is a $\tilde{\omega}$-confidence set for the parameter vector x_i corresponding to asset i. Since the residual errors $\{\epsilon_i :\ i = 1, \ldots, n\}$ are assumed to be independent, it follows that

$$\mathcal{S}(\tilde{\omega}) = \mathcal{S}_1(\tilde{\omega}) \times \mathcal{S}_2(\tilde{\omega}) \times \cdots \times \mathcal{S}_n(\tilde{\omega})$$

is a $\tilde{\omega}^n$-confidence set for (μ, V).

Let $\mathcal{S}_m(\tilde{\omega})$ denote the projection of $\mathcal{S}(\tilde{\omega})$ along the vector μ, i.e.,

$$\mathcal{S}_m(\tilde{\omega}) = \{\mu :\ \mu = \mu_0 + v, |v_i| \le \gamma_i, i = 1, \ldots, n\}, \tag{12}$$

where

$$\mu_{0,i} = \bar{\mu}_i,\ \gamma_i = \sqrt{(m+1)(A^T A)^{-1}_{11} c_{m+1}(\tilde{\omega})s_i^2},\ i = 1, \ldots, n. \tag{13}$$

We immediately see that

$$\mathrm{P}\left(\mu \in \mathcal{S}_m(\tilde{\omega})\right) \ge \mathrm{P}\left((\mu, V) \in \mathcal{S}(\tilde{\omega})\right) = \tilde{\omega}^n. \tag{14}$$

Let $Q = (e_2, e_3, \ldots, e_{m+1})^T \in \Re^{m\times(m+1)}$ be a projection matrix that projects x_i along V_i, where $e_i \in \Re^{m+1}$ is the ith coordinate vector for $i = 2, \ldots, m+1$. Define the projection $\mathcal{S}_v(\tilde{\omega})$ of $\mathcal{S}(\tilde{\omega})$ along V as follows:

$$\mathcal{S}_v(\tilde{\omega}) = \left\{V :\ V = V_0 + W,\ \|W_i\|_G \le \rho_i, i = 1, \ldots, n\right\},$$

where

$$V_0 = (\bar{V}_1, \cdots, \bar{V}_n), \tag{15}$$

$$G = (Q(A^TA)^{-1}Q^T)^{-1} = BB^T - \frac{1}{p}(Be)(Be)^T, \tag{16}$$

$$\rho_i = \sqrt{(m+1)c_{m+1}(\tilde{\omega})s_i^2}, \; i = 1, \ldots, n. \tag{17}$$

We easily observe that

$$\mathrm{P}\left(V \in \mathcal{S}_v(\tilde{\omega})\right) \geq \mathrm{P}\left((\mu, V) \in \mathcal{S}(\tilde{\omega})\right) = \tilde{\omega}^n. \tag{18}$$

Evidently, $\mathcal{S}_m(\tilde{\omega})$ and $\mathcal{S}_v(\tilde{\omega})$ are in the form of (2) and (3), respectively. Goldfarb and Iyengar [15] set $\mathcal{S}_m = \mathcal{S}_m(\tilde{\omega})$ and $\mathcal{S}_v = \mathcal{S}_m(\tilde{\omega})$, and claim that $\mathcal{S}_m$ and $\mathcal{S}_v$ are $\tilde{\omega}^n$-confidence sets for μ and V, respectively. However, in view of (14) and (18), we immediately see that $\mathcal{S}_m$ and $\mathcal{S}_v$ have at least $\tilde{\omega}^n$-confidence levels, but their actual confidence levels are unknown and can be much higher than $\tilde{\omega}^n$. Also, using the relation $\mathcal{S}(\tilde{\omega}) \subseteq \mathcal{S}_m(\tilde{\omega}) \times \mathcal{S}_v(\tilde{\omega})$, we have

$$\mathrm{P}\left((\mu, V) \in \mathcal{S}_m(\tilde{\omega}) \times \mathcal{S}_v(\tilde{\omega})\right) \geq \mathrm{P}\left((\mu, V) \in \mathcal{S}(\tilde{\omega})\right) = \tilde{\omega}^n.$$

Hence, viewed as a joint uncertainty set of (μ, V), $\mathcal{S}_m(\tilde{\omega}) \times \mathcal{S}_v(\tilde{\omega})$ has at least $\tilde{\omega}^n$-confidence level, but its actual confidence level is unknown and can be much higher than the desired $\tilde{\omega}^n$. Thus, the robust model based on $\mathcal{S}_m(\tilde{\omega}) \times \mathcal{S}_v(\tilde{\omega})$ can be too conservative. In addition, as observed from computational experiments, the robust portfolio corresponding to this uncertainty set is highly non-diversified, in other words, it concentrates on a few assets only. One possible interpretation for this phenomenon is that $\mathcal{S}_m(\tilde{\omega})$ has a box-type structure. From this point of view, this is an inherent drawback of the uncertainty structure proposed in [15]. To combat these drawbacks, we introduce a "joint" ellipsoidal uncertainty set for (μ, V) in Section 3, and show that it can be constructed from a statistical approach.

3 Joint uncertainty set and robust portfolio selection models

In this section, we consider the same factor model of asset returns as described in Section 2. In particular, we first introduce a "joint" ellipsoidal uncertainty set for the model parameters (μ, V). A statistical procedure, motivated by [15], is also proposed for constructing such an uncertainty set. Finally, we discuss several robust portfolio selection problems for this uncertainty set.

For the remainder of the paper, we assume that all notations are defined in Section 2, unless defined explicitly otherwise.

As discussed in Section 2, the "separable" uncertainty set $\mathcal{S}_m(\tilde{\omega}) \times \mathcal{S}_v(\tilde{\omega})$ of (μ, V) proposed in [15] has several drawbacks. To overcome these drawbacks, we consider a "joint" ellipsoidal uncertainty set with ω-confidence level in the form of

$$\mathcal{S}_{\mu,v} \equiv \mathcal{S}_{\mu,v}(\omega) = \left\{ (\mu, V) : \sum_{i=1}^{n} \frac{(\bar{x}_i - x_i)^T (A^T A)(\bar{x}_i - x_i)}{s_i^2} \leq \hat{c}(\omega) \right\} \tag{19}$$

for some $\hat{c}(\omega)$. We next propose a statistical procedure, motivated by [15], for constructing such an uncertainty set for (μ, V).

Recall from Section 2 that the random variable $\mathscr{Y}^i$ is distributed according to the F-distribution with $m+1$ degrees of freedom in the numerator and $p-m-1$ degrees of freedom in the denominator for $i = 1, \dots, n$. This fact together with (6) and the assumption that the residual errors $\{\epsilon_i : i = 1, \dots, n\}$ are independent, implies that the random variables $\{\mathscr{Y}^i : \; i = 1, \dots, n\}$ are independently and identically distributed ($i.i.d.$). It follows from a standard statistical result (e.g., see [23]) that the mean and standard deviation of the random variable $\mathscr{Y}^i$ for $i = 1, \dots, n$ are

$$\mu_F = \frac{p-m-1}{p-m-3}, \qquad \sigma_F = \sqrt{\frac{2(p-m-1)^2(p-2)}{(m+1)(p-m-3)^2(p-m-5)}}, \tag{20}$$

respectively, provided that $p > m+5$, which usually holds in practice. Using the central limit theorem (e.g., see [19]), we conclude that the distribution of

$$Z_n = \frac{\sum\limits_{i=1}^{n} \mathscr{Y}^i - n\mu_F}{\sigma_F \sqrt{n}}$$

converges towards the standard normal distribution $\mathcal{N}(0,1)$ as $n \to \infty$. Notice that when n approaches a couple dozen, the distribution of Z_n is very nearly $\mathcal{N}(0,1)$. Given a confidence level ω, let $c(\omega)$ be a critical value for a standard normal variable Z, i.e., $\mathrm{P}\left(Z \leq c(\omega)\right) = \omega$. Thus, we have

$$\lim_{n \to \infty} \mathrm{P}\left(Z_n \leq c(\omega)\right) = \omega.$$

Hence, for a relative large n,

$$\mathrm{P}\left(\frac{\sum\limits_{i=1}^{n} \mathscr{Y}^i - n\mu_F}{\sigma_F \sqrt{n}} \leq c(\omega) \right) \approx \omega,$$

which together with (9) implies that

$$\mathrm{P}\left(\sum_{i=1}^{n} \frac{(\bar{x}_i - x_i)^T (A^T A)(\bar{x}_i - x_i)}{s_i^2} \leq (m+1)(c(\omega)\sigma_F \sqrt{n} + n\mu_F) \right) \approx \omega.$$

Using this result and the relation $x_i = (\mu_i, V_{1i}, V_{2i}, \ldots, V_{mi})^T$, we see that the set

$$\left\{(\mu, V):\ \sum_{i=1}^{n} \frac{(\bar{x}_i - x_i)^T (A^T A)(\bar{x}_i - x_i)}{s_i^2} \le (m+1)(c(\omega)\sigma_F \sqrt{n} + n\mu_F)\right\} \tag{21}$$

is an uncertainty set for (μ, V) with ω-confidence level when n is relatively large (say a couple dozen), where μ_F and σ_F are defined in (20). Evidently, this uncertainty set is in the form of (19) with $\hat{c}(\omega) = (m+1)(c(\omega)\sigma_F\sqrt{n} + n\mu_F)$. We next consider the case where n is relatively small. Recall that $\mathscr{Y}^i$ $(i = 1, \ldots, n)$ are *i.i.d.* F-distribution. Thus, we can find a $h(\omega)$ based on simulation techniques (e.g., see [22]) so that $\mathrm{P}(\sum_{i=1}^{n} \mathscr{Y}^i \le h(\omega)) \approx \omega$. Letting $\hat{c}(\omega) = (m+1)h(\omega)$ and using (9), we easily see that the set $\mathcal{S}_{\mu,V}$ defined in (19) with such $\hat{c}(\omega)$ is an ω-confidence uncertainty set for (μ, V). Thus, roughly speaking, for a given $\omega > 0$, an uncertainty set for (μ, V) in the form of (19) with ω-confidence level can be constructed by the above statistical procedure.

Throughout the rest of the paper, we assume that $\mathcal{S}_{\mu,v}$ is a "joint" ellipsoidal uncertainty set for (μ, V) given by (19) with ω-confidence level. This together with (19) and the assumption that A has full column rank, implies that

$$\hat{c}(\omega) > 0, \quad \text{if } \omega > 0. \tag{22}$$

We next introduce several robust portfolio selection problems for the uncertainty set $\mathcal{S}_{\mu,v}$. Indeed, an investor's position in this market is described by a portfolio $\phi \in \Re^n$, where the ith component ϕ_i represents the fraction of total wealth invested in ith asset. The return r_ϕ on the portfolio ϕ is given by

$$r_\phi = r^T\phi = \mu^T\phi + f^T V\phi + \epsilon^T\phi \sim \mathcal{N}(\phi^T\mu, \phi^T(V^T F V + D)\phi), \tag{23}$$

and hence, the mean and variance of the return of the portfolio ϕ are

$$\mathrm{E}[r_\phi] = \phi^T\mu, \qquad \mathrm{Var}[r_\phi] = \phi^T(V^T F V + D)\phi, \tag{24}$$

respectively. For convenience, we assume throughout the paper that short sales are not allowed, i.e., $\phi \ge 0$. Let

$$\Phi = \{\phi :\ e^T\phi = 1, \phi \ge 0\}. \tag{25}$$

The objective of the *robust maximum return* problem is to maximize the worst case expected return subject to a constraint on the worst case variance, i.e., to solve the following problem

$$\begin{array}{lll} \max\limits_{\phi} & \min\limits_{(\mu,V)\in\mathcal{S}_{\mu,v}} & \mathrm{E}[r_\phi] \\ \text{s.t.} & \max\limits_{(\mu,V)\in\mathcal{S}_{\mu,v}} & \mathrm{Var}[r_\phi] \le \lambda, \\ & \phi \in \Phi. & \end{array} \tag{26}$$

A closely related problem, the *robust minimum variance* problem, is the "dual" of (26). The objective of this problem is to minimize the worst case variance of the portfolio subject to a constraint on the worst case expected return on the portfolio. It can be formulated as

$$\begin{array}{lll} \min\limits_{\phi} & \max\limits_{(\mu,V)\in\mathcal{S}_{\mu,v}} & \mathrm{Var}[r_\phi] \\ \text{s.t.} & \min\limits_{(\mu,V)\in\mathcal{S}_{\mu,v}} & \mathrm{E}[r_\phi] \geq \beta, \\ & \phi \in \Phi. & \end{array} \tag{27}$$

We next address a drawback associated with the robust model (26). Let

$$\mathcal{S}_\mu = \{\mu:\ (\mu,V)\in\mathcal{S}_{\mu,v} \text{ for some } V\}, \quad \mathcal{S}_V = \{V:\ (\mu,V)\in\mathcal{S}_{\mu,v} \text{ for some } \mu\}.$$

In view of (24), we easily observe that (26) is equivalent to

$$\begin{array}{lll} \max\limits_{\phi} & \min\limits_{(\mu,V)\in\mathcal{S}_\mu\times\mathcal{S}_V} & \mathrm{E}[r_\phi] \\ \text{s.t.} & \max\limits_{(\mu,V)\in\mathcal{S}_\mu\times\mathcal{S}_V} & \mathrm{Var}[r_\phi] \leq \lambda, \\ & \phi \in \Phi. & \end{array} \tag{28}$$

Hence, the uncertainty set of (μ, V) used in (26) is essentially $\mathcal{S}_\mu \times \mathcal{S}_V$. Recall that $\mathcal{S}_{\mu,v}$ has ω-confidence level, and hence, we have

$$\mathrm{P}\left((\mu,V)\in\mathcal{S}_\mu\times\mathcal{S}_V\right) \geq \mathrm{P}\left((\mu,V)\in\mathcal{S}_{\mu,v}\right) = \omega.$$

Using this relation, we immediately conclude that $\mathcal{S}_\mu \times \mathcal{S}_V$ has at least ω-confidence level, but its actual confidence level is unknown and can be much higher than the desired ω. Hence, the robust model (26) can be too conservative even though $\mathcal{S}_{\mu,v}$ has the desired confidence level ω. Using a similar argument, we easily see that the robust model (27) also has this drawback. Thus, the robust models (26) and (27) are not suitable for the uncertainty set $\mathcal{S}_{\mu,v}$.

To combat the drawback of the robust model (26), we establish the following two lemmas.

Lemma 3.1 *Let*

$$\lambda^l = \min_{\phi\in\Phi}\max_{V\in\mathcal{S}_V} \mathrm{Var}[r_\phi],$$

and let $\phi^*(\lambda)$ denote an optimal solution of the problem (26) for any $\lambda > \lambda^l$. Then, $\phi^*(\lambda)$ is also an optimal solution of the following problem

$$\max_{\phi\in\Phi}\ \min_{(\mu,V)\in\mathcal{S}_\mu\times\mathcal{S}_V} \mathrm{E}[r_\phi] - \theta\mathrm{Var}[r_\phi], \tag{29}$$

for some $\theta \geq 0$.

Proof. Recall that the problem (26) is equivalent to the problem (28). It implies that $\phi^*(\lambda)$ is also an optimal solution of (28). Let

$$f(\phi) = \min_{(\mu,V)\in\mathcal{S}_\mu\times\mathcal{S}_V} \mathrm{E}[r_\phi], \quad g(\phi) = \max_{(\mu,V)\in\mathcal{S}_\mu\times\mathcal{S}_V} \mathrm{Var}[r_\phi].$$

In view of (24), we see that $f(\phi)$ is concave and $g(\phi)$ is convex over the compact convex set Φ. For any $\lambda > \lambda^l$, we easily observe that: i) the problem (28) is feasible and its optimal value is finite; and ii) there exists a $\phi \in \Phi$ such that $g(\phi) < \lambda$. Hence, by Proposition 5.3.1 of Bertsekas [6], we know that there exists at least one Lagrange multiplier $\theta \geq 0$ such that $\phi^*(\lambda)$ solves the problem (29) for such θ. ■

We observe that the converse of Lemma 3.1 also holds.

Lemma 3.2 *Let $\phi^*(\theta)$ denote an optimal solution of the problem (29) for any $\theta \geq 0$. Then, $\phi^*(\theta)$ is also an optimal solution of the problem (26) for*

$$\lambda = \max_{V \in \mathcal{S}_V} \mathrm{Var}[r_{\phi^*(\theta)}]. \tag{30}$$

Proof. We first observe that $\phi^*(\theta)$ is a feasible solution of the problem (26) with λ given by (30). Now assume for a contradiction that there exits a $\phi \in \Phi$ such that

$$\max_{(\mu,V)\in\mathcal{S}_{\mu,v}} \mathrm{Var}[r_\phi] \leq \lambda, \quad \min_{(\mu,V)\in\mathcal{S}_{\mu,v}} \mathrm{E}[r_\phi] > \min_{(\mu,V)\in\mathcal{S}_{\mu,v}} \mathrm{E}[r_{\phi^*(\theta)}],$$

or equivalently,

$$\max_{V\in\mathcal{S}_V} \mathrm{Var}[r_\phi] \leq \lambda, \quad \min_{\mu\in\mathcal{S}_\mu} \mathrm{E}[r_\phi] > \min_{\mu\in\mathcal{S}_\mu} \mathrm{E}[r_{\phi^*(\theta)}]$$

due to (24). These relations together with (24) and (30) imply that, for such ϕ,

$$\begin{aligned} \min_{(\mu,V)\in\mathcal{S}_\mu\times\mathcal{S}_V} \mathrm{E}[r_\phi] - \theta\mathrm{Var}[r_\phi] &= \min_{\mu\in\mathcal{S}_\mu} \mathrm{E}[r_\phi] - \theta \max_{V\in\mathcal{S}_V} \mathrm{Var}[r_\phi], \\ &> \min_{\mu\in\mathcal{S}_\mu} \mathrm{E}[r_{\phi^*(\theta)}] - \theta \max_{V\in\mathcal{S}_V} \mathrm{Var}[r_{\phi^*(\theta)}], \\ &= \min_{(\mu,V)\in\mathcal{S}_\mu\times\mathcal{S}_V} \mathrm{E}[r_{\phi^*(\theta)}] - \theta\mathrm{Var}[r_{\phi^*(\theta)}], \end{aligned}$$

which is a contradiction to the fact that $\phi^*(\theta)$ is an optimal solution of the problem (29). Thus, the conclusion holds. ■

In view of Lemmas 3.1 and 3.2, we conclude that the problems (26) and (29) are equivalent. We easily observe that the conservativeness of the robust model (29) (or equivalently (26)) can be alleviated if we replace $\mathcal{S}_\mu \times \mathcal{S}_V$ by $\mathcal{S}_{\mu,v}$ in (29). This leads to the *robust maximum risk-adjusted return (RAR)* portfolio selection problem with the uncertainty set $\mathcal{S}_{\mu,v}$:

$$\max_{\phi\in\Phi} \min_{(\mu,V)\in\mathcal{S}_{\mu,v}} \mathrm{E}[r_\phi] - \theta\mathrm{Var}[r_\phi], \tag{31}$$

where $\theta \geq 0$ represents the risk-aversion parameter.

In contrast to the robust models (26) and (27), the robust maximum RAR model (31) has a clear advantage that the confidence level of the uncertainty set (or equivalently, the robustness of the model) is controllable and no projection of $\mathcal{S}_{\mu,v}$ into the subspaces of μ and V is involved. In Section 4, we show that the robust maximum RAR problem (31) can be reformulated as a cone programming problem. For the ease of future reference, we refer to the model resulted from (31) by replacing $\mathcal{S}_{\mu,v}$ by $\{(\bar{\mu}, \bar{V})\}$ as the classical maximum RAR portfolio selection problem. Thus, in terms of (24), it can be written as

$$\max_{\phi\in\Phi} \ \bar{\mu}^T\phi - \theta\phi^T(\bar{V}^T F\bar{V} + D)\phi. \tag{32}$$

4 Cone programming for robust portfolio selection problem

In this section, we show that the robust maximum RAR portfolio selection problem (31) can be reformulated as a cone programming problem.

In view of (23), the robust maximum RAR problem (31) can be written as

$$\max_{\phi\in\Phi} \left\{ \min_{(\mu,V)\in\mathcal{S}_{\mu,v}} \left\{\mu^T\phi - \theta\phi^T V^T FV\phi\right\} - \theta\phi^T D\phi \right\}. \tag{33}$$

By introducing auxiliary variables ν and t, the problem (33) can be reformulated as

$$\begin{array}{ll} \max\limits_{\phi,\nu,t} & \nu - \theta t \\ \text{s.t.} & \min\limits_{(\mu,V)\in\mathcal{S}_{\mu,v}} \left\{\mu^T\phi - \theta\phi^T V^T FV\phi\right\} \geq \nu, \\ & \phi^T D\phi \leq t, \\ & \phi \in \Phi. \end{array} \tag{34}$$

We next aim to reformulate the inequality

$$\min_{(\mu,V)\in\mathcal{S}_{\mu,v}} \left\{\mu^T\phi - \theta\phi^T V^T FV\phi\right\} \geq \nu \tag{35}$$

as linear matrix inequalities (LMIs). Before proceeding, we introduce two lemmas that will be used subsequently.

We now state a lemma about the $\mathscr{S}$-procedure. For a discussion about the $\mathscr{S}$-procedure and its applications, see Boyd et al. [7].

Lemma 4.1 *Let $F_i(x) = x^T A_i x + 2b_i x + c_i$, $i = 0, \ldots, p$ be quadratic functions of $x \in \Re^n$. Then $F_0(x) \leq 0$ for all x such that $F_i(x) \leq 0$, $i = 1, \ldots, p$, if there exists $\tau_i \geq 0$ such that*

$$\sum_{i=1}^{p} \tau_i \begin{pmatrix} c_i & b_i^T \\ b_i & A_i \end{pmatrix} - \begin{pmatrix} c_0 & b_0^T \\ b_0 & A_0 \end{pmatrix} \succeq 0.$$

Moreover, if $p = 1$ then the converse holds if there exists x_0 such that $F_1(x_0) < 0$.

In the next lemma, we state one simple property of the standard Kronecker product, denoted as $\otimes$. For its proof, see [18].

Lemma 4.2 *If $H \succeq 0$ and $K \succeq 0$, then $H \otimes K \succeq 0$.*

In the following lemma, we show that the inequality (35) can be reformulated as LMIs.

Lemma 4.3 *Let $\mathcal{S}_{\mu,v}$ be defined in (19) for $\omega > 0$. Then, the inequality (35) is equivalent to the following LMIs*

$$\begin{pmatrix} \tau R - 2\theta S \otimes \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} & \tau h + q \\ \tau h^T + q^T & \tau\eta - 2\nu \end{pmatrix} \succeq 0, \qquad (36)$$
$$\begin{pmatrix} 1 & \phi^T \\ \phi & S \end{pmatrix} \succeq 0, \quad \tau \geq 0,$$

where

$$R = \begin{pmatrix} \frac{A^TA}{s_1^2} & & \\ & \ddots & \\ & & \frac{A^TA}{s_n^2} \end{pmatrix} \in \Re^{[(m+1)n]\times[(m+1)n]}, \quad \eta = \sum_{i=1}^n \bar{x}_i^T \left(\frac{A^TA}{s_i^2}\right) \bar{x}_i - \hat{c}(\omega), \qquad (37)$$

$$h = \begin{pmatrix} -\frac{A^TA\bar{x}_1}{s_1^2} \\ \vdots \\ -\frac{A^TA\bar{x}_n}{s_n^2} \end{pmatrix} \in \Re^{(m+1)n}, \quad q = \begin{pmatrix} \phi_1 \\ 0 \\ \vdots \\ \phi_n \\ 0 \end{pmatrix} \in \Re^{(m+1)n} \qquad (38)$$

(here, 0 denotes the m-dimensional zero vector).

Proof. Given any $(\nu, \theta, \phi) \in \Re \times \Re \times \Re^n$, we define

$$H(\mu, V) = -\mu^T\phi + \theta\phi^T V^T F V \phi + \nu.$$

As in (4), let

$$x_i = (\mu_i, V_{1i}, V_{2i}, \ldots, V_{mi})^T, \; i = 1, \ldots, n.$$

We now view $H(\mu, V)$ as a function of $x = (x_1, \ldots, x_n) \in \Re^{(m+1)n}$. Then we easily see that

$$\frac{\partial H}{\partial x_i} = \begin{pmatrix} -\phi_i \\ 2\theta\phi_i F V \phi \end{pmatrix}, \quad \frac{\partial^2 H}{\partial x_i \partial x_j} = \begin{pmatrix} 0 & 0 \\ 0 & 2\theta\phi_i\phi_j F \end{pmatrix}, \; i, j = 1, \ldots, n.$$

Using these relations and performing the Taylor series expansion for $H(\mu, v)$ at $x = 0$, we obtain that

$$H(\mu, V) = \frac{1}{2} \sum_{i,j=1}^n x_i^T \begin{pmatrix} 0 & 0 \\ 0 & 2\theta\phi_i\phi_j F \end{pmatrix} x_j + \sum_{i=1}^n \begin{pmatrix} -\phi_i \\ 0 \end{pmatrix}^T x_i + \nu. \qquad (39)$$

Recall that $\mathcal{S}_{\mu,v}$ is given by (19). Hence, $\mathcal{S}_{\mu,v}$ can be written as

$$\mathcal{S}_{\mu,v} = \left\{ (\mu, V) : \sum_{i=1}^{n} x_i^T \left(\frac{A^T A}{s_i^2} \right) x_i - 2 \sum_{i=1}^{n} \left(\frac{A^T A \bar{x}_i}{s_i^2} \right)^T x_i + \sum_{i=1}^{n} \bar{x}_i^T \left(\frac{A^T A}{s_i^2} \right) \bar{x}_i - \hat{c}(\omega) \leq 0 \right\}. \tag{40}$$

Since $\omega > 0$, we further see from (22) that $\hat{c}(\omega) > 0$, and hence, we conclude that $x = \bar{x}$ strictly satisfies the inequality given in (40). In view of (39), (40) and Lemma 4.1, we conclude that $H(\mu, V) \leq 0$ for all $(\mu, V) \in \mathcal{S}_{\mu,v}$ if and only if there exists a $\tau \in \Re$ such that

$$\tau \begin{pmatrix} R & h \\ h^T & \eta \end{pmatrix} - \begin{pmatrix} E & -q \\ -q^T & 2\nu \end{pmatrix} \succeq 0, \quad \tau \geq 0, \tag{41}$$

where R, η, h and q are defined in (37) and (38), respectively, and E is given by

$$E = (E_{ij}) \in \Re^{[(m+1)n] \times [(m+1)n]}, \quad E_{ij} = \begin{pmatrix} 0 & 0 \\ 0 & 2\theta\phi_i\phi_j F \end{pmatrix} \in \Re^{(m+1)\times(m+1)}, \; i, j = 1, \ldots, n.$$

In terms of Kronecker product $\otimes$, we easily see that

$$E = 2\theta(\phi\phi^T) \otimes \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix}.$$

This together with Lemma 4.2 and the fact that $F \succeq 0$, implies that (41) holds if and only if

$$\begin{pmatrix} \tau R - 2\theta S \otimes \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} & \tau h + q \\ \tau h^T + q^T & \tau\eta - 2\nu \end{pmatrix} \succeq 0, \quad S \succeq \phi\phi^T, \quad \tau \geq 0. \tag{42}$$

Using the well-known Schur Complement Lemma, we further observe that (42) holds if and only if (36) holds. Thus, it follows that $H(\mu, V) \leq 0$ for all $(\mu, V) \in \mathcal{S}_{\mu,v}$ if and only if (36) holds. The conclusion immediately follows from this result and the fact that the inequality (35) holds if and only if $H(\mu, V) \leq 0$ for all $(\mu, V) \in \mathcal{S}_{\mu,v}$.

■

In the following theorem, we show that the robust maximum RAR portfolio selection problem (31) can be reformulated as a cone programming problem.

Theorem 4.4 *Let $\mathcal{S}_{\mu,v}$ be defined in (19) for $\omega > 0$. Then, the robust maximum RAR portfolio* *selection problem (31) is equivalent to*

$$
\begin{array}{rcccl}
\max\limits_{\phi,S,\tau,\nu,t} & & \nu - \theta t & & \\
s.t. & \left(\begin{array}{cc} \tau R - 2\theta S \otimes \left(\begin{array}{cc} 0 & 0 \\ 0 & F \end{array}\right) & \tau h + q \\ \tau h^T + q^T & \tau\eta - 2\nu \end{array}\right) & \succeq & 0, & \\
& \left(\begin{array}{cc} 1 & \phi^T \\ \phi & S \end{array}\right) & \succeq & 0, & \\
& \left(\begin{array}{c} 1+t \\ 1-t \\ 2D^{1/2}\phi \end{array}\right) & \in & \mathcal{L}^{n+2}, & \\
& \tau \geq 0, \quad \phi & \in & \Phi, &
\end{array}
\qquad (43)
$$

where Φ, R, η, h *and* q *are defined in (25), (37) and (38), respectively, and* $\mathcal{L}^k$ *denotes the* k*-dimensional second-order cone given by*

$$
\mathcal{L}^k = \left\{ z \in \Re^k : \; z_1 \geq \sqrt{\sum_{i=2}^{k} z_i^2} \right\}.
$$

Proof. We observe that the inequality $\phi^T D \phi \leq t$ is equivalent to

$$
\left(\begin{array}{c} 1+t \\ 1-t \\ 2D^{1/2}\phi \end{array}\right) \in \mathcal{L}^{n+2},
$$

which together with Lemma 4.3, implies that (34) is equivalent to (43). The conclusion immediately follows from this result and the fact that (31) is equivalent to (34). ∎

In the following theorem, we establish a basic property for the problem (43).

Theorem 4.5 *Assume that* $0 \neq F \succeq 0$, $\omega > 0$ *and* $\theta > 0$. *Then, the problem (43) and its dual problem are both strictly feasible, and hence, both problems are solvable and the duality gap is zero.*

Proof. Let $\mathrm{ri}(\cdot)$ denote the relative interior of the associated set. We first show that the problem (43) is strictly feasible. In view of (25), we immediately see that $\mathrm{ri}(\Phi) \neq \emptyset$. Let $\phi^0 \in \mathrm{ri}(\Phi)$, and let $t^0 \in \Re$ such that $t^0 > (\phi^0)^T D \phi^0$. Then we easily observe that

$$
\left(\begin{array}{c} 1+t^0 \\ 1-t^0 \\ 2D^{1/2}\phi^0 \end{array}\right) \in \mathrm{ri}(\mathcal{L}^{n+2})
$$

Now, let S^0 be a $n \times n$ symmetric matrix such that $S^0 \succ \phi^0(\phi^0)^T$. Then, by Schur Complement Lemma, one has

$$
\left(\begin{array}{cc} 1 & (\phi^0)^T \\ \phi^0 & S^0 \end{array}\right) \succ 0.
$$

Using the assumption that A has full column rank, we observe from (37) that $R \succ 0$. Hence, there exists a sufficiently large $\tau^0 > 0$ such that

$$M \equiv \tau^0 R - 2\theta S^0 \otimes \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} \succ 0. \tag{44}$$

Now, let ν^0 be sufficiently small such that

$$\tau^0\eta - 2\nu^0 - (\tau^0 h + q^0)^T M^{-1}(\tau^0 h + q^0) > 0,$$

where $q^0 = (\phi_1^0, 0, \ldots, \phi_n^0, 0)^T \in \Re^{(m+1)n}$ (here, 0 denotes the m-dimensional zero vector). This together with (44) and Schur Complement Lemma, implies that

$$\begin{pmatrix} \tau^0 R - 2\theta S^0 \otimes \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} & \tau^0 h + q^0 \\ (\tau^0 h + q^0)^T & \tau^0\eta - 2\nu^0 \end{pmatrix} \succ 0.$$

Thus, we see that the point $(\phi^0, S^0, \tau^0, \nu^0, t^0)$ is a strictly feasible point of the problem (43).

We next show that the dual of the problem (43) is also strictly feasible. Let

$$X^1 = \begin{pmatrix} X_{11}^1 & X_{12}^1 \\ X_{21}^1 & X_{22}^1 \end{pmatrix}, \quad X^2 = \begin{pmatrix} X_{11}^2 & X_{12}^2 \\ X_{21}^2 & X_{22}^2 \end{pmatrix}, \quad x^3 = \begin{pmatrix} x_1^3 \\ x_2^3 \\ x_3^3 \end{pmatrix}$$

be the dual variables corresponding to the first three constraints of the problem (43), respectively, where $X_{11}^1 \in \Re^{[(m+1)n]\times[(m+1)n]}$, $X_{12}^1 \in \Re^{(m+1)n}$, $X_{22}^2 \in \Re^{n\times n}$, X_{21}^2, $x_3^3 \in \Re^n$, X_{22}^1, X_{11}^2, x_1^3, $x_2^3 \in \Re$. Also, let $x^4 \in \Re$ be the dual variable corresponding to the constraint $e^T\phi = 1$. Then, we easily see that the dual of the problem (43) is

$$\begin{array}{llcl} \min\limits_{X^1, X^2, x^3, x^4} & X_{11}^2 + x_1^3 + x_2^3 + x^4 & & \\ \text{s.t.} & -2\Psi(X_{12}^1) - 2X_{21}^2 - 2D^{1/2}x_3^3 + x^4 e & \geq & 0, \\ & 2\theta \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} \odot X_{11}^1 - X_{22}^2 & = & 0, \\ & -\begin{pmatrix} R & h \\ h^T & \eta \end{pmatrix} \bullet X^1 & \geq & 0, \\ & -x_1^3 + x_2^3 & = & -\theta, \\ & 2X_{22}^1 & = & 1, \\ & X^1 \succeq 0, \ X^2 \succeq 0, \ x^3 & \in & \mathcal{L}^{n+2}, \end{array} \tag{45}$$

where $\Psi : \Re^{(m+1)n} \to \Re^n$ is defined as $\Psi(x) = (x_1, x_{m+2}, \ldots, x_{(n-2)(m+1)+1}, x_{(n-1)(m+1)+1})^T$ for any $x \in \Re^{(m+1)n}$, and

$$\begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} \odot X \equiv \left(\begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} \bullet X_{ij} \right) \in \Re^{n\times n} \tag{46}$$

for any $X = (X_{ij}) \in \Re^{[(m+1)n]\times[(m+1)n]}$ with $X_{ij} \in \Re^{(m+1)\times(m+1)}$ for $i, j = 1, \ldots, n$. We now construct a strictly feasible soultion (X^1, X^2, x^3, x^4) of the dual problem (45). Let $x^3 = (\theta, 0, \ldots, 0) \in \Re^{n+2}$. It clearly satisfies the constraint $-x^3_1 + x^3_2 = -\theta$, and moreover, $x^3 \in \mathrm{ri}(\mathcal{L}^{n+2})$ due to $\theta > 0$. Next, let

$$X^1 = \frac{1}{2(1+\gamma)} \left[\begin{pmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_n \\ 1 \end{pmatrix} \begin{pmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_n \\ 1 \end{pmatrix}^T + \gamma I \right]. \tag{47}$$

We easily see that $X^1_{22} = 1/2$. Since $\omega > 0$, we know from (22) that $\hat{c}(\omega) > 0$. This together with (37), (38) and (47), implies that

$$\begin{aligned} -\begin{pmatrix} R & h \\ h^T & \eta \end{pmatrix} \bullet X^1 &= -\tfrac{1}{2(1+\gamma)} \left[R \bullet \begin{pmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_n \end{pmatrix} \begin{pmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_n \end{pmatrix}^T + 2h^T \begin{pmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_n \end{pmatrix} + \eta + \gamma \begin{pmatrix} R & h \\ h^T & \eta \end{pmatrix} \bullet I \right] \\ &= -\tfrac{1}{2(1+\gamma)} \left[-\hat{c}(\omega) + \gamma \begin{pmatrix} R & h \\ h^T & \eta \end{pmatrix} \bullet I \right] \succ 0 \end{aligned}$$

and $X^1 \succ 0$ for sufficiently small positive γ. Now, let

$$X^2_{22} = 2\theta \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} \odot X^1_{11}. \tag{48}$$

We next show that $X^2_{22} \succ 0$. Indeed, using (46) and the assumption that $0 \neq F \succeq 0$, we easily see that

$$\begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} \odot I \succ 0. \tag{49}$$

Further, for any $u \in \Re^n$, we have

$$\begin{aligned} u^T \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} \odot \left[\begin{pmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_n \end{pmatrix} \begin{pmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_n \end{pmatrix}^T \right] u &= \sum_{i,j} \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} \bullet (u_i u_j \bar{x}_i \bar{x}_j^T), \\ &= \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} \bullet \left(\sum_{i,j} u_i u_j \bar{x}_i \bar{x}_j^T \right), \\ &= \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} \bullet \left((\sum_i u_i \bar{x}_i)(\sum_i u_i \bar{x}_i)^T \right) \ \geq \ 0, \end{aligned}$$

and hence,

$$\begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} \odot \left[\begin{pmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_n \end{pmatrix} \begin{pmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_n \end{pmatrix}^T \right] \succeq 0.$$

This together with (47)-(49) and the assumption $\theta > 0$, implies that $X^2_{22} \succ 0$. Letting $X^2_{12} = 0$ and $X^2_{11} = 1$, we immediately see that $X^2 \succ 0$. We also observe that, for sufficiently large x^4, (X^1, X^2, x^3, x^4) also strictly satisifes the first constraint of (45) and hence, it is a strictly feasible solution of the dual problem (45). The remaining proof directly follows from strong duality.

■

In view of Theorem 4.5, we conclude that the problem (43) can be efficiently solved by standard primal-dual interior point method solvers (e.g., SeDuMi [24] and SDPT3 [26]).

5 Computational results

In this section, we present preliminary computational experiments for the robust maximum RAR portfolio selection problems. We conduct two types of computational tests. The first type of tests are based on simulated data (see Subsection 5.1), and the second type of tests use real market data (see Subsection 5.2). The main objectives of these computational tests are two-fold: one is to compare the performance of the *classical* maximum RAR problem (32) and its *robust* counterpart (33); another is to compare the performance of the robust maximum RAR problems with our "joint" uncertainty structure (19) and Goldfarb and Iyengar's "separable" uncertainty structure (2)-(3). All computations are performed using SeDuMi V1.1R2 [24] within Matlab R2006a. Throughout this section, the diversification number of a portfolio is defined as the number of its components that are above 10^{-2}. The symbols "NOM", "LROB" and "GIROB" are used to label the classical portfolio, and the robust portfolios corresponding to our "joint" uncertainty structure and Goldfarb and Iyengar's "separable" uncertainty structure, respectively.

5.1 Computational results for simulated data

In this subsection, we conduct computational tests for simulated data. The data is similarly simulated as in Section 7 of [15]. We fix the number of assets $n = 50$ and the number of factors $m = 5$. A symmetric positive definite factor covariance matrix F is randomly generated. This factor covariance matrix is assumed to be known and fixed. The nominal factor loading matrix V is also randomly generated. The covariance matrix D of the residual returns ϵ is assumed to be certain and set to $D = 0.1\ \mathrm{diag}(V^T F V)$, i.e., it is assumed that the linear model explains 90% of the asset variance. The nominal asset returns μ_i are chosen independently according to a uniform distribution on $[0.5\%, 1.5\%]$. Next, we generate a sequence of asset and factor return vectors r and f according to the market model (1) for an investment period of length $p = 90$.

In the first part of simulation tests, we compare the performance of the *classical* maximum RAR problem (32) and its *robust* counterpart (33) with our "joint" uncertainty structure (19) as the risk aversion parameter θ ranges from 0 to 10^4. Given a desired confidence level ω, our "joint" uncertainty set $\mathcal{S}_{\mu,v}$ is built according to (21). The computational results for

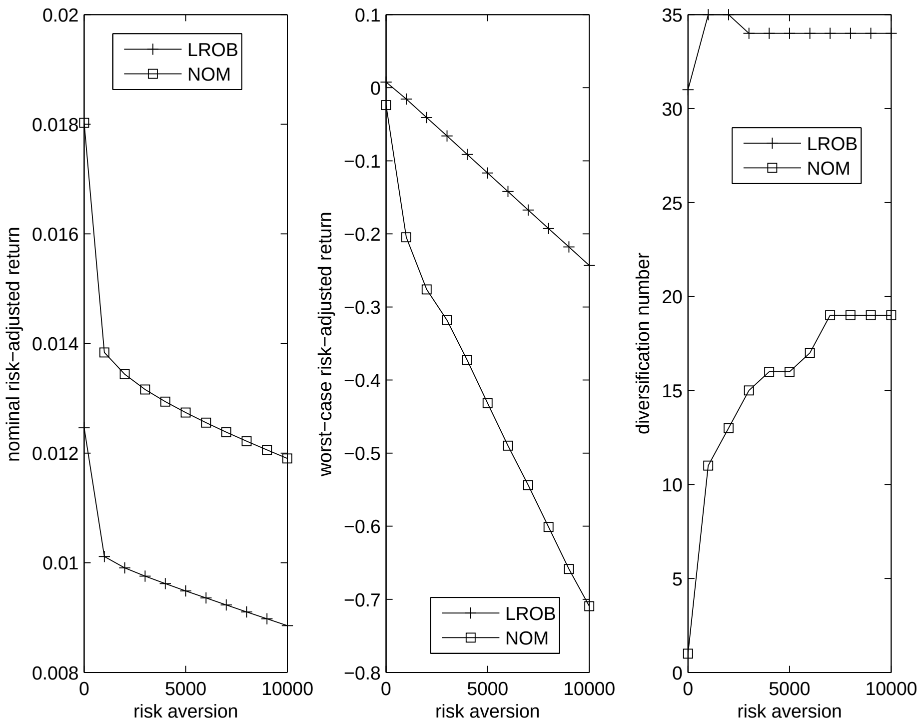

Figure 1: Performance of portfolios for $\omega = 0.05$.

$\omega = 0.05, 0.50, 0.95$ are shown in Figures 1-3, respectively. In each of these three figures, the first plot is about the nominal RAR of the robust and classical portfolios. The nominal RAR of any portfolio ϕ is given by

$$\bar{\mu}^T \phi - \theta \phi^T (\bar{V}^T F \bar{V} + D)\phi, \tag{50}$$

and the classical portfolio maximizes it over the set Φ. The second plot in Figures 1-3 is about the worst-case RAR of the robust and classical portfolios. The worst-case RAR of any portfolio ϕ is given by

$$\min_{(\mu,V)\in\mathcal{S}_{\mu,v}} \mu^T \phi - \theta \phi^T (V^T F V + D)\phi, \tag{51}$$

and the robust portfolio maximizes it over the set Φ. The last plot in Figures 1-3 is about the diversification number of the robust and classical portfolios. We observe that the nominal RAR of the robust portfolio is only slightly worse than that of the classical one, but the worst-case RAR of the robust portfolio is much better than that of the classical one. This phenomenon becomes even more prominent as ω increases. Indeed, when ω is increased, the difference of their worst-case RARs increases faster than that of their nominal RARs. In addition, we observe that the robust portfolio is more diversified than the classical one, and moreover, its diversification is fairly stable with respect to ω, in other words, its diversification number does not change much as ω increases.

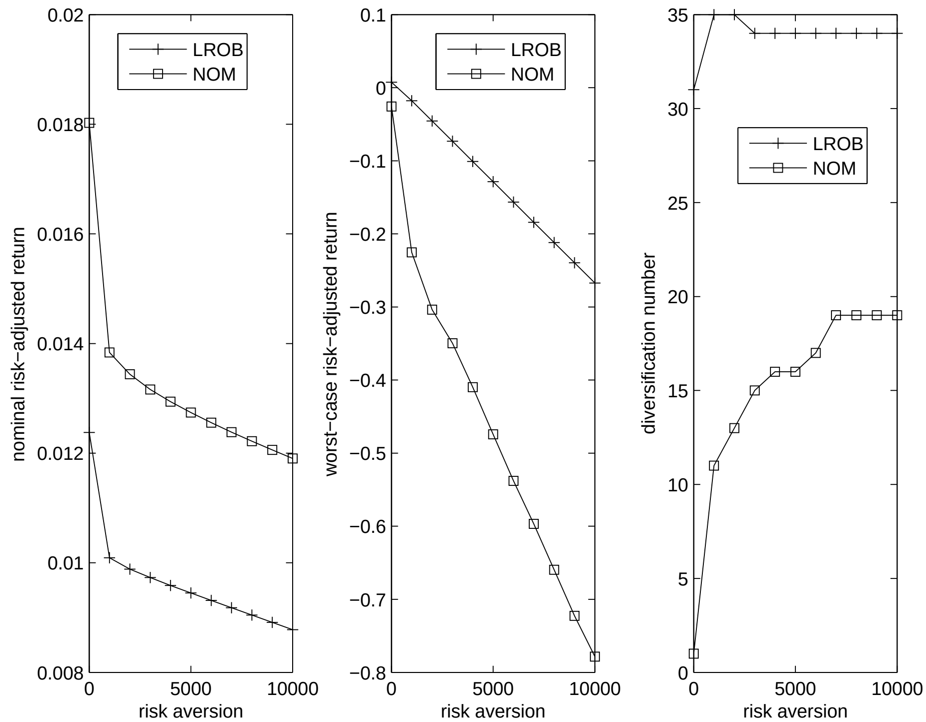

Figure 2: Performance of portfolios for $\omega = 0.5$.

In the second part of simulation experiments, we compare the performance of the robust maximum RAR problems with our "joint" uncertainty structure (19) and Goldfarb and Iyengar's "separable" uncertainty structure (2)-(3) as the risk aversion parameter θ ranges from 0 to 10^4. Given a desired confidence level $\omega > 0$, our "joint" uncertainty set $\mathcal{S}_{\mu,v}$ is built according to (21), and Goldfarb and Iyengar's "separable" uncertainty set $\mathcal{S}_m \times \mathcal{S}_v$ is built according to (12)-(17) with $\tilde{\omega} = \omega^{1/n}$. From Section 2, we know that $\mathcal{S}_m \times \mathcal{S}_v$ has at least ω-confidence level. Also, it shall be mentioned that the robust RAR problem with the uncertainty set $\mathcal{S}_m \times \mathcal{S}_v$ can be reformulated and solved as a second-order cone program (see [15]). The computational results for $\omega = 0.05, 0.50, 0.95$ are shown in Figures 4-6, respectively. In each of these three figures, the left plot is about the diversification number of the robust portfolios. The right plot in Figures 4-6 is about the wealth growth rate over next p periods, i.e., the periods indexed by $p+1, \ldots, 2p$. Let $R \in \Re^{n \times p}$ denote the asset returns for next p periods. Assume that R is randomly generated according to the same distribution as r. Let ϕ_r be a robust portfolio computed from the data of the previous p periods, i.e., the periods indexed by $1, \ldots, p$. Assume that ϕ_r is used for the investment over next p periods. The wealth growth rate over next p periods is defined as

$$\left(\Pi_{1 \le k \le p}(e + R_k)\right)^T \phi_r - 1, \tag{52}$$

where $e \in \Re^n$ denotes the all-one vector and $R_k \in \Re^n$ denotes the kth column of R for $k = 1, \ldots, p$. We observe that the diversification of these robust portfolios is insensitive

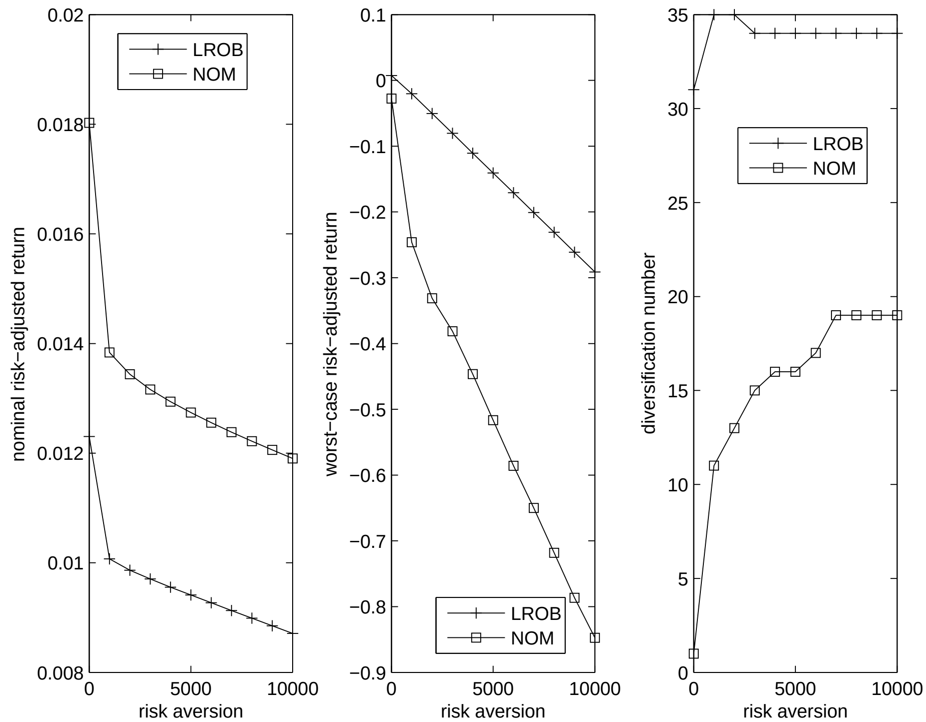

Figure 3: Performance of portfolios for $\omega = 0.95$.

with respect to ω. Indeed, for $\omega = 0.05, 0.5, 0.95$, the diversification number of our robust portfolio is around 34, and that of Goldfarb and Iyengar's is around 4. Evidently, our robust portfolio is fairly diversified, but Goldfarb and Iyengar's is highly non-diversified. One possible interpretation for this phenomenon is that our uncertainty set $\mathcal{S}_{\mu,v}$ is ellipsoidal, but Goldfarb and Iyengar's uncertainty set $\mathcal{S}_m \times \mathcal{S}_v$ is partially box-type, in particular, $\mathcal{S}_m$ has a box structure (see (12)). It seems that the ellipsoidal uncertainty structure leads to much more diversified robust portfolio than the fully or partially box-type one. In addition, we observe that for $\omega = 0.05$, the wealth growth rate of our robust portfolio is slightly worse than that of Goldfarb and Iyengar's, but for $\omega = 0.5, 0.95$, our wealth growth rate is much better than Goldfarb and Iyengar's. We now provide an interpretation for this phenomenon. Recall that the uncertainty set $\mathcal{S}_m \times \mathcal{S}_v$ has at least ω-confidence level, but its actual confidence level is unknown and it can be much higher than the desired ω. When ω is small, this may enhance the robustness of the model and the resulting robust portfolio can have better performance than the one corresponding to an ω-confidence uncertainty set (e.g., $\mathcal{S}_{\mu,v}$). However, for a relatively large ω, the uncertainty set $\mathcal{S}_m \times \mathcal{S}_v$ can be much over confident, and the corresponding robust RAR model can be too conservative.

In the previous experiment, we have already observed that the robust RAR model based on Goldfarb and Iyengar's "separable" $\mathcal{S}_m \times \mathcal{S}_v$ can be conservative. A natural question is how much conservative it can be. Unfortunately, there is some technical difficulty involved in answering this question directly. Instead, we next examine this conservativeness indirectly.

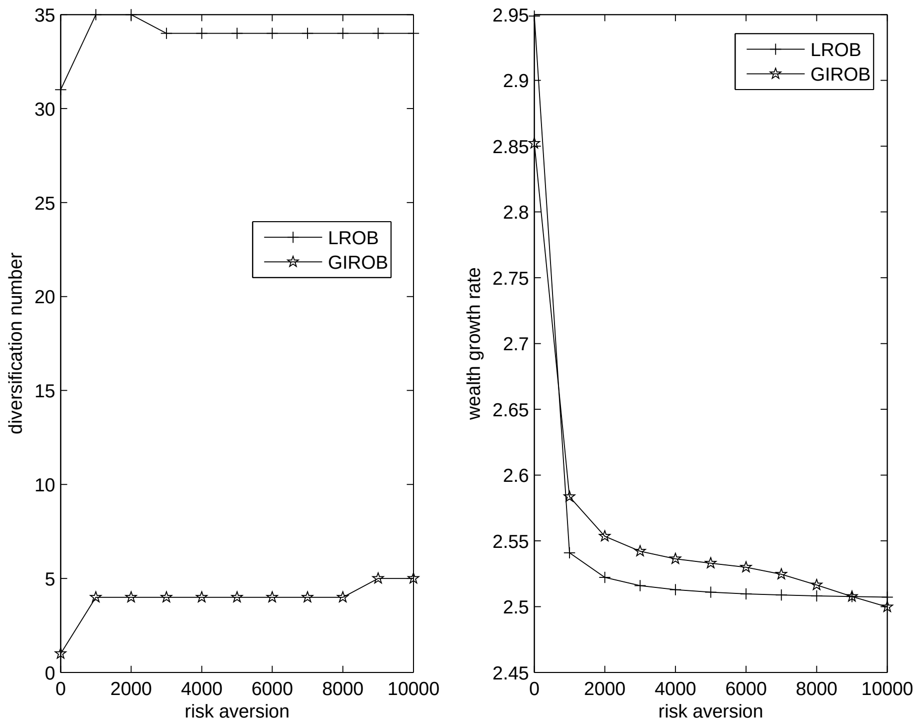

Figure 4: Performance of portfolios for $\omega = 0.05$.

Given an $\omega > 0$, let $\mathcal{S}_{\mu,v}$ be built according to (21), and hence $\mathcal{S}_{\mu,v}$ has ω-confidence level. Let $\tilde{\mathcal{S}}_m$ be a Cartesian product of the projections of $\mathcal{S}_{\mu,v}$ along μ_i for $i = 1, \ldots, n$, and let $\tilde{\mathcal{S}}_v$ be a Cartesian product of the projections of $\mathcal{S}_{\mu,v}$ along V_i for $i = 1, \ldots, n$. It follows from (21) that

$$\begin{aligned}
\tilde{\mathcal{S}}_m &= \{\mu : \ \mu = \mu_0 + v, \ |v_i| \leq \gamma_i, \ i = 1, \ldots, n\}, \\
\tilde{\mathcal{S}}_v &= \{V : \ V = V_0 + W, \ \|W_i\|_G \leq \rho_i, \ i = 1, \ldots, n\},
\end{aligned}$$

where

$$\begin{aligned}
\gamma_i &= \sqrt{(A^T A)^{-1}_{11} \tilde{c}(\omega) s_i^2}, \quad \rho_i = \sqrt{\tilde{c}(\omega) s_i^2}, \ i = 1, \ldots, n, \\
\tilde{c}(\omega) &= (m+1)(c(\omega)\sigma_F \sqrt{n} + n\mu_F),
\end{aligned}$$

and μ_0, V_0, G are defined in (13)-(16), and μ_F, σ_F are defined in (20). We easily observe that: i) the uncertainty set $\tilde{\mathcal{S}}_m \times \tilde{\mathcal{S}}_v$ has the identical uncertainty structure as $\mathcal{S}_m \times \mathcal{S}_v$; ii) the approach for building $\tilde{\mathcal{S}}_m \times \tilde{\mathcal{S}}_v$ is similar to the one for $\mathcal{S}_m \times \mathcal{S}_v$; and iii) $\tilde{\mathcal{S}}_m \times \tilde{\mathcal{S}}_v$ has at least ω-confidence level due to

$$\mathrm{P}\left((\mu, V) \in \tilde{\mathcal{S}}_m \times \tilde{\mathcal{S}}_v\right) \geq \mathrm{P}\left((\mu, V) \in \mathcal{S}_{\mu,v}\right) = \omega,$$

and its actual confidence level is unknown. Because of these key similarities, the conservativeness of the robust RAR model with the uncertainty set $\tilde{\mathcal{S}}_m \times \tilde{\mathcal{S}}_v$ will reflect that of the one corresponding to $\mathcal{S}_m \times \mathcal{S}_v$.

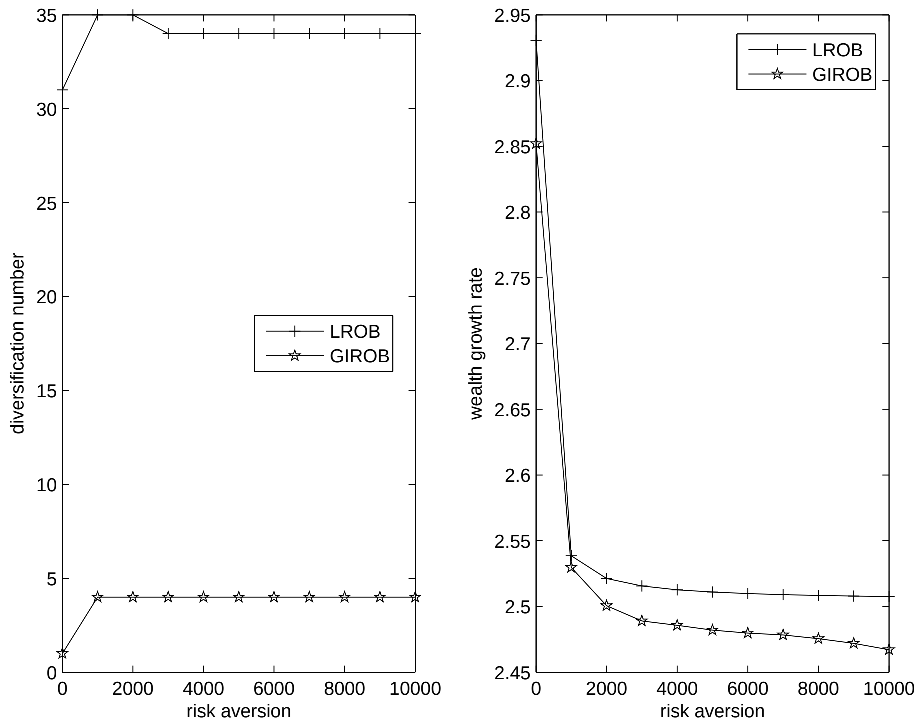

Figure 5: Performance of portfolios for $\omega = 0.5$.

We now examine the conservativeness of the robust maximum RAR model based on $\tilde{\mathcal{S}}_m \times \tilde{\mathcal{S}}_v$ by comparing its performance with that of the one based on $\mathcal{S}_{\mu,v}$ as the risk aversion parameter θ ranges from 0 to 10^4. The symbol "PLROB" is used to label the robust portfolio corresponding to the uncertainty set $\tilde{\mathcal{S}}_m \times \tilde{\mathcal{S}}_v$. The computational results for the confidence levels $\omega = 0.05,\ 0.50,\ 0.95$ are shown in Figures 7-9, respectively. In each of these three figures, the first plot is about the nominal RAR of the robust portfolios. The nominal RAR is computed according to (50). The second plot in Figures 7-9 is about the worst-case RAR of the robust portfolios. The worst-case RAR is computed according to (51). The third plot in these figures is about the wealth growth rate over next p periods of the investment using the robust portfolios. The wealth growth rate is computed according to (52). The last plot in Figures 7-9 is about the diversification number of the robust portfolios. From the first three plots in these figures, we observe that the worst-case RAR, nominal RAR and wealth growth rate of the robust portfolio corresponding to $\tilde{\mathcal{S}}_m \times \tilde{\mathcal{S}}_v$ are much worse than those corresponding to $\mathcal{S}_{\mu,v}$ for all three confidence levels. It implies that the robust portfolio corresponding to $\tilde{\mathcal{S}}_m \times \tilde{\mathcal{S}}_v$ is generally more conservative than the one corresponding to $\mathcal{S}_{\mu,v}$. In addition, we observe from the last plot in Figures 7-9 that the robust portfolio corresponding to $\tilde{\mathcal{S}}_m \times \tilde{\mathcal{S}}_v$ is highly non-diversified, but the one corresponding to $\mathcal{S}_{\mu,v}$ is fairly diversified.

Recall that there are several aforementioned key similarities between $\mathcal{S}_m \times \mathcal{S}_v$ and $\tilde{\mathcal{S}}_m \times \tilde{\mathcal{S}}_v$. Based on the above discussion, we can conclude that: i) the robust model based on Goldfarb and Iyengar's uncertainty set $\mathcal{S}_m \times \mathcal{S}_v$ can be too conservative; and ii) the uncertainty

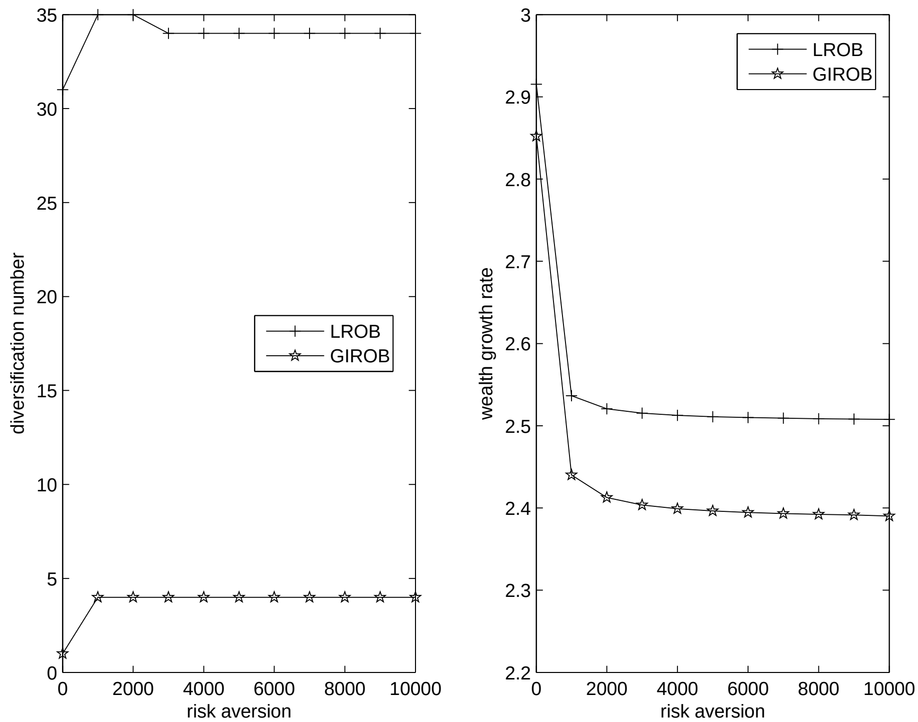

Figure 6: Performance of portfolios for $\omega = 0.95$.

structure of their type leads to highly non-diversified robust portfolios. We will further see these drawbacks in the experiments of real market data in Subsection 5.2.

5.2 Computational results for real market data

In this subsection, we perform some experiments on real market data for the classical maximum RAR model and its robust counterpart corresponding to our "joint" and Goldfarb and Iyengar's "separable" uncertainty structures. The universe of assets that are chosen for investment are those currently ranked at the top of each of 10 industry categories by Fortune 500 in 2006. In total there are $n = 47$ assets in this set (see Table 1). The set of factors are 10 major market indices (see Table 2). The data sequence consists of daily asset returns from July 25, 2002 through May 10, 2006. It shall be mentioned that the data for this paper was collected on May 11, 2006. The most recent data available on that day was the one on May 10, 2006.

A complete description of the experimental procedure is as follows. The entire data sequence is divided into investment periods of length $p = 90$ days. For each investment period t, the factor covariance matrix F is computed based on the factor returns of the previous p trading days, and the variance d_i of the residual return is set to $d_i = s_i^2$, where s_i^2 is given by (8). Given an $\omega > 0$, our "joint" uncertainty set $\mathcal{S}_{\mu,v}$ is built according to (21), and Goldfarb and Iyengar's "separable" uncertainty set $\mathcal{S}_m \times \mathcal{S}_v$ is built according to (12)-(17) with

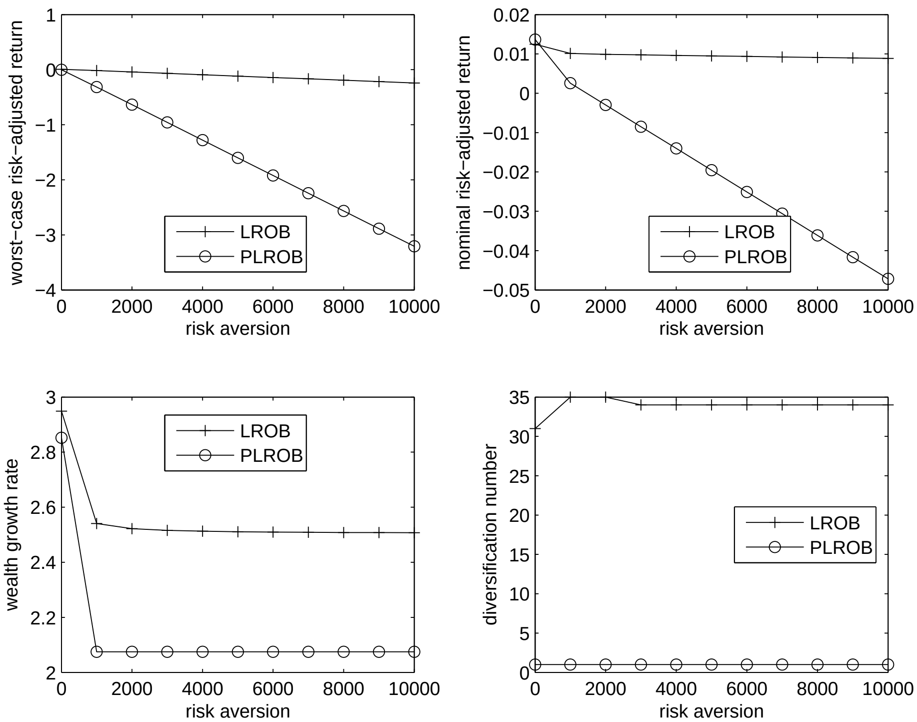

Figure 7: Performance of portfolios for $\omega = 0.05$.

$\tilde{\omega} = \omega^{1/n}$. The robust portfolios are computed by solving the robust maximum RAR model corresponding to those uncertainty sets. The classical portfolio is computed according to (32). The resulting portfolios are held constant for the investment at each period t.

Since both the robust and classical strategies require a block of data of length $p = 90$ to construct uncertainty sets or estimate the parameters, the first investment period indexed by $t = 1$ starts from $(p+1)$th day. The time period July 25, 2002 – May 10, 2006 contains 11 periods of length $p = 90$, and hence in all there are 10 investment periods. Given a sequence of portfolios $\{\phi^t\}_{t=1}^{10}$, the corresponding overall wealth growth rate is defined as

$$\Pi_{1\leq t\leq 10}\left[\Pi_{tp\leq k\leq (t+1)p}(e+r_k)\right]^T \phi_r^t - 1,$$

and the average diversification number is defined as

$$\frac{1}{10}\sum_{t=1}^{10} I(\phi^t),$$

where $I(\phi^t)$ denotes the diversification number of the portfolio ϕ^t.

We next report the performance of the classical portfolio and the robust portfolios corresponding to our "joint" uncertainty set $\mathcal{S}_{\mu,v}$, and Goldfarb and Iyengar's "separable" uncertainty set $\mathcal{S}_m \times \mathcal{S}_v$ as the risk aversion parameter θ ranges from 0 to 10^4. The computational results for the confidence levels $\omega = 0.05, 0.5, 0.95$ are shown in Figures 10-12, respectively.

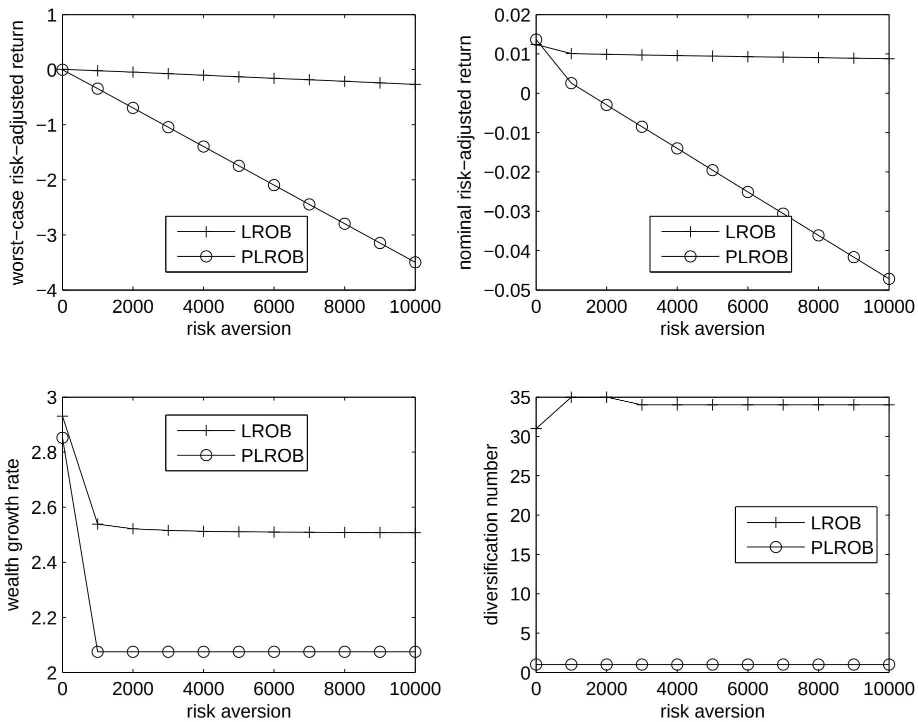

Figure 8: Performance of portfolios for $\omega = 0.5$.

In each of Figures 10-12, the first plot is about the average diversification number of those portfolios, and the second plot is about the overall wealth growth rate over next $p = 10$ periods of the investment using those portfolios. We observe that our robust portfolio is fairly diversified, but Goldfarb and Iyengar's is highly non-diversified. Also, the investment using the robust portfolios has better overall wealth growth rate than the one using the classical portfolio. Furthermore, the overall wealth growth rate of the investment based on our robust portfolio is higher than the one using Goldfarb and Iyengar's. Thus, in terms of diversification and overall wealth growth rate, our robust portfolio has better performance than Goldfarb and Iyengar's. In addition, it is interesting to see that the performance of both robust portfolios is not much sensitive with respect to ω.

An important aspect of any investment strategy is the cost of realizing it. We are interested in comparing the costs of implementing the aforementioned investment strategies. For this purpose, we will quantify the average transaction cost of a sequence of portfolios $\{\phi^t\}_{t=1}^{10}$ by

$$\frac{1}{9}\sum_{t=2}^{10} \|\phi^t - \phi^{t-1}\|_1.$$

(see also the discussion in [15]). In Figures 13-15, we report the average transaction costs of the investments using the classical and robust portfolios for the confidence levels $\omega = 0.05, 0.5, 0.95$, respectively. We easily observe that the investment based on our robust portfolio incurs much lower average transaction cost than the ones using the classical portfolio and

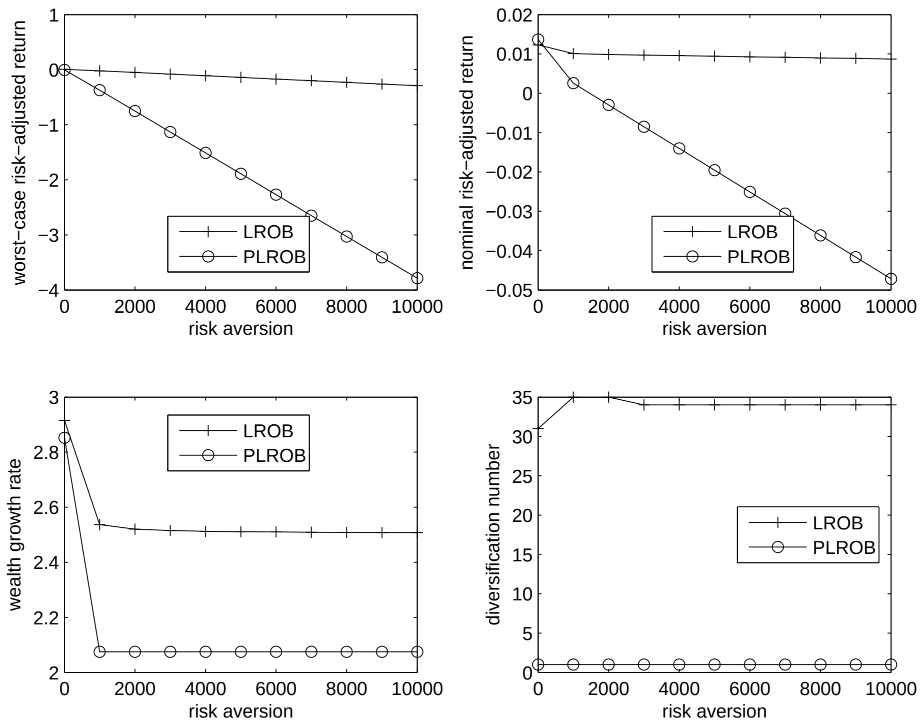

Figure 9: Performance of portfolios for $\omega = 0.95$.

Goldfarb and Iyengar's robust portfolio. Also, we see that the average transaction cost for the robust portfolio is fairly stable with respect to ω.

6 Concluding Remarks

In this paper, we considered the factor model of asset returns. By exploring the correlations of the mean return vector μ and factor loading matrix V, we proposed a statistical approach for constructing a "joint" uncertainty set $\mathcal{S}_{\mu,v}$ for (μ, V). The robust RAR portfolio selection model was also studied. From the computational experiments, we observe that the robust RAR model with our uncertainty set $\mathcal{S}_{\mu,v}$ is less conservative than the one based on Goldfarb and Iyengar's "separable" uncertainty set $\mathcal{S}_m \times \mathcal{S}_v$. In particular, our robust portfolio has much better performance than Goldfarb and Iyengar's in terms of wealth growth rate and transaction cost. Moreover, $\mathcal{S}_{\mu,v}$ is ellipsoidal, but $\mathcal{S}_m \times \mathcal{S}_v$ is partially box-type. A consequence of this fact is that the robust portfolio corresponding to $\mathcal{S}_{\mu,v}$ is fairly diversified, but the one based on $\mathcal{S}_m \times \mathcal{S}_v$ is highly non-diversified. Even though we only considered the robust RAR model in this paper, we expect that the similar results will also hold for other robust portfolio selection models, e.g., robust maximum Sharpe ratio and robust value-at-risk models (see [15]). The extension of this work to other robust models will be left for future research.

In addition, the results of this paper strongly rely on the assumptions in the model (1). We

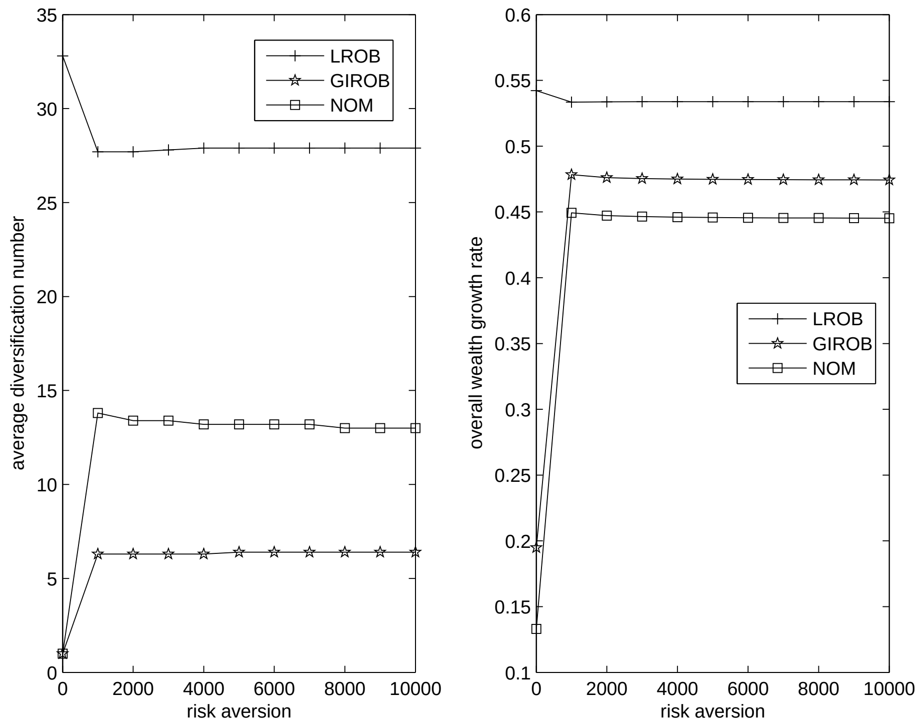

Figure 10: Performance of portfolios for $\omega = 0.05$.

believe that the accuracy of the model has much influence on the performance of the resulting robust portfolio. Thus, it would be interesting to investigate: i) how to select suitable factors for the model? ii) how to adjust the existing factor returns such that the resulting model is more accurate?

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The author is grateful to Antje Berndt for stimulating discussions on selecting suitable factors and providing me the real market data. The author is also indebted to Garud Iyengar for making his code available to me. We also gratefully acknowledge comments from Dimitris Bertsimas, Victor DeMiguel, Darinka Dentcheva, Andrzej Ruszczyński and Reha Tütüncü at the 2006 INFORMS Annual Meeting in Pittsburgh, USA.

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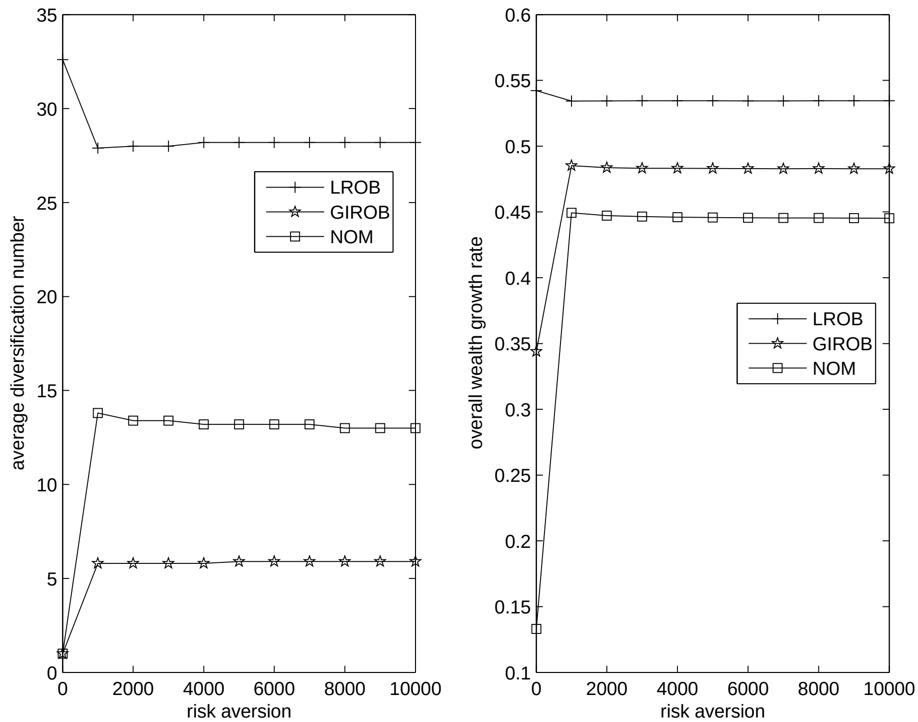

Figure 11: Performance of portfolios for $\omega = 0.5$.

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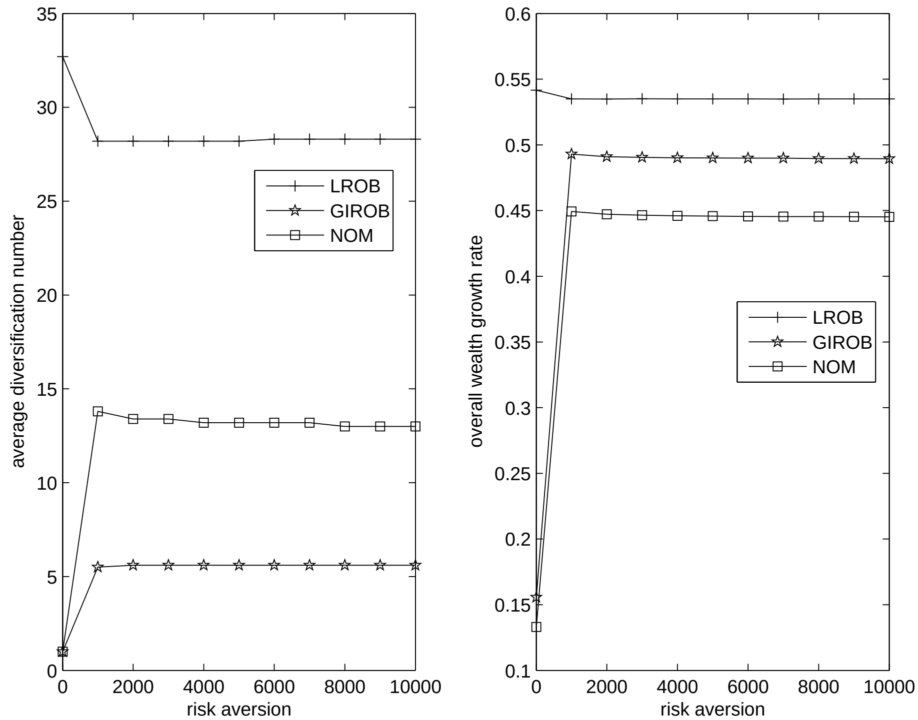

Figure 12: Performance of portfolios for $\omega = 0.95$.

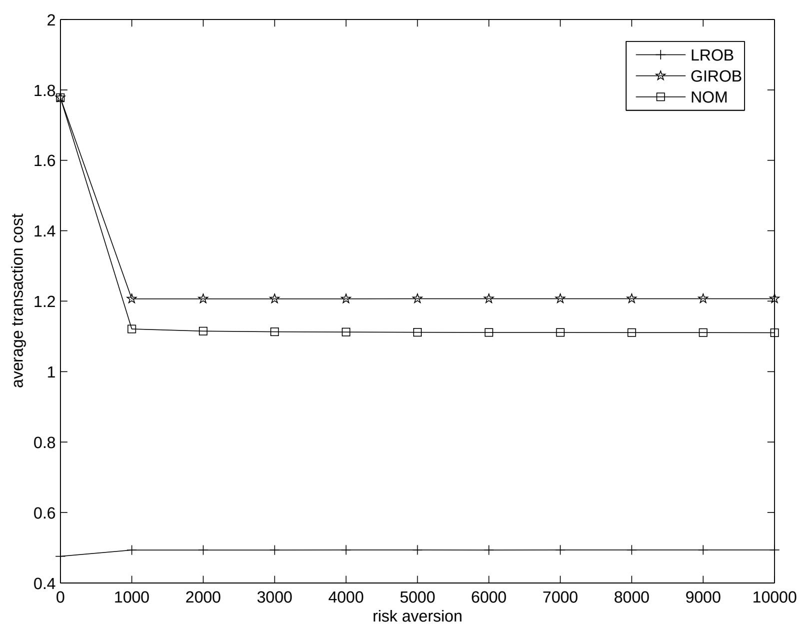

Figure 13: Average cost of portfolios for $\omega = 0.05$.

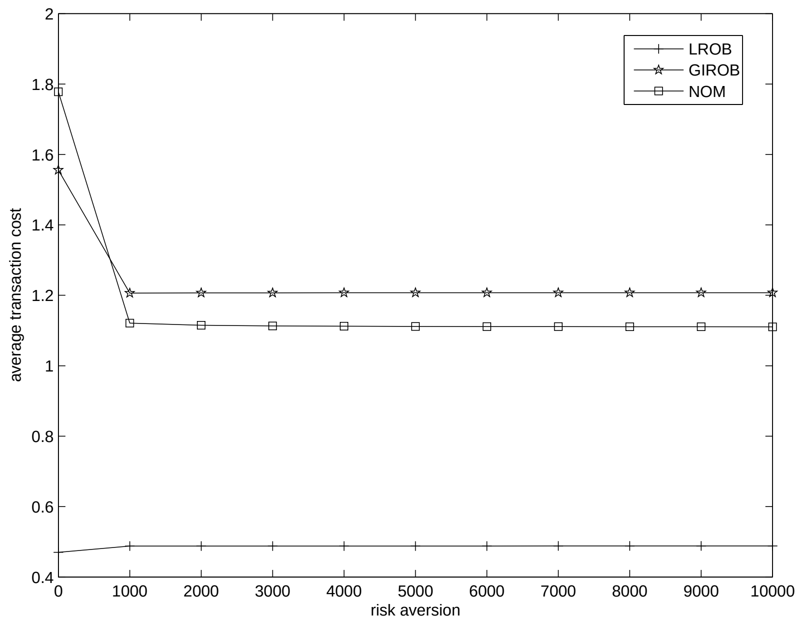

Figure 14: Average cost of portfolios for $\omega = 0.5$.

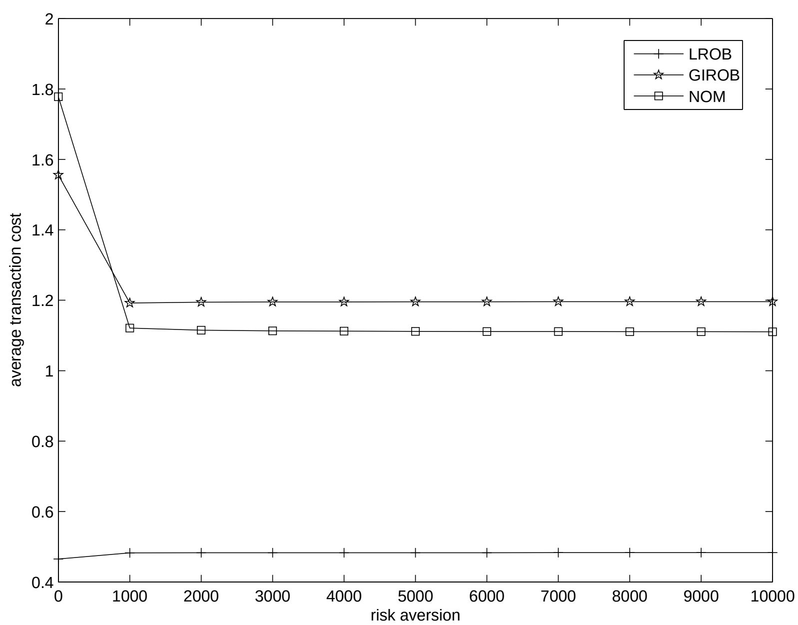

Figure 15: Average cost of portfolios for $\omega = 0.95$.

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Table 1: Assets

Aerospace and Defense		Telecommunications	
BA	Boeing Corp.	VZ	Verizon Communications
UTX	United Technologies	T	AT&T
LMT	Lockheed Martin	S	Sprint Nextel
NOC	Northrop Grumman	CMCSK	Comcast
HON	Honeywell Intl.	BLS	BellSouth
Semiconductors and Other Electronic Components		Computer Software	
INTC	Intel Corp.	MSFT	Microsoft
TXN	Texas Instruments	ORCL	Oracle
SANM	Sanmina-SCI	CA	CA
SLR	Solectron	ERTS	Electronic Arts
JBL	Jabil Circuit	SYMC	Symantec
Computers and Office Equipment		Pharmaceuticals	
IBM	Intl. Business Machines	PFE	Pfizer
HPQ	Hewlett-Packard	JNJ	Johnson & Johnson
DELL	Dell	ABT	Abbott Laboratories
XRX	Xerox	MRK	Merck
AAPL	Apple Computer	BMY	Bristol-Myers Squibb
Network and Other Communications Equipment		Chemicals	
MOT	Motorola	DOW	Dow Chemical
CSCO	Cisco Systems	DD	DuPon
LU	Lucent Technologies	LYO	Lyondell Chemical
QCOM	Qualcomm	PPG	PPG Industries
Electronics and Electrical Equipment		Utilities (Gas & Electric)	
EMR	Emerson Electric	DUK	Duke Energy
WHR	Whirlpool	D	Dominion Resources
ROK	Rockwell Automation	EXC	Exelon
SPW	SPX	SO	Southern
		PEG	Public Service Enterprise Group

Table 2: Factors

DJCMP65	Dow Jones Composite 65 Stock Average
DJINDUS	Dow Jones Industrials
DJUTILS	Dow Jones Utilities
DJTRSPT	Dow Jones Transportation
FRUSSL2	Russell 2000
NASA100	Nasdaq 100
NASCOMP	Nasdaq Composite
NYSEALL	NYSE Composite
S&PCOMP	S&P 500 Composite
WILEQTY	Dow Jones Wilshire 5000 Composite