One Class Nonsmooth Dyscrete Step Control Problem

Shahlar F.Maharramov. Yasar University, Faculity Science Department of Mathematics, Izmir 535500, Turkey, (email: shahlar.maharramov@yasar.edu.tr)

Abstract

In this paper a survey and refinement of its recent results in the discrete optimal control theory are presented. The step control problem depending on a parameter is investigated. No smoothness of the cost function φ is assumed and new versions of the discrete maximum principle for the step control problem are derived.

Key words. Optimal control Problems, subdifferential, superdifferential, Maximum principle, step control system.

1. Introduction

Some applied problems in fields such as economy, military defense, and chemistry are inherently multistage problems in nonsmooth optimization. In such problems, there are several stages which are characterized by their own equations, controls, phase coordinates, constants, etc. Usually these stages can be connected to each other by additional conditions. Here problems will be considered where these relations are given by switching points which are controlled by a given parameter. These multistage processes will be called step control systems or discrete systems with varying structure.

Example 1 (see reference $[8]$): A car moves according to the law $\dot{x} = y,\ \dot{y} = ug_1(y),\ u \in U$ at the time interval $\Delta_1 = [t_0, t_1]$, and according to $\dot{x} = y,\ \dot{y} = ug_2(y),\ u \in \sigma(y(t_1))$ at the time interval $\Delta_2 = [t_1, T]$. The initial and final time moments t_0 and T are fixed while instant t_1 is not fixed. The set $U = [0,1]$ and the functions $g_1,\ g_2,\ \sigma$ are positive and differentiable in R^1. The car starts from the origin $(x^0, y^0) = (0,0)$. The state variables x and y are assumed to be continuous on the whole interval $\Delta = [0, T]$. It is required to maximize $x(T)$. To find the necessary optimality conditions we have to build Hamilton-Pontryagin functions for each step and derive optimality condition at the switching moment t_1 and steps Δ_1 and Δ_2 In this example, switching moments are interesting for us because at the switching point we have to derive the optimality condition. By using increment formula and conjugate systems we can get the necessary condition for this step system.

Example 2: Consider a rocket with two types of engines that work consecutively. With work of the second engine depends on the first one. Moreover, the rocket moves from one controlling area to a second one that changes all the structure (controls, functions, conditions, etc.). For the smooth case some articles were published previously $[1,4,8,11,18,19]$. In [1, 6, 9, 10] the authors had gained necessary optimality condition of first order and investigated singular control, time with delay, sufficient optimality condition as a type Krotov for discrete switching optimal control problem.

In, [4] the author does not make any assumptions about the number of switches nor about the mode sequence, they are determined by the solution of the problem. Sufficient and necessary optimality conditions for optimality are formulated for the second optimization problem. If they exist, bang-bang-type solutions of the embedded optimal control problem are solutions of the original problem. Otherwise, suboptimal solutions are obtained via the Chattering lemma by the author. In [18] the author develops a computational method for solving an optimal control problem which is governed by a switched dynamical time system with time delay. Then, we derive the required gradient of the cost function which is obtained via solving a number of delay differential equation forward in time. On this basis author solved this problem as a mathematical programming.All this results dedicated in the smooth case optimal switching control problem (in all these papers the cost functional is smooth).In the present paper, the author's main aim is to formulate necessary optimality conditions for nonsmooth case and the switching points which depend on certain parameters, by using Mordukhovich cone (see, e.g.,[2,13,14]). To start our discussion, first we have describe certain points about nonsmooth analysis.

2. Tools of Nonsmooth Analysis

Given a nonempty set $\Omega \subset R^n$, consider the associated distance function

$dist(x;\Omega) = \inf_{u\in\Omega}\|x-w\|$ and define Euclidean projector of x to Ω by

$\Pi(x;\Omega) := \{w \in \Omega \mid \|x-w\| = dist(x;\Omega)\}$.

If the set Ω is closed, then the set $\Pi(x;\Omega)$ is nonempty for every $x \in R^n$.

This nonconvex cone to closed sets and corresponding subdifferential of lower semicontinuous extended–real –valued functions satisfying these requirements were introduced by Mordukhovich in the beginning of 1975, who was not familiar with Clarke's constructions at that time. The initial motivation came from the intention to derive necessary optimality conditions for optimal control problems with endpoint geometric constraints by passing to the limit from free endpoint control problems, which are much easier to handle. This was published in [15] (first in Russian and then translated into English), where the original normal cone definition was given in finite dimensional spaces by

$N(x;\Omega) := \operatorname*{Lim\,sup}_{x\to\bar{x}}[cone(x-\Pi(x;\Omega))]$, via the Euclidean projector, while the basic subdifferential $\partial\varphi(\bar{x})$ was defined geometrically via the normal cone to the epigraph of φ.Here it is assumed that φ is a real valued finite function and basic subdifferential defined is defined

$\partial\varphi(\bar{x}) := \{x^* \in R^n \, (x^*,-1) \in N((\bar{x},\varphi(\bar{x})); epi\varphi)\}$.

Here $epi\varphi := \{(x,\mu) \in R^{n+1} | \mu \geq \varphi(x)\}$ and is called the epigraph of given extended real valued function. Note that this cone is nonconvex (see, ref$[2]$) and for the locally Lipschitzian functions convex hull of subdifferential have a Clarke generalized gradient, $\bar{\varphi}_k(x^0) = co\partial\varphi(x^0)$.If φ_k is lower semicontinuous around x, then its basic subdifferential can be shown by: $\partial\varphi(x^0) = \operatorname*{Lim\,sup}_{x\xrightarrow{\varphi}\bar{x}} \hat{\partial}\varphi(x)$.Here, $\hat{\partial}\varphi\,(x^0) := \left\{ x^* \in R^n Li\min f \dfrac{\varphi(u)-\varphi(x)-\langle x^*, u-x\rangle}{|u-x|} \geq 0 \right\}$

is the Frechet subdifferential. By using plus-minus symmetric constructions, we can write

$\partial^+\varphi(x^0) := -\partial(-\varphi)(x^0), \hat{\partial}^+\varphi(x^0) := -\partial(-\hat{\varphi})(x^0)$

which are called basic superdifferential and Frechet superdifferential, respectively. Here

$\hat{\partial}\varphi^+\,(x^0) := \left\{ x^* \in R^n Lim\sup \dfrac{\varphi\,(x)-\varphi\,(x^0)-\langle x^*, x-x^0\rangle}{|x-x^0|} \geq 0 \right\}$.For a Locally Lipschitzian function subdifferential and superdifferential may be different. For example, if we take $\varphi(x) = |x|$ on R, then $\partial\varphi(0) = [-1,1]$ but $\hat{\partial}\varphi(0) = \{-1,1\}$.

If φ is Lipschitz continuous around point x^0 then, the strictly differentiability of the function φ at x^0 (see $[2,13]$) are equivalent to $\partial\varphi(x^0)=\partial^+\varphi(x^0)=\{\nabla\varphi(x^0)\}$. If $\partial\varphi(x^0)=\hat{\partial}\varphi(x^0)$ then, this function lower regular at x^0. Symmetrically we can give upper regularity of the function at the point by using definitions of superdifferential and Frechet superdifferential. Also if the extended-real-valued function is Lipschitz continuous around the given point and upper regular at this point then the Freshet superdifferential is not empty. Furthermore it is equal to Clarke generalized subdifferential at this point (for proof, see $[16]$). By using all these nonsmooth analysis tools, we will try to find the superdifferential form of the necessary optimality condition for the step discrete system.

3. Necessary optimality condition

Consider a controlling process, which is described by the following discrete system with varying structure: Minimize $$S(u,v)=\sum_{i=1}^{3}\varphi_i(x_i(t_i)) \qquad (2.1)$$

subject to

$$x_i(t+1)=f_i(t,x_i(t),u_i(t)),\ t\in T_i=\{t_{i-1},t_{i-1}+1,...,t_i-1\},\ i=1,2,3, \qquad 2.2)$$

$$\left.\begin{aligned} x_1(t_0)&=g_1(v_1)\\ x_i(t_{i-1})&=g_i(x_{i-1}(t_{i-1}),v_i),\quad i=2,3\end{aligned}\right\} \qquad (2.3)$$

$$u_i(t)\in U_i\subset R^r,\ t\in T_i,\ i=1,2,3. \qquad (2.4)$$

$$v_i\in V_i,\ i=1,2,3. \qquad (2.5)$$

Here, $v_i, i=1,2,3$ are q-dimensional controlling parameters and $V_i\subseteq R^q$, $i=1,2,3$, i.e. $v_i\in V_i$, $i=1,2,3$.

For these equations it is clear that the system's conditions are described in 3 stages (for a rocket entering from space to the atmosphere and then into water). In any stage, the system is described by its equation, controls, switching points, and controlling parameters for switching points. In case there are no switching points, we can apply the Pontryagin's maximum principle to any part of the system, but in this case it is difficult. For this we have to get new conditions of the switching points. In this problem, $g_1:R^q\to R^n$ are assumed to be at least twice continuously differentiable vector-valued functions, $g_i:R^n\times R^q\to R^n$ are given at least twice continuously differentiable vector-valued functions, i=2,3, $f_i:R\times R^n\times R^r\to R^n$ are given continuous, at least twice continuously partially differentiable vector-valued functions with respect to x, $\varphi_i:R^n\to R$ are given at least functions. We don't assume any smoothness on the cost functional φ_i $i=1,2,3$, u_i(t): R → $U_i\subset R^r$ are controls and $v_i\in V_i\subset R^q$ are controlling parameters. The sets U_i, V_i, are assumed to be nonempty and bounded. The pair $(u_i^0(t),v_i^o)$ which take volume from these sets are called admissible control. A pair $(u_i^0(t),v_i^o)$ with the properties (2.4) and (2.5) is called admissible. The triple $(u_i^0(t),v_i^o,x_i^0(t))$ is an admissible process. For the fixed admissible control $(u_i^0(t),v_i^o)$ we introduce the following notation:

$$H_i(t, x_i, u, \Psi_i^0) = \Psi_i^{0'}(t) \cdot f_i(t, x_i, u_i),$$

$$\Delta_{u_i} H_i[t] \equiv H_i(t, x_i^0(t), u_i(t), \Psi_i^0(t)) - H_i(t, x_i^0(t), u_i^0(t), \Psi_i^0(t)),$$

$$\frac{\partial H_i[t]}{\partial x_i} = \frac{\partial H_i(t, x_i^0(t), u_i^0(t), \psi_i^0(t))}{\partial x_i}, \ \Delta_{v_1} g_1[v_1] \equiv g_1(v_1) - g_1(v_1^0)$$

$$\Delta_{v_i} g_i(x_{i-1}^0(t_{i-1}), v_i^0) \equiv g_i(x_{i-1}^0(t_{i-1}), v_i) - g_i(x_{i-1}^0(t_{i-1}), v_i^0), i = 2,3$$

$$L_1(v_1, \Psi_1^0(t_0 - 1)) = \Psi_1^{0'}(t_0 - 1) g_1(v_1),$$

$$L_2(x_1(t_1), v_2, \Psi_2^0(t_1 - 1)) = \Psi_2^{0'}(t_1 - 1) g_2(x_1(t_1), v_2)$$

$$L_3(x_2(t_2), v_3, \Psi_3^0(t_2 - 1)) = \Psi_3^{0'}(t_2 - 1) g_3(x_2(t_2), v_3).$$

Theorem 1*: Assume that* $\varphi_i : R^n \to R$ *is finite at* $x_i^0(t_i)$ *and* $\hat{\partial}\varphi(x^0(t_i) \neq 0)$*. If the sets*

$$f_i(t_i x_i^0(t), U_i) = \{\alpha_i : \alpha_i = f_i(t_i x_i^0(t), u_i), u_i \in U\}, \ i = 1,2,3$$

$$g_1(V_1) = \{\alpha_4 : \alpha_4 = g_1(v_1), v_1 \in V_1\}$$

$$g_i(x_{i-1}^0(t_{i-1}), V_i) = \{\alpha_i : \alpha_i = g_i(x_{i-1}^0(t_{i-1}), v_i), v_i \in V\}, \quad i = 2,3$$

are convex, then for the optimality of an admissible control $(u^0(t), v^0)$ *in the problem described (2.1)-(2.5) problem it is necessary that for any* $x_i^* \in \hat{\partial}\varphi(x^0(t_i))$ *the following conditions are true:*

Discrete maximum principle for the control

$$\sum_{t=t_{i-1}}^{t_i - 1} \Delta_{u_{i(t)}} H_i[t] \leq 0, \text{ for all } u_i(t) \in U_i, \quad i = 1,2,3, \ t \in T_i \tag{2.6}$$

Discrete maximum principle for the controlling parameter $v_i^0, \quad i = 1, 2,3$

$$\max_{v_1 \in V_1} L_1(v_1, \psi_1^0(t_0 - 1)) = L_1(v_1^0, \psi_1^0(t_0 - 1)) \tag{2.7}$$

$$\max_{v_i \in V_i} L_i(x_{i-1}^0(t_{i-1}), v_i, \psi_i^0(t_{i-1} - 1)) = L_i(x_{i-1}^0(t_{i-1}), v_i^0, \psi_i^0(t_{i-1} - 1)), i = 2,3 \tag{2 .8}$$

where $\psi(.)$ *is adjoin trajectory and satisfying (2.11) systems. If the set* $f_i(t, x^0, U)$ *is convex, then the necessary optimality condition is global over all* $u_i \in U_i$

*Proof***:** In the control problem, one of the methods to get the necessary optimality conditions is to use the increment formula. For this, we have to calculate the increment formula, to find a conjugate system for the corresponding problems and use an analog of needle variations in the continuous case. The rest of the increment formula can be estimated using the step method.

For the optimal pair $(u^0(t), v^0)$ we can write increment of the functional following form

$\Delta S(u^0, v^0) = \sum_{i=1}^{3} [\varphi_i(x_i(t_i)) - \varphi_i(x_i^0(t_i))] \geq 0$. Using nonsmooth analysis tools we can write that, for any $x_i^* \in \hat{\partial}^+\varphi(x_i^0(t_i))$ we can write

$\varphi_i(x_i(t_i))-\varphi_i(x_i^0(t_i))\leq \left\langle x_i^*, (\Delta x_i^0(t)\right\rangle + 0(\Delta x_i^0(t))$. Then the increment of the functional takes the following form: $\Delta S(u^0,v^0)=\sum_{i=1}^{3}\left\langle x_i^*,\Delta x_i^0(t)\right\rangle+0(\Delta x_i^0(t))$.

Let multiply both sides of the equation (1) $\psi_i(t)$ and summing up from i =1 to 3.By using this and nonsmooth analysis definition, Taylor's increment formula after some calculation we can write the increment of the functional at an arbitrary admissible pair $(u_i(t),v_i)$ as

$$\Delta S(u^0,v^0)=\sum_{i=1}^{3}\left[\left\langle x_i^*,\Delta x_i(t)\right\rangle\right]+\sum_{i=1}^{3}\sum_{t=t_{i-1}}^{t_i-1}\psi_i^{0'}(t-1)\Delta x_i(t)-\sum_{i=1}^{3}\sum_{t=t_{i-1}}^{t_i-1}\left[H_i(t,\bar{x}_i(t),\bar{u}_i(t),\psi_i^0(t))-\right.$$

$$\left.-H_i(t,x_i^0(t),u_i^0(t),\psi_i^0(t))\right]+\sum_{i=1}^{3}\psi_i^{0'}(t_i-1)\Delta x_i(t_i)-\psi_1^{0'}(t_0-1)\Delta_{\bar{v}_1}g_1(v_1^0)-\psi_2^{0'}(t_1-1)\times$$

$$\times[g_2(\bar{x}_1(t_1),\bar{v}_2)-g_2(x_1^0(t_1),v_2^0)]-\psi_3^{0'}(t_2-1)[g_3(\bar{x}_2(t_2),\bar{v}_3)-g_2(x_2^0(t_2),v_3^0)]=$$

$$=\sum_{i=1}^{3}\left[\varphi_i(\bar{x}_i(t_i))-\varphi_i(x_i^0(t))\right]+\sum_{i=1}^{3}\sum_{t=t_{i-1}}^{t_i-1}\psi_i^{0'}(t-1)\Delta x_i(t)-\sum_{i=1}^{3}\sum_{t=t_{i-1}}^{t_i-1}\Delta_{\bar{u}_i}H_i[t]-$$

$$-\sum_{i=1}^{3}\sum_{t=t_{i-1}}^{t_i-1}\left[H_i(t,\bar{x}_i(t),\bar{u}_i(t),\psi_i^0(t))-H_i(t,x_i(t),\bar{u}_i(t),\psi_i^0(t))\right]+\sum_{i=1}^{3}\psi_i^{0'}(t_i-1)\Delta x_i(t_i)-$$

$$-\Delta_{\bar{v}_1}L_1(v_1^0,\psi_1^0(t-1))-\Delta_{\bar{v}_2}L_2(x_1^0(t),v_2^0,\psi_2^0(t_1-1))-\Delta_{\bar{v}_3}L_3(x_2^0(t_2),v_3^0,\psi_3^0(t_2-1))-$$

$$-\left[L_2(\bar{x}_1(t_1),\bar{v}_2,\psi_2^0(t_1-1))-L_2(x_1^0(t_1),\bar{v}_2,\psi_2^0(t_1-1))\right]-\left[L_3(\bar{x}_2(t_2),\bar{v}_3,\psi_3^0(t_2-1))-\right.$$

$$\left.-L_3(x_2^0(t_2),\bar{v}_3,\psi_3^0(t_2-1))\right]. \qquad (2.9)$$

where by definition

$$\eta_1(u^0,v^0;\Delta u,\Delta v)=\sum_{i=1}^{3}\sum_{t=t_{i-1}}^{t_{i-1}}\frac{\partial\Delta_{\bar{u}_i}H'[t]}{\partial x}\Delta x_i(t)-o_3(\|\Delta x_1(t_1)\|)-o_4(\|\Delta x_2(t_2)\|)-$$

$$-\frac{\partial\Delta_{\bar{v}_2}L_2(x_1^0(t_1),v_2^0,\psi_2^0(t_1-1))}{\partial x_1}\Delta x_1(t_1)-\frac{\partial\Delta_{\bar{v}_3}L_3(x_2^0(t_2),v_3,\psi_2^0(t_2-1))}{\partial x_2}\Delta x_2(t_2)+$$

$$+\sum_{i=1}^{3}o_1^{(i)}\|\Delta x_i(t_i)\|-\sum_{i=1}^{3}\sum_{t=t_{i-1}}^{t_i-1}o_2^{(i)}\|\Delta x_i(t)\|. \qquad 2.10)$$

Here $o_i(\cdot),\quad i=1,..,8$ are defined by the expansions

$$\varphi_i(\bar{x}_i(t_i))-\varphi_i(x_i(t_i))=\frac{\partial\varphi_i'(x_i^0(t_i))}{\partial x_i}\Delta x_i(t_i)+o_1^{(i)}(\|\Delta x_i(t_i)\|),\ i=\overline{1,3}$$

$$H_i(t,\bar{x}_i(t),\bar{u}_i(t),\psi_i^0(t))-H_i(t,x_i^0(t),\bar{u}_i\ (t),\psi_i^0(t))=\frac{\partial H_i'(t,x_i^0(t),\bar{u}_i(t),\psi_i^0(t))}{\partial x_i}\times$$

$$\times\Delta x_i(t)+o_2^{(i)}(\|\Delta_i(t)\|),\ i=\overline{1,3}$$

$$L_2(\bar{x}_1(t_1),\bar{v}_2,\psi_2^0(t_1-1))-L_2(x_1^0(t_1),\bar{v}_2,\psi_2^0(t_1-1))=\frac{\partial L_2'(x_1^0(t_1),\bar{v}_2,\psi_2^0(t_1-1))}{\partial x_1}\Delta x_1(t_1)+$$

$$+o_3\left(\left\|\Delta x_1(t_1)\right\|\right),$$

$$L_3\left(\bar{x}_2(t_2),\bar{v}_3,\psi_3^0(t_2-1)\right)-L_3\left(x_2^0(t_2),\bar{v}_3,\psi_3^0(t_2-1)\right)=\frac{\partial L_3'\left(x_2^0(t_2),\bar{v}_3,\psi_3^0(t_2-1)\right)}{\partial x_2}\Delta x_2(t_2)+$$

$$+o_4\left(\left\|\Delta x_2(t_2)\right\|\right),$$

Now take $\Psi_i^0(t),\quad i=1,2,3,$ as solutions of the following linear difference equations

$$\left.\begin{array}{c}\Psi_i^0(t-1)=\dfrac{\partial H_i[t]}{\partial x_i},\ i=1,2,3,\ t\in T_i\\ \Psi_1^0(t_1-1)=-x_1^*+\dfrac{\partial L_2\left(x_1^0(t_1),v_2^0,\psi_2^0(t_1-1)\right)}{\partial x_1}\\ \Psi_2^0(t_2-1)=-x_2^*+\dfrac{\partial L_3\left(x_2^0(t_2),v_3^0,\psi_3^0(t_2-1)\right)}{\partial x_1}\\ \Psi_3^0(t_3-1)=-x_3^*\end{array}\right\} \tag{2.11}$$

the increment formula (14) reduces to a simpler one:

$$\Delta S\left(u^0,v^0\right)=-\sum_{i=1}^{3}\sum_{t=t_{i-1}}^{t_i-1}\Delta_{\bar{u}_i}H[t]-\Delta_{\bar{v}_1}L_1\left(v_1^0,\psi_1^0(t_0-1)\right)-\Delta_{\bar{v}_2}L_2\left(x_1^0(t),v_2^0,\psi_2^0(t_1-1)\right)-$$

$$-\Delta_{\bar{v}_3}L_3\left(x_2^0(t_2),v_3^0,\psi_3^0(t_2-1)\right)+\eta_1\left(u^0,v^0;\Delta u,\Delta v\right) \tag{2.12}$$

Let $\left(u_i^{\,o}(t),v_i^{\,o}\right)$ *be an optimal pair, and assume that the sets of admissible velocities are convex along the process* $\left(u_i(t),v_i,x_i(t)\right)$*, i. e., the sets*

$$f_i\left(t,x_i^0(t),U_i\right)=\left\{\alpha_i:\alpha_i=f_i\left(t,x_i^0(t),u_i\right),u\in U_i\right\},\ i=1,2,3,$$

$$g_1(V_1)=\left\{\alpha_4:\alpha_4=g_1(v_1),v_1\in V_1\right\},$$

$$g_i\left(x_{i-1}^0(t_{i-1}),V_i\right)=\left\{\alpha_{i+3}:\alpha_{i+3}=g_i\left(x_{i-1}^0(t_{i-1}),v_i\right),v\in V_i\right\},\ \text{i=2,3},$$

are convex. Let $\varepsilon\in[0,1]$ be an arbitrary number. Denote the increment of the optimal pair by

$$\Delta u_i(t;\varepsilon)=u_i(t;\varepsilon)-u_i^0(t),\quad t\in T_i,\quad i=1,2,3, \tag{2.13}$$

(2.13)

$$\Delta v_i(\varepsilon)=v_i(\varepsilon)-v_i^0,\ i=1,2,3,$$

Then, by convexity, for each $u_i(t)\in U_i$, $v_i\in V$, $t\in T_i$, $i=1,2,3$, there are $u_i(t,\varepsilon)\in U_i$, $v_i(\varepsilon)\in V_i$, $i=1,2,3$ such that

$$\Delta_{u_{i(t:\varepsilon)}}f_i[t]=\varepsilon\Delta_{u_{i(t)}}f_i[t],\ i=1,2,3,$$

$$\Delta_{v_{1(\varepsilon)}}g_1\left(v_1^0\right)=\varepsilon\Delta_{v_1}g_1\left(v_1^0\right),$$

$$\Delta_{v_{i(\varepsilon)}}g_i\left(x_{i-1}^0(t_{i-1}),v_i^0\right)=\varepsilon\Delta_{v_i}g_i\left(x_{i-1}^0(t_{i-1}),v_i^0\right),\ i=2,3.$$

Equation (2.10) introduces an increment of the solution $x_i(t)$ *which is denoted by* $\left\{\Delta x_i(t;\varepsilon),i=1,2,3\right\}$*. Using the step methods, we can prove that* $\left\|\Delta x_i(t;\varepsilon)\right\|\le Z_{11}\varepsilon$, $t\in T_i\cup t_i$,

$i=1,2,3$. *Using these estimates in (2.12) it can easily be seen that the necessary optimality condition is* $\Delta S(u^0, v^0) \geq 0$.

Theorem 2: *Assume that* φ_i *is Lipschitz continuous around at* $x^0{}_i$, *upper regular at this point and the sets*

$$f_i(t_i x_i^0(t), U_i) = \{\alpha_i : \alpha_i = f_i(t_i x_i^0(t), u_i), u_i \in U\}, \; i=1,2,3$$

$$g_1(V_1) = \{\alpha_4 : \alpha_4 = g_1(v_1), v_1 \in V_1\}$$

$$g_i(x_{i-1}^0(t_{i-1}), V_i) = \{\alpha_i : \alpha_i = g_i(x_{i-1}^0(t_{i-1}), v_i), v_i \in V\}, \; i=2,3$$

are convex. Then for the optimality of an admissible control $(u^0(t), v^0)$ *in the problem given through (2.1)-(2.5), it is necessary that for any* $x_i^* \in \overline{\partial}\varphi(x^0(t_i))$ *the following conditions are true:*

Discrete maximum principle for the control $u_i^0(t), i=1,2,3$

$$\sum_{t=t_{i-1}}^{t_i-1} \Delta_{u_{i(t)}} H_i[t] \leq 0 \text{, for all } u_i(t) \in U_i, i=1,2,3, \; t \in T_i \tag{2.14}$$

Discrete maximum principle for the controlling parameter $v_i^0, i=1,2,3$ (2.15)

$$\max_{v_i \in V_i} L_i(x_{i-1}^0(t_{i-1}), v_i, \psi_i^0(t_{i-1}-1)) = L_i(x_{i-1}^0(t_{i-1}), v_i^0, \psi_i^0(t_{i-1}-1)), i=2,3 \tag{2.16}$$

where $\psi(.)$ *is adjoin trajectory and satisfies the system described under* (*2.11*).

It is easy to prove this theorem by using the tools of nonsmooth analysis given above. It should be noted that the system of linear difference equations (2.11) is the conjugate system for the problem (2.1)-(2.5). If we take smoothness on the cost functional φ_i then we can get some following corollary and anologies of Pontryagin maximum principle.

4. Necessary optimality conditions using the linearzing principle as an analogue of Euler equation in nonsmooth case.

If the cost functional is differentiable, the functions f_i, g_i have also partial derivatives with respect to u_i, v_i , respectively, and the sets U_i and $\mathrm{V_i}$ are convex, then another necessary optimality condition can be obtained using the linearzing maximum principle of Pontryagin. The proof of the next following corollaries to a large extent similar to the proof of Theorem 1 and is omitted. For the proof the interested reader is referred to the thesis [12].

Corollary 1*:(The superdifferential form of linearzing maximum principle). If the sets* U_i, V_i *are convex, then, for the optimality of the pair* $(u^0(t), v^0)$, *it is necessary that the following inequalities hold.*

$$\sum_{t=t_{i-1}}^{t_i-1} \frac{\partial H_i'[t]}{\partial u_i}(u_i(t) - u_i^0(t)) \leq 0 \tag{3.1}$$

for all $u_i(t) \in U_i$, $t \in T_i$, $i=1,2,3$

$$\frac{\partial L_1'\left(v_1^0,\psi_1^0(t_0-1)\right)}{\partial v_1}\left(v_1-v_1^0\right)\le 0\,, \tag{3.2}$$

for all $\mathrm{v}_1 \in \mathrm{V}_1$,

$$\frac{\partial L_i'\left(x_{i-1}^0(t_{i-1}),v_i^0,\Psi_i^0(t_{i-1}-1)\right)}{\partial v_i}\left(v_i-v_i^0\right)\le 0,\quad \textit{for all } v_i\in V_i,\, i=2,3. \tag{3.3}$$

In the case of openness of the sets $U_i, V_i \;, i=1,2,3$ *also using Euler's equation one can derive the necessary optimality conditions:*

Corollary 2 (An analogue of Euler equation): *If the sets* $U_i \;, V_i$ *are open, then for the optimality of the pair* $\left(u^0(t), v^0\right)$*, it is necessary that following equations hold*

$$\frac{\partial H_i^{'}[t]}{\partial u_i},\quad t\in T_i,\quad i=1,2,3 \quad . \tag{3.4}$$

$$\frac{\partial L_1^{'}\left(v_1^0,\psi_1^0(t_0-1)\right)}{\partial v_1}=0\,, \qquad 3.5)$$

$$\frac{\partial L_i^{'}\left(x_{i-1}^0(t_{i-1}),v_i^0,\psi_i^0(t_{i-1}-1)\right)}{\partial v_i}=0\,,\ \text{i=1,2,3} \qquad (3.6\ )$$

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