Monotonicity of Löwner Operators and Its Applications to Symmetric Cone Complementarity Problems*

Lingchen Kong

Department of Applied Mathematics, Beijing Jiaotong University,Beijing 100044, P.R. China, konglchen@126.com

Levent Tunçel

Department of Combinatorics and Optimization, Faculty of Mathematics, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada, ltuncel@math.uwaterloo.ca

Naihua Xiu

Department of Applied Mathematics, Beijing Jiaotong University, Beijing 100044, P.R. China nhxiu@center.njtu.edu.cn

This paper focuses on monotone Löwner operators in Euclidean Jordan algebras and their applications to the symmetric cone complementarity problem (SCCP). We prove necessary and sufficient conditions for locally Lipschitz Löwner operators to be monotone, strictly monotone and strongly monotone. We also study the relationship between monotonicity and operator-monotonicity of Löwner operators. As a by-product of our results, we establish a new class of C-functions for SCCP, which is an extension of the Mangasarian class of NCP-functions for the nonlinear complementarity problem, and present some characterizations of the C-functions for SCCP under certain assumptions.

Key words: Löwner operator, Euclidean Jordan algebra, Monotonicity, Symmetric cone complementarity problem, C-function

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1. Introduction. Consider a real-valued scalar function $g : (a, b) \to \mathbb{R}$. Such a function can be used to define an analogous operator G on the n-by-n symmetric matrices over the reals. That is, if x has the spectral decomposition

$$x = \sum_{i=1}^{n} \lambda_i(x) u_i u_i^T$$

then

$$G(x) := \sum_{i=1}^{n} g\left(\lambda_i(x)\right) u_i u_i^T,$$

where $\lambda_i(x)$ and u_i $(i = 1, 2, \cdots, n)$ are the eigenvalues and the corresponding eigenvectors of x, respectively. The domain of g implies a corresponding domain for G.

In 1934, Löwner [27] gave a characterization for when the operator G (obtained from such a scalar function g) is monotone with respect to the partial order induced by the cone of positive semidefinite matrices: G is monotone with respect to this partial order over its domain if and only if for every $\alpha_1, \alpha_2, \cdots, \alpha_n \in (a, b)$, all distinct, the n-by-n matrix with ijth entry

$$\frac{g(\alpha_i) - g(\alpha_j)}{\alpha_i - \alpha_j}$$

is positive semidefinite and every entry of it is nonnegative (when $i = j$ above, the ratio is interpreted as the derivative of g; so, in Löwner's theorem, the differentiability is "built in"). This very fundamental result connects monotonicity of g, the cone of positive semidefinite matrices and the monotonicity of operators on the underlying algebra of symmetric matrices over the reals. Following the terminology used in the book by Bhatia [1], we call such monotonicity property the *Operator-Monotonicity* of the function G. It is different from the usual concept of monotonicity of a vector-valued function coined by Kachurovskii [19]. The latter plays a crucial role in the theory and algorithms for complementarity problems, variational inequality problems and equilibrium problems [9, 17, 29, 36].

In 1984, Korányi [21] gave a full generalization of Löwner's theorem on operator-monotonicity of G to the setting where the underlying algebra is a Euclidean Jordan Algebra and the partial order is the one induced by the underlying symmetric cone (the cone of squares); see Section 2 for the details. Along this direction, Sun and Sun [41] studied the differentiability and semismoothness of the operator G above. We will follow the terminology of Sun and Sun and call such operators *Löwner Operators*.

On another front, in the area of nonlinear complementarity problems (NCP), there is an elegant result of Mangasarian going back to 1976, providing a very general tool for reformulation of NCPs based on the strict monotonicity of a scalar function [32]. Given $\theta : \mathbb{R} \to \mathbb{R}$, strictly increasing with $\theta(0) = 0$, Mangasarian class of NCP-functions is defined as $\phi_M : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$,

$$\phi_M(a, b) := \theta\left(|a - b|\right) - \theta(a) - \theta(b).$$

Tseng [44] asked whether Mangasarian class of NCP-functions above can be generalized to complementarity functions (C-functions) for the semidefinite complementarity problems.

Our main interest in this paper is to study monotonicity of Löwner operator G in $\mathcal{J}$ and its application to the symmetric cone complementarity problem (SCCP for short), which is to find a vector $x \in \mathcal{J}$ such that

$$x \in K,\ F(x) \in K,\ \langle x, F(x)\rangle = 0, \tag{1}$$

where $\mathcal{J}$ is the Euclidean Jordan algebra, K is the symmetric cone in $\mathcal{J}$, and $F : \mathcal{J} \to \mathcal{J}$ is a continuous function. SCCP provides a simple, natural, and unified framework for various existing complementarity problems, such as the nonnegative orthant nonlinear complementarity problem, the second-order cone complementarity problem (SOCCP), the semidefinite complementarity problem (SDCP). It has wide applications in engineering, economics, management science, and other fields; see the recent studies [7, 8, 11, 12, 13, 16, 22, 23, 24, 25, 26, 30, 31, 37, 38, 43, 45].

The main results in this paper are as follows. We prove that the nondifferentiable and locally Lipschitz Löwner operator G is (strictly/strongly) monotone on $\mathcal{J}$ if and only if the scalar function g is (strictly/strongly) increasing on $\mathbb{R}$. This result subsumes some closely related recent results which assumed the stronger condition of semismoothness. We also give an important application of the result which ties in the theorems of Löwner, Korányi and Mangasarian. Namely, we generalize Mangasarian class of NCP-functions to all SCCPs. Therefore, not only does our result answer Tseng's question in the affirmative, but it also establishes the positive answer in the more general and unifying setting of symmetric cones.

This paper is organized as follows. In the next section, we establish the preliminaries and cover the needed background and related work in the literature. In Section 3, we prove that given a locally Lipschitz function G on $\mathcal{J}$, it is (strictly/strongly) monotone if and only if g is (strictly/strongly) increasing. In Section 4, we investigate the relationship between monotonicity and operator-monotonicity for G on the Euclidean Jordan algebras. Utilizing our characterization of the strict monotonicity of Löwner operators, in Section 5, we extend the Mangasarian class of NCP-functions for the nonlinear complementarity problem [32] to the symmetric cone complementarity problem.

The following notation will be used throughout this paper. Let $\mathcal{X}$ and $\mathcal{Y}$ be two finite dimensional inner product spaces over the real field $\mathbb{R}$ with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\|\cdot\|$. For a given set $S \subseteq \mathcal{X}$, $\text{int}(S)$, $\text{cl}(S)$ and $\text{conv}(S)$ denote the interior, the closure and convex hull of S, respectively. We write $x \succeq_K y$ (respectively, $x \succ_K y$) to mean $x - y \in K$ (respectively, $x - y \in \text{int}(K)$) for vectors $x, y \in \mathcal{J}$. Also, we write $A \succeq B$ (respectively, $A \succ B$) to mean $A - B$ is positive semidefinite (respectively, positive definite) for operators A and B from $\mathcal{J}$ into itself. Let I be the identity operator from $\mathcal{J}$ into itself, i.e., $Ix = x$ for all $x \in \mathcal{J}$. For an operator A from $\mathcal{J}$ into itself, A^T denotes the adjoint operator of A.

2. Preliminaries. A *Euclidean Jordan algebra* is a triple $(\mathcal{J}, \langle\cdot,\cdot\rangle, \circ)$, where $(\mathcal{J}, \langle\cdot,\cdot\rangle)$ is a finite-dimensional inner product space over real field $\mathbb{R}$ and $(x, y) \to x \circ y : \mathcal{J} \times \mathcal{J} \to \mathcal{J}$ is a bilinear mapping which satisfies the following conditions:

(i) $x \circ y = y \circ x$ for all $x, y \in \mathcal{J}$,
(ii) $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$ for all $x, y \in \mathcal{J}$ where $x^2 := x \circ x$ and
(iii) $\langle x \circ y, z\rangle = \langle x, y \circ z\rangle$ for all $x, y, z \in \mathcal{J}$.

We call $x \circ y$ the *Jordan product* of x and y. In addition, we assume that there is an element e such that $x \circ e = e \circ x = x$ for all $x \in \mathcal{J}$, which is called the *identity* element in $\mathcal{J}$. Define *the set of squares* as $K := \{x^2 : x \in \mathcal{J}\}$. It is well known that K is a *symmetric cone*. That is, K is a closed, convex, self-dual cone with nonempty interior and for any two elements belonging to its interior $x, y \in \mathrm{int}(K)$, there exists an invertible linear transformation $\Lambda : \mathcal{J} \to \mathcal{J}$ such that $\Lambda(K) = K$ and $\Lambda(x) = y$.

An element $c \in \mathcal{J}$ is *idempotent* if $c^2 = c \neq 0$. It is also *primitive* if it cannot be written as a sum of two idempotents. *A complete system of orthogonal idempotents* is a finite set $\{c_1, c_2, \cdots, c_k\}$ of idempotents with $c_i \circ c_j = 0$ $(i \neq j)$ and $\sum_{i=1}^{k} c_i = e$. A complete system of orthogonal primitive idempotents is called a *Jordan frame* of $\mathcal{J}$. In the Euclidean Jordan algebra $\mathcal{J}$, for any element $x \in \mathcal{J}$, let $m(x)$ be the smallest positive integer such that the set $\{e, x, x^2, \cdots, x^m\}$ is linearly dependent. Then $m(x)$ is said to be the *degree* of x which is denoted by $\deg(x)$. The *rank* of $\mathcal{J}$ denoted by $\mathrm{rank}(\mathcal{J})$ is defined as $\mathrm{rank}(\mathcal{J})$:=$\max\{\deg(x) : x \in \mathcal{J}\}$. Let $\dim(\mathcal{J})$ denote the dimension of $\mathcal{J}$. Obviously, $\mathrm{rank}(\mathcal{J}) \leq \dim(\mathcal{J})$.

For any element $x \in \mathcal{J}$, we have the following important spectral decomposition theorems.

Spectral Decomposition Type I (Theorem III.1.1, [6])) Let $\mathcal{J}$ be a Euclidean Jordan algebra. Then for $x \in \mathcal{J}$ there exist unique real numbers $\mu_1(x), \mu_2(x), \cdots, \mu_{\bar{r}}(x)$, all distinct, and a unique complete system of orthogonal idempotents $\{b_1, b_2, \cdots, b_{\bar{r}}\}$ such that

$$x = \mu_1(x)b_1 + \cdots + \mu_k(x)b_{\bar{r}}.$$

Spectral Decomposition Type II (Theorem III.1.2, [6])) Let $\mathcal{J}$ be a Euclidean Jordan algebra of rank r. Then for every vector $x \in \mathcal{J}$ there exist a Jordan frame $\{c_1, c_2, \cdots, c_r\}$ and real numbers $\lambda_1(x), \lambda_2(x), \cdots, \lambda_r(x)$, the eigenvalues of x, such that

$$x = \lambda_1(x)c_1 + \lambda_2(x)c_2 + \cdots + \lambda_r(x)c_r. \tag{2}$$

We call (2) the *spectral decomposition* of x.

Strictly speaking, the Jordan frame $\{c_1, c_2, \cdots, c_r\}$ and the unique complete system of orthogonal idempotents $\{b_1, b_2, \cdots, b_{\bar{r}}\}$ in the above spectral decomposition theorems depend on x. We do not write this dependence explicitly for the sake of simplicity in notation. Let $\sigma(x)$ be the set consisting of all distinct eigenvalues of x. It follows that $\sigma(x)$ contains at least one element and at most r. For each $\mu_i(x) \in \sigma(x)$, denoting $N_i(x) = \{j : \lambda_j(x) = \mu_i(x)\}$, we can verify that $\bar{r}$ is the number of elements in $\{b_i : \mu_i(x) \in \sigma(x)\}$ and

$$b_i = \sum_{j \in N_i(x)} c_j.$$

Detailed treatments of Euclidean Jordan algebras can be found in Koecher's 1962 lecture notes [20] and in the monograph by Faraut and Korányi [6]. Summaries can be found in the articles [7, 11, 38, 43].

Given a real interval (a, b) with $a < b$ $(a, b \in \mathbb{R} \cup \{-\infty\} \cup \{+\infty\})$, we denote by $\mathcal{J}(a, b)$ the set of all x in $\mathcal{J}$ such that $ae \prec_K x \prec_K be$, and call it the *open box* in $\mathcal{J}$. Letting $g : (a, b) \to \mathbb{R}$ be a real-valued function, for $x \in \mathcal{J}(a, b)$ with $x = \sum_{j=1}^{r} \lambda_j(x) c_j = \sum_{i=1}^{\bar{r}} \mu_i(x) b_i$, we define a vector-valued function associated with the Euclidean Jordan algebra by

$$G(x) := \sum_{j=1}^{r} g(\lambda_j(x)) c_j = g(\lambda_1(x)) c_1 + g(\lambda_2(x)) c_2 + \cdots + g(\lambda_r(x)) c_r \tag{3}$$

or

$$G(x) := \sum_{j=1}^{\bar{r}} g(\mu_j(x)) b_j = g(\mu_1(x)) b_1 + g(\mu_2(x)) b_2 + \cdots + g(\mu_{\bar{r}}(x)) b_{\bar{r}}. \tag{4}$$

When $g(t)$ is taken as $t_+ := \max\{0, t\}, t_- := \min\{0, t\}$, or $|t| := t_+ - t_-$ $(t \in \mathbb{R})$, the Löwner function becomes the well-known metric projection function on $\mathcal{J}$

$$x_+ := \sum_{i=1}^{r} (\lambda_i(x))_+ c_i, \quad x_- := \sum_{i=1}^{r} (\lambda_i(x))_- c_i, \text{ or } |x| := \sum_{i=1}^{r} |\lambda_i(x)| c_i.$$

It is easy to verify that

$$x_+ \in K, \; x_- = -(-x)_+ \in (-K), \tag{5}$$

and

$$x_+ \circ x_- = 0, \; x = x_+ + x_-, \text{ and } |x| = x_+ - x_-. \tag{6}$$

Similarly, we can define the vector-valued functions $\sqrt{x}$, x^{-1}, x^n, $\ln x$, $\exp(x)$, etc. by the 1-dimensional functions $\sqrt{t}$, t^{-1}, t^n, $\ln t$, $\exp(t)$, etc., respectively.

The functions of form (3) or (4) are the Löwner operators under the framework of Euclidean Jordan algebra. They have special properties and important applications in electrical networks, elementary particles, etc.; see, e.g., [1, 15] for the details. In [27], Löwner studied differentiability and operator-monotonicity of $G(\cdot)$ for the case that $\mathcal{J}$ is the space of real symmetric matrices with its special structure. Korányi [21] extended Löwner's results to the setting of a general $\mathcal{J}$. For nonsmooth analysis of $G(\cdot)$ on the Euclidean Jordan algebra associated with symmetric matrices, see [4, 5, 34]; and on the Euclidean Jordan algebra associated with the second-order cone, see [3, 34, 40]. More recently, Sun and Sun [41] studied analyticity, differentiability and semismoothness of Löwner operators under the framework of Euclidean Jordan algebras. They showed that $G(\cdot)$ is (semismooth) continuously differentiable at a point x if and only if for each $j \in \{1, 2, \cdots, r\}$, $g(\cdot)$ is (semismooth) continuously differentiable at $\lambda_j(x)$.

We continue to review some concepts and properties from Euclidean Jordan algebras and Löwner operators, which will be used in the sequel. Let us recall the Peirce decomposition of Euclidean Jordan algebras. Let $\{c_1, c_2, \cdots, c_r\}$ be a Jordan frame of $\mathcal{J}$. For $i, j \in \{1, 2, \cdots, r\}$, define the subspaces

$$J_{ii} := \{y \in \mathcal{J} : y \circ c_i = y\}, \text{ and } J_{ij} := \left\{y \in \mathcal{J} : y \circ c_i = \frac{1}{2}y = y \circ c_j\right\}, \ i \neq j.$$

Then, we have the following result (see, Theorem IV.2.1 in [6]).

Theorem 1 *Let $\mathcal{J}$ be a Euclidean Jordan algebra of rank r and $\{c_1, c_2, \cdots, c_r\}$ be a given Jordan frame in $\mathcal{J}$. Then space $\mathcal{J}$ is the orthogonal direct sum of spaces J_{ij} $(i \leq j)$. Furthermore,*

(i) $J_{ij} \circ J_{ij} \subseteq J_{ii} + J_{jj}$;

(ii) $J_{ij} \circ J_{jk} \subseteq J_{ik}$, *if* $i \neq k$;

(iii) $J_{ij} \circ J_{kl} = \{0\}$, *if* $\{i, j\} \bigcap \{k, l\} = \emptyset$.

For each $x \in \mathcal{J}$, we define the corresponding *Lyapunov transformation* $L(x) : \mathcal{J} \to \mathcal{J}$ by $L(x)y = x \circ y$ for all $y \in \mathcal{J}$. Thus, $L(x)$ is a symmetric operator with respect to the inner product in the sense that $\langle L(x)y, z\rangle = \langle y, L(x)z\rangle$ for all $y, z \in \mathcal{J}$. We say two elements $x, y \in \mathcal{J}$ *operator commute* if $L(x)L(y) = L(y)L(x)$. Lemma X.2.2 [6] (or Theorem 27 in [38]) gives the following characterization of operator commutativity.

Lemma 2 *The elements x, y of a Euclidean Jordan algebra of rank r operator commute if and only if x and y share a Jordan frame $\{c_1, c_2, \cdots, c_r\}$.*

Applying the above lemma, for a given Jordan frame $\{c_1, c_2, \cdots, c_r\}$, we have that c_i, c_j operator commute and $L(c_i)L(c_j) = L(c_j)L(c_i)$ for every $i, j \in \{1, 2, \cdots, r\}$. Similarly, b_i and b_j operator commute and $L(b_i)L(b_j) = L(b_j)L(b_i)$ for every $i, j \in \{1, 2, \cdots, \bar{r}\}$.

For each $x \in \mathcal{J}$, define the transformation $Q(x) := 2L^2(x) - L(x^2)$ which is called the *quadratic representation* of $\mathcal{J}$. The following useful property about quadratic representations is well-known (see for instance Sturm [42]). We provide a proof for the convenience of the reader.

Lemma 3 *Let K be a symmetric cone in $\mathcal{J}$. For any $x \in K$, we have*

$$\langle h, Q(x)h\rangle \geq 0, \quad \forall h \in \mathcal{J}.$$

Proof. For any fixed $x \in K$, it follows from the Spectral Decomposition Type II that

$$x = \lambda_1(x)c_1 + \lambda_2(x)c_2 + \cdots + \lambda_r(x)c_r,$$

where the eigenvalues $\lambda_1(x), \lambda_2(x), \cdots, \lambda_r(x)$ of x are all nonnegative. For the Jordan frame $\{c_1, c_2, \cdots, c_r\}$, using Theorem 1, we can write any element $h \in \mathcal{J}$ as

$$h = \sum_{i=1}^{r} h_i c_i + \sum_{1 \le j < l \le r} h_{jl},$$

where $h_i \in \mathbb{R}\ (i = 1, 2, \cdots, r)$ and $h_{jl} \in J_{jl}\ (1 \le j < l \le r)$. It follows from Theorem 9 in [41] that

$$\langle h, Q(x)h \rangle = \sum_{i=1}^{r} \lambda_i(x) h_i^2 \|c_i\|^2 + \sum_{1 \le j < l \le r} \lambda(x)_j \lambda(x)_l \|h_{jl}\|^2.$$

The conclusion follows immediately. □

Let C be an open set and $H : C \subseteq \mathcal{X} \to \mathcal{Y}$ be a locally Lipschitz function on C. By Rademacher's theorem, H is almost everywhere differentiable (in the sense of Fréchet) in C. Suppose D_H is the set of points in C where H is differentiable. Let $H'(x)$, which is a linear mapping from $\mathcal{X}$ to $\mathcal{Y}$, denote the *derivative* of H at $x \in C$ if H is differentiable at x, and $\nabla H(x)$ denote the *Jacobian* of H at x specified by $\nabla H(x) := H'(x)^T$, in the sense of $\langle y, \nabla H(x)z \rangle = \langle H'(x)y, z \rangle$ for all $y \in \mathcal{X}$ and $z \in \mathcal{Y}$. Then, the *Clarke generalized Jacobian* of H at x is defined by

$$\partial H(x) := \text{conv}\left\{ \lim_{\bar{x} \to x, \bar{x} \in D_H} \nabla H(\bar{x}) \right\}.$$

An element of $\partial H(x)$ is sometimes called a subgradient. We observe that $\partial H(x) = \{\nabla H(x)\}$ if H is smooth (continuously differentiable) at x. Using $\partial H(x)$, we can define semismoothness and strong semismoothness of H.

Definition 4 *A directionally differentiable and locally Lipschitz function $H : C \subseteq \mathcal{X} \to \mathcal{Y}$ is said to be semismooth at $x \in C$ if*

$$V^T d - H'(x; d) = o(\|d\|)$$

for any $d \neq 0, d \in \mathcal{X}$ sufficiently small and $V \in \partial H(x + d)$, where

$$H'(x, d) := \lim_{t \downarrow 0} \frac{H(x + td) - H(x)}{t}$$

is the directional derivative of H at x along the direction d. In particular, if $o(\|d\|)$ can be replaced by $O(\|d\|^2)$, function H is said to be strongly semismooth.

Semismoothness was originally introduced by Mifflin [33] for functionals. Qi and Sun [35] extended the definition of semismoothness to vector-valued functions and developed a systematic theory that employs semismoothness in the analysis of the superlinear convergence of Newton's method for solving systems of nondifferentiable equations (see [35]).

Suppose that the scalar function g is locally Lipschitz on the interval (a,b). Let $\tau := (\tau_1, \tau_2, \cdots, \tau_r)^T \in \mathbb{R}^r$ with $\tau_i \in (a,b)$, and ∂g denote the subdifferential of g in the sense of Clarke. Define *the first generalized divided difference* $\partial g^{[*]}$ of g at τ as the set of all $r \times r$ symmetric matrices and the element $g^{[1]}(\tau) \in \partial g^{[*]}$ with ij-th entry $(g^{[1]}(\tau))_{ij}$ given by $[\tau_i, \tau_j]'_g$ for $i,j = 1,2,\cdots,r$, where

$$[\tau_i, \tau_j]'_g := \begin{cases} \frac{g(\tau_i)-g(\tau_j)}{\tau_i-\tau_j} & \text{if } \tau_i \neq \tau_j, \\ w_i & \text{if } \tau_i = \tau_j, \end{cases}$$

with some $w_i \in \partial g(\tau_i)$.

Based on derivative expression of differentiable Löwner function by Korányi [21] and further studies on differentiability and semismoothness of Löwner function by Sun and Sun [40], Kong, Sun and Xiu [22] derived the Clarke generalized Jacobian of semismooth Löwner function $G(\cdot)$, and gave sufficient conditions for any element $V \in \partial G(x)$ to be positive semidefinite at $x \in \mathcal{J}$. Observing carefully the proofs of these two results, we found that they are still valid when "semismoothness" property of $G(\cdot)$ is weakened to "local Lipschitzian" property of $G(\cdot)$. The strengthened versions of these results are stated in the following two theorems for the convenience of the reader.

Theorem 5 *Let $x = \sum_{j=1}^{r} \lambda_j(x) c_j = \sum_{i=1}^{\bar{r}} \mu_i(x) b_i(x)$ be in $\mathcal{J}(a,b)$. If g is locally Lipschitz at $\lambda_j(x)$ for each $j \in \{1,2,\cdots,r\}$, then the function $G(x)$ is locally Lipschitz at x, and the Clarke generalized Jacobian $\partial G(x)$ of $G(x)$ is a set of symmetric linear operators given by*

$$\partial G(x) = \mathrm{conv}\left\{ \lim_{h \to 0, x+h \in D_G} \nabla G(x+h) \right\}.$$

Furthermore,

$$\partial G(x) = 2 \sum_{i \neq j,\ i,j=1}^{\bar{r}} [\mu_i(x), \mu_j(x)]_g L(b_j(x)) L(b_i(x)) + \sum_{i=1}^{\bar{r}} \partial g(\mu_i(x)) Q(b_i(x)) \quad (7)$$

or equivalently

$$\partial G(x) = \left\{ 2 \sum_{i \neq j,\ i,j=1}^{r} [\lambda_i(x), \lambda_j(x)]'_g L(c_i) L(c_j) + \sum_{i=1}^{r} [\lambda_i(x), \lambda_i(x)]'_g Q(c_i) \right\}. \quad (8)$$

Theorem 6 *Let* $x = \sum_{j=1}^{r} \lambda_j(x)c_j = \sum_{i=1}^{\bar{r}} \mu_i(x)b_i(x)$ *be in* $\mathcal{J}(a,b)$*. If* g *is locally Lipschitz at* $\lambda_j(x)$ *for each* $j \in \{1,2,\cdots,r\}$ *and* $\partial g(\alpha) \subseteq \mathbb{R}_+$ *(*$\partial g(\alpha) \subseteq \mathbb{R}_{++}$*) for all* $\alpha \in (a,b)$*, then the function* $G(x)$ *is locally Lipschitz at* x *and the element* $V \in \partial G(x)$ *is positive semidefinite (positive definite). Here* $\partial G(x)$ *is given by (7) or (8). Moreover, when* $\partial g(\alpha) \subseteq \mathbb{R}_{++}$*, there exists a scalar* $\gamma(x) > 0$ *such that* $V \succeq \gamma(x)I \succ 0$*.*

We end this section with the following lemma.

Lemma 7 *(Theorem 3.5, [23]) For a Euclidean Jordan algebra* $\mathcal{J}$*,* $\mathrm{rank}(\mathcal{J}) = \dim(\mathcal{J})$ *if and only if there is the unique Jordan frame in* $\mathcal{J}$*.*

3. Monotonicity. In this section we study the monotonicity of Löwner functions. Let $F: D \to \mathcal{J}$ be a function on a subset $D \in \mathcal{J}$. We say that F is *monotone* on D if

$$\langle x - y, F(x) - F(y)\rangle \geq 0, \quad \forall\ (x,y) \in D \times D,$$

F is *strictly monotone* on D if

$$\langle x - y, F(x) - F(y)\rangle > 0, \quad \forall\ (x,y) \in D \times D \text{ with } x \neq y,$$

and F is *strongly monotone with modulus* $\mu > 0$ on D if

$$\langle x - y, F(x) - F(y)\rangle \geq \mu\|x - y\|^2, \quad \forall\ (x,y) \in D \times D.$$

Clearly, when $\mathcal{J} = \mathbb{R}$ and $D = (a,b)$, monotonicity, strict monotonicity and strong monotonicity of F reduce to increasing, strictly increasing and strongly increasing properties of the 1-dimensional function, respectively.

We state the following important lemma, which provides necessary and/or sufficient conditions for a locally Lipschitz function F to be monotone, strictly monotone and strongly monotone. For further details, see [14, 18, 28, 36] and the references therein.

Lemma 8 *Let* $F: D \to \mathcal{J}$ *be locally Lipschitz, where* D *is a nonempty convex open subset of* $\mathcal{J}$*. Then the following hold:*

(a) F *is monotone on* D *if and only if for every* $x \in D$ *the subgradients* $A \in \partial F(x)$ *are positive semidefinite.*

(b) If the subgradients $A \in \partial F(x)$ *are positive definite for every* $x \in D$*, then* F *is strictly monotone on* D*.*

(c) F *is strongly monotone on* D *if and only if for all* $x \in D$ *the subgradients* $A \in \partial F(x)$ *are uniformly positive definite.*

Proof. We only need to prove the Part (c), because the Parts (a) and (b) were obtained in Propositions 2.1-2.2 in [28].

Part (c): " $\Rightarrow$ " Suppose that F is strongly monotone with modulus $\mu > 0$ on D. Setting $T(x) := F(x) - \mu x$, we can obtain from the definition of strong monotonicity that $T(x)$ is monotone on D. Thus, using Part (a), for every $x \in D$ the subgradients in $\partial T(x)$ are positive semidefinite. Observing that $\partial T(x) = \partial F(x) - \mu I$, for every operator $A \in \partial F(x)$, we have $B := A - \mu I \succeq 0$ (since $B \in \partial T(x)$, and $T(x)$ is monotone). This implies that $A \succeq \mu I$. Hence, the desired conclusion follows immediately.

" $\Leftarrow$ " For the converse, suppose that for all $x \in D$ the subgradients $A \in \partial F(x)$ are uniformly positive definite. That is, there exists a positive scalar μ such that $A - \mu I \succeq 0$. It follows from Part (a) that $F(x) - \mu x$ is monotone. Using the definition of monotonicity, we have

$$\langle x - y, [F(x) - \mu x] - [F(y) - \mu y] \rangle \geq 0, \quad \forall\ (x, y) \in D \times D.$$

In other words,

$$\langle x - y, F(x) - F(y) \rangle - \langle x - y, \mu(x - y) \rangle \geq 0, \quad \forall\ (x, y) \in D \times D.$$

This says that F is strongly monotone with modulus $\mu > 0$ on D. $\square$

The following is the main result of this section.

Theorem 9 *Let g be a locally Lipschitz function from (a, b) into $\mathbb{R}$, and let G be the corresponding Löwner operator from $\mathcal{J}(a, b)$ into $\mathcal{J}$. Then the following hold:*

(a) G is monotone on $\mathcal{J}(a, b)$ if and only if g is monotone on (a, b).

(b) G is strictly monotone on $\mathcal{J}(a, b)$ if and only if g is strictly monotone on (a, b).

(c) G is strongly monotone with modulus $\mu > 0$ on $\mathcal{J}(a, b)$ if and only if g is strongly monotone with modulus $\mu > 0$ on (a, b).

Proof. Part (a): " $\Rightarrow$ " It is trivial from the definition of monotonicity.

" $\Leftarrow$ " Since $g(\cdot)$ is locally Lipschitz, by Theorem 5 the Clarke generalized Jacobian $\partial G(\cdot)$ is given by (7) or (8). Since g is monotone, the subdifferential $\partial g(t) \subseteq \mathbb{R}_+$ for every $t \in (a, b)$. Hence, the conclusion follows immediately from Theorem 6 and Lemma 8(a).

Part (b): " $\Rightarrow$ " It is similar to the proof of Part (a).

" $\Leftarrow$ " For any $x, y \in \mathcal{J}(a, b)$, in order to conclude that $\langle y - x, G(y) - G(x) \rangle > 0,\ \forall\ x \neq y$, we consider the following two cases.

<u>Case 1</u>: If x, y operator commute, using Lemma 2, there exists a Jordan frame $\{e_1, \cdots, e_r\}$ such that

$$x = \sum_{i=1}^{r} x_i e_i\ , \quad y = \sum_{i=1}^{r} y_i e_i.$$

Note that by strict monotonicity of g, $\langle s-t, g(s)-g(t)\rangle > 0$ for any scalars $s \neq t$. Also note that $x \neq y$ if and only if there exists $i \in \{1, 2, \cdots, r\}$ such that $x_i \neq y_i$. Thus, we derive that

$$\begin{aligned}
& \langle y-x, G(y)-G(x)\rangle \\
= & \langle \sum_{i=1}^{r} y_i e_i - \sum_{i=1}^{r} x_i e_i, G\left(\sum_{i=1}^{r} y_i e_i\right) - G\left(\sum_{i=1}^{r} x_i e_i\right)\rangle \\
= & \sum_{i=1}^{r} \langle y_i - x_i, g(y_i) - g(x_i)\rangle \\
> & \ 0.
\end{aligned}$$

Case 2: If x, y do not operator commute, then setting $h := y-x$, we have $y = x+h$. For any $z \in [x, y]$, it is easy to show that z and h do not operator commute either. Set $z = \sum_{j=1}^{r} \lambda_j(z) c_j(z) = \sum_{i=1}^{\bar{r}} \mu_i(z) b_i(z)$ as in Section 1 with $N_i(z) := \{j : \lambda_j(z) = \mu_i(z)\}$ and $b_i(z) := \sum_{j \in N_i(z)} c_j(z)$. Then it follows from Spectral Decomposition Type I and the argument after Spectral Decomposition Type II that

$$h = \sum_{i=1}^{r} h_i c_i(z) + \sum_{1 \leq j < l \leq r} h_{jl} = \sum_{i=1}^{\bar{r}} \sum_{k \in N_i(z)} h_k c_k(z) + \sum_{1 \leq j < l \leq \bar{r}} \sum_{m \in N_j(z), k \in N_l(z), m<k} h_{mk}$$

with

$$\sum_{1 \leq j < l \leq \bar{r}} \sum_{m \in N_j(z), k \in N_l(z), m<k} h_{mk} \neq 0. \tag{9}$$

Then, by equation (7) of Theorem 5, we have for every $w_i \in \partial g(\mu_i(z))$ $(i = 1, 2, \cdots, \bar{r})$ and the corresponding element $V \in \partial G(z)$,

$$\begin{aligned}
& \langle h, Vh\rangle \\
= & \langle h, 2\sum_{i \neq j, i,j=1}^{\bar{r}} [\mu_i(z), \mu_j(z)]_g L(b_j(z)) L(b_i(z)) h\rangle + \langle h, \sum_{i=1}^{\bar{r}} w_i Q(b_i(z)) h\rangle \\
= & \sum_{1 \leq j < l \leq \bar{r}} [\mu_j(z), \mu_l(z)]_g \left\| \sum_{m \in N_j(z), k \in N_l(z), m<k} h_{mk} \right\|^2 + \sum_{i=1}^{\bar{r}} w_i \langle h, Q(b_i(z)) h\rangle,
\end{aligned}$$

where the last equality holds by Spectral Decomposition Type I and the fact

$$\begin{aligned}
L(b_j(z)) L(b_i(z)) h &= b_j(z) \circ (b_i(z) \circ h) = \frac{1}{4} \sum_{m \in N_j(z), k \in N_i(z), m \neq k} h_{mk} \\
&= \frac{1}{2} \sum_{m \in N_j(z), k \in N_i(z), m<k} h_{mk}.
\end{aligned}$$

Using $[\mu_j(z), \mu_l(z)]_g > 0$ for $1 \le j < l \le \bar{r}$ and $w_i \ge 0$ for $i = 1, 2, \cdots, \bar{r}$, and applying Lemma 3, we derive that

$$\langle h, Vh \rangle = 0 \Rightarrow \sum_{1 \le j < l \le \bar{r}} [\mu_j(z), \mu_l(z)]_g \left\| \sum_{m \in N_j(z), k \in N_l(z), m<k} h_{mk} \right\|^2 = 0.$$

Moreover,

$$\sum_{1 \le j < l \le \bar{r}} \left\| \sum_{m \in N_j(z), k \in N_l(z), m<k} h_{mk} \right\|^2 = 0,$$

a contradiction to (9). So, in this case, $\langle h, Vh \rangle > 0$. By the mean value theorem, we have

$$G(y) - G(x) \in \mathrm{conv}\{\partial G(z)(y - x) : z \in [x, y]\}.$$

Hence,

$$\langle y - x, G(y) - G(x) \rangle \in \mathrm{conv} \left\{ \langle h, Vh \rangle : V \in \bigcup_{z \in [x,y]} \partial G(z) \right\}.$$

The last inclusion implies in this case that

$$\langle y - x, G(y) - G(x) \rangle > 0, \quad \forall x \neq y.$$

Combining the above two cases, we conclude the proof of Part (b).

Part (c): " $\Rightarrow$ " It is similar to the proof of Part (a).

" $\Leftarrow$ " Let $q(t) := g(t) - \mu$ for every $t \in (a, b)$. Then $q(\cdot)$ is monotone on the interval (a, b) by the assumption. Applying the sufficiency result of Part (a), we observe that $Q(x)$, the corresponding Löwner operator of $q(\cdot)$, is monotone on the box set $\mathcal{J}(a, b)$ (it is easy to verify that $Q(x) = G(x) - \mu I$). The desired conclusion therefore follows immediately. □

Note the following facts for 1-dimensional functions,

(i) $\ln t$, $-t^{-1}$ and t^α ($\alpha > 0$) are strictly monotone on $\mathbb{R}_{++}$;

(ii) $t|t|$, t^{2n+1} ($n = 0, 1, 2, \cdots$) and c^t with $c > 1$ are strictly monotone on $\mathbb{R}$;

(iii) t_+ and t_- are monotone on $\mathbb{R}$.

By applying Theorem 9, we can easily obtain some examples of monotone Löwner operators associated with the Euclidean Jordan algebra.

Example 10 *(i) Löwner operators* $\ln x$, $-x^{-1}$ *and* x^α ($\alpha > 0$) *are strictly monotone on* $\mathrm{int}(K)$*;*

(ii) Löwner operators $x|x|$, x^{2n+1} ($n = 0, 1, 2, \cdots$) and c^x with $c > 1$ are strictly monotone on $\mathcal{J}$;

(iii) Löwner operators x_+ and x_- are monotone on $\mathcal{J}$.

We end this section with the following two remarks.

(i) In the above theorem, we employ the assumption that g is locally Lipschitz although it is not directly related to monotonicity. However, at this point, trying to further weaken this assumption seems to be of limited use. Indeed, in the field of variational inequalities and complementarity problems, such an assumption is often satisfied. For instance, the projection residual function and Fischer-Burmeister function for NCP are semismooth, and of course, are locally Lipschitz. So, this assumption is not that restrictive in our context.

(ii) Another direction of strengthening the above theorem would be to use more general notions for monotonicity, such as pseudo-monotonicity. However, we next show that "monotonicity" in Theorem 9 cannot be replaced by "pseudo-monotonicity."

Let $F : D \to \mathcal{J}$ be a function on a subset $D \in \mathcal{J}$. We say that F is *pseudo-monotone* on D if for all $(x, y) \in D \times D \subseteq \mathcal{J} \times \mathcal{J}$,

$$\langle F(x), y - x\rangle \geq 0 \;\Rightarrow\; \langle F(y), y - x\rangle \geq 0.$$

When $\dim(\mathcal{J}) = 1$, Proposition 9.4 in [14] implies that a function $f : \mathbb{R} \to \mathbb{R}$ is pseudo-monotone on $\mathbb{R}$ if and only if there exist disjoint consecutive intervals (possibly empty) I_1, I_2 and I_3 such that $I_1 \cup I_2 \cup I_3 = \mathbb{R}$ and f is negative on I_1, zero on I_2 and positive on I_3.

Consider the Löwner operator $G(x)$ acting on $\mathbb{R}^n$, induced by the function $g : \mathbb{R} \to \mathbb{R}$, where g is continuously differentiable, and defined as

$$g(t) \begin{cases} > 0 & \text{if} \quad t > 0, \\ = 0 & \text{if} \quad t = 0, \\ < 0 & \text{if} \quad t < 0, \end{cases}$$

with $g(-2) = -10$, $g(-1) = -5$, $g(2) = 10$, and $g(4) = 1$. It is evident that g is pseudo-monotone on $\mathbb{R}$. However, taking $x = (2, -2, 0, \cdots, 0)^T$ and $y = (4, -1, 0, \cdots, 0)^T$, we have

$$\langle G(x), y - x\rangle = \langle \begin{pmatrix} g(2) \\ g(-2) \\ g(0) \\ \vdots \\ g(0) \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \rangle = 10 > 0,$$

meanwhile,

$$\langle G(y), y-x\rangle = \langle \begin{pmatrix} g(4) \\ g(-1) \\ g(0) \\ \vdots \\ g(0) \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \rangle = -3 < 0.$$

4. Operator-Monotonicity. Let us start with the definition of operator-monotonicity for a vector-valued function from a subset $D \subseteq \mathcal{J}$ into $\mathcal{J}$.

Definition 11 *F is said to be* operator-monotone *on D if for all $x, y \in D$ with $x \succeq_K y$, we have $F(x) \succeq_K F(y)$.*

We are ready to look at the connection between the monotonicity of g or Löwner operator G, and the operator-monotonicity of G.

Theorem 12 *Let g be a real-valued function from (a,b) into $\mathbb{R}$, and let G be the corresponding Löwner operator on $\mathcal{J}(a,b)$. If G is operator-monotone, then G is monotone.*

Proof. It follows immediately from the definition of operator-monotonicity that g is monotone. The desired conclusion follows from Theorem 9 (a). □

However, monotonicity of a vector-valued function does not necessarily imply the operator-monotone property. In order to illustrate this, we need a result introduced by Korányi [21] in a simple formally real Jordan algebra. Here, a simple formally real Jordan algebra means that it is simple (which is not the direct sum of two Euclidean Jordan algebras) and formally real ($x^2 + y^2 = 0 \Leftrightarrow x = 0, y = 0$). For further details, see [6].

Lemma 13 *(Theorem, [21]) Let $\mathcal{J}$ be a simple formally real Jordan algebra of rank r and let g be a real-valued function defined on the interval (a,b), and let G be the corresponding Löwner operator on $\mathcal{J}(a,b)$. Then G is operator-monotone if and only if for every choice of $x_1, \cdots, x_r$ in (a,b) the matrix $g^{[1]} = ([x_j, x_k])_{r\times r}$ is positive semidefinite and every entry of it is nonnegative.*

It is well known that every Euclidean Jordan algebra is the Cartesian product of several simple Euclidean Jordan algebras [6]. Therefore, if $\mathcal{J}$ is not a simple Euclidean Jordan algebra, the conclusion of the lemma above should be somewhat modified as follows: G is operator-monotone if and only if for each simple Euclidean Jordan algebra $\mathcal{J}_\nu$ in $\mathcal{J}$ and for every choice of $x_1, \cdots, x_{r_\nu}$ in (a,b), the matrix $g^{[1]} = ([x_j, x_k])_{r_\nu \times r_\nu}$ is positive semidefinite and every entry of it is nonnegative, where r_ν is the rank of $\mathcal{J}_\nu$.

Applying the modified lemma, we verify easily that some monotone Löwner functions in Example 10 are operator-monotone:

(i) $\tau x + \gamma$ on $\mathcal{J}$ for every $\tau \geq 0$ and $\gamma \in \mathbb{R}$;

(ii) $-x^{-1}$ on $\mathrm{int}(K)$;

(iii) x^r on $\mathrm{int}(K)$ for every $0 \leq r \leq 1$.

The third result above is an extension of the well-known "Löwner-Heinz inequality" in the space of real symmetric matrices. It can be proven in other ways; see, e.g., Theorem V.1.9 and Exercise V.1.10 in [1], Proposition 3.14 in [2] and their proofs. However, the following two theorems show that some of them are not operator-monotone.

Theorem 14 *Let $g = t^\alpha$ for $t \in \mathbb{R}$ and $\alpha \geq \alpha_0$ with a fixed parameter $\alpha_0 > 1$, and let $G = x^\alpha$ be the corresponding Löwner operator in a Euclidean Jordan algebra $\mathcal{J}$ of rank r. If $r < \dim(\mathcal{J})$, then x^α is not operator-monotone on* $\mathrm{int}(K)$.

Proof. It follows from Propositions III.4.4 in [6] that every Euclidean Jordan algebra is, in a unique way, a direct sum of simple Euclidean Jordan algebras. It therefore follows from Theorem 3.3 and Theorem 3.5 in [23] that in $\mathcal{J}$ with rank $r < \dim(\mathcal{J})$ there exists a simple Euclidean Jordan algebra (a simple formally real Jordan algebra) whose Jordan frame is not unique. Without loss of generality, let $\mathcal{J}$ be a simple formally real Jordan algebra. For any given Jordan frame $\{c_1, c_2, \cdots, c_r\}$, choose $x \in \mathrm{int}(K)$ with

$$x = \sum_{i=1}^{r} x_i c_i = \left[(\alpha + \alpha^2)^{\frac{1}{\alpha-1}} + \alpha\right] c_1 + c_2 + x_3 c_3 + \cdots + x_r c_r,$$

i.e., $x_1 = (\alpha + \alpha^2)^{\frac{1}{\alpha-1}} + \alpha, x_2 = 1$ and $x_3, \cdots, x_r > 0$. It is easy to see that the 2×2 matrix

$$\begin{pmatrix} g'(x_1) & [x_1, x_2]_g \\ [x_1, x_2]_g & g'(x_2) \end{pmatrix}$$

is not positive semidefinite. In fact, since $g'(x_1) = \alpha x_1^{\alpha-1}$, $g'(x_2) = \alpha$, and $[x_1, x_2]_g = \frac{x_1^\alpha - 1}{x_1 - 1}$, we have

$$\begin{aligned}
& g'(x_1)g'(x_2) - ([x_1, x_2]_g)^2 \\
= \; & \alpha^2 x_1^{\alpha-1} - \left(\frac{x_1^\alpha - 1}{x_1 - 1}\right)^2 \\
= \; & \left(\frac{1}{x_1 - 1}\right)^2 \left[\alpha^2 x_1^{\alpha-1}(x_1 - 1)^2 - (x_1^\alpha - 1)^2\right] \\
= \; & \left(\frac{1}{x_1 - 1}\right)^2 \left[(-2\alpha^2 x_1^\alpha + 2x_1^\alpha) - (x_1^{2\alpha} - \alpha^2 x_1^{\alpha+1} - \alpha^2 x_1^{\alpha-1}) - 1\right] \\
= \; & \left(\frac{1}{x_1 - 1}\right)^2 \left[-2(\alpha^2 - 1)x_1^\alpha - x_1^{\alpha-1}(x_1^{\alpha+1} - \alpha^2 x_1^2 - \alpha^2) - 1\right] \\
< \; & -\left(\frac{1}{x_1 - 1}\right)^2 \left[x_1^{\alpha-1}((x_1^{\alpha-1} - \alpha^2)x_1^2 - \alpha^2)\right] \\
< \; & 0,
\end{aligned}$$

where the first inequality holds by $\alpha^2 - 1 > 0$, and the second by $x_1^{\alpha-1} > \alpha + \alpha^2$ and $(x_1^{\alpha-1} - \alpha^2)x_1^2 - \alpha^2 > [(\alpha + \alpha^2) - \alpha^2]\alpha^2 - \alpha^2 = \alpha^3 - \alpha^2 > 0$. The desired conclusion follows from Lemma 13. $\square$

Using the same proof technique as above, we establish the next theorem.

Theorem 15 *Let $g = \alpha^t$ for $t \in \mathbb{R}$, $\alpha \geq \alpha_0$ with a fixed parameter $\alpha_0 > 1$, and let $G = \alpha^x$ be the corresponding Löwner operator in a Euclidean Jordan algebra $\mathcal{J}$ of rank r. If $r < \dim(\mathcal{J})$, then α^x is not operator-monotone on* $\mathrm{int}(K)$.

To end this section, we state a sufficient condition which guarantees the equivalence between monotonicity and operator-monotonicity of the Löwner operator G.

Theorem 16 *Let g be a real-valued function from (a, b) into $\mathbb{R}$, and let G be the corresponding Löwner operator on $\mathcal{J}(a, b)$ in a Euclidean Jordan algebra $\mathcal{J}$ of rank r. If $r = \dim(\mathcal{J})$, then G is operator-monotone on $\mathcal{J}(a, b)$ if and only if g is increasing on (a, b).*

Proof. It follows from Lemma 7 that when $r = \mathrm{rank}(\mathcal{J}) = \dim(\mathcal{J})$ there is a unique Jordan frame $\{c_1, c_2, \cdots, c_r\}$ in $\mathcal{J}$. Hence, any two elements $x, y \in \mathcal{J}$ operator commute with expressions

$$x = \sum_{i=1}^{r} x_i c_i, \text{ and } y = \sum_{i=1}^{r} y_i c_i.$$

Therefore, $x \succeq_K y$ if and only if $x_i \geq y_i$ for all $i \in \{1, 2, \cdots, r\}$. At the same time, $G(x) \succeq_K G(y)$ if and only if $g(x_i) \geq g(y_i)$ for all $i \in \{1, 2, \cdots, r\}$. The desired conclusion follows. $\square$

Note that the sufficient condition above implies that the underlying cone is isomorphic to the nonnegative orthant; so, the theorem is very restricted.

5. Application: Mangasarian C-function. Recall the Mangasarian class of NCP-functions for nonlinear complementarity problems, which is defined as $\phi_M : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

$$\phi_M(a, b) := \theta(|a - b|) - \theta(a) - \theta(b),$$

where θ is a strictly increasing function from $\mathbb{R}$ into $\mathbb{R}$ with $\theta(0) = 0$. In this section, we establish Mangasarian class of C-functions for SCCP. Here, a function $\Phi : \mathcal{J} \times \mathcal{J} \to \mathcal{J}$ is said to be a *C-function for SCCP* (see [9, 17] and references therein) if it satisfies

$$\Phi(x, y) = 0 \iff x \in K,\ y \in K,\ \langle x, y \rangle = 0.$$

As in the case of NCP, using this C-function, SCCPs can be completely reformulated as systems of nonlinear equations. Hence, they can be solved by employing the powerful tools of nonlinear (smooth or nonsmooth) equations theory.

For any $x \in \mathcal{J}$, if $x = \sum_{i=1}^{r} \lambda_i c_i$, then using the above function θ we define the corresponding Löwner operator $\Theta(\cdot)$ by

$$\Theta(x) := \sum_{j=1}^{r} \theta(\lambda_j(x)) c_j. \tag{10}$$

We further define the function Φ_M from $\mathcal{J} \times \mathcal{J}$ into $\mathcal{J}$ by

$$\Phi_M(x, y) := \Theta(|x - y|) - \Theta(x) - \Theta(y). \tag{11}$$

When $\mathcal{J} = \mathbb{R}^n$ and $K = \mathbb{R}^n_+$, Φ_M is the NCP-function given by Mangasarian [32]. Before presenting our main result, we need to recall a proposition, which summarizes some equivalent reformulations related to problem (1).

Proposition 17 *(Proposition 6, [11]) Let K be a symmetric cone in $\mathcal{J}$. For $x, y \in \mathcal{J}$ and $\mu \in \mathbb{R}$, the following conditions are equivalent:*

(a) $x \in K, \quad y \in K, \quad$ *and* $\quad \langle x, y \rangle = 0$;
(b) $x \in K, \quad y \in K, \quad$ *and* $\quad x \circ y = 0$;
(c) $x + y \in K, \quad$ *and* $\quad x \circ y = 0$;
(d) $x - (x - \mu y)_+ = 0 \quad$ *for any fixed* $\mu > 0$;
(e) $x + y - \sqrt{x^2 + y^2} = 0$.

In each case, the elements x and y operator commute.

The following theorem shows that Φ_M is a class of C-functions for SCCP, which is one of our principal results in this paper.

Theorem 18 *Let $\mathcal{J}$ be a Euclidean Jordan algebra of rank r, and K be the symmetric cone in $\mathcal{J}$. If Θ given by (10) has the strict monotonicity property over $\mathcal{J}$, then the following statements are equivalent:*

(a) $x \in K, \quad y \in K, \quad$ *and* $\quad x \circ y = 0$.

(b) $\Phi_M(x, y) = 0$.

Proof. "$(a) \Rightarrow (b)$" Since (a) holds, the elements x, y operator commute by Proposition 17. Thus, there is a Jordan frame $\{e_1, \cdots, e_r\}$ such that

$$x = \sum_{i=1}^{r} x_i e_i \text{ and } y = \sum_{i=1}^{r} y_i e_i.$$

So, $x \circ y = \sum_{i=1}^{r} x_i y_i e_i$, and (a) implies that $x_i \geq 0, y_i \geq 0$ and $x_i y_i = 0$ for all $i = 1, 2, \cdots, r$. Then

$$\theta(|x_i - y_i|) - \theta(x_i) - \theta(y_i) = 0 \ \text{ for all } \ i = 1, 2, \cdots, r,$$

from which we obtain

$$\begin{aligned}
\Phi_M(x,y) &= \Theta(|x-y|) - \Theta(x) - \Theta(y) \\
&= \sum_{i=1}^{r} \theta(|x_i - y_i|)e_i - \sum_{i=1}^{r} \theta(x_i)e_i - \sum_{i=1}^{r} \theta(y_i)e_i \\
&= \sum_{i=1}^{r} [\theta(|x_i - y_i|) - \theta(x_i) - \theta(y_i)]e_i \\
&= 0.
\end{aligned}$$

"$(b) \Rightarrow (a)$" Let the spectral decompositions of $x - y$ and $|x - y|$ be given by

$$x - y = \sum_{i=1}^{r} z_i e_i \quad \text{and} \quad |x - y| = \sum_{i=1}^{r} |z_i| e_i, \tag{12}$$

where $\{e_1, \cdots, e_r\}$ is a Jordan frame in $\mathcal{J}$. Then, by Theorem 1 we have

$$x = \sum_{i=1}^{r} x_i e_i + \sum_{1 \le j < l \le r} x_{jl}, \;\; y = \sum_{i=1}^{r} y_i e_i + \sum_{1 \le j < l \le r} y_{jl}, \tag{13}$$

where $x_i, y_i \in \mathbb{R}$, and $x_{jl}, y_{jl} \in J_{jl}$. Furthermore,

$$\langle \sum_{1 \le j < l \le r} x_{jl}, u \rangle = 0, \quad \langle \sum_{1 \le j < l \le r} y_{jl}, u \rangle = 0 \tag{14}$$

for any $u \in \text{span}\{e_1, \cdots, e_r\}$. Comparing (12) with (13), we obtain that

$$\sum_{1 \le j < l \le r} x_{jl} = \sum_{1 \le j < l \le r} y_{jl}. \tag{15}$$

Take u in (14) as

$$u := \sum_{i=1}^{r} \theta(|z_i|)e_i - \sum_{i=1}^{r} \theta(x_i)e_i - \sum_{i=1}^{r} \theta(y_i)e_i.$$

Then, by the definition of Θ and $0 = \Phi_M(x,y) = \Theta(|x-y|) - \Theta(x) - \Theta(y)$, we conclude that

$$\begin{aligned}
u &= \Theta(|x-y|) - \Theta(\sum_{i=1}^{r} x_i e_i) - \Theta(\sum_{i=1}^{r} y_i e_i) \\
&= \Theta(x) + \Theta(y) - \Theta(\sum_{i=1}^{r} x_i e_i) - \Theta(\sum_{i=1}^{r} y_i e_i),
\end{aligned}$$

from which we deduce using (14) and (15) that

$$\begin{aligned} 0 &= \langle \sum_{1\le j<l\le r} x_{jl}, u\rangle \\ &= \langle \sum_{1\le j<l\le r} x_{jl}, \Theta(x)-\Theta(\sum_{i=1}^r x_ie_i)+\Theta(y)-\Theta(\sum_{i=1}^r y_ie_i)\rangle \\ &= \langle x-\sum_{i=1}^r x_ie_i, \Theta(x)-\Theta(\sum_{i=1}^r x_ie_i)\rangle + \langle y-\sum_{i=1}^r y_ie_i, \Theta(y)-\Theta(\sum_{i=1}^r y_ie_i)\rangle. \end{aligned}$$

This along with the strict monotonicity of Θ yields

$$x-\sum_{i=1}^r x_ie_i = 0, \quad y-\sum_{i=1}^r y_ie_i = 0.$$

Hence, $|x-y| = \sum_{i=1}^r |x_i-y_i|e_i$ and

$$\begin{aligned} 0 &= \Phi_M(x,y) = \Theta(|x-y|)-\Theta(x)-\Theta(y) \\ &= \sum_{i=1}^r [\theta(|x_i-y_i|)-\theta(x_i)-\theta(y_i)]e_i. \end{aligned}$$

Thus, for all $i=1,2,\cdots,r$ we have

$$\theta(|x_i-y_i|)-\theta(x_i)-\theta(y_i) = 0.$$

The desired result follows immediately from that of Mangasarian [32]. □

In the above theorem, we employed the strict monotonicity of Θ on $\mathcal{J}$. The following example implies that this condition is necessary and that it cannot be replaced by the monotonicity of Θ on $\mathcal{J}$. Let θ be an increasing function from $\mathbb{R}$ into $\mathbb{R}$ defined as

$$\theta(t) := \begin{cases} 1 & \text{if } \quad t\ge 1, \\ t & \text{if } 0\le t<1, \\ 0 & \text{if } \quad t<0. \end{cases}$$

We consider Mangasarian NCP-function $\phi_M(a,b)$. It is easy to verify that any real pair (a,b) with $a\ge 1$ and $b\le 0$ solves the equation $\phi_M(a,b)=0$.

However, the condition in Theorem 18 that Θ has the strict monotonicity property over $\mathcal{J}$ can be replaced by the weaker assumption that Θ is monotone over $\mathcal{J}$ and strictly monotone in a neighborhood of the origin in $\mathcal{J}$.

Theorem 19 *Let $\mathcal{J}$ be a Euclidean Jordan algebra of rank r, and K be the symmetric cone in $\mathcal{J}$. If Θ given by (10) is monotone over $\mathcal{J}$ and strictly monotone in a neighborhood of the origin in $\mathcal{J}$, then the following statements are equivalent:*

(a) $x \in K, \quad y \in K, \quad$ *and* $\quad x \circ y = 0$.

(b) $\Phi_M(x, y) = 0$.

Proof. Let θ be increasing on $\mathbb{R}$ with $\theta(0) = 0$ and strictly increasing on the interval $(-\varepsilon, \varepsilon)$, where $\varepsilon > 0$. It suffices to prove the following conclusion holds:

$$\theta(|s-t|) - \theta(s) - \theta(t) = 0 \Leftrightarrow s \geq 0, t \geq 0, st = 0.$$

It is clear that $s \geq 0, t \geq 0, st = 0 \Rightarrow \theta(|s-t|) - \theta(s) - \theta(t) = 0$. So, we only need to show that

$$\theta(|s-t|) - \theta(s) - \theta(t) = 0 \Rightarrow s \geq 0, t \geq 0, st = 0.$$

In fact, from the definition of θ above, we derive readily that $\theta(t) > 0$ if and only if $t > 0$, while $\theta(t) < 0$ if and only if $t < 0$. Noting that

$$|s-t| > \begin{cases} 0 & \text{if } \ s \neq t, \\ \max\{s, t\} & \text{if } \ st < 0, \end{cases}$$

from $0 \leq \theta(|s-t|) = \theta(s) + \theta(t)$, we obtain that $s, t \geq 0$. We conclude that $\theta(|s-t|) = \theta(s) + \theta(t)$ if and only if $st = 0$. □

In many solution methods for nonlinear equations, such as Newton and quasi-Newton methods, differentiability or semismoothness, and the Jacobian or the Clarke generalized Jacobian, of the equations, are often required at each iterate of the underlying algorithm and at the solution points. The following theorem gives sufficient conditions for the Mangasarian C-function Φ_M to be differentiable or strongly semismooth, and derives the corresponding Jacobian or the Clarke generalized Jacobian.

Theorem 20 *Let $\mathcal{J}$ be a Euclidean Jordan algebra of rank r, and K be the symmetric cone in $\mathcal{J}$. For the Mangasarian C-function Φ_M given by (11), the following statements hold:*

(a) If θ is strongly semismooth everywhere, then Φ_M is strongly semismooth everywhere and has the Clarke generalized Jacobian $\partial\Phi_M(x, y)$ given by

$$\text{conv}\left[(\partial|x-y|)^T \partial\Theta(|x-y|) - \partial\Theta(x), -(\partial|x-y|)^T \partial\Theta(|x-y|) - \partial\Theta(y)\right].$$

(b) If θ is continuously differentiable and $\theta'(0) = 0$, then Φ_M is continuously differentiable, with the Jacobian $\nabla\Phi_M(x, y)$ given by

$$\left[(\partial|x-y|)^T \nabla\Theta(|x-y|) - \nabla\Theta(x), -(\partial|x-y|)^T \nabla\Theta(|x-y|) - \nabla\Theta(y)\right].$$

Proof. Part (a): Since θ is strongly semismooth everywhere, from Theorem 17 of Sun and Sun [41] on the strong semismoothness of Löwner function, we know that Θ given by (10) is strongly semismooth everywhere. Noting that $\Theta(|x-y|)$ is a composition of two functions $\Theta(z)$ and $z := |x-y|$, and $|x-y|$ is strongly semismooth everywhere, we conclude Part (a) from Theorem 19 in [10].

Part (b): It suffices to prove that $\Gamma(x, y) := \Theta(|x-y|)$ is continuously differentiable at $x = y$ in $\mathcal{J}$. Since θ is continuously differentiable at any $z \in \mathcal{J}$ and $\theta'(0) = 0$, from Theorem 13 of Sun and Sun [41] on the differentiability of Löwner function, we derive that $\Theta(z)$ is continuously differentiable at any $z \in \mathcal{J}$, and $\nabla\Theta(0) = 0$ (which can also be implied by (7) or (8)). Hence,

$$\partial\Gamma(x, y)|_{x=y} = [(\partial|x-y|)^T\nabla\Theta(|x-y|), -(\partial|x-y|)^T\nabla\Theta(|x-y|)]|_{x=y} = \{0\}.$$

Combining with $\lim_{x\neq y, x-y\to 0} \nabla\Gamma(x, y) = 0$, we complete the proof of Part (b). $\square$

Applying Theorems 9, 18 and 20, we can construct many C-functions for SCCP. The simplest of these is $\theta(t) = t, t \in \mathbb{R}$. In this case, we have

$$\Phi_{M_1}(x, y) = |x-y| - x - y = 2[(x-y)_+ - x], \quad \forall x, y \in \mathcal{J},$$

which is a constant multiple of the projection residual function for SCCP, and strongly semismooth on $\mathcal{J} \times \mathcal{J}$. If we take $\theta(t) = t^3, t \in \mathbb{R}$, then we obtain the C-function

$$\Phi_{M_2}(x, y) = |x-y|^3 - x^3 - y^3, \quad \forall x, y \in \mathcal{J},$$

which is twice continuously differentiable on $\mathcal{J} \times \mathcal{J}$. It is interesting that if we take $\theta(t) = t|t|, t \in \mathbb{R}$, then we have

$$\begin{aligned}
\Phi_{M_3}(x, y) &= (x-y)^2 - x \circ |x| - y \circ |y| \\
&= x \circ (x - |x|) + x \circ (y - |y|) - 2x \circ y \\
&= 2(x \circ x_- + y \circ y_- - x \circ y) \\
&= 2\left[-x \circ y + \frac{1}{2\beta}(x_-^2 + y_-^2)\right], \quad \beta = \frac{1}{2},
\end{aligned}$$

which is the same as the C-function in the case of $\beta = \frac{1}{2}$ by Kong and Xiu [24], and continuously differentiable and strongly semismooth everywhere.

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