The Value of Information in Inventory Management

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Abstract

Inventory management traditionally assumes the precise knowledge of the underlying demand distribution and a risk-neutral manager. New product introduction does not fit this framework because (i) not enough information is available to compute probabilities and (ii) managers are generally risk-averse. In this work, we analyze the value of information for two-stage inventory management in a robust optimization framework; the precise distribution of the demand is not available, but the decision-maker receives advance information on the success (high, low or moderate) of the product and knows range forecasts for the demand in each scenario. We derive closed-form expressions for the optimal order quantities and provide insights into the impact of the cost parameters. We also compute the critical probability of the baseline scenario, for which the manager switches between pre-defined optimal strategies, and justify this behavior in the light of a risk-return tradeoff. Numerical experiments are very encouraging; performance improves by 10 to 50% when the manager considers three demand subregions rather than one and minimizes the worst-case cost.

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1 Introduction

Inventory management has been widely studied in the literature when the distribution of the random demand is precisely known (Bramel and Simchi-Levi [2], Porteus [12]). Such probabilistic approaches generally assume that the managers are risk-neutral. When a product is newly introduced to the market, however, the data needed to compute demand forecasts is scarce, which makes it difficult to estimate probability distributions accurately, and over-reliance on wrong guesses might yield dire outcomes. For instance, Honda announced in June 2007 that it would discontinue its Accord Hybrid model after only three years of commercialization and cited management’s failure to anticipate the importance of gas mileage, as opposed to horsepower, for hybrids-buying motorists as the main reason for the car’s weak sales [18]; in the video game industry “Nintendo’s Wii is hard to find in stores because the company miscalculated demand” [16]. Even well-established product lines face significant sources of randomness: according to industry analysts, Dell’s global market share fell by 8.7% to 13.9% in 2006 while Hewlett-Packard’s rose by 23.9% to 17.4%, because Dell anticipated neither the move of U.S. customers from desktops to notebooks, nor the flagging sales of business computers as companies evaluated Microsoft Vista ([5], [6]).

Moreover, managers often exhibit high levels of risk aversion due to their limited understanding of the market response at such an early stage of the product lifecycle. Risk aversion is well-documented in the portfolio management literature, but has also been observed repeatedly in the operations management community: in June 2007, the Wall Street Journal devoted seven full pages to an advertising section on “Perspectives on Managing Risk” [17], noting that “whereas risk was rarely talked about in boardrooms five years ago, today it has come to the forefront of the agenda.” A natural way to incorporate risk preferences in inventory management would be to use the same approach as in finance and rely on the expected value of utility functions, which are nondecreasing and concave in the objective. Unfortunately, utility functions are difficult to articulate in practice, and thus are unlikely to be adopted by decision-makers. Researchers routinely assume utilities with
linear risk tolerance, such as power and exponential functions, simply because these are convenient to manipulate; however, they might not capture the true preferences of the manager. A more intuitive risk criterion, the probability that the objective exceeds some threshold, suffers from tractability issues; computing this probability will in general require multi-variate integration. The reader is referred to, for instance, Eeckhoudt et. al. [4], Lau [9], and more recently, Chen et. al. [3] for studies of risk-averse inventory management.

A few practitioners have attempted to describe randomness in a non-probabilistic manner to match the information at their disposal. Experts (see Nahmias [10], Simchi-Levi et. al. [13] and Sheffi [14]) have identified the following two principles as fundamental to the practice of modern management under uncertainty: (i) point forecasts are meaningless (because always wrong) and should be replaced by range forecasts, (ii) aggregate forecasts are more accurate than individual ones, as independent sources of uncertainty tend to cancel each other out. As an example, Hewlett-Packard has utilized these insights to decrease the cost of manufacturing its DeskJet printers; production of the sure component of the demand – the lowest value of the range – is assigned to a cheap, slow supplier in Singapore and production of the other units is allocated to an expensive, fast plant in Vancouver once demand is revealed (Sheffi [14], pp.99-100). Such practitioners-driven strategies establish the importance, from an industry perspective, of quantitative business models that are tailored to the level of information available and suggest that descriptions of uncertainty based on range forecasts are a pertinent alternative to probabilistic frameworks. Such a model represents the central theme of our research.

The framework proposed here builds upon the following idea: although the manager is unable to compute the demand distribution for a new product accurately, focus groups, advance orders, reviews in the specialized press or other sources of information allow him to estimate a range for the demand, as well as an interval within that range where the demand is likely to fall with known probability. This modest amount of information can be utilized to divide the demand region into several sub-intervals; here we consider three of them: low, moderate (baseline) and
high demand. Using three scenarios to represent demand stochasticity is a common assumption in the practice-oriented inventory management literature (see Graves et. al. [7], Kim and Gunther [8], Spengler et. al. [15]). Considering higher numbers of scenarios is possible, but in practice it becomes more and more difficult to assess their probabilities accurately, which runs against the purpose of the work. Benefiting from advance information, e.g., knowing in which interval the demand will fall before it is realized, helps the manager improve system performance by adjusting his strategy to the demand scenario while minimizing the worst-case cost in each demand region. While range forecasts have become the cornerstone of decision-making under uncertainty using robust optimization (Bertsimas and Thiele [1]), it is natural for practitioners to wonder how to incorporate additional information beyond the demand support. The framework we propose here complements the robust optimization approach to inventory management presented by Bertsimas and Thiele [1] by (i) incorporating an element of adaptability and (ii) analyzing the probabilities for each subregion that allow the decision-maker to take most advantage of this information structure. The originality of this work lies in our combining elements of robust optimization and of stochastic optimization to devise an approach most relevant to practitioners. We do not require the demand distribution to be symmetric; however, the two extreme scenarios are defined so that they have the same probability. Our focus is on understanding what the decision-maker needs to know about the middle (baseline) scenario in order to take the most effective action. Our problem setting differs from Perakis and Roels [11] in major aspects, as they use a minimax regret objective and compare the value of various types of information such as the mean, variance, or shape of the distribution.

Contributions. We make the following contributions to the literature.

- We analyze in depth inventory management models with advance information when demands are correlated over two time periods. This corresponds to a situation where it becomes known early on whether the product is going to be a failure, a success, or “just average”, and the manager can order before and after this information becomes available.
• We derive the optimal order quantities in closed form, highlight the impact of the cost parameters and emphasize the value of the breakpoint (in the probability of the baseline scenario) for which the optimal strategy changes. This helps the manager understand how to design scenarios, i.e., what type of information he should be looking for. A central message of this work is that the probabilities of the three scenarios must be chosen wisely for the decision-maker to gain the most benefits.

• We test the proposed model via extensive simulation experiments and illustrate the risk-return tradeoff. Numerical results suggest that the approach with three demand subregions decreases the cost by 10-50% compared to the case with only one range forecast. We also discuss the impact of risk aversion on the optimal strategy and explain its managerial implications.

**Structure.** The rest of the paper is structured as follows. In Section 2, we present problem generalities along with the notation. Section 3 investigates the two-period inventory management problem. We devote Section 4 to the experimental results. Finally, we conclude our analysis in Section 5.

## 2 Problem Generalities

We assume that the decision-maker does not know the precise distribution of the demand, and that instead he gains advance information on customers’ interest in the product through a scenario-dependent range forecast. We consider three possible scenarios: low, moderate and high demands; this corresponds to demand for a new product, which either meets, exceeds or fails below expectations. Specifically, we assume that the demand at each time period falls within a bounded interval, i.e., \([\bar{d}_1 - \hat{D}_2, \bar{d}_1 + \hat{D}_2]\), where \(\bar{d}_1\) is the nominal demand and \(\hat{D}_2\) and \(\hat{D}_3\) capture upside and downside forecast uncertainty before advance information is revealed. This demand forecast is divided into three disjoint sub-intervals, called demand regions; we assume that the manager knows the probability of each scenario being realized. Ordering cost is linear in the amount ordered and holding and
backorder costs are linear in the amounts in inventory (computed at the end of the time period), respectively. Replenishment lead times are equal to zero, so that items arrive immediately after being ordered. The objective is to minimize the total cost for the worst-case demand within each range forecast. There are no fixed costs, so that the optimal strategy in each demand sub-region is known to be basestock.

The scenarios corresponding to low and high probabilities are defined to have the same probability; note that this does not assume that demand is symmetric. Terminology is as follows.

**Region 1:** This is the baseline scenario. Demand falls in $[\bar{d}_1 - \hat{d}_3, \bar{d}_1 + \hat{d}_2]$ with probability $\pi_1$. The range forecast can be interpreted as a confidence interval with probability $\pi_1$. As a result, $\hat{d}_2$ and $\hat{d}_3$ vary when $\pi_1$ varies.

**Region 2:** This is the region of high demand. The range forecast is $[\bar{d}_1 + \hat{d}_2, \bar{d}_1 + \hat{D}_2]$ and demand falls in it with probability $\pi_2$.

**Region 3:** This is the region of low demand. The range forecast is $[\bar{d}_1 - \hat{D}_3, \bar{d}_1 - \hat{d}_3]$ and demand falls in it with probability $\pi_3 = \pi_2$.

Our motivation in studying this problem is to investigate the value of additional information on the decision-maker’s strategy in two-stage models and to compare the associated optimal costs, in a setting that is well-suited to the information available in practice.

We use the following notation:
$d_i$: demand in period $i$, $i = 1, 2$,

$q_i$: order quantity at the beginning of period $i$, $i = 1, 2$,

$x_i$: inventory at the beginning of period $i$, $i = 1, 2$ (with $x_{i+1} = x_i + q_i - d_i$),

$c$: unit ordering cost,

$h$: unit holding cost per period,

$b$: unit backorder cost per period,

$\pi_j$: probability that the random demand of a period is within the bounds of region $j$, $j = 1, 2, 3$,

$S_j$: basestock level for demand values that fall in interval $j$, $j = 1, 2, 3$.

Initial inventory is zero. As is common practice, we assume that the unit backorder cost exceeds the unit ordering cost.

3 The Sequential Inventory Management Problem

3.1 Problem setup

We assume the system is exposed to random demand for two consecutive time periods. In each period, an order is placed, then demand in that time period is revealed and inventory holding and backorder costs are computed. The objective of the decision-maker is to minimize the worst-case total cost over the two planning periods. Here, we receive advance information in the following sense: the first order is placed, then the demand scenario (i.e., success of the product) is revealed, and finally demand is realized twice. Demand in the second time period always falls within the same subinterval as demand in the first time period; we call this case “correlated demand.” This is motivated by practical situations where demands in each time period depend on one common driver, the overall success of the product, as determined by reviews in the press, word-of-mouth buzz, advance orders, and so on.
The problem is formulated as follows:

\[
\Gamma = \min_{q_1 \geq 0} \left[ cq_1 + \pi_1 \Psi_1 + \pi_2 \Psi_2 + \pi_3 \Psi_3 \right], \tag{1}
\]

where:

\[
\begin{align*}
\Psi_1 &= \max_{d_1 - d_3 \leq d_1 \leq d_1 + d_2} \left\{ \max \{h(q_1 - d_1), -b(q_1 - d_1)\} + \min_{S_1} \left\{ \max_{d_1 - d_3 \leq d_2 \leq d_1 + d_2} \left\{ c \max \{0, S_1 - x_2\} \right\} \right. \right. \\
&\left. \left. \quad + \max \{h(x_2 + \max \{0, S_1 - x_2\} - d_2), -b(x_2 + \max \{0, S_1 - x_2\} - d_2)\} \right\} \right\} \tag{2} \\
\Psi_2 &= \max_{d_1 + d_2 \leq d_1 \leq d_1 + \hat{D}_2} \left\{ \max \{h(q_1 - d_1), -b(q_1 - d_1)\} + \min_{S_2} \left\{ \max_{d_1 + d_2 \leq d_2 \leq d_1 + \hat{D}_2} \left\{ c \max \{0, S_2 - x_2\} \right\} \right. \right. \\
&\left. \left. \quad + \max \{h(x_2 + \max \{0, S_2 - x_2\} - d_2), -b(x_2 + \max \{0, S_2 - x_2\} - d_2)\} \right\} \right\} \tag{3} \\
\Psi_3 &= \max_{d_1 - \hat{D}_3 \leq d_1 \leq d_1 - \hat{d}_3} \left\{ \max \{h(q_1 - d_1), -b(q_1 - d_1)\} + \min_{S_3} \left\{ \max_{d_1 - \hat{D}_3 \leq d_2 \leq d_1 - \hat{d}_3} \left\{ c \max \{0, S_3 - x_2\} \right\} \right. \right. \\
&\left. \left. \quad + \max \{h(x_2 + \max \{0, S_3 - x_2\} - d_2), -b(x_2 + \max \{0, S_3 - x_2\} - d_2)\} \right\} \right\} \tag{4}
\end{align*}
\]

We now compute the optimal basestock levels in the second stage.

**Theorem 3.1** The optimal order-up-to levels are determined by:

\[
\begin{align*}
S_1^* &= \bar{d}_1 + \frac{b \hat{d}_2 - h \hat{d}_3}{b + h}, \\
S_2^* &= \bar{d}_1 + \hat{d}_2 + \frac{b}{b + h} (\hat{D}_2 - \hat{d}_2), \\
S_3^* &= \bar{d}_1 - \hat{d}_3 - \frac{h}{b + h} (\hat{D}_3 - \hat{d}_3).
\end{align*}
\]

**Proof:** We now simplify Equations (2)-(4). By decomposing the equation for \(\Psi_1\) into two parts, we write:

\[
\Psi_1 = \max_{d_1 - d_3 \leq d_1 \leq d_1 + d_2} \left\{ \max \{h(q_1 - d_1), -b(q_1 - d_1)\} + \Omega_1 \right\},
\]

where
\[ \Omega_1 = \min_{S_1} \left\{ \max_{d_i - d_3 \leq d_2 \leq d_1 + d_2} \left\{ c \max\{0, S_1 - x_2\} + \max\left\{ h(\max\{x_2, S_1\} - d_2), -b(\max\{x_2, S_1\} - d_2)\right\} \right\} \right\}. \]

**Case i.** \( S_1 \leq x_2 \)

\[ \Omega_1 = \max \left\{ h(x_2 - (\hat{d}_1 - \hat{d}_3)), -b(x_2 - (\hat{d}_1 + \hat{d}_2)) \right\} \]

**Case ii.** \( S_1 \geq x_2 \)

\[ \begin{align*}
\Omega_1 &= \min_{S_1} \left\{ \max_{d_i - d_3 \leq d_2 \leq d_1 + d_2} \left\{ c(S_1 - x_2) + \max\left\{ h(S_1 - d_2), -b(S_1 - d_2)\right\} \right\} \right\} \\
&= \min_{S_1} \left\{ c(S_1 - x_2) + \max\left\{ h[S_1 - (\hat{d}_1 - \hat{d}_3)], -b[S_1 - (\hat{d}_1 + \hat{d}_2)]\right\} \right\}
\end{align*} \]

This yields:

\[ \begin{align*}
S_1^* &= \hat{d}_1 + \frac{bd_2 - h\hat{d}_3}{b + h}, \\
\Omega_1 &= c(S_1^* - x_2) + \frac{bh}{b + h}(\hat{d}_2 + \hat{d}_3), \\
\Psi_1 &= \left\{ \max_{\hat{d}_1 - \hat{d}_3 \leq d_1 \leq \hat{d}_1 + \hat{d}_2} \left\{ \max\{h(q_1 - d_1), -b(q_1 - d_1)\} + \Omega_1 \right\} \right\} \\
&= \left\{ \max_{\hat{d}_1 - \hat{d}_3 \leq d_1 \leq \hat{d}_1 + \hat{d}_2} \left\{ \max\{|h - c|(q_1 - d_1), -|b + c|(q_1 - d_1)\} \right\} + cS_1^* + \frac{bh}{b + h}(\hat{d}_2 + \hat{d}_3) \right\}.
\end{align*} \]

By decomposing the equation for \( \Psi_2 \) into two parts and with similar notations, we obtain:

\[ \begin{align*}
S_2^* &= \hat{d}_1 + \hat{d}_2 + \frac{b}{b + h}(\hat{D}_2 - \hat{d}_2), \\
\Omega_2 &= c(S_2^* - x_2) + \frac{bh}{b + h}(\hat{D}_2 - \hat{d}_2), \\
\Psi_2 &= \left\{ \max_{\hat{d}_1 + \hat{d}_2 \leq d_1 \leq \hat{d}_1 + \hat{D}_2} \left\{ \max\{h(q_1 - d_1), -b(q_1 - d_1)\} + \Omega_2 \right\} \right\} \\
&= \left\{ \max_{\hat{d}_1 + \hat{d}_2 \leq d_1 \leq \hat{d}_1 + \hat{D}_2} \left\{ \max\{|h - c|(q_1 - d_1), -|b + c|(q_1 - d_1)\} \right\} + cS_2^* + \frac{bh}{b + h}(\hat{D}_2 - \hat{d}_2) \right\}.
\end{align*} \]

(5)
Finally:

\[ S_3^* = \tilde{d}_1 - \hat{d}_3 - \frac{h}{b+h}(\tilde{D}_3 - \hat{d}_3), \]
\[ \Omega_3 = c(S_3^* - x_2) + \frac{bh}{b+h}(\tilde{D}_3 - \hat{d}_3), \]
\[ \Psi_3 = \begin{cases} 
\max_{d_1 - \tilde{D}_3 \leq d_1 \leq \tilde{d}_1 - \hat{d}_3} \{ \max\{h(q_1 - d_1), -b(q_1 - d_1)\} + \Omega_3 \} \\
\max_{d_1 - \tilde{D}_3 \leq d_1 \leq \tilde{d}_1 - \hat{d}_3} \{ \max\{h - c\}(q_1 - d_1), -[b + c](q_1 - d_1)\} 
\end{cases} + cS_3^* + \frac{bh}{b+h}(\tilde{D}_3 - \hat{d}_3). \] (7)

We now reinject these results in the cost function and compute the optimal order in the first time period. To analyze the optimal objective, we must distinguish between two cases: (i) the unit ordering cost of an item is less than the holding cost (Section 3.2), and (ii) the holding cost is smaller than the unit ordering cost (Section 3.3). The following analysis holds when ordering occurs in each scenario in the second time period. This matches the application where the company produces items mid-season, i.e., the initial order is never sufficient to satisfy the demand for the whole season. This is not a restrictive assumption, as there is no fixed cost, that is, no incentive to order only once; the manager will never benefit from producing everything before the season begins and refusing to take advantage of the advance information.

### 3.2 Optimal policy when \( c < h \)

When \( c < h \), the piecewise linear functions defining the \( \Psi_i, i = 1, 2, 3 \), have one decreasing piece and one increasing piece. Hence, their maximum over an interval is reached at one of the extremities of that interval. We reformulate the cost function as:

\[ \Gamma = \min_{q_1 \geq 0} \left[ cq_1 + \pi_1 \Psi_1 + \pi_2 \Psi_2 + \pi_3 \Psi_3 \right], \]
where:

\[
\Psi_1 = \begin{cases} 
-(b + c)q_1 + (2c + b)(\bar{d}_1 + \bar{d}_2) + \frac{h(b-c)}{b+h}(\bar{d}_3 - \hat{d}_3), & \text{if } q_1 \leq \bar{d}_1 + \bar{d}_2 + \frac{c - h}{b+h}(\bar{d}_2 + \hat{d}_3), \\
(h - c)q_1 + (2c - h)(\bar{d}_1 - \hat{d}_3) + \frac{b(h + c)}{b+h}(\bar{d}_2 + \hat{d}_3), & \text{if } q_1 \geq \bar{d}_1 + \bar{d}_2 + \frac{c - h}{b+h}(\bar{d}_2 + \hat{d}_3),
\end{cases}
\]

\[
\Psi_2 = \begin{cases} 
-(b + c)q_1 + (2c + b)(\bar{d}_1 + \bar{d}_2) + \left((b + c) + \frac{b(h + c)}{b+h}\right)(\hat{D}_2 - \hat{d}_2), & \text{if } q_1 \leq \bar{d}_1 + \bar{d}_2 + \frac{c + b}{b+h}(\hat{D}_2 - \hat{d}_2), \\
(h - c)q_1 + (2c - h)(\bar{d}_1 + \hat{d}_2) + \frac{b(h + c)}{b+h}(\hat{D}_2 - \hat{d}_2), & \text{if } q_1 \geq \bar{d}_1 + \bar{d}_2 + \frac{c + b}{b+h}(\hat{D}_2 - \hat{d}_2),
\end{cases}
\]

and:

\[
\Psi_3 = \begin{cases} 
-(b + c)q_1 + (2c + b)(\bar{d}_1 - \hat{d}_3) + \frac{h(b-c)}{b+h}(\hat{D}_3 - \hat{d}_3), & \text{if } q_1 \leq \bar{d}_1 - \hat{d}_3 + \frac{c - h}{b+h}(\hat{D}_3 - \hat{d}_3), \\
(h - c)q_1 + (2c - h)(\bar{d}_1 - \hat{d}_3) + \left((h - c) + \frac{h(b-c)}{b+h}\right)(\hat{D}_3 - \hat{d}_3), & \text{if } q_1 \geq \bar{d}_1 - \hat{d}_3 + \frac{c - h}{b+h}(\hat{D}_3 - \hat{d}_3).\end{cases}
\]

Lemma 3.2 provides the order of the three breakpoints, which will help us in our study of the piecewise linear objective function.

**Lemma 3.2** Assume $c < h$. Then the following inequalities hold:

\[
\bar{d}_1 - \hat{d}_3 + \frac{c - h}{b+h}(\bar{D}_3 - \hat{d}_3) < \bar{d}_1 + \bar{d}_2 + \frac{c - h}{b+h}(\bar{d}_2 + \hat{d}_3) < \bar{d}_1 + \bar{d}_2 + \frac{c + b}{b+h}(\hat{D}_2 - \hat{d}_2). \tag{8}
\]

*Proof:* Follows by simple algebraic manipulations, using that $c - h < 0$. \qed

Invoking Lemma 3.2, we rewrite the objective function as follows. (Recall that $\pi_2 = \pi_3$ by construction.)

\[
\Gamma = \begin{cases} 
\min_{q_1}\{-bq_1\} + \kappa_1, & \text{if } 0 \leq q_1 \leq \bar{d}_1 - \hat{d}_3 + \frac{c - h}{b+h}(\bar{D}_3 - \hat{d}_3), \\
\min_{q_1}\{[-b + (b + h)\pi_2]q_1\} + \kappa_2, & \text{if } \bar{d}_1 - \hat{d}_3 + \frac{c - h}{b+h}(\bar{D}_3 - \hat{d}_3) \leq q_1 \leq \bar{d}_1 + \bar{d}_2 + \frac{c - h}{b+h}(\bar{d}_2 + \hat{d}_3), \\
\min_{q_1}\{(h - (b + h)\pi_2)q_1\} + \kappa_3, & \text{if } \bar{d}_1 + \bar{d}_2 + \frac{c - h}{b+h}(\bar{d}_2 + \hat{d}_3) \leq q_1 \leq \bar{d}_1 + \bar{d}_2 + \frac{c + b}{b+h}(\hat{D}_2 - \hat{d}_2), \\
\min_{q_1}\{hq_1\} + \kappa_4, & \text{if } \bar{d}_1 + \bar{d}_2 + \frac{c + b}{b+h}(\hat{D}_2 - \hat{d}_2) \leq q_1,
\end{cases}
\]

where $\kappa_1, \kappa_2, \kappa_3,$ and $\kappa_4$ are the constant terms of the cost function, given above. Theorem 3.3 provides the optimal order quantity.

**Theorem 3.3** Assume $c < h$. 

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If $h \leq b$, the optimal order quantity, $q^*_1$, is given by:

$$
q^*_1 = \begin{cases} \\
\bar{d}_1 + \hat{d}_2 + \frac{c+b}{b+h}(\hat{D}_2 - \hat{d}_2), & \text{if } 0 \leq \pi_1 < \frac{b-h}{b+h}, \\
\bar{d}_1 + \hat{d}_2 + \frac{c-h}{b+h}(\hat{d}_2 + \hat{d}_3), & \text{if } \frac{b-h}{b+h} < \pi_1 \leq 1.
\end{cases}
$$

If $b < h$, the optimal order quantity, $q^*_1$, is given by:

$$
q^*_1 = \begin{cases} \\
\bar{d}_1 - \hat{d}_3 + \frac{c-h}{b+h}(\hat{D}_3 - \hat{d}_3), & \text{if } 0 \leq \pi_1 < \frac{h-b}{b+h}, \\
\bar{d}_1 + \hat{d}_2 + \frac{c-h}{b+h}(\hat{d}_2 + \hat{d}_3), & \text{if } \frac{h-b}{b+h} < \pi_1 \leq 1.
\end{cases}
$$

**Proof:** The proof follows from studying the slope of the piecewise linear cost function.

(i) The function achieves its minimum value at the first critical point, $q^*_1 = \bar{d}_1 - \hat{d}_3 + \frac{c-h}{b+h}(\hat{D}_3 - \hat{d}_3)$, if and only if the slope of the piecewise cost function is positive in the second and third intervals. That is: $-b + (b+h)\pi_2 > 0$ and $h - (b+h)\pi_2 > 0$, which yields:

$$
\frac{-h-b}{b+h} < \pi_1 < \frac{h-b}{b+h}.
$$

(ii) The function achieves its minimum value at the second critical point, $q^*_1 = \bar{d}_1 + \hat{d}_2 + \frac{c-h}{b+h}(\hat{d}_2 + \hat{d}_3)$, if and only if the slope of the piecewise cost function is negative in the second interval and positive in the third interval. That is: $-b + (b+h)\pi_2 < 0$ and $h - (b+h)\pi_2 > 0$, which yields:

$$
\pi_1 > \max \left\{ \frac{h-b}{b+h}, \frac{-h-b}{b+h} \right\}.
$$

(iii) The function achieves its minimum value at the third critical point, $q^*_1 = \bar{d}_1 + \hat{d}_2 + \frac{c+b}{b+h}(\hat{D}_2 - \hat{d}_2)$, if and only if the slope of the piecewise cost function is all negative in the second and third intervals. That is: $-b + (b+h)\pi_2 < 0$ and $h - (b+h)\pi_2 < 0$, which yields:

$$
\frac{h-b}{b+h} < \pi_1 < \frac{-h-b}{b+h}.
$$

**Remark:** The fraction $\left| \frac{b-h}{b+h} \right|$, which captures the imbalance between holding and backorder
costs, plays a critical role in understanding the optimal strategy. In particular, if the probability \( \pi_1 \) of demand falling within the baseline region exceeds that threshold, i.e., the baseline scenario represents the demand range with high enough probability, the optimal order \( q_1^* \) is independent of the total upside and downside amounts of uncertainty \( \hat{D}_2 \) and \( \hat{D}_3 \) and the relationship (greater or smaller) between holding and backorder costs. It is also smaller than the maximum demand in that demand region, \( \bar{d}_1 + \hat{d}_2 \), by an amount proportional to the difference between ordering and holding costs, and the width of the baseline scenario, \( \hat{d}_2 + \hat{d}_3 \).

### 3.3 Optimal policy when \( h < c \)

In this section, we assume that the unit holding cost is smaller than the unit ordering cost. The piecewise linear functions defining the \( \Psi_i, i = 1, 2, 3 \), are non-decreasing, and the maximum over a demand interval is reached for the highest demand in that interval. We rewrite the cost function as follows.

\[
\Gamma = \min_{q_1 \geq 0} \left[ cq_1 + \pi_1 \Psi_1 + \pi_2 \Psi_2 + \pi_3 \Psi_3 \right],
\]

where:

\[
\Psi_1 = \max \left\{ (h - c)[q_1 - (\bar{d}_1 + \hat{d}_2)], -(b + c)[q_1 - (\bar{d}_1 + \hat{d}_2)] \right\} + \omega_1,
\]

\[
\Psi_2 = \max \left\{ (h - c)[q_1 - (\bar{d}_1 + \hat{D}_2)], -(b + c)[q_1 - (\bar{d}_1 + \hat{D}_2)] \right\} + \omega_2,
\]

\[
\Psi_3 = \max \left\{ (h - c)[q_1 - (\bar{d}_1 - \hat{d}_3)], -(b + c)[q_1 - (\bar{d}_1 - \hat{d}_3)] \right\} + \omega_3,
\]

where \( \omega_1, \omega_2, \) and \( \omega_3 \) are the constant terms of the cost functions. The cost function \( \Gamma \) is a piecewise linear function with three breakpoints.
Specifically:

\[
\Gamma = \begin{cases} 
\min_{q_1} \{-bq_1\} + \nu_1, & \text{if } 0 \leq q_1 \leq \bar{d}_1 - \hat{d}_3, \\
\min_{q_1} \{[-b + (b + h)\pi_2]q_1\} + \nu_2, & \text{if } \bar{d}_1 - \hat{d}_3 \leq q_1 \leq \bar{d}_1 + \hat{d}_2, \\
\min_{q_1} \{[h - (b + h)\pi_2]q_1\} + \nu_3, & \text{if } \bar{d}_1 + \hat{d}_2 \leq q_1 \leq \bar{d}_1 + \hat{D}_2, \\
\min_{q_1} \{hq_1\} + \nu_4 & \text{if } \bar{d}_1 + \hat{D}_2 \leq q_1,
\end{cases}
\]

where:

\[
\nu_1 = (b + c)[\bar{d}_1 + \pi_1\hat{d}_2 + \pi_2(\hat{D}_2 - \hat{d}_3)] + \pi_1\omega_1 + \pi_2\omega_2 + \pi_3\omega_3,
\]

\[
\nu_2 = (b + c)[(\pi_1 + \pi_2)\bar{d}_1 + \pi_1\hat{d}_2 + \pi_2\hat{D}_2] - (h - c)\pi_2(\bar{d}_1 - \hat{d}_3) + \pi_1\omega_1 + \pi_2\omega_2 + \pi_3\omega_3,
\]

\[
\nu_3 = -(h - c)[(\pi_1 + \pi_2)\bar{d}_1 + (\pi_1\hat{d}_2 - \pi_2\hat{d}_3)] + (b + c)\pi_2(\bar{d}_1 + \hat{D}_2) + \pi_1\omega_1 + \pi_2\omega_2 + \pi_3\omega_3,
\]

\[
\nu_4 = -(h - c)[\bar{d}_1 + \pi_1\hat{d}_2 + \pi_2(\hat{D}_2 - \hat{d}_3)] + \pi_1\omega_1 + \pi_2\omega_2 + \pi_3\omega_3.
\]

Theorem 3.4 provides the optimal order for this parameter setting.

**Theorem 3.4** Assume \( h < c < b \). Then the optimal order quantity, \( q_1^* \), is given by:

\[
q_1^* = \begin{cases} 
\bar{d}_1 + \hat{D}_2 & \text{if } 0 \leq \pi_1 < \frac{b - h}{b + h}, \\
\bar{d}_1 + \hat{d}_2 & \text{if } \frac{b - h}{b + h} < \pi_1 \leq 1.
\end{cases}
\]

**Proof:** The proof follows from studying the slope of the piecewise linear cost function and similar to the one given for Theorem 3.3. \( \square \)

**Remark:** Here, the optimal order no longer depends explicitly on the unit cost parameters \( c, b \) and \( h \). The fraction \( \frac{b - h}{b + h} \) plays again a critical role in understanding the optimal solution; if \( \pi_1 \) is smaller than this threshold value the optimal order is the maximum demand \( \bar{d}_1 + \hat{D}_2 \), otherwise it is equal to the maximum demand \( \bar{d}_1 + \hat{d}_2 \) in Region 1. This suggests that the manager should take into account the value of \( \frac{b - h}{b + h} \) in mind when he designs demand scenarios. A baseline scenario
that represents the demand range with, say, 60% confidence, does not lead to solutions of similar quality if \( \frac{b-h}{b+h} \) equals, for instance, 0.2 or 0.8.

4 Experimental Results

We now perform extensive simulation experiments to investigate the performance of the system under various values of \( \pi_1 \) as well as of the cost parameters \( c, h \) and \( b \). We distinguish between three experimental settings: i. \( h < c < b \) ii. \( c < h < b \) iii. \( c < b < h \). We consider two possible distributions for the demand, namely, the Normal and symmetric Triangular distributions. Reported statistics are based on 1,000 replications. Because demand is symmetric, we use the notation \( \hat{D}_2 = \hat{D}_3 = \hat{D} \) and \( \hat{d}_2 = \hat{d}_3 = \hat{d} \).

4.1 Case \( h < c < b \)

The set of cost parameters used in the experiments is given in Table 1. In the first setting, we keep the holding cost significantly low, whereas in the second setting we assume it takes a moderately high value. Figures 1 and 2 show the simulation results when the demand follows a triangular distribution (specifically, Triangular(10, 50, 90)) and Figures 3-4 depict the results when the demand is Normally distributed (specifically, Normal(50, 10^2)). Note that \( \pi_1 = 1 \) corresponds to the case where there is no advance information about the demand, by definition of the confidence intervals (\( \hat{d}_1 \) becomes equal to \( \hat{D} \) and Region 1 covers the entire demand range.) On the other hand, when \( \pi_1 \neq 0 \), we obtain 2 subregions, \([\hat{d}_1 - \hat{D}, \hat{d}_1] \) and \([\hat{d}_1, \hat{d}_1 + \hat{D}] \), since \( \hat{d} \rightarrow 0 \).

<table>
<thead>
<tr>
<th>( h )</th>
<th>( c )</th>
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<td>Value</td>
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<tr>
<td>1</td>
<td>[1, 52]</td>
<td>3</td>
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<tr>
<td>17</td>
<td>[17, 51]</td>
<td>2</td>
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Figure 1: Average scaled cost for demand distribution Triangular(10, 50, 90), $h = 1, b = 52$.

Figure 2: Average scaled cost for demand distribution Triangular(10, 50, 90), $h = 17, b = 52$. 
Figure 3: Average scaled cost for demand distribution $\text{Normal}(50, 10^2)$, $h = 1$, $b = 52$.

Figure 4: Average scaled cost for demand distribution $\text{Normal}(50, 10^2)$, $h = 17$, $b = 52$. 
We make the following observations.

1. Having three demand subregions improves system performance (i.e., reduces cost) by between 12% and 25%.

2. When $h$ is significantly smaller than $b$, the logistics cost is a convex function of $\pi_1$ achieving a minimum at 0.4 and 0.6, regardless of the value of $c$, for Triangularly and Normally distributed demands, respectively (see Figures 1 and 3). Furthermore, the curve is quite flat around the optimal $\pi_1^*$, providing robust solutions for $\pi_1$.

3. When the holding cost is moderately high, the logistics cost achieves its minimum at 0.55 and 0.6, regardless of the value of $c$, for Triangularly and Normally distributed demand cases, respectively (see Figures 2 and 4). However, we now observe a sharp decrease in the cost function around the optimal $\pi_1^*$ value. This decrease in the cost function can be explained by the piecewise structure of the optimal order quantity $q_1^*$ given in Theorem 3.4. In the first
run of experiments (when \( h \ll b \)), the critical ratio is given by \( \frac{b-h}{b+h} = \frac{52-1}{52+1} \approx 1 \), which in turn implies that \( q_1^* = \bar{d}_1 + \hat{D} \) for all values of \( \pi_1 \). In the second scenario, we have \( h = 17 \), resulting a critical ratio of \( \frac{b-h}{b+h} = \frac{52-17}{52+17} \approx 1/2 \). Therefore, the optimal first-period order quantity is given by the equation \( q_1^* = \bar{d}_0 + \hat{D} \) for \( \pi_1 < 1/2 \), and by \( q_1^* = \bar{d}_0 + \hat{d}_0 \) otherwise. Due to this instantaneous change in the optimal order quantity, the cost function exhibits a jump at \( \pi_1 \approx 1/2 \). Figure 5 displays this shift in the order quantity at \( \pi_1 = 1/2 \) for \( h = 17 \), in the Normally distributed demand case. Note that the order quantity is independent of \( c \) and the graph is same for all \( c \) values in the range \([17, 52] \).

### 4.2 Case \( c < h < b \)

The set of cost parameters used in this set of experiments is given in Table 2. The optimal order quantity is characterized by the first part of Theorem 3.3. Figures 6 and 7 show the simulation results when the demand follows a Triangular distribution (specifically, Triangular(10, 50, 90)) and Figures 8-9 depict the results when the demand is Normally distributed (specifically, Normal(50, 10^2)). We make the following observations.

1. Having three demand subregions improves the system performance (i.e., reduces cost) by 10-50%. Improvements decrease as the holding cost increases. This matches the intuition since increasing the holding cost reduces the potential benefits that can be achieved through the optimal strategy.

2. When \( h \) is significantly smaller than, or very close to, the value of \( b \) (i.e., \( 1 = h \ll b = 52 \) or
$h \geq 49$, the logistics cost is a smooth convex function of $\pi_1$. This is due to the structure of the optimal order given in Theorem 3.3. When $h$ is very small or very large compared to $b$, the piecewise nature of the optimal order quantity with respect to $\pi_1$ disappears, resulting in a smooth cost function that does not exhibit any sharp decrease. For instance, $h = b = 52$ results in a critical ratio of $\frac{b-h}{b+h} = 0$, which in turn makes the optimal order quantity equal to $q_1^* = \bar{d}_1 + \hat{d} + \frac{2(c-h)}{(b+h)} \hat{d}$ regardless of the value of $\pi_1$. As a result, we observe a smooth convex cost function.

Figure 6: Average scaled cost for demand distribution Triangular(10, 50, 90), $c = 1$, $b = 52$.

### 4.3 $c < b < h$

The set of cost parameters used in this class of experiments is given in Table 3. The optimal order used in this set of experiments is characterized by the second part of Theorem 3.3. As in the previous sections, we use Normal and Triangular distributions with the same parameters as before. We present the results for the parameter set in the first row of Table 3; the results for
Figure 7: Average scaled cost for demand distribution Triangular(10, 50, 90), $c = 17$, $b = 52$.

Figure 8: Average scaled cost for demand distribution Normal(50, $10^2$), $c = 1$, $b = 52$. 

Figure 9: Average scaled cost for demand distribution Normal(50, 10^2), c = 17, b = 52.

the parameters in the second row are similar and omitted here as they do not provide any further insights.

Table 3: Parameter values.

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<td>52</td>
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</table>

Figures 10 and 11 show the simulation results when the demand follows a Triangular distribution (specifically, Triangular(10, 50, 90)) and a Normal distribution (specifically, Normal(50, 10^2)), respectively. We make the following observations.

1. Having three demand subregions improves the system performance by 14-52% in terms of cost reduction. The improvement increases as the backorder cost increases, due to the increase in potential gains that can be achieved through a better inventory control.
2. When $b$ is significantly smaller than, or very close to, the value of $h$ (i.e., $1 = b \ll h = 52$ or $b \geq 49$), the logistics cost is a smooth convex function of $\pi_1$. This is due to the structure of the optimal order given in Theorem 3.3. When $b$ is very small, or large, compared to $h$, the piecewise structure of the optimal order with respect to $\pi_1$ disappears, resulting in a smoother cost function that does not exhibit any sharp decrease.

Figure 10: Average scaled cost for demand distribution Triangular(10, 50, 90), $c = 1$, $h = 52$.

4.4 Impact of risk aversion

In previous sections, we have analyzed the average system performance by experimenting over various $\pi_1$ values for several sets of cost parameters. We have observed that the average cost function is generally smooth as long as the optimal order quantity is determined by a single equation, and that the cost function experiences a jump at the point where the optimal order quantity equation changes, that is, at $\pi_1 = \frac{b-h}{b+h}$, which we call the Critical Risk Ratio (CRR). For any given $c, h, \text{ and } b$, Theorems 3.3 and 3.4 provide two different equations for the optimal order quantity,
one for $\pi_1$ values below the CRR ($Q^*_{\pi_1<CRR}$) and another for $\pi_1$ values above the CRR ($Q^*_{\pi_1>CRR}$). Experimental results in previous sections suggest that using $Q^*_{\pi_1>CRR}$ rather than $Q^*_{\pi_1<CRR}$, regardless of the value of $\pi_1$, might provide lower average cost figures. To confirm this, we perform additional simulation experiments. Specifically, we consider a Normally distributed demand and use parameter values of $c = 17$, $b = 52$ and $h$ in the range $[17, 51]$. We use the order quantity $Q^*_{\pi_1<CRR}$ even if the value of $\pi_1$ is greater than the CRR, and vice versa; Figure 12 shows the difference in the average costs realized using the two order quantities (i.e., $\bar{C}(Q^*_{\pi_1<CRR}) - \bar{C}(Q^*_{\pi_1>CRR})$). We observe that the order quantity $Q^*_{\pi_1>CRR}$ does indeed result in a lower expected cost, even for $\pi_1 < CRR$, regardless of the system parameters.

So what is the benefit for the manager to use the other order quantity, $Q^*_{\pi_1<CRR}$, when $\pi_1 < CRR$? To answer this question, we investigate the amount of risk associated with each order quantity; we quantify risk using the standard deviation of the cost. Figures 13-15 show the standard deviation of the total cost under the order quantities 1 and 2 (by order quantity 1 and 2 we refer
Figure 12: Average cost difference \((\bar{C}(Q_{\pi_1<CRR}^*) - \bar{C}(Q_{\pi_1>CRR}^*))\) for demand distribution Normal\((50, 10^2)\), \(c = 17\), \(b = 52\).

to \(Q_{\pi_1<CRR}^*\) and \(Q_{\pi_1>CRR}^*\), respectively).

Figure 13: Standard deviation of cost \((\sigma_{C(Q_{\pi_1<CRR}^*)} \text{ versus } \sigma_{C(Q_{\pi_1>CRR}^*)})\) for demand distribution Normal\((50, 10^2)\), \(c = 17\), \(b = 52\) and \(h\) is at its low level.

Our first observation from these results is that the standard deviation of the cost achieved for
the order quantity $Q^*_{\pi_1<CRR}$ is significantly lower than the standard deviation obtained for the order quantity $Q^*_{\pi_1>CRR}$, in particular for small values of $\pi_1$. This means that, although the strategy based on $Q^*_{\pi_1>CRR}$ has lower (and hence better) expected cost, using the order quantity $Q^*_{\pi_1>CRR}$ for small $\pi_1$ values is much riskier than using the order quantity $Q^*_{\pi_1<CRR}$. When the difference between holding and backorder costs, $|b-h|$, is large, we observe that the standard deviation of the cost achieved for $Q^*_{\pi_1<CRR}$ (order quantity 1) is always below the standard deviation of the cost.
achieved for $Q_{\pi_1 > \text{CRR}}$ (order quantity 2). In this case, the value of the critical risk ratio is large, which in turn is reflected by the order quantity 1, $Q_{\pi_1 < \text{CRR}}$, being optimal in Theorems 3.3 and 3.4 over a large range of $\pi_1$ values. On the other hand, when the holding and backorder costs are close from each other, the CRR goes to zero, which makes the use of order quantity 1 suboptimal in our model. In Figure 15, we see that the risk associated with order quantity 1 exceeds the risk associated with order quantity 2. Thus, $Q_{\pi_1 < \text{CRR}}$ loses its attractiveness in terms of risk protection and we switch from $Q_{\pi_1 < \text{CRR}}$ to $Q_{\pi_1 > \text{CRR}}$.

5 Conclusions

We have analyzed the value of information for two-stage inventory management in a robust optimization framework. The precise distribution of the demand is not known, but the decision-maker receives advance information on the success (high, low or moderate) of the product and has range forecasts for the demand in each scenario. We have derived the optimal order quantities in closed form; an advantage of the proposed approach is that it provides insights into the impact of the cost parameters on the optimal solution. We have shown the existence of breakpoints, where the manager switches between pre-defined optimal strategies, depending on the probability of the baseline scenario. These breakpoints play a critical role in understanding the optimal amount of information; in particular, when holding and backorder costs are of the same order of magnitude but distinct, our numerical experiments suggest that the manager should define the baseline scenario and the corresponding range forecast so that the probability of that scenario equals the critical ratio $\frac{b-h}{b+h}$.

The proposed three-scenario approach significantly decreases the total expected cost compared to the single-scenario case, where the decision-maker only has one range forecast. Our analysis indicates that a few well-chosen range forecasts can substantially improve system performance in robust inventory management.
References


