A Filled Function Method with One Parameter for $R^n$ Constrained Global Optimization *

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Abstract. For $R^n$ constrained global optimization problem, a new auxiliary function with one parameter is proposed for escaping the current local minimizer. First, a new definition of filled function is given and under mild assumptions, it is proved to be a filled function. Then a new algorithm is designed according to the theoretical analysis and some preliminary numerical results are given to demonstrate the efficiency of this global method for $R^n$ constrained global optimization.

Key Words. Local minimizer, Global minimizer, Constrained global optimization, Filled function method.

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1. Introduction

Optimization theory and methods have been widely used with real life application for many years. Yet, most classical optimization techniques only find local optimizers. Recently, the focus of the studies of Optimization theory and methods shifts to global optimization. Numerous algorithms have been developed for finding a global optimal solution to problems with nonlinear, nonconvex stucture and can be found in the surveys by Horst and Pardalos [6], Ge [1] as well as in the introductory textbook by Horst et al.[3]. These global optimization methods can be classified into two groups: stochastic(see [5]) and deterministic methods(see[1], [3], [4]). The methods of ”filled function” and ”tunnel function” belong to the latter category.

There are many papers about filled function method devoted to unconstrained global optimization in the open literatures, for example, [1], [2], [8],[9],[7],[10],[11], etc. However, up to now, to our best knowledge, there are fewer literatures for finding a global minimizer on $R^n$ constrained global optimization problem.

In this paper, we consider the following global optimization:

$$\min_{x \in S} f(x)$$

where $S = \{x \in R^n : g(x) \leq 0\}$, $g(x) = (g_1(x),...,g_m(x))$, and propose a new filled function method for solving problem $(P)$. The method iterates from one local minimum to a better one. In each

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iteration, we construct a filled function, and a local minimizer of the filled function then leads to a new solution of reduced objective function value.

To proceed further, this paper requires the following assumptions:

- Problem $P$ has at least one global minimizer. The sets of global minimizers and local minimizers are denoted by $G(P)$ and $L(P)$, respectively.
- $L^* = \{ x \in L(P) : f(x) = f(x^*) \}$ is a closed bounded set, where $x^* \in L(P)$.
- The number of minimizers of $(P)$ can be infinite, but the number of the different value of minimizer of $(P)$ is finite.
- It require $f(x), \nabla f(x), g_i(x), \nabla g_i(x), i = 1, \ldots, m$ to be Lipschitz continuously differentiable over $\mathbb{R}^n$, i.e. there exist $L_f > 0, L_{\nabla f} > 0, L_{g_i}, L_{\nabla g_i}, i = 1, \ldots, m$, such that for any $x, y \in \mathbb{R}^n$, the following hold true:
  \[
  |f(x) - f(y)| \leq L_f \|x - y\|, \quad \|\nabla f(x) - \nabla f(y)\| \leq L_{\nabla f}\|x - y\|
  \]
  \[
  |g_i(x) - g_i(y)| \leq L_{g_i}\|x - y\|, \quad \|\nabla g_i(x) - \nabla g_i(y)\| \leq L_{\nabla g_i}\|x - y\|, \quad i = 1, \ldots, m.
  \]

What’s more, we give a new definition of filled function as the follows: Suppose $x^*$ is a current local minimizer of $(P)$. $P(x, x^*)$ is said to be a filled function of $f(x)$ at $x^*$, if it satisfies the following properties:

- (P1) $x^*$ is a strict local maximizer of $P(x, x^*)$ on $\mathbb{R}^n$;
- (P2) $\nabla P(x, x^*) \neq 0$, for any $x \in S_1 \backslash \{x^*\}$ or $x \in R^n \backslash S$, where $S_1 = \{ x \in S : f(x) \geq f(x^*) \}$;
- (P3) If $x^*$ is not a global minimizer, then there exists at least one point $x^*_1 \in S_2$ such that $x^*_1$ is a local minimizer of $P(x, x^*)$.

Adopting the concept of the filled function, a global optimization problem can be solved via a two-phase cycle. In Phase I, we start from a given point and use any local minimization procedure to locate a local minimizer $x^*$ of $f(x)$. In Phase II, we construct a filled function at $x^*$ and minimize it in order to identify a point $x^*_1 \in S$ with $f(x^*_1) < f(x^*)$. If such an $x^*_1$ is obtained, we can then use $x^*_1$ as the initial point in Phase I again, and find a better minimizer $x^{**}$ of $f(x)$ with $f(x^{**}) < f(x^*)$. This process repeats until the time when minimizing a filled function does not yield a better solution. The current local minimum will be then taken as a global minimizer of $f(x)$.

This paper is organized as follows. Following this introduction, a new filled function is proposed and the properties of the new filled function are investigated. In Section 3, a solution algorithm is suggested and preliminary numerical experiments are performed. Finally, some conclusions are drawn in Section 4.

2. A new filled function and its properties

Consider the one parameter filled function for problem $(P)$

\[
T(x, x^*, \rho) = -\|x - x^*\|^4 - \rho(\max(0, f(x) - f(x^*)))^3 + \sum_{i=1}^{m}[\max(0, g_i(x) - g_i(x^*))]^3
\]
\[
+ \frac{1}{\rho}(\exp[\min(0, \max(f(x) - f(x^*), g_i(x), i = 1, \ldots m))]^3 - 1)
\] (2)

where $x^*$ is a current local minimizer of $f(x)$, $\| \cdot \|$ indicates the Euclidean vector norm.

The following theorems show that $T(x, x^*, \rho)$ is a filled function when $\rho > 0$ small enough.
Theorem 2.1 Assume that \( x^* \) is a local minimizer of (P), then \( x^* \) is a strict local maximizer of \( T(x, x^*, \rho) \).

**Proof.** Since \( x^* \) is a local minimizer of (P), there exists a neighborhood \( N(x^*, \sigma^*) \) of \( x^*, \sigma^* > 0 \) such that
\[
f(x) \geq f(x^*), g(x^*) \leq 0, g(x) \leq 0
\]
for all \( x \in N(x^*, \sigma^*) \cap S \).

Therefore, for all \( x \in N(x^*, \sigma^*) \cap S, x \neq x^* \), it holds
\[
\{\min[0, \max(f(x) - f(x^*), g(x))]\} = 0,
\]
and we have
\[
T(x, x^*, \rho) = -\|x - x^*\|^4 - \rho([\max(0, f(x) - f(x^*))]^3 + \sum_{i=1}^n([\max(0, g_i(x) - g_i(x^*))]^3)
< 0 = T(x^*, x^*, \rho).
\]

For all \( x \in N(x^*, \sigma^*) \cap (R^n \setminus S) \), there exists at least one index \( i_0 \in 1, \cdots, n \) such that \( g_{i_0}(x) \geq 0 \). Therefore, we have
\[
\{\min[0, \max(f(x) - f(x^*), g(x))]\} = 0
T(x, x^*, \rho) = -\|x - x^*\|^4 - \rho([\max(0, f(x) - f(x^*))]^3 + \sum_{i=1}^n([\max(0, g_i(x) - g_i(x^*))]^3)
< 0 = T(x^*, x^*, \rho).
\]

From the above discussion, we obtain that \( x^* \) is a strict local maximizer of \( T(x, x^*, \rho) \), and this completes the proof. ■

**Remark:** Obviously, for all \( x \in N(x^*, \sigma^*) \), \( \sigma^* > 0 \), we have
\[
\{\min[0, \max(f(x) - f(x^*), g(x))]\} = 0
\]
\[
\nabla T(x, x^*, \rho) = -4\|x - x^*\|^2(x - x^*) - 3\rho(\nabla f(x)[\max(0, f(x) - f(x^*))]^2
+ \sum_{i=1}^m \nabla g_i(x)[\max(0, g_i(x) - g_i(x^*))]^2]
\]
\[
\nabla^2 T(x, x^*, \rho) = -4\|x - x^*\|^2E - 8(x - x^*)(x - x^*)^\top
- 3\rho|\nabla^2 f(x)[\max(0, f(x) - f(x^*))]^2 + 2\nabla f(x)\nabla f(x)^\top \max(0, f(x) - f(x^*))
+ \sum_{i=1}^m \nabla^2 g_i(x)[\max(0, g_i(x) - g_i(x^*))]^2 + 2 \sum_{i=1}^m \nabla g_i(x)\nabla g_i(x)^\top \max(0, g_i(x) - g_i(x^*))
\]
where \( E \) is an identity matrix.

For any \( d \in D = \{d \in R^n : \|d\| = 1\} \), if
\[
0 < \rho < \frac{4}{3L_f^2L_{\nabla^2 f} + 3 \sum_{i=1}^m L_{g_i}^2L_{\nabla^2 g_i}},
\]
then we have
\[
d^T \nabla^2 T(x, x^*, \rho)d = -4\|x - x^*\|^2 - 8((x - x^*)^Td)^2
- 3\rho|d^T \nabla^2 f(x)d[\max(0, f(x) - f(x^*))]^2 + 2(d^T \nabla f(x))^2 \max(0, f(x) - f(x^*))
+ \sum_{i=1}^m d^T \nabla^2 g_i(x)d[\max(0, g_i(x) - g_i(x^*))]^2 + \sum_{i=1}^m 2(d^T \nabla g_i(x))^2 \max(0, g_i(x) - g_i(x^*))]
\leq -4\|x - x^*\|^2 - 3\rho|d^T \nabla^2 f(x)d[\max(0, f(x) - f(x^*))]^2
+ \sum_{i=1}^m d^T \nabla^2 g_i(x)d[\max(0, g_i(x) - g_i(x^*))]^2
\leq -4\|x - x^*\|^2 + 3\rho|L_f^2L_{\nabla^2 f} + \sum_{i=1}^m L_{g_i}^2L_{\nabla^2 g_i}||x - x^*\|^2 < 0,
\]
where \( L_{\nabla^2 f_i}, i = 1, 2, \ldots, m \) are maximum of \( \| \nabla^2 f(x) \| \) and \( \| \nabla^2 g_i(x) \| \) on \( N(x^*, \sigma^*) \), respectively.

Therefore, \( \nabla^2 T(x, x^*, \rho) \) is negative definite, \( x^* \) is an isolated local maximizer of \( T(x, x^*, \rho) \) and \( \nabla T(x, x^*, \rho) \neq 0 \) for all \( x \in N(x^*, \sigma^*) \setminus \{x^*\} \).

Theorem 2.1 reveals that the proposed new filled function satisfies property (P1).

The following theorem shows that when confined on \( S_1 \setminus \{x^*\} \) or \( R^n \setminus S \) not on \( R^n \), \( T(x, x^*, \rho) \) is continuously differentiable and \( \nabla T(x, x^*, \rho) \neq 0 \).

**Theorem 2.2** Assume that \( x^* \) is a local minimizer of \( (P) \) and \( x \in S_1 \setminus \{x^*\} \) or \( x \in R^n \setminus S \), then for \( \rho > 0 \) small enough, it holds \( \nabla T(x, x^*, \rho) \neq 0 \).

**Proof.** Since \( x^* \) is a local minimizer of \( (P) \), by Theorem 2.1, there exists a neighborhood \( N(x^*, \sigma^*) \) such that \( f(x) \geq f(x^*) \) and \( g(x) \leq 0 \) hold true for any \( x \in N(x^*, \sigma^*) \cap S \).

Consider the following three situations:

(1): \( x \in N(x^*, \sigma^*) \).
In this case, by the remark, it holds \( \nabla T(x, x^*, \rho) \neq 0 \)

\[
(2): x \in S_1 \setminus N(x^*, \sigma^*) \), and \( f(x) \geq f(x^*) \).
In this case, we have \( \| x - x^* \| \geq 1 \)

\[
\nabla T(x, x^*, \rho) = -4\| x - x^* \|^2(x - x^*) - 3\rho(\nabla f(x)[\max(0, f(x) - f(x^*))])^2 \]

\[
+ \sum_{i=1}^{m} \nabla g_i(x)[\max(0, g_i(x) - g_i(x^*))^2].
\]

\[
(x - x^*)^T \nabla T(x, x^*, \rho) = -4\| x - x^* \|^4 - 3\rho[(x - x^*)^T \nabla f(x)(f(x) - f(x^*))^2]
\]

\[
+ \sum_{i=1}^{m} (x - x^*)^T \nabla g_i(x)[\max(0, g_i(x) - g_i(x^*))^2]
\]

\[
- 4\| x - x^* \|^4 - 3\rho[(x - x^*)^T (\nabla f(x) - \nabla f(x^*))(f(x) - f(x^*))^2]
\]

\[
+ (x - x^*)^T \nabla f(x^*)(f(x) - f(x^*))^2
\]

\[
+ \sum_{i=1}^{m} (x - x^*)^T \nabla g_i(x^*)[\max(0, g_i(x) - g_i(x^*))^2]
\]

\[
+ \sum_{i=1}^{m} (x - x^*)^T (\nabla g_i(x) - \nabla g_i(x^*))[\max(0, g_i(x) - g_i(x^*))^2]
\]

\[
\leq -4\| x - x^* \|^4 + 3\rho L_{g_i}^2 L_{\nabla f} \| x - x^* \|^4 + \sum_{i=1}^{m} L_{g_i}^2 \| x - x^* \|^4
\]

\[
+ L_{f}^2 \| \nabla f(x^*) \| \| x - x^* \|^3 \| x - x^* \|^2
\]

\[
+ \sum_{i=1}^{m} L_{g_i}^2 \| \nabla g_i(x^*) \| \| x - x^* \|^2
\]

\[
+ \sum_{i=1}^{m} L_{g_i}^2 \| \nabla g_i(x^*) \| \| x - x^* \|^2 \frac{\| x - x^* \|^2}{\sigma^2}
\]

\[
\leq \| x - x^* \|^4(-4 + 3\rho L_{f}^2 L_{\nabla f} + \sum_{i=1}^{m} L_{g_i}^2 L_{\nabla g_i})
\]

\[
+ \frac{L_{g_i}^2 \| \nabla f(x^*) \|}{\sigma^2} + \sum_{i=1}^{m} \frac{L_{g_i}^2 \| \nabla g_i(x^*) \|}{\sigma^2}
\]

\[
< 0.
\]

Where \( \rho > 0 \) should be small enough.

(3): \( x \in R^n \setminus \{N(x^*, \sigma^*) \cup S \} \).
In this case, by considering two situations: \( f(x) \geq f(x^*) \) and \( f(x) < f(x^*) \) and using the way similar to the proof of this theorem in the second case, we can obtain that \( \nabla T(x, x^*, \rho) \neq 0 \).

Therefore, for any \( x \in S_1 \setminus \{x^*\} \) or \( x \in R^n \setminus S \), it holds \( \nabla T(x, x^*, \rho) \neq 0 \). ■

Theorem 2.2 reveals that the proposed new filled function satisfies property (P2).
Theorem 2.3 Assume that \( x^* \) is not a global minimizer of \((P)\) and that \( \text{cl} S = \text{cl} \mathcal{S}\) holds, then there exists a point \( x_0^* \in \mathcal{S}_2 \) such that \( x_0^* \) is a local minimizer of \( T(x, x^*, \rho) \) when \( \rho > 0 \) is small enough.

Proof. Since \( x^* \) is a local but not global minimizer of \((P)\), there exists another local minimizer of \( f(x), x_1^* \), such that \( f(x_1^*) < f(x^*) \) and \( g(x_1^*) \leq 0 \).

Consider the following two cases:

Case (1): \( S \) is a bounded closed set.
In this case, by the assumption \( \text{cl}(\text{int} \mathcal{S}) = \text{cl} \mathcal{S} \), there exists another point \( x_2^* \in \{x_1^*, \delta\} \cap \text{int} \mathcal{S} \) such that
\[
\begin{aligned}
f(x_2^*) &< f(x^*),
g(x_2^*) < 0.
\end{aligned}
\]
Where \( \delta > 0 \) sufficiently small and these two expressions imply that
\[
\begin{aligned}
\exp[\min(0, \max(f(x_2^*) - f(x^*), g_i(x_2^*), i = 1,...m))]^3 - 1 < 0,
\end{aligned}
\]
\[
\begin{aligned}
T(x_2^*, x^*, \rho) &= -\|x_2^* - x^*\|^4 - \rho(\sum_{i=1}^{m} [\max(0, g_i(x_2^*) - g_i(x^*))]^3) \\
&\quad + \frac{1}{\rho}(\exp[\max(f(x_2^*) - f(x^*), g_i(x_2^*), i = 1,...m)]^3 - 1)).
\end{aligned}
\]
On the other hand, for any \( x \in \partial \mathcal{S} \), there exists at least one index \( i_0 \in \{1,...,m\} \) such that \( g_{i_0}(x) = 0 \).
Therefore, it is easy to obtain
\[
\min(0, \max(f(x) - f(x^*), g_i(x), i = 1,...m)) = 0.
\]
\[
\begin{aligned}
T(x, x^*, \rho) &= -\|x - x^*\|^4 - \rho(\sum_{i=1}^{m} [\max(0, g_i(x) - g_i(x^*))]^3) \\
&\quad + (\max(f(x) - f(x^*), 0))^3).
\end{aligned}
\]
All the above relations imply that for any \( x \in \partial \mathcal{S} \) with small \( \rho > 0 \), we must have
\[
T(x, x^*, \rho) > T(x_2^*, x^*, \rho).
\]
Clearly, \( \partial \mathcal{S} \) is a closed bounded set and \( \mathcal{S} \setminus \partial \mathcal{S} \) is an open bounded set. There must exist one point \( x_0^* \in \mathcal{S} \setminus \partial \mathcal{S} \) such that
\[
\min_{x \in \mathcal{S}} T(x, x^*, \rho) = T(x_0^*, x^*, \rho) = \min_{x \in \mathcal{S} \setminus \partial \mathcal{S}} T(x, x^*, \rho),
\]
where obviously we have \( x_0^* \in \text{int} \mathcal{S} \) and \( f(x_0^*) < f(x^*) \).

Case (2): \( \mathcal{S} \) is an unbounded closed set.


In this case, by Assumption 2, \( L^{x_1} = \{ x \in L(P) : f(x) = f(x_1^*) \} \) is a bounded closed set.

Denote \( B(\theta, 1) = \{ x \in R^n : \| x - x^* \| \leq 1 \} \) and let \( \varepsilon_0 > 0 \) be small enough, it is easy to know that

\[
L^{x_1}_{\varepsilon_0} = L^{x_1} + \varepsilon_0 B(\theta, 1)
\]

is also a closed bounded set.

By the condition of \( cl(intS) = clS \), there exists one point \( x_2^* \in L^{x_1}_{\varepsilon_0} \cap intS \) such that

\[
f(x_1^*) \leq f(x_2^*) < f(x^*), g(x_2^*) < 0.
\]

These expressions imply that

\[
T(x_2^*, x^*, \rho) = -\| x_2^* - x^* \|^4 - \rho \left( \sum_{i=1}^{m} \left[ \max(0, g_i(x_2^*) - g_i(x^*)) \right]^3 \right) + \frac{1}{\rho} \left( \exp[\max(f(x_2^*) - f(x^*), g_i(x_2^*), i = 1, \ldots, m)]^3 - 1 \right) < 0.
\]

Therefore, we must have

\[
\min_{x \in L^{x_1}_{\varepsilon_0}} T(x, x^*, \rho) = T(x_0^*, x^*, \rho) \leq T(x_2^*, x^*, \rho) < 0.
\]

Next, we will show that the following two situations can not occur: \( x_0^* \in L^{x_1}_{\varepsilon_0} \cap \partial S \) or \( x_0^* \in L^{x_1}_{\varepsilon_0} \cap (R^n \setminus S) \).

By contradiction, if \( x_0^* \in L^{x_1}_{\varepsilon_0} \cap \partial S \) or \( x_0^* \in L^{x_1}_{\varepsilon_0} \cap (R^n \setminus S) \), then there exists one index \( i_0 \in \{1, \ldots, m\} \) such that \( g_{i_0}(x) = 0 \) or \( g_{i_0}(x) > 0 \), this means

\[
\min(0, \max(f(x) - f(x^*), g_i(x), i = 1, \ldots, m)) = 0.
\]

and

\[
T(x_0^*, x^*, \rho) = -\| x_0^* - x^* \|^4 - \rho \left( \sum_{i=1}^{m} \left[ \max(0, g_i(x_0^*) - g_i(x^*)) \right]^3 \right) + \left( \max(f(x_0^*) - f(x^*), 0) \right)^3.
\]

Since \( x_0^* \in L^{x_1}_{\varepsilon_0}, \| x_0^* - x^* \|, \max(0, g_i(x_0^*) - g_i(x^*)) \) and \( \max(f(x_0^*) - f(x^*), 0) \) must be bounded. So, if \( \rho > 0 \) is chosen to be small enough, we will have

\[
T(x_0^*, x^*, \rho) > T(x_2^*, x^*, \rho)
\]

which contradicts with \( T(x_0^*, x^*, \rho) \leq T(x_2^*, x^*, \rho) \). Therefore, it must have \( x_0^* \in L^{x_1}_{\varepsilon_0} \cap intS \) and \( f(x_0^*) < f(x^*) \).

Theorem 2.3 clearly state that the proposed filled function satisfies property (P3) and Theorems 2.1–2.3 state that the proposed filled function satisfies properties (P1)-(P3).

**Theorem 2.4** If \( x^* \) is a global minimizer of (P), then \( T(x, x^*, \rho) < 0 \) for all \( x \in S \setminus \{x^*\} \).

**Proof.** Since \( x^* \) is a global minimizer of (P), we have \( f(x) \geq f(x^*) \) for all \( x \in S \setminus \{x^*\} \). Thus, by Theorem 2.1, \( T(x, x^*, \rho) < 0 \) for all \( x \in S \setminus \{x^*\} \).  ■

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3. Solution algorithm and preliminary numerical implementation

The theoretical properties of the proposed filled function $T(x, x^*, \rho)$ were discussed in the last section. The development in this section is to suggest an algorithm and perform preliminary numerical experiments to give an initial feeling of the efficiency of the proposed algorithm.

3.1. Algorithm

Initialization Step
1. Choose a tolerance $\epsilon > 0$, e.g., set $\epsilon := 10^{-4}$.
2. Choose a fraction $\hat{\rho} > 0$, e.g., set $\hat{\rho} := 0.1$.
3. Choose a lower bound of $r$ such that $\rho_L > 0$, e.g., set $\rho_L := 10^{-6}$.
4. Select an initial point $x_1 \in S$.
5. Set $k := 1$.

Main Step
1. Starting from an initial point $x_1 \in S$, minimize $f(x)$ and obtain the first local minimizer $x_1^* = f(x)$.
2. Choose a set of initial points $\{x_{k+1}^i : i = 1, 2, \cdots, m\}$ such that $x_{k+1}^i \in S \setminus N(x_k^*, \sigma_k)$ for some $\sigma_k > 0$.
3. Let $\rho := 1$.
4. Set $i := 1$.
5. If $i \leq m$, then set $x := x_{k+1}^i$ and go to 6; Otherwise, the algorithm is incapable of finding a better minimizer starting from the initial points, $\{x_{k+1}^i\}$. The algorithm stops and $x_1^*$ is taken as a global minimizer.
6. If $f(x) < f(x_1^*)$ and $x \in S$, then use $x$ as an initial point for a local minimization method to find $x_{k+1}^*$ such that $f(x_{k+1}^*) < f(x_{k+1}^i)$, set $x_{k}^i = x_{k+1}^*$, $k := k + 1$ and go to 2; Otherwise, go to 7.
7. If $\|T(x, x_1^*, \rho)\| \geq n\epsilon$, go to Inner Circular Step $1^0$; Otherwise, go to 8.
8. Reduce $\rho$ by setting $\rho := \hat{\rho} \times \rho$.
   (a) If $\rho \geq \rho_L$, then go to 4.
   (b) Otherwise, go to 5.

Inner Circular Step

$1^0$ Let $x_{k'} := x$, $g_{k'} = \nabla T(x_{k'}, x_1^*, \rho)$, $k' := 1$.
$2^0$ Let $d_{k'} = -g_{k'}$.
$3^0$ Get the step size $\alpha_{k'}$, and let $x_{k'+1} = x_{k'} + \alpha_{k'}d_{k'}$. If $x_{k'+1}$ attains the boundary of $S$ during minimization, then set $i := i + 1$ and go to Main Step 5; Otherwise go to $4^0$.
$4^0$ If $f(x_{k'+1}) < f(x_1^*)$, go to Main Step 6; Otherwise, go to 5.
$5^0$ Let $g_{k'+1} = \nabla T(x_{k'+1}, x_1^*, \rho)$, go to $6^0$.
$6^0$ If $k' > n$, let $x = x_{k'+1}$, go to $1^0$; Otherwise, go to $7^0$.
$7^0$ Let $d_{k'+1} = -g_{k'+1} + \beta_{k'}d_{k'}$.

where $\beta_{k'} = \frac{g_{k'+1}^T(g_{k'+1} - g_{k'})}{\gamma_{k'}^2g_{k'}}$. Let $k' := k' + 1$, go to $3^0$.

The motivation and mechanism behind the algorithm are explained below.

A set of $m = 4n + 4$ initial points is chosen in Step 2 to minimize the filled function. We set the initial points symmetric about the current local minimizer. For example, when $n = 2$, the initial
3.2. Preliminary numerical experiment

To it that terminated immediately and returned to the process of minimization of from Theorem 2.3 that this point $S$ to neither taken only on the set $R^n$ to minimize the original objective function one can use the new

Problem 2

The global minimum solutions:

Step 3 finds a new $x$ in the direction $d_{k'}$ such that $T(x, x^*, \rho)$ can reduce to certain extent. Then, one can use the new $x$ to minimize the filled function again.

In Step 3, use $\alpha_{k'} = 0.1$ as the step size.

Recall from Theorem 2.3 that the value of $\rho$ should be selected small enough. Otherwise, there could be no better minimizer of $T(x, x^*, \rho)$. Thus, the value of $\rho$ is decreased successively in Step 8 of the solution process if no better solution is found when minimizing the filled function. If all the initial points were chosen to calculate and $\rho$ reaches its lower bound $\rho_L$ and no better solution is found, the current local minimizer is taken as a global minimizer.

In Inner Circular Step, we use PRP Conjugate Gradient Method to get the search direction. In fact, it can also use other local method to get the search direction.

Remark: Please notes that from the above algorithm, the process of minimization of $T(x, x^*, \rho)$ is taken only on the set $S_1$ or $R^n \setminus S$ on which $T(x, x^*, \rho)$ is differentiable. Once a point $x'$ belonging to neither $S_1$ nor $R^n \setminus S$ is obtained, then the process of minimization of $T(x, x^*, \rho)$ should be terminated immediately and returned to the the process of minimization of $f(x)$. So We only see to it that $T(x, x^*, \rho)$ is differentiable on the set $S_1$ or $R^n \setminus S$ rather than on the whole $R^n$.

3.2. Preliminary numerical experiment

Although the focus of this paper is more theoretical than computational, we still test our algorithm on several global minimization problems to have an initial feeling of the potential practical value of the filled function algorithm. All the numerical experiments are implemented in Fortran 95, under Windows XP and Pentium (R) 4 CPU 2.80GMHZ. The Fortran 95 function 'nconf' in module 'IMSL' is used in the algorithm to find local minimizers of (P).

Problem 1

$$f(x) = x_1^2 + x_2^2 - \cos(17x_1) - \cos(17x_2) + 3$$

s.t

$$g_1(x) = (x_1 - 2)^2 + x_2^2 - 1.6^2 \leq 0.$$  
$$g_2(x) = x_1^2 + (x_2 - 3)^2 - 2.7^2 \leq 0.$$  

The global minimum solutions: $x^* = (0.7250289, 0.3991602)$, and $f^* = 1.837504$.

Problem 2

$$f(x) = (4 - 2.1x_1^2 + \frac{x_1^4}{3})x_1^2 + x_1x_2 + (-4 + 4x_2^2)x_2^2$$
\[
g(x) = -\sin(4\pi x_1) + 2\sin^2(2\pi x_2) \leq 0
\]
\[-1 \leq x_1, x_2 \leq 1
\]
The global minimum solutions: \(x^* = (0.1090018, -0.62233971)\), and \(f^* = -0.9711032\).

**Problem 3**

\[
f(x) = -25(x_1 - 2)^2 - (x_2 - 2)^2 - (x_3 - 1)^2 - (x_4 - 4)^2 - (x_5 - 1)^2 - (x_6 - 4)^2
\]
s.t
\[
g_1(x) = (x_3 - 3)^2 + x_4 \geq 4
\]
\[
g_2(x) = (x_5 - 3)^2 + x_6 \geq 4
\]
\[
g_3(x) = x_1 - 3x_2 \leq 2
\]
\[
g_4(x) = -x_1 + x_2 \leq 2
\]
\[
g_5(x) = x_1 + x_2 - 6 \leq 0
\]
\[
g_6(x) = x_1 + x_2 \geq 2
\]
\[
0 \leq x_1, x_4 \leq 6
\]
\[
0 \leq x_2 \leq 8
\]
\[
1 \leq x_3, x_5 \leq 5
\]
\[
0 \leq x_6 \leq 10
\]
The global minimum solutions: \(x^* = (5.0000000, 0.9999999, 5.000000, 0.0000000, 5.0000000, 10.000000)\), and \(f^* = -310.0000\).

The main iterative results are summarized in tables for each function. The symbols used are shown as follows:

- \(x^0_k\): The \(k\)-th initial point
- \(k\): The iteration number in finding the \(k\)-th local minimizer
- \(x^*_k\): The \(k\)-th local minimizer
- \(f(x^*_k)\): The function value of the \(k\)-th local minimizer

<table>
<thead>
<tr>
<th>(k)</th>
<th>(x^0_k)</th>
<th>(x^*_k)</th>
<th>(f(x^*_k))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(2,1.5)</td>
<td>(1.835084,1.100764)</td>
<td>5.612519</td>
</tr>
<tr>
<td>2</td>
<td>(1.516314,1.182714)</td>
<td>(0.4396141,0.3538329)</td>
<td>1.982739</td>
</tr>
<tr>
<td>3</td>
<td>(0.7104309,0.3993877)</td>
<td>(0.7250289,0.3991602)</td>
<td>1.837504</td>
</tr>
</tbody>
</table>

Table 1. **Problem 1**
Table 2. Problem 2

<table>
<thead>
<tr>
<th>k</th>
<th>$x_k^*$</th>
<th>$x_k^*$</th>
<th>$f(x_k^*)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(0.5,-0.9)</td>
<td>(0.5734327,-0.8912419)</td>
<td>-0.064501</td>
</tr>
<tr>
<td>2</td>
<td>(0.1714329,-0.8912776)</td>
<td>(0.1248393,-0.8749353)</td>
<td>-0.7654098</td>
</tr>
<tr>
<td>3</td>
<td>(0.1248175,-0.6229354)</td>
<td>(0.1090018,-0.6223397)</td>
<td>-0.9711032</td>
</tr>
</tbody>
</table>

Table 3. Problem 3

<table>
<thead>
<tr>
<th>k</th>
<th>$x_k^*$</th>
<th>$x_k^*$</th>
<th>$f(x_k^*)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>(5.000000,1.000000,5.105109,3.997957,5.000000,3.997957)</td>
<td>(5.000000,1.000000,5.000000,3.997957,5.000000,3.997957)</td>
<td>-274.0000</td>
</tr>
<tr>
<td>3</td>
<td>(5.000000,1.000000,5.000000,0.1104090,5.000000,10.11041)</td>
<td>(5.000000,1.000000,5.000000,0.000000,5.000000,10.00000)</td>
<td>-310.0002</td>
</tr>
</tbody>
</table>

4. Conclusions

In this paper, we first give a definition of a filled function for constrained minimization problem and construct a new filled function with one parameter. And then design an elaborate solution algorithm based on this filled function. Finally, we perform some numerical experiments to give initial impression of the potentiality of this filled function method. Of course, the efficiency of this filled function method relies on the efficiency of the local minimization method for constraint optimization problem. Meanwhile, from the preliminary numerical results, we can see that algorithm can move successively from one local minimum to another better one, but in most cases, we have to use more time to judge the current point is a global minimizer than to find a global minimizer. Recently, global optimality conditions have been derived in [12] for quadratic binary integer programming problems. However, A global optimality conditions for continuous variables are still an open problem, in general. The criterion of a global minimizer will provide solid stopping conditions for a continuous filled function method.

References


