Publication: Optimization by the Fixed-Point Method  
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Petaling Jaya, Selangor, Malaysia

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Foreword

Joseph Fourier considered a linear programming problem as early as 1823. In 1911 a method involving vertex-to-vertex movement along edges of the feasible polyhedron (as in the simplex method) was suggested as a way to solve a problem that involved optimization. In 1928 John von Neumann published his paper on game theory and in 1939 the Russian mathematician Kantorovich published *Mathematical Methods for Organization and Planning of Production*, in which broad classes of economic scheduling tasks are presented as mathematical optimization problems. In 1944 John von Neumann in collaboration with the Austrian economist Oskar Morgenstern published *The Theory of Games and Economic Behaviour*, and in 1947 von Neumann conjectured the equivalence of linear programs and matrix games, introduced the important concept of duality, and made several proposals for the numerical solution of linear programming and game problems. Serious interest by other mathematicians began in 1948 with the rigorous development of duality and related matters. We are told that Kantorovich’s proposals remained mostly unknown both in the Soviet Union and elsewhere for nearly two decades, nevertheless it appears more likely that his work was used by GOSPLAN and the Soviets didn’t bother to tell others very much about their planning methods.

Another strand is that the common history of the computer is not that reasonable: the notion of a stored program computer dates back to Countess Ada Lovelace and Charles Babbage, and it is not reasonable to say that Babbage was restricted in funding when we now know that he was involved in cryptographic work and was getting his gears cut in one of the Scandinavian countries - it appears more likely that the money flowed freely and Babbage did not die a frustrated man. There was a move to electronic switching before WW2 using thermionic valves and one can be quite certain that these valves would have been miniaturised during the war years or maybe earlier. Thus it is reasonable to posit the existence of a limited number of digital computers during WW2 for use in cryptography, weapons design and for the solution of economic allocation problems. In fact there are some references to the use of computers during WW2, specifically at Bletchley Park.

Thus the version of the history suggesting that “... linear programming was comparatively unknown before 1947.” and “No work of any significance was carried out before this date ...” [6] appears to be of doubtful validity. Allowing for the WW2 beginning to wind down in 1945 and demobilization taking maybe two years (and beginning with the lower ranks) we have military operations researchers probably only just returning to civilian life in 1947, so any burst of activity on OR during the war years would reasonably only see the light of day when the practitioners had the freedom, time and the permission to talk about it. Also, apart from permission and time to write papers on the subject, the war severely limited journal publication and communication between researchers so it is reasonable to believe that linear programming did not just burst upon the world in 1947, but dawned slowly in the Soviets with the implementation of a command economy and in armed forces and corporations maybe from as early as 1927. Following WW2 the use of the Simplex Method became widespread first in the US military and then industry; currently large problems are constructed and solved.
With the increase in the size of problems has come renewed interest in other solution methods besides the Simplex Method; this book considers such methods, and one method in detail.

Jalaluddin Abdullah @ Julian Graham Morris

2011/03/20
Preface

In the name of God, the Benificent, the Merciful

This book is on the fixed-point method of solving optimization problems, beginning with Chapter 2 which introduces some general results from linear algebra. Chapter 3 is on general fixed-point theory; in this chapter the notion of a swapping matrix is introduced - this matrix acts on idempotent symmetric matrices and is used in the construction of what is called a semi-unitary matrix which has linear properties in the region close to fixed-points associated with solutions to the LP problem. The fixed-points of the semi-unitary matrix and convergence to such fixed-points is investigated.

Chapter 4 makes explicit the classificatory nature of the fixed-point approach. The classificatory stage of an investigation is regarded in the literature of research methodology as the first stage of a proper investigation, but since classification in the work of Pyle, Cline, and Bruni is implicit, it might have escaped operations researchers that this stage of the investigation of the LP problem had been completed rather elegantly, and that the resulting fixed-point formulation was therefore particularly worthy of further analysis. The aim of Chapter 4 is to detail this first stage and then to extend the approach to algorithms for the solution of both linear and non-linear programs. More specifically, for linear programming pivotal results include the duality theorem of von Neumann [27], and the Karush-Kuhn-Tucker Theorem [23, op cit]. The classificatory implications of this result have been explored in the work of Pyle et al [31] in which they reduce the LP problem to a form with certain invariant properties and then apply the theory of polyhedral cones in computing solutions. Here lie some of the origins of the fixed-point formulation of a linear program, leading to the work of Bruni [9], where it finds simpler expression.

These authors applied the duality theorem to implicitly classify large sets of LP’s as essentially equivalent under affine transformation of the solution space; Chapter 4 shows that their work can be regarded as creating equivalence classes of similar linear programs and a canonical representative for each such class; primal and dual invariant problems are constructed and they are shown to be representatives of the equivalence classes to which they belong.

In Chapter 5 the fixed-point problem is constructed by combining the primal and dual invariant problems to form a fixed-point problem - the problem of finding the non-negative fixed-points of an idempotent symmetric matrix. This approach combines both the invariant primal and invariant dual in one balanced equation.

In Chapter 6 the key notion of proximality is introduced and it is shown that, in the context of the LP fixed-point problem, proximality implies linear behaviour; this lays the groundwork for the construction of a solution method which combines a converging series approach with regression to ensure termination after a finite number of steps. In the following Chapter 7 the solution method is applied to a number of LP problems.

Finally, and in contrast to the Karush-Kuhn-Tucker conditions which are not operational in nature,
Chapter 8 contains a generalization of the approach to non-linear convex optimization problems in which operational conditions for an optimum are derived.

Appendix A contains a spectral analysis of the product of projection matrices, being relevant to the convergence analysis of Chapter 3.3.1.

Appendix B is on computational issues and program specification.

The programming language Scheme, which is a dialect of LISP, is used throughout the book. Interpreters/compilers, including DrScheme and Gambit for the Macintosh, PC, and UNIX are available. The exercises are Scheme based. All example programs run under DrScheme, which is available on the internet.

The book has grown out of research carried out in the Department of Economics at the University of Birmingham, England under Dr. Aart Heesterman leading to a thesis entitled *Fixed Point Algorithms for Linear Programming* [1]. A number of staff at Universiti Teknologi Malaysia assisted me while writing an earlier version of the book and in this respect I am particularly grateful for the use of library facilities at Perpustakaan Sultanah Zanariah, to Dr. Yusof Yaakob (Publishing), Encik Aziz Yaakob in the Computer Centre, and to Hj. Mohammad Shah, Dr. Ali Abdul Rahman for checking an earlier draft, and Encik Muhammad Hisyam Lee Wee Yee in the Mathematics Department, for support in getting \LaTeX{} running well, to Allahyarham Professor Mohd. Rashidi bin Md. Razali (late of the Mathematics Department) for encouragement, to Dr. Ibrahim Ngah and Abdul Razak and others who have built up and maintained the Resource Centre of Fakulti Alam Bina, and to all those who have developed and maintained the \TeX{} typesetting system including Andrew Trevor for Oz\TeX{}, Richard Koch *et al* for \TeX{}Shop, and Christiaan M. Hofman Adam R. Maxwell and Michael O. McCracken for Skim.

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List of Notation

Changes in notation between chapters are indicated in the introductions to chapters. Blocks of text are sometimes marked by brackets. These are used at the beginning and end points of sub-sections of proofs.

Matrices and operators are represented by uppercase letters, and vectors by lower-case letters which may be boldface.

Some symbols’ meaning changes, generally from general to specific; such symbols have more than one entry in the following table.

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<td>$d$</td>
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<td>$\lor$</td>
<td>logical OR</td>
<td>throughout</td>
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<td>$\land$</td>
<td>logical AND</td>
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<td>$\lor$</td>
<td>vector lattice operator</td>
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<td>$z$</td>
<td>$2m \times 1$ vector</td>
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Chapter 1

Introduction

1.1 Problem Statement

Operations research (OR) involves the study and control of activities and has thus been the object of considerable analysis throughout man’s history. One of the central problems to have been delineated is the linear programming (LP) problem, which involves maximizing a linear expression subject to linear inequality constraints is:

Given an \( m \times n \) matrix \( A \), and vectors \( b \) and \( c \),

\[
\text{find the vectors } \ x \ \text{which maximize } c^T x, \ \text{subject to } A x \geq b. \quad (1.1)
\]

The more than or equal sign signifies that each LP constraint is of this form; thus we are concerned essentially with homogeneous systems of inequalities where each line of the matrix equation has a binary relation of the form \( \geq \). The case of equality, where the \( i^{th} \) constraint is of the form \( a_i x = b_i \), where \( a_i \) denotes the \( i^{th} \) row of \( A \), can be allowed for by including the additional line \( a_i x \leq b_i \), or by solving the equality for one of the variables and substituting it out, or (in the context of this book) by projecting it out - which may on occasions be the computationally most stable approach.

The author has become a little tired of non-orthogonal definitions, so "optimal" here means what he reckons it should mean (optimal of course) that is a feasible vector which is the best possible.

There are a number of formulations of the LP problem (including the one above) which can be shown to be equivalent:

\[
\begin{align*}
\text{max } & \{ c^T x : A x \leq b \} & \quad (a) \\
\text{min } & \{ c^T x : A x \geq b \} & \quad (b) \\
\text{max } & \{ c^T x : A x \leq b, x \geq 0 \} & \quad (c) \\
\text{min } & \{ c^T x : A x \geq b, x \geq 0 \} & \quad (d) \\
\text{max } & \{ c^T x : A x = b, x \geq 0 \} & \quad (e) \\
\text{min } & \{ c^T x : A^T x = b, x \geq 0 \} & \quad (f)
\end{align*}
\]
For details refer to [37, p. 91].

1.2 Historical Context

1.2.1 Origins

The origins of the linear programming problem date back to Johann Bernoulli in 1717, who was working on theoretical mechanics. He introduced the concept of a virtual velocity: Given a region \( R \), the vector \( y \) is a virtual velocity in the point \( x^* \in R \) if there exists a curve \( C \) in \( R \) starting in \( x \) such that the half line \( \{ x^* + \lambda y : \lambda \geq 0 \} \) is tangent to \( C \). Assuming that

\[
\text{for any virtual velocity } y, \\
\text{the vector } -y \text{ is also a virtual velocity}
\]  

(1.3)

the virtual velocity principle states that a mass point at \( x^* \) subject to force \( b \) is in equilibrium iff \( b^T y = 0 \) (refer to Figure 1.1).

\[
\begin{align*}
R & \quad \text{\textbullet} \quad x^* + \lambda y \\
\downarrow & \quad \downarrow \\
\text{C} & \quad \text{\textbullet} \quad b
\end{align*}
\]

Figure 1.1: Johann Bernoulli’s Virtual Velocity

Lagrange [24] observed that if the region is given by

\[
R = \{ x : f_1(x) = \cdots = f_n(x) = 0 \},
\]  

(1.4)

where \( f_i : \mathbb{R}^n \mapsto \mathbb{R} \) and the gradients \( \nabla f_i, i = 1, \ldots, n \) are linearly independent, then condition 1.3 is satisfied and the principle of virtual velocity can be written as

\[
\nabla f_1(x^*)^T y = \cdots = \nabla f_n(x^*)^T y = 0,
\]

where

\[
\nabla f_i = \begin{bmatrix}
\frac{\partial f_i}{\partial x_1} \\
\vdots \\
\frac{\partial f_i}{\partial x_n}
\end{bmatrix}.
\]

As was observed by Fourier [15], the condition \( Ax \leq b \) defines a convex set, while the objective function \( c^T x \) can be regarded as defining a set of parallel hyperplanes. The optimum solution has the
1.2. HISTORICAL CONTEXT

hyperplane from this set with the maximum value of $c^T x$ which osculates the convex set of feasible solutions.

Fourier defined the feasible region along the lines of Equation 1.4, but allowed inequality constraints. He generalized the principle of virtual velocity to:

$$A \text{ mass point in } x^* \text{ subject to force } b, \text{ whose position is restricted to } R, \text{ is in equilibrium iff } b^T y \leq 0.$$ 

Thus Fourier’s contribution to the analysis of optimization problems goes beyond simply contributing directly in the form of a solution algorithm, as he extended the work of Johann Bernoulli and Lagrange on optimization subject to continuously differentiable constraints to the case where a function is subject to inequality constraints. He also pioneered the link between the theory of inequalities and the theory of polyhedra, thus spanning the algebraic and geometric perspectives. Fourier appears to be the first to suggest the geometric idea of navigating between vertices along the edges of the convex polyhedron associated with the feasible solution set, until an optimum is reached.

The first solution of the linear programming problem is also due to Fourier; it is now known as Fourier-Motzkin elimination, and is of considerable analytic importance as a generalization of the procedure is used as a tool for proving results leading to Farkas’ Lemma, and the transposition theorems of Gordan and Stiemke [23]. (Refer also to [38, Ch. 1])

Poussin in the years 1910-11 designed the algebraic analogue of Fourier’s geometric method for solving the linear programming problem [30]. Dantzig in 1951 gave this algebra an efficient tableau form and called it the Simplex Method [11].

1.2.2 Inefficiency of the Simplex Method

The branch of mathematics which considers the efficiency of algorithms is called complexity analysis; we introduce a few simple notions from this branch at this point.

The basic arithmetic operations are taken to be addition, multiplication, and division. If the number of basic arithmetic operations required for an algorithm to compute the answer to a problem increases as a polynomial in the parameters which describe the size of the problem, then the problem is said to be polynomial-time.

Under certain conditions the Simplex Method is not efficient - it is possible for the current solution to do a “tour” of many or even all of the vertices of the polyhedron defined by the feasible set of solutions [22], or to cycle repeatedly through a sequence of vertices without reaching an optimum. In fact the Simplex Method is not a polynomial-time algorithm, and so in recent times (the last forty years or so) there has been research into new algorithms. To be fair, the problem of cycling can be removed, and the non polynomial-time nature appears not to affect certain “average” problems.
CHAPTER 1. INTRODUCTION

Through this renewed research a number of algorithms have been developed which are polynomial-time, including Khachian’s ellipsoid method \[21\], Karmarkar’s simplex method \[20\], and Renegar’s centreing algorithm \[34\].

In summary, interest in the LP problem has not abated, for a number of reasons, which are detailed:

1. The non polynomial-time nature of the Simplex Method: This classical and most common method of solution, is now known to be an inefficient algorithm, due to the publication in 1972 of an example where the algorithm tours all the vertices of the feasible set, showing that it is not a polynomial-time algorithm \[22\]. The desire to solve larger problems, and this discovery of deficiency in the Simplex Method have lead to new efforts to find efficient ways of solving LPs, and to a resurgence in analysis of the problem.

2. The ability to represent a number of OR problems as LP’s - for example an important early problem in OR, the transportation problem, can be written as a linear program.

3. The advent of parallel processing machines and the possibility that new algorithms may be suitable for hybrid analog-digital machines.

1.3 Applications of Linear Programming

The formulation of practical problems as linear programs dates back at least to work by G.B. Dantzig in 1947. He published his work Programming in a Linear Structure the following year. Actual formulation of problems as linear programs can confidently be assumed to date back to at least the Second World War.

Common examples of problems which can be formulated as LP’s are now given.

1.3.1 The Product Mix Problem

Mashino Corporation produces mountain bikes in \(n\) variants \(v_1, \ldots, v_n\). Each variant sells at a fixed price, leading to profits \(s_1, \ldots, s_n\). To produce one unit of variant \(i\), \(l_i\) man hours of labour are needed and \(m_i\) resource units.

The corporation is constrained in that it has only \(l\) man-hours available, and can deliver at most \(m\) units of material per hour.

We see that labour required to produce \(x_i\) units of variant \(i\) is \(l_i x_i\), so total labour requirement is \(l_1 x_1 + \cdots + l_n x_n \leq l\). Similarly material requirement is \(m_1 x_1 + \cdots + m_n x_n \leq m\), and profit is \(s_1 x_1 + \cdots + s_n x_n\).
The company wishes to maximize profit, so the problem can be written concisely as

Find \( x_1, \ldots, x_n \) which maximize \( s_1 x_1 + \cdots + s_n x_n \) subject to

\[
l_1 x_1 + \cdots + l_n x_n \leq l \quad \text{and} \quad m_1 x_1 + \cdots + m_n x_n \leq m,
\]

where \( x_1, \ldots, x_n \geq 0 \).

Setting

\[
\begin{bmatrix}
  x_1 \\
  \vdots \\
  x_n
\end{bmatrix} = x,
\begin{bmatrix}
l_1 & \cdots & l_n \\
m_1 & \cdots & m_n
\end{bmatrix} = A,
\begin{bmatrix}
l \\
m
\end{bmatrix} = b,
\begin{bmatrix}
s_1 \\
\vdots \\
s_n
\end{bmatrix} = c,
\]

shows that the problem 1.5 can be formulated as a linear program.

A similar problem involves arriving at the most economical purchase decision for animal feed, given the dietary requirements of the animals and the prices and composition of the feed brands.

### 1.3.2 The Transportation Problem

The transportation problem can be regarded as the problem of minimizing the cost of transporting a commodity from \( m \) sources to \( n - 1 \) destinations, where the cost of moving a commodity unit from source \( i \) to destination \( j \) is \( b_{ij} \) currency units. The problem was formulated by F.H. Hitchcock in 1941 [17]. The reader may refer to [33, Ch. 3] for the mathematical description of, and classical solutions to this problem.

From an economic viewpoint it is not necessary to assume that total supply is equal to total demand as one may exceed the other in a disequilibrium situation. With \( w_{ij} \) commodity units transported from source \( i \) to destination \( j \) we have the unbalanced transportation problem

\[
\text{minimize} \quad \sum_{i=1}^{m} \sum_{j=1}^{n-1} b_{ij} w_{ij} \quad \text{subject to}
\]

\[
\sum_{j=1}^{n-1} w_{ij} \leq s_i \quad i = 1, \ldots, m \quad \text{(supply constraints)}
\]

\[
\sum_{i=1}^{m} w_{ij} \geq d_j \quad j = 1, \ldots, n - 1 \quad \text{(demand constraints)}
\]

\( w_{ij} \geq 0 \quad \forall i, j. \)

If supply exceeds demand then a dummy destination (with index \( j = n \)) is introduced with demand equal to the excess, then the condition \( \sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j \) will obtain, and the system can be written as

\[
\text{minimize} \quad \sum_{i=1}^{m} \sum_{j=1}^{n} b_{ij} w_{ij} \quad \text{subject to}
\]

\[
\sum_{j=1}^{n} w_{ij} = s_i \quad i = 1, \ldots, m \quad \text{(supply constraints)}
\]

\[
\sum_{i=1}^{m} w_{ij} = d_j \quad j = 1, \ldots, n \quad \text{(demand constraints)}
\]

\( w_{ij} \geq 0 \quad \forall i, j, \)

which is the standard transportation problem. If demand exceeds supply we add a dummy source with supply equal to the excess demand.
Writing

\[ A^T_\tau = \begin{bmatrix} I_m & \otimes & I_n \\ 1_m & \otimes & I_n \end{bmatrix}, \quad b = \begin{bmatrix} b_{11} \\ \vdots \\ b_{1n} \\ \vdots \\ b_{m1} \\ \vdots \\ b_{mn} \end{bmatrix}, \quad c = \begin{bmatrix} s \\ d \end{bmatrix}, \quad and \quad w = \begin{bmatrix} w_{11} \\ \vdots \\ w_{1n} \\ \vdots \\ w_{m1} \\ \vdots \\ w_{mn} \end{bmatrix}, \]

where \( \mathbf{1}_i \) is an \( i \times 1 \) vector of 1’s and \( \otimes \) is the Kronecker product (considered in detail in Section 2.2.6), it can be seen that the transportation problem has the representation as a linear program of the form

\[
\text{minimize } b^T w \text{ s.t. } A^T_\tau w = c, \quad w \geq 0,
\]

which is a formulation of type 1.2 (f).
Chapter 2

Background Theory

2.1 Function Pairs

A function is said to be surjective, or onto, if its image is equal to its codomain; a function is said to
be bijective if it is one-to-one and onto - that is 1:1 and surjective. Two functions are said to be a
function-pair if the domain of each is the codomain of the other; a function-pair, say \{t, u\}, is said to be
a \textit{regular} function-pair if \(tut = t\) and \(utu = u\). \(^1\)

Given a function-pair \(\{t, u\}\), with \(t : V \rightarrow W, u : W \rightarrow V, v \in V\) and \(w \in W\), elements of the form \(v = wu\)
or \(w = vt\) are called \textit{central}; sets comprising central elements are called central.

\textbf{Definition 1} \hspace{1em} An element is \textit{central} in the domain of one function of a function-pair if the element is
in the range of the other function of the pair.

Functions of the form \(t_c = t|_V\) with codomain \(Vt\), and \(u_c = u|_W\) with codomain \(Wu\) are called \textit{central}
functions, and function-pairs of the form \(\{t_c, u_c\}\) are called \textit{central function-pairs}.

\textbf{Lemma 2.1.1} \hspace{1em} (a) \(t_c\) is bijective,
(b) \(u_c\) is bijective,
(c) \(V_c = W_c u\),
(d) \(W_c = V_c t\), and
(e) \(t_c\) and \(u_c\) are mutual inverses.
(f) \(v \in V\) is central \iff \(v = wu\); \(V_c = Vtu\),
(g) \(w \in W\) is central \iff \(w = ut\); \(W_c = Wut\),

\begin{proof}
(a) \(t_c\) is 1:1: \((v_1, v_2 \in Wu) \land (v_1t = v_2t) \Rightarrow (v_1 = w_1u) \land (v_2 = w_2u) \land (v_1t = v_2t)\)
\Rightarrow \((v_1 = w_1u) \land (v_2 = w_2u) \land (w_1t = w_2ut) \Rightarrow (v_1 = w_1u) \land (v_2 = w_2u) \land (w_1utu = w_2utu)\)
\Rightarrow \((v_1 = w_1u) \land (v_2 = w_2u) \land (w_1u = w_2u) \Rightarrow v_1 = v_2\) ; \(t_c\) is onto \(W_c\) : \(\text{image}(t_c) = V_c t_c = V_c t \subseteq Vt = W_c = Vt \subseteq Wtu = V_c t = V_c t_c = \text{image}(t_c)\), that is \(\text{image}(t_c) = W_c\), so \(t_c\) is bijective. (b)
\end{proof}

\(^1\)This usage is motivated by semigroup nomenclature. \[10\]
The proof is similar to (a), (c) $V_c = Wu = Wtu \subseteq Vtu = Wc \subseteq Wu = V_c$, so $V_c = Wc$, (d) the proof is similar to (c). (e) For $v \in V_c, v = wu \exists w \in W \Rightarrow vt_cu_c = wtu_c = wtu_c = wtu = wu = v$, so $t_cu_c$ is the identity map on $V_c$, similarly it is found that $u_c t_c$ is the identity map on $W_c$, so $t$ and $u$ are mutual inverses. (f) $v$ is central $\iff v = wu \exists w \in W \Rightarrow (vt = wtu) \land (v = wu) \exists w \in W \Rightarrow (vtu = wtu = wu) \land (v = wu) \exists w \in W \Rightarrow v = vtu \Rightarrow v = wu$ where $w = vt \iff v$ is central, that is $v$ is central iff $v = vtu$. (g) proof is exactly similar to that of (f) \hfill \square

Lemma 2.1.2
Given the regular function-pair \{f, \bar{f}\}
with $f : X \to X$ and $\bar{f} : X \to X$,
(a) $x$ is central iff $x = xf$.
(b) $x$ is central iff $x = x\bar{f}f$.

Proof: (a) $x$ is central $\iff x = xf \exists \bar{x} \in X \Rightarrow (xf = x\bar{f}f) \land (x = x\bar{f}) \exists \bar{x} \in X \Rightarrow (xf = x\bar{f}f) \land (x = x\bar{f}) \exists \bar{x} \in X \Rightarrow x = xf \Rightarrow x = x\bar{f}$ where $\bar{x} = xf \iff x$ is central, that is $x$ is central iff $x = xf$. (b) proof is similar. \hfill \square

In the following lemma we are interested in conditions on the domains of a regular function-pair, where the image under either function of any element satisfying the condition corresponding to the domain of the function satisfies the condition corresponding to the domain of the other function of the pair.

Lemma 2.1.3
Given the regular function-pair \{f, \bar{f}\}
with $f : X \to X$ and $\bar{f} : X \to X$,
$X_1 \subseteq X$, and $\bar{X}_1 \subseteq X$,
with $X_1f \subseteq X_1$, $\bar{X}_1 \bar{f} \subseteq X_1$,
(a) $(X_1 \cap Xf)f = X_1 \cap Xf$
(b) $(X_1 \cap Xf)\bar{f} = X_1 \cap \bar{X}f$
(c) $f_1 = f : X_1 \cap Xf \to X_1 \cap Xf$ is a bijection
(d) $\bar{f}_1 = \bar{f} : \bar{X}_1 \cap Xf \to \bar{X}_1 \cap \bar{X}f$ is a bijection
(e) $f_1$ and $\bar{f}_1$ are mutual inverses.

Proof: (a) and (b) $(X_1 \cap Xf)f \subseteq X_1f \cap Xf \subseteq X_1f \cap Xf \subseteq X_1 \cap Xf,$
that is

$$(X_1 \cap Xf)f \subseteq X_1 \cap Xf, \tag{2.1}$$

and similarly

$$(X_1 \cap Xf)\bar{f} \subseteq X_1 \cap \bar{X}f, \tag{2.2}$$

So

$$(X_1 \cap Xf)f \subseteq (X_1 \cap Xf)f, \tag{2.3}$$

and

$$(X_1 \cap Xf)\bar{f} \subseteq (X_1 \cap Xf)\bar{f}, \tag{2.4}$$
Now \( f \) and \( f' \) are 1:1 on \( \mathcal{X}_f \) and \( Xf \) respectively, so

\[
X_1 \cap \mathcal{X}_f \subseteq (X_1 \cap Xf)f, \tag{2.5}
\]

and

\[
\mathcal{X}_1 \cap Xf \subseteq (X_1 \cap \mathcal{X}_f)f, \tag{2.6}
\]

From (2.1) and (2.6)

\[
(X_1 \cap \mathcal{X}_f)f = X_1 \cap Xf, \tag{2.7}
\]

and from (2.5) and (2.2)

\[
(X_1 \cap Xf)f = X_1 \cap \mathcal{X}_f, \tag{2.8}
\]

(c) that \( f_1 \) is well defined follows from (a), that it is 1:1 follows from it being a restriction of \( f \), that it is onto also follows from (a), so it is a bijection. (d) the proof is similar to that of (c). (e) follows from \( f \) and \( f' \) being mutual inverses. \( \square \)

### 2.2 Matrices

#### 2.2.1 The Identity

A right identity \( I_r \) satisfies \( XI_r = X \) for all \( X \), while a left identity satisfies \( I_lX = X \) for all \( X \). So if we have a left identity \( L \) and a right identity \( R \) then \( LR = R \) and \( LR = L \), so \( L = R \). Thus the matrix \( I \) with 1’s on its diagonal and zeros elsewhere is the unique identity.

#### 2.2.2 The Inverse

The matrix \( R \) is a right inverse of \( A \) if

\[
AR = I \tag{2.9}
\]

The matrix \( L \) is a left inverse of \( A \) if

\[
LA = I \tag{2.10}
\]

From Equation 2.9 we have \( LAR = L \) and from Equation 2.10 we have \( LAR = R \) and so \( L = R \), thus if a matrix \( A \) has both a left and a right inverse then it has a unique inverse which we denote by \( A^{-1} \) and we say that \( A \) is regular. Note that for \( A \) and \( B \) regular \( (B^{-1}A^{-1})AB = B^{-1}(A^{-1}A)B = B^{-1}B = I \) and \( AB(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AA^{-1} = I \), so \( (AB)^{-1} = B^{-1}A^{-1} \) for \( A \) and \( B \) regular. Thus the set of regular matrices is closed under multiplication.
2.2.3 Involution

With $\overline{\alpha}$ denoting the complex conjugate of $\alpha$ - that is, for $a, b \in \mathbb{R}, a + ib = a - ib$, the matrix transpose, or more generally, an involution, $*$, satisfies

1. $(A^*)^* = A$,
2. $(A + B)^* = A^* + B^*$,
3. $(AB)^* = B^* A^*$,
4. $(\alpha A)^* = \overline{\alpha} A^*$.

A matrix $X$ is said to be Hermitian if $X^* = X$ [35, p. 178]; a matrix $X$ is said to be idempotent if $X^2 = X$. A matrix $X$ is said to be unitary if $XX^* = X^*X = I$.

The involution used from this point in this book is the matrix transpose, denoted by a superscript $T$.

We identify a column vector with a matrix having only one column; we define $\|x\|^2 = x^T x$.

**Lemma 2.2.1** Given $\|Xy\|^2 = \|y\|^2$, $y = Xy \Leftrightarrow y^T X y = y^T y$.

Proof: $y = Xy \Leftrightarrow \|y - Xy\|^2 = 0 \Leftrightarrow (y - Xy)^T (y - Xy) = 0 \Leftrightarrow y^T y - y^T X y - y^T X^T y + y^T X^T X y = 0 \Leftrightarrow y^T y - 2y^T X y + y^T y = 0 \Leftrightarrow 2y^T y - 2y^T X y = 0 \Leftrightarrow y^T y - y^T X y = 0 \Leftrightarrow y^T X y = y^T y$. \(\Box\)

2.2.4 Unitary and Semi-Unitary Matrices

The matrix $X$ is said to be unitary if $XX^T = X^T X = I$.

The unitary matrices form a group under matrix multiplication since

$(XY)(XY)^T = XYY^T X^T = XX^T = I$, and

$(XY)^T (XY) = Y^T X^T X Y = Y^T Y = I$.

A matrix $X$ is said to be semi-unitary w.r.t. an Hermitian idempotent $Q$ if $XX^T = X^T X = Q$; $X$ is said to be $Q$-unitary. \(\footnote{Naimark [25, p. 111-112] introduces the similar notion of partially isometric operators.} \)

**Lemma 2.2.2**

(a) $X$ is $Q$-unitary $\Leftrightarrow X^T$ is $Q$-unitary.

(b) $X$ is $Q$-unitary $\Rightarrow XQ = QX = X$.

(c) The set of $Q$-unitary matrices of the same dimension is closed under matrix multiplication.
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(a) Obvious.
(b) \( X \) is \( Q \)-unitary \( \Rightarrow \) \( XX^T = X^TX = Q \Rightarrow (XX^T X = QX) \land (X^TX = Q) \)
\( \Rightarrow XQ = QX; (QX - X)(QX - X)^T = (QX - X)(XTQ - X^T) \)
\( = (QX - X)(X^T - X^T) = QXX^TQ - QXX^T - XXTQ + XXT \)
Proof:
\( = Q^3 - Q^2 + Q = Q - Q + Q = 0 \Rightarrow (QX - X)(QX - X)^T = 0 \)
\( \Rightarrow QX = X. \)
(c) Suppose \( X_1 \) and \( X_2 \) are \( Q \)-unitary matrices of the same dimension,
then \( (X_1X_2)^T(X_1X_2) = X_2^TX_1^TX_1X_2 = X_2^TQX_2 = X_2^T X_2 Q = Q^2 = Q, \)
and \( (X_1X_2)(X_1X_2)^T = X_1X_2X_2^TX_1^T = X_1QX_1^T = QX_1X_1^T = Q^2 = Q. \)

2.2.5 The Moore-Penrose Pseudo-Inverse

The Moore-Penrose pseudo-inverse is used extensively in the following chapters. Given a matrix \( X \),
define
\[
X^+ = \lim_{\delta \to 0} \{X^T(XX^T + \delta^2I)^{-1}\}.
\]

It can be shown that \( X^+ \) is well-defined [4, p. 19, Theorem 3.4]; it is called the Moore-Penrose pseudo-inverse (MPPI) of \( X \).

A non-constructive definition is as follows [4, p. 28, Theorem 3.9]:
\( Y = X^+ \) if and only if
\[
\begin{align*}
XYX &= X, \\
YXY &= Y, \\
(XY)^T &= XY, \text{ and } \\
(YX)^T &= YX.
\end{align*}
\]

The characterization given by Equation Set 2.11 is much used for verifying that the inverse has been correctly computed.

2.2.5.1 Properties

If the Moore-Penrose inverse of a matrix exists then it is unique, since if we assume \( X \) and \( Y \) are
two Moore-Penrose inverses of \( Z \), then \( X = XZX = XZX = XXTZT = XX^T(ZTY^TZ^T) = X(X^TZT)(Y^TZ^T) = X(ZX)^T(ZY) = X(ZX)(ZY) = (XXZ)Y = XYZ = XYZY \\
= (XZ)^T(YZ)Y = (XZ)^T(YZ)Y = ZTX^TZY^T = (Z^T X^TX^T)Y^T \)
\( = (ZX)^TZ^TY^T = Z^TY^T = (YZ)^TY = YZY = Y. \)

Note that if a matrix \( X \) has an inverse then the inverse is the MPGI since 1. \( XX^{-1}X = X(X^{-1}X) = XI = X. \)
2. \( X^{-1}XX^{-1} = X^{-1}(XX^{-1}) = X^{-1}I = X^{-1}. \)
3. \( XX^{-1} = I \) which is symmetric.
4. \( X^{-1}X = I \) which is symmetric.

\textbf{Lemma 2.2.3} If \( X^+ \) exists then \( X^T+ \) also exists and \( X^T+ = X^+T. \)
Proof: 1. \(X^TX^+X^T = (XX^+)^T = X^T\).
2. \(X^TX^TX^+T = (X^+XX^+)T = X^+T\).
3. \((X^TX^+)T = X^+X = (X^+)^T = X^TX^+T\).
4. \((X^+TX^)T = XX^+ = (XX^+)T = X^+TX^T\), so \(X^T = X^+T\). \(\square\)

**Lemma 2.2.4** \((XX^T)^+ = X^TX^+\).

1. \((XX^T)(XX^+X^+) = X(X^TX^+X^+) = X(X^TX^+)X^+ = X^+X^X^+\) which is symmetric.
2. \((X^+X^+)(XX^T) = X^TX^+X^X^+ = X^TX^+X^+\) which is symmetric.

Proof:
3. From 1 we have \((XX^T)(XX^+^+) = XX^+
\Rightarrow (XX^T)(XX^+X^+) = (XX^+)(XX^T) = (XX^+X)X^T = XX^T.
4. Again from 1 we have \((XX^T)(XX^+X^+) = XX^+
\Rightarrow (XX^T)(XX^+X^+) = (XX^+X^+)(XX^+) = X^T(X^+X^+) = X^+X^+\). \(\square\)

**Lemma 2.2.5**

(a) \((XX^T)^+X^T = X^+T\).
(b) \((XX^T)^+X = X^+\).
(c) \(X(X^T)^+ = X^+\).
(d) \(X^T(XX^T)^+ = X^+\).

Proof:
(a) \((XX^T)^+X^T = X^+X^T + X^T = X^+(XX^+)^T = X^+X^X^+ = X^+\).
(b) Replace \(X\) with \(X^T\) in result (a).
(c) Transpose result (a).
(d) Transpose result (b). \(\square\)

**Lemma 2.2.6**

(a) \(X^TXX^+ = X^T\).
(b) \(X^+XX^T = X^T\).
(c) \(XX^+X^T = X^T\).

Proof:
(a) \(X^TXX^+ = (X^TX)^T(T) = [(XX^+)^T]T = (XX^+)^T = X^T\).
(b) \(X^+XX^T = (X^+)^T = (X^+X)^T = X^T\).
(c) \(XX^+X^T = (X^+X)^T = (X^+X)^T = X^+X^T = X^T = X^T\). \(\square\)

**Lemma 2.2.7** \(X^+Y = 0 \iff X^TY = 0\).

\(X^+Y = 0 \Rightarrow XX^+Y = 0 \Rightarrow (XX^+)^T = X^TY = 0 \Rightarrow X^TX^TY = 0 \Rightarrow X^TY = 0 \Rightarrow X^TY = 0\).

Proof:
\(X^TY = 0 \iff X^TY = 0 \iff X^Ty = 0 \iff X^+X^+Y = 0 \iff X^+Y = 0 \iff X^TY = 0 \iff X^T = X^T = X^T\). \(\square\)

**Lemma 2.2.8** \(X\) is idempotent symmetric iff \(I - X\) is idempotent symmetric.

Proof: If \(X\) is idempotent symmetric then \((I - X)^T = I^T - X^T = I - X\) and \((I - X)^2 = I - 2X + X^2 = I - 2X + I = I - X\), so \(I - X\) is idempotent symmetric; since \(I - (I - X) = X\) the converse holds. \(\square\)

**Lemma 2.2.9** \(((I - XX^+)^T)^+X = 0\).
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\[ ((I - XX^+) Y)^+ X = ((I - XX^+) Y) + ((I - XX^+) Y)((I - XX^+) Y)^+ X \]
\[ = ((I - XX^+) Y)^+ ((I - XX^+) Y)((I - XX^+) Y)^T X \]
\[ = ((I - XX^+) Y)^+ ((I - XX^+) Y)^T((I - XX^+) Y)^T X \]

Proof: \[ ((I - XX^+) Y)^+ ((I - XX^+) Y)^T Y^T (I - XX^+) X \]
\[ = ((I - XX^+) Y)^+ ((I - XX^+) Y)^T Y^T (X - XX^+ X) \]
\[ = ((I - XX^+) Y)^+ ((I - XX^+) Y)^T Y^T (X - X) = 0 \]

2.2.5.2 Existence

This leaves existence to be established, and to this end we establish: any symmetric matrix has an MPGI, since such a matrix can be written in the form \( X = Y D Y^{-1} \) where \( Y^{-1} = Y^T \) and \( D \) is a diagonal matrix, in which case \( X^\dagger = Y D Y^{-1} \) (how do we form the MPGI of a diagonal matrix?) is the MPGI of \( X \) since

1. \( XX^\dagger X = (YD^{-1})(YD^T Y^{-1})(YD^{-1}) = YD(Y^{-1}Y)D^+(Y^{-1}Y)DY^{-1} = YDD^+DY^{-1} = YDY^{-1} = X \),
2. \( X^\dagger XX^\dagger = (YD^+Y^{-1})(YD^{-1})(YD^+Y^{-1}) = YD^+(Y^{-1}Y)D(Y^{-1}Y)D^+Y^{-1} = YD^+DD^+Y^{-1} = YD^+Y^{-1} = X^\dagger \),
3. \( X^\dagger = (YD^{-1})(YD^+Y^{-1}) = YD(Y^{-1}Y)D^+Y^{-1} = YDD^+Y^{-1} = YD^+DD^+Y^T \),
4. \( X^\dagger X = (YD^+Y^{-1})(YD^{-1}) = YD^+(Y^{-1}Y)DY^{-1} = YD^+DY^{-1} = YD^+DY^T \),

which is symmetric.

So \((X^T X)^+\) exists and we have existence of the MPGI for arbitrary matrices.

2.2.5.3 Geometric Insights

The matrix \( XX^+ \) is symmetric and idempotent as, from the definition of the pseudo-inverse, \((XX^+)^2 = XX^+XX^+ = XX^+ \) (since \( XX^+X = X \)), while property (c) above asserts the symmetry of \( XX^+ \).

Regarded as a linear transformation, \( X^+X \) is the orthogonal projection onto the row space \( \mathcal{L}(X) \) of \( X \), while \( XX^+ \) is the orthogonal projection onto the column space \( \mathcal{R}(X) \) of \( X \).

We say that a projection matrix \( P \) whose domain is vector space \( V \) is a projection if it satisfies

\[ P^2 v = P v \quad \forall v \in \mathcal{V} \]

From this it follows that \( P^2 \mathcal{V} = P \mathcal{V} \).

We define a partial order on projections with the same domain as follows: Given projections \( P \) and \( Q \) with the same domain \( \mathcal{D} \), \( P \geq Q \) if the image of \( P \) is a superset of the image of \( Q \) - that is

\[ P \geq Q \iff P \mathcal{D} \supseteq Q \mathcal{D} \].
Lemma 2.2.10  Given projections $P$ and $Q$, if $P \geq Q$ then $PQ$ and $QP$ are projections.

Proof: $P \geq Q \Rightarrow PV \supseteq QV \
\Rightarrow \forall v \in V \exists v₁ s.t PV₁ = Qv \
\Rightarrow \forall v \in V \exists v₁ s.t PPv₁ = PQv and QPV₁ = QQv \
\Rightarrow \forall v \in V \exists v₁ s.t PQV₁ = QPV and QPV₁ = Qv \
\Rightarrow PQ = Q \Rightarrow PQP = PQ \land QQP = QP$

Lemma 2.2.11  If $P \geq Q$ then $PQ = QP = Q$.

Proof: $P \geq Q \Rightarrow PD \supseteq QD \
\Rightarrow QDP \supseteq QD \
\Rightarrow \forall v \in V \exists v₁ s.t QPV₁ = QPV and QPV₁ = Qv \
\Rightarrow PQ = Q \Rightarrow PQP = PQ \land QQP = QP$

2.2.5.4  Computation

The following result is required to derive the pseudo-inverse of matrices which occur in Chapters 5.A and Chapter 3.3.2:

If $C_{m+1} = [C_m|c_{m+1}]$ then

(a) $C_{m+1}^+ = \left[ \frac{C_m^+[I - C_mC_m^+k_{m+1}^T]}{k_{m+1}^T} \right]$,

where

(b) $k_{m+1} = \frac{(I - C_mC_m^+)c_{m+1}}{\|I - C_mC_m^+c_{m+1}\|^2}$,

if $(I - C_mC_m^+)c_{m+1} \neq 0$,

(c) $C_m^+ = \frac{C_m^+[C_m^+c_{m+1}]}{1 + \|C_m^+c_{m+1}\|^2}$,

otherwise.

Proof: By verification of the four conditions above (refer to Albert [4]).

2.2.5.5  The Invariant Framework

The major use of the variables detailed here is the construction of the invariant problems in Chapter 4

Define

(a) $\mathfrak{A} = AA^+$, and
(b) $b = (I - AA^+)b$,
(c) $c = A^+c$,
(d) $\mathfrak{D} = I - AA^+$.
Note that $\mathfrak{A}$ and $\mathfrak{D}$ are symmetric idempotents in view of the latter two non-constructive conditions for the pseudo-inverse. It is a straightforward matter, using the definitions above and the nonconstructive characterization of the Moore-Penrose pseudo-inverse, to show that

\[
\begin{align*}
\mathfrak{A}^T &= \mathfrak{A} \quad (a) & \mathfrak{A}b &= 0 \quad (f) \\
\mathfrak{A}^2 &= \mathfrak{A} \quad (b) & \mathfrak{D}b &= b \quad (g) \\
\mathfrak{D}^T &= \mathfrak{D} \quad (c) & \mathfrak{A}c &= c \quad (h) \\
\mathfrak{D}^2 &= \mathfrak{D} \quad (d) & \mathfrak{D}c &= 0 \quad (i) \\
\mathfrak{AD} &= \mathfrak{DA} = 0 \quad (e) & b^Tc &= 0 \quad (j)
\end{align*}
\]

\section*{2.2.6 The Kronecker Product}

The results of this section are needed for the analysis of the transportation model, which was begun in Section 1.3.2 and will be continued in the following chapters.

Given an $m \times n$ matrix $X = (x_{ij})$ and a $p \times q$ matrix $Y = (y_{k\ell})$, the Kronecker product $X \otimes Y$ is defined by the $mp \times nq$ matrix

\[
(X \otimes Y)_{(i-1)p+k,(j-1)q+\ell} = x_{ij}y_{k\ell},
\]

that is

\[
X \otimes Y = \begin{pmatrix}
x_{11}Y & \cdots & x_{1n}Y \\
\vdots & \ddots & \vdots \\
x_{m1}Y & \cdots & x_{mn}Y
\end{pmatrix}
\]

From [29, p. 919] we have, for conformable matrices,

1. $\otimes$ is associative,
2. $X \otimes (Y + Z) = X \otimes Y + X \otimes Z$,
3. $(W \otimes X)(Y \otimes Z) = (WY) \otimes (XZ)$,
4. $(X \otimes Y)^T = X^T \otimes Y^T$,
5. $(X \otimes Y)^T = X^T \otimes Y^T$,
6. For $X$, $Y$ square, non-singular, $(X \otimes Y)^{-1} = X^{-1} \otimes Y^{-1}$,
7. For $X$, $Y$ square, $\text{trace}(X \otimes Y) = (\text{trace}X)(\text{trace}Y)$,
8. $\text{rank}(X \otimes Y) = (\text{rank}X)(\text{rank}Y)$, and
9. Each of the sets of normal, Hermitian, positive definite, and unitary matrices is closed under the Kronecker product.

\textbf{Lemma 2.2.13} \((X \otimes Y)^+ = X^+ \otimes Y^+\).
Proof: We verify the four conditions for the pseudo-inverse:

1. \((X \otimes Y)(X^+ \otimes Y^+)(X \otimes Y) = (XX^+) \otimes (YY^+)(X \otimes Y) = (XX^+X) \otimes (YY^+Y) = X \otimes Y. \)

2. \((X^+ \otimes Y^+)(X \otimes Y)(X^+ \otimes Y^+) = (X^+X \otimes Y^+Y)(X^+ \otimes Y^+) = (X^+XX^+ \otimes Y^+YY^+) = X^+ \otimes Y^+. \)

3. \([((X \otimes Y)(X^+ \otimes Y^+))^T = (XX^+ \otimes (YY^+))^T = (XX^+)^T \otimes (YY^+)^T = (XX^+) \otimes (YY^+) = (X \otimes Y)(X^+ \otimes Y^+). \)

4. Similar to 3. \(\Box\)

2.2.7 Applications

2.2.7.1 Solution of Linear Equations

The idea which is implicit here is that if a concise solution of the general linear equation appears to require the pseudo-inverse, then perhaps a concise (and precise) solution of the linear programming problem might also require the pseudo-inverse.

The general linear equation can be written in matrix form as

\[ Ax = b \]

where \(A\) is an \(m \times n\) matrix, \(x\) is an \(n \times 1\) matrix (that is an \(n\)-dimensional column vector) and \(b\) is an \(m \times 1\) matrix (that is an \(m\)-dimensional column vector). Now we write

\[
A = \begin{bmatrix}
a_1 & a_2 & \cdots & a_n
\end{bmatrix}
\]

so the above equation becomes

\[
\begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix},
\]

that is

\[ x_1a_1 + x_2a_2 + \cdots + x_na_n = b. \]

Thus we are asking whether there is a linear combination of the vectors \(a_1\) to \(a_n\) equal to \(b\).

Alternatively: does \(b\) lie in the space generated by \(a_1\) to \(a_n\)? There are two possibilities

1. \(AA^+b = b\), that is \(b\) is in the space generated by \(a_1\) to \(a_n\); further

   (a) \(A^+A = I\), in which case \(a_1\) to \(a_n\) are linearly independent and there is only one solution
   \[ x_s = A^+b = (A^TA)^{-1}A^Tb \] (the inverse exists!) or
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(b) $A^+A \neq I$, in which case $a_1$ to $a_n$ are linearly dependent and there are multiple solutions of the form

$$x_s = A^+b + \mathcal{N}(A) = A^+b + (I - A^+A)y, y \in \mathbb{R}^n.$$  

2. $AA^+b \neq b$, that is $b$ is not in the space generated by $a_1$ to $a_n$ in which case there are no solutions.

It is, however, possible to obtain a regression solution $\hat{x} = A^+b$ for which $A\hat{x}$ is close to $b$ - an approach which dates back to Gauss.

Case 1: $(AA^+)b = b \Rightarrow A(A^+b) = b$, so $A^+b$ is certainly a solution. Further, if $y$ is any other solution then $Ay = b \Rightarrow AA^+Ay = AA^+b \Rightarrow Ay = AA^+b \Rightarrow A(y - A^+b) = 0 \Rightarrow A^+A(y - A^+b) = 0 \Rightarrow (I - A^+A)(y - A^+b) = y - A^+b \Rightarrow (I - A^+A)y = y - A^+b \Rightarrow y = A^+b + (I - A^+A)y = y \in A^+b + (I - A^+A)z, z \in \mathbb{R}^n$. Conversely, $y = A^+b + (I - A^+A)z, z \in \mathbb{R}^n \Rightarrow Ay = AA^+b + A(I - A^+A)z, z \in \mathbb{R}^n \Rightarrow Ay = AA^+b \Rightarrow Ay = b$. Thus in this case (i.e. $(AA^+)b = b$) the solutions are precisely of the form $A^+b + (I - A^+A)z, z \in \mathbb{R}^n$. The exhaustive sub-cases (a) and (b) follow immediately.

Case 2. If there exists a solution $x$ then $Ax = b \Rightarrow (A^+Ax = A^+b) \land (Ax = b) \Rightarrow (AA^+Ax = AA^+b) \land (Ax = b) \Rightarrow (Ax = AA^+b) \land (Ax = b) \Rightarrow AA^+b = b$. Thus $AA^+b \neq b$ implies the problem has no solution. □

Specifically what is needed for our purposes is simply a summary of Case 1 above:

**Lemma 2.2.14** $Ax = b \Leftrightarrow x \in \{A^+b + (I - A^+A)y, y \in \mathbb{R}^n\}$

Proof: $Ax = b \Rightarrow Ax = b$ and $x = A^+Ax + (I - A^+A)x \Rightarrow x \in A^+b + (I - A^+A)\mathbb{R}^n$. □

Refer also to W. Kahan[19].

2.2.7.2 LP’s with Equality Constraints

Consider the problem

$$\begin{align*}
\text{maximize} & \quad c^T x \\
\text{subject to} & \quad A_1 x \geq b_1 \\
& \quad A_2 x = b_2
\end{align*}$$

Since all solutions will be of the form $A_2^+b_2 + (I - A_2^+A_2)y$ we can write the objective function as

$$c^T[A_2^+b_2 + (I - A_2^+A_2)y] = [(I - A_2^+A_2)c]^T y + c^TA_2^+b_2$$

and the inequality as

$$A_1[A_2^+b_2 + (I - A_2^+A_2)y] \geq b_1 \Rightarrow A_1A_2^+b_2 + A_1(I - A_2^+A_2)y \geq b_1$$

$$\Rightarrow A_1(I - A_2^+A_2)y \geq b_1 - A_1A_2^+b_2.$$
Thus the problem can be rewritten as

\[
\begin{align*}
\text{maximize} & \quad \left[(I - A_2^+ A_2) c\right]^T y + c^T A_2^+ b_2 \\
\text{subject to} & \quad [A_1(I - A_2^+ A_2)] y \geq b_1 - A_1 A_2^+ b_2.
\end{align*}
\]

If the problem is

\[
\begin{align*}
\text{maximize} & \quad c^T x \\
\text{subject to} & \quad A_1 x \geq b_1, \\
& \quad x \geq 0, \\
& \text{and} \quad A_2 x = b_2
\end{align*}
\]

then we obtain the LP

\[
\begin{align*}
\text{maximize} & \quad \left[(I - A_2^+ A_2) c\right]^T y + c^T A_2^+ b_2 \\
\text{subject to} & \quad [A_1(I - A_2^+ A_2)] y \geq b_1 - A_1 A_2^+ b_2 \text{ and} \\
& \quad -(I - A_2^+ A_2)] y \geq A_2^+ b_2.
\end{align*}
\]

### 2.3 The Gram-Schmidt Function

The aim of this section is to develop regression theory from an algebraic rather than geometric perspective for the reason that computers have been taught more algebra than geometry; in Appendix B.1 this perspective will be seen to facilitate the development of an algorithm for affine minimization.

The reader may notice that in this section life is made easier by the assumption that for the field of real numbers \( a/0 = 0 \). Note that by so doing we are in the company of Isabelle/HOL. The author’s advice to anyone who might question this assumption is to

1. within the context of the field axioms prove that \( a/0 = 0 \) is undecidable (or at least fail to prove that \( a/0 \neq 0 \)), reflect on the consequences of Gödel’s first incompleteness theorem, and then relax, or

2. prove that \( a/0 \) is false, then contact the author of this book and the Isabelle/HOL group, preferably providing dunce’s hats for us to wear.

#### 2.3.1 The Elementary Function

Our aim here is to make precise the vector function underlying Gram-Schmidt orthogonalization and to delineate some important properties of this function. We need this theory in Chapter 3.3.2 to develop an efficient algorithm for computing a solution to the fixed-point problem of Chapter 5.

Note that in Isabelle/HOL we denote the inner product of two column vectors \( x \) and \( y \) by \( x^T y = x^T y \), we denote the vector \( x \) scaled by the real number \( r \) by \( r \cdot x \), and we take \( a/0 = 0 \) for all \( a \in \mathbb{R} \), leaving the reader to satisfy himself that this arrangement is not inconsistent with field rules.
2.3. THE GRAM-SCHMIDT FUNCTION

2.3.1.1 Definition

Given vectors \( x \) and \( y \) from the same space, define the regression function \( \triangleright \) and residual (or Gram-Schmidt) function \( \triangleleft \) as follows:

\[
\begin{align*}
x \triangleright y &= yy^+x \\
x \triangleleft y &= (I - yy^+)x
\end{align*}
\]

Alternatively we have

\[
\begin{align*}
x \triangleright y &= x^T y \\
x \triangleleft y &= x - x \triangleright y
\end{align*}
\]

2.3.1.2 Properties

Lemma 2.3.1

\[
\begin{align*}
a. & \quad x \triangleleft 0 = x \\
b. & \quad 0 \triangleleft x = 0 \\
c. & \quad x \triangleleft x = 0 \\
d. & \quad (\lambda x) \triangleleft y = \lambda (x \triangleleft y),
\end{align*}
\]

Lemma 2.3.2

\[
\begin{align*}
(x \triangleleft y)^T (x \triangleright y) &= 0 .
\end{align*}
\]

Proof:

\[
\begin{align*}
g. & \quad x^T (y \triangleleft x) = x^T (I - xx^+)y = (x^T - x^T xx^+)y = (x^T - x^T) y = 0^T y = 0 . \\
h. & \quad x \triangleleft (y \triangleleft x) = (I - (y \triangleleft x)(y \triangleleft x)^+)x = (I - (I - xx^+)y((I - xx^+)y)^+)x \\
&= x - (I - xx^+)y((I - xx^+)y)^+x = x - (I - xx^+)y0 = x - 0 = x \\
i. & \quad x \triangleleft y = 0 \Rightarrow x - (x^T y/(y^T y))y = 0 \Rightarrow x = (x^T y/(y^T y))y \\
&\Rightarrow \exists \lambda \in \mathbb{R} \text{ s.t. } x = \lambda y \Rightarrow x \triangleleft y = (\lambda y) \triangleleft y = \lambda (y \triangleleft y) = \lambda 0 = 0 \Rightarrow x \triangleleft y = 0 .
\end{align*}
\]

Note that, in view of result (d), there will be elision in some of the following proofs as we will avoid the use of brackets when we mean \((\lambda \cdot x) \triangleleft y\).

Lemma 2.3.2

\[
(x \triangleleft y)^T (x \triangleright y) = 0 .
\]
Proof: $(x \triangleleft y)^T (x \triangleright y) = ((I - yy^+)x)^T(yy^+x) = x^T(I - yy^+)yy^+x = x^T(yy^+ - yy^+yy^+)x = x^T(yy^+ - yy^+)x = 0$, \hfill \Box

Figure 2.1 sums up the relationship between $x \triangleleft y$ and $x \triangleright y$.

Figure 2.1: Elementary Gram-Schmidt Function

We define $x \triangleleft [y, z] = (x \triangleleft y) \triangleleft (z \triangleleft y)$, and $x \triangleleft \{ y \} = x \triangleleft y$

The following lemma is, in the author’s opinion, sufficiently important as to be is referred to as the fundamental lemma of regression.

**Lemma 2.3.3**

(a) $x \triangleleft [y, y] = x \triangleleft y$

(b) $x \triangleleft [y, z] = x \triangleleft [z, y]$
(a) $x \lhd [y, y] = (x \lhd y) \lhd (y \lhd y) = (x \lhd y) \lhd 0 = x \lhd y$.
(b) If $y = 0$ then $(x \lhd y) \lhd (z \lhd y) = (x \lhd 0) \lhd (z \lhd 0)$
v
If $z = 0$ we similarly find the theorem true (the theorem is symmetric in $y$ and $z$).
If neither $y$ nor $z$ is zero then let $\beta = y/\|y\|$ and $\gamma = z/\|z\|$; that is
we normalize $y$ and $z$ so $\beta, \beta = \gamma, \gamma = 1$, and since these factors are non-zero
we can apply Lemma 2.3.1(b):
\[
(x \lhd y) \lhd (z \lhd y) = [x \lhd (\|y\|\beta)] \lhd [z \lhd (\|y\|\beta)] = (x \lhd \beta) \lhd (z \lhd \beta)
\]
\[
= (x \lhd \beta) \lhd ([\|z\|\gamma] \lhd \beta) = (x \lhd \beta) \lhd ([\|z\|\gamma \lhd \beta])
\]
\[
= (x \lhd \gamma \lhd \beta) = [x - (x, \beta)\beta] \lhd (\gamma \lhd \beta)
\]
\[
(x \lhd \gamma \lhd \beta) = [x - (x, \beta)\beta] \lhd (\gamma \lhd \beta)
\]
\[
= x - \frac{x, [\gamma - (\gamma, \beta)\beta]}{[\gamma - (\gamma, \beta)\beta, [\gamma - (\gamma, \beta)\beta]} [\gamma - (\gamma, \beta)\beta] - (x, \beta)\beta
\]
\[
= x - \frac{x, [\gamma - (\gamma, \beta)\beta]}{1 - (\beta, \gamma)^2} [\gamma - (\beta, \gamma)\beta] - (x, \beta)\beta
\]
\[
= x + \frac{x, [\gamma - (\gamma, \beta)\beta]}{1 - (\beta, \gamma)^2} [\gamma - (\beta, \gamma)\beta] - (x, \beta)\beta
\]
\[
= x + \frac{x, (\beta, \gamma)\beta - (x, \gamma)\gamma - (x, \beta)\beta + (x, \beta)\beta (\beta, \gamma)\gamma - (x, \beta)\beta + (x, \beta)\beta (\beta, \gamma)\gamma - (x, \beta)\beta}{1 - (\beta, \gamma)^2}
\]
\[
= x + \frac{x, (\beta, \gamma)\beta + (x, \beta)\gamma (\gamma, \beta)\gamma - (x, \gamma)\gamma - (x, \beta)\beta}{1 - (\beta, \gamma)^2}
\]
which is symmetric in $\beta$ and $\gamma$, and is therefore equal to $(x \lhd z) \lhd (y \lhd z)$, which completes the proof. \hfill \Box

Note that Lemma 2.3.3 shows that the definition
\[
x \lhd \{y, z\} = (x \lhd y) \lhd (z \lhd y)
\]
is well-defined; we extend this definition in the next sub-section.

Note that $x \lhd [y, z] = (I - [y : z][y : z]^+)x$.

For the geometric interpretation of the above result please refer to Figure 2.2.

### 2.3.2 The Multiple Function

#### 2.3.2.1 Definition

Let the brackets $[\ ]$ denote sequences, then we define
\[
[x_1, x_2, \ldots, x_n] \lhd y = [x_1 \lhd y, x_2 \lhd y, \ldots, x_n \lhd y] \text{ for } n \geq 1
\]
We identify a sequence with only one element with the element itself, so
\[
x \triangle [y] = x \triangle y
\]
\[
x \triangleright [y] = x \triangleright y
\]
and define
\[
x \triangle [y_1, \ldots, y_n] = (x \triangle y_n) \triangle ([y_1, \ldots, y_{n-1}] \triangle y_n)
\]
\[
x \triangleright [y_1, \ldots, y_n] = x - x \triangleright [y_1, \ldots, y_n]
\]
for \( n \geq 2 \) \hspace{1cm} (2.19)

which, in view of Definition 2.17, implies
\[
x \triangle [y_1, \ldots, y_n] = (x \triangle y_n) \triangle [y_1 \triangle y, \ldots, y_{n-1} \triangle y_n]
\]
\hspace{1cm} (2.20)

Note that Equation 2.20 is recursive and that Equation 2.18 is the starting condition.

2.3.2.2 Set Property

Lemma 2.3.3 states that \((x \triangle y) \triangle (z \triangle y) = (x \triangle z) \triangle (y \triangle z)\); this means that the definition \( x \triangle \{y, z\} = (x \triangle y) \triangle (z \triangle y) \) is well-defined, as the order of \( y \) and \( z \) does not affect the result. This result is generalized to:

**Theorem 2.3.4** \( x_1 \triangle [x_2, \ldots, x_n] = x_1 \triangle [x_{\pi(2)}, \ldots, x_{\pi(n)}] \), \( n \geq 2 \) where \( \pi \) is a permutation of \( \{2, \ldots, n\} \)

**Proof:** Assume true for all \( m < n \) then
\[
x_1 \triangle [x_2, \ldots, x_n] = (x_1 \triangle x_n) \triangle [x_2 \triangle x_n, \ldots, x_{n-1} \triangle x_n] \hspace{1cm} (2.20)
\]
2.3. THE GRAM-SCHMIDT FUNCTION

which, under the above assumption,

\[\begin{align*}
&= (x_1 \lhd x_n) \lhd [x_{\pi(2)} \lhd x_n, \ldots, x_{\pi(n-1)} \lhd x_n] \\
&= (2.20) \quad x_1 \lhd [x_{\pi(2)}, \ldots, x_{\pi(n-1)}, x_n]
\end{align*}\]

Thus \(x_2\) to \(x_{n-1}\) can be permuted arbitrarily. Further

\[\begin{align*}
x_1 \lhd [x_2, \ldots, x_n] &\quad (2.20) \quad (x_1 \lhd x_n) \lhd [x_2 \lhd x_n, \ldots, x_{n-1} \lhd x_n] \\
&\quad (2.20) \quad ([x_1 \lhd x_{n-1}) \lhd (x_n \lhd x_n)] \lhd [(x_2 \lhd x_{n-1}) \lhd (x_{n-1} \lhd x_n)]_{i=2, \ldots, n-2} \\
&\quad (L 2.3.3) \quad ([(x_1 \lhd x_{n-1}) \lhd (x_n \lhd x_n)] \lhd [(x_2 \lhd x_{n-1}) \lhd (x_{n-1} \lhd x_n)]_{i=2, \ldots, n-2} \\
&\quad (2.20) \quad (x_1 \lhd x_{n-1}) \lhd [(x_2 \lhd x_{n-1}) \lhd \ldots \lhd (x_{n-2} \lhd x_{n-1}), (x_{n-1} \lhd x_{n-1})] \\
&\quad (2.20) \quad x_1 \lhd [x_2, \ldots, x_{n-2}, x_n, x_{n-1}]
\end{align*}\]

so \(x_{n-1}\) and \(x_n\) can be permuted. Thus the terms \(x_2\) to \(x_n\) can be permuted arbitrarily. □

It follows that the definition

\[x_1 \lhd \{x_2, \ldots, x_n\} = x_1 \lhd [x_2, \ldots, x_n] \quad (2.21)\]

is well-defined for all \(n\), and that we may write Equation 2.20 as

\[x \lhd \{y_1, \ldots, y_n\} = (x \lhd y_n) \lhd \{y_1 \lhd y_n, \ldots, y_{n-1} \lhd y_n\} \quad (2.22)\]

The final result is on two stage regression:

**Theorem 2.3.5** For \(n \geq 3\), \(x_1 \lhd \{x_2, \ldots, x_n\} = (x_1 \lhd \{x_2, \ldots, x_{n-1}\}) \lhd \{x_n \lhd \{x_2, \ldots, x_{n-1}\}\}

Proof: The proof is by induction. First note that the theorem is true for \(n = 3\) since \(x_1 \lhd \{x_2, x_3\} = (x_1 : x_2) : (x_3 : x_2)\). Now for fixed \(n \geq 4\) suppose the theorem is true for the number \(n-1\), that is T2.3.5 (n-1), then

\[x_1 \lhd \{x_2, \ldots, x_n\} = x_1 \lhd \{x_2, \ldots, x_{n-2}, x_n, x_{n-1}\}\]
\[ (x_1 : x_{n-1}) \triangleleft \{ x_2 : x_{n-1}, \ldots, x_{n-2} : x_{n-1}, x_n : x_{n-1} \} \]

\[ ((x_1 : x_{n-1}) \triangleleft \{ x_2 : x_{n-1}, \ldots, x_{n-2} : x_{n-1} \}) \]
\[ \triangleleft ((x_n : x_{n-1}) \triangleleft \{ x_2 : x_{n-1}, \ldots, x_{n-2} : x_{n-1} \}) \]
\[ (T \text{ 2.3.5 (n-1)}) \]
\[ = (x_1 \triangleleft \{ x_2, \ldots, x_{n-1} \}) \triangleleft (x_n \triangleleft \{ x_2, \ldots, x_{n-1} \}) \]

In closing this section we note how minimal are the assumptions: from Theorem 2.3.4 and definition 2.20 we have derived most of the core of regression theory. The rest of the derivation of the core is left as an exercise; for example proving that \( X : (Y \cup Z) = (X : Y) : (Z : Y) = (X : Z) : (Y : Z) \), where \( X, Y \) and \( Z \) are sets of vectors.

### 2.4 Affine Spaces

#### 2.4.1 Definition

Given a vector space \( V \) over a field \( K \). A non-empty set \( A \) is said to be an affine space if there is a fixed \( p \in A \) and arbitrary \( v_1, v_2 \in V \), and an addition \(+ : p \times V \rightarrow V\) satisfying the conditions

1. \( (p + v_1) + v_2 = p + (v_1 + v_2) \),
2. For any \( q \in A \), there exists a unique vector \( v \in V \) such that \( q = p + v \).

\( K \) is called the coefficient field.

An affine combination of vectors \( x_1 \cdots x_k \) is a linear combination \( \lambda_1 x_1 + \cdots + \lambda_k x_k \) where \( \lambda_1 + \cdots + \lambda_k = 1 \).

An affine subspace of a vector space \( V \) is a subset closed under affine combinations of vectors in the subset. For the set \( P = \{ p_1, p_2, \ldots, p_n \} \), we define the affine hull \( \text{aff}(P) \) of \( P \) to be the set of affine combination of elements of \( P \), that is

\[ \left\{ \sum_{i=1}^{n} \alpha_i p_i \middle| \sum_{i=1}^{n} \alpha_i = 1 \right\} \]

This set is the smallest affine space containing \( P \).

If \( V \) is a vector space, \( V_1 \) is a proper subspace of \( V \) and \( p \in V \setminus V_1 \) then \( A_1 = p + V_1 \) is an affine space. Note that if \( \sum \alpha = 1 \) and \( a_i \in A_1 \) then \( \sum \alpha_i a_i \in A_1 \). More specifically, the line running through two points in an affine space is a subset of the space.
Lemma 2.4.1 If $A$ is of finite dimension then $A$ is closed.

Proof $A$ is of the form $p + W$ where $W$ is a finite subspace which is necessarily closed. Since the map $w \mapsto p + w$ is a continuous bijection, $A = p + W$ inherits the property of being closed. \( \square \)

Thus, for affine space $A$, if $a_i \in A, i = 1, \cdots, \infty$ and $\lim_{i \to \infty} a_i = a$, then $a \in A$.

2.4.2 The Affine Minimum

Here we compute the point of minimum norm in the affine hull of a set of points. This result is needed for computing a fixed-point in Chapter 3.3.2.

The problem$^3$

\[
\text{Minimize } \| x_1 g_1 + \cdots + x_n g_n \|^2 \text{ subject to } x_1 + \cdots + x_n = 1
\]

can be written as

\[
\text{Minimize } \| Gx \|^2 \text{ subject to } 1^T x - 1 = 0,
\]

where $x = [x_1 \cdots x_n]^T$ and $G = [g_1 \cdots g_n]$.

Note that $x_1 g_1 + \cdots + x_n g_n = Gx$, and that $Gx_1 = Gx_2$, $\iff G(x_1 - x_2) = 0$, $\iff x_1 - x_2 \in \mathcal{L}(G)^\perp$. So the component of $x$ lying in $[\mathcal{L}(G)]^\perp$ has no effect on $Gx$, and $x$ may be required to be an element of $\mathcal{L}(G)$, which condition may be written as

\[
G^T y = x \tag{2.23}
\]

where $y$ is unconstrained so, in view of Lemma 2.2.6,

\[
G^+ Gx = x. \tag{2.24}
\]

This leads to the Lagrangian

\[
\mathcal{L} = \| G G^T y \|^2 + \lambda(1^T G^T y - 1).
\]

Differentiating w.r.t. $y$:

\[
\frac{\partial \mathcal{L}}{\partial y} = 2(GG^T)^2 y + \lambda G 1 = 0 \Rightarrow (GG^T)^2 y = -\lambda G 1/2
\]

\[
\Rightarrow (G^T)^+ (GG^T)^2 y = -\lambda (GG^T)^+ G 1/2 \Rightarrow [(GG^T)^+ G] G^T y = -\lambda (GG^T)^+ G 1/2
\]

---

$^3$The author would like to thank Professor Heinz Neudecker of the University of Amsterdam for providing a solution to this problem [26]. The calculations here are similar to those cited.
\[(L \ 2.2.5 \ b) \Rightarrow G^T G^T y = -\lambda G^T + 1/2 \quad (2.23) \Rightarrow G^T x = -\lambda G^T + 1/2\]

\[\Rightarrow G^T G^T x = -\lambda G^T G^T + 1/2 \Rightarrow (G^+ G)^T x = -\lambda (G^+ G)^T 1/2\]

\[\Rightarrow G^+ G x = -\lambda G^+ G 1/2\]

\[\Rightarrow x = -\lambda G^+ G 1/2 \quad (2.24)\]

\[\Rightarrow 1^T x = -\lambda 1^T G^+ G 1/2\]

\[\Rightarrow \lambda = \frac{2}{1^T G^+ G 1} \text{ if } G 1 \neq 0 \quad (2.26)\]

(If \(G 1 = 0\), \(\|G x\|^2\) is minimized by taking \(x = 1/n\).) Substituting (2.26) into (2.25):

\[x = G^+ G 1/1^T G^+ G 1 \Rightarrow\]

\[G x = GG^+ G 1/1^T G^+ G 1 = G 1/\|G^+ G 1\|^2 \quad (2.27)\]

### 2.4.3 Scheme Functions

The file linalg.scm contains a number of functions; usage is as follows

<table>
<thead>
<tr>
<th>name</th>
<th>example</th>
<th>semantics</th>
</tr>
</thead>
<tbody>
<tr>
<td>vector</td>
<td>'(1 2 3)</td>
<td>VECTOR</td>
</tr>
<tr>
<td>vec+</td>
<td>(vec+ '(1 2 3) '(4 5 6))</td>
<td>addition</td>
</tr>
<tr>
<td>vec-</td>
<td>(vec- '(1 2 3) '(4 5 6))</td>
<td>subtraction</td>
</tr>
<tr>
<td>s*vec</td>
<td>(s*vec 4.5 '(1 2 3))</td>
<td>scalar mult.</td>
</tr>
<tr>
<td>vec/s</td>
<td>(vec/s '(1 2 3) 7)</td>
<td>scalar div.</td>
</tr>
<tr>
<td>matrix</td>
<td>'((1 2 3) (4 5 6))</td>
<td>MATRIX</td>
</tr>
<tr>
<td>idmat</td>
<td>(idmat 4)</td>
<td>identity</td>
</tr>
<tr>
<td>mat+</td>
<td>(mat+ '((1 2 3) (2 3 4)) '((6 7 4) (2 -1 7)))</td>
<td>addition</td>
</tr>
<tr>
<td>mat-</td>
<td>(mat- '((1 2 3) (2 3 4)) '((6 7 4) (2 -1 7)))</td>
<td>subtraction</td>
</tr>
<tr>
<td>s*mat</td>
<td>(s*mat 2.5 '((6 7 4) (-5 -1 6)))</td>
<td>scalar mult.</td>
</tr>
<tr>
<td>mat/s</td>
<td>(s*mat '((6 7 4) (-5 -1 6)) 2.5)</td>
<td>scalar div.</td>
</tr>
<tr>
<td>mat*s</td>
<td>(mat*s '((1 2 3) (2 3 4)) '((6 7 4) (2 -1 7)))</td>
<td>multiplication</td>
</tr>
<tr>
<td>transpose</td>
<td>(transpose '((1 2 3) (2 3 4)))</td>
<td>transpose</td>
</tr>
<tr>
<td>ginverse</td>
<td>(ginverse '((1 2 3) (4 5 6)))</td>
<td>mpgi</td>
</tr>
</tbody>
</table>
2.5 The Original Problems

An object is a dual of another object if it is the mirror image in some sense of that object. The most important property of this image is that the image of the image of an object be the original object itself. We generally require this of any dual - that the dual of the dual be the identity map. In this chapter we deal with a pair of problems as an asymmetric “dual” - a pair which doesn’t satisfy this image property; nevertheless the term is used as closely related problems are dual in nature.

We define the binary operator \( \leq \) on two sets \( A, B \subseteq \mathbb{R} \), the real numbers, as follows: \( A \leq B \) means that for all \( x \in A \) and \( y \in B \), \( x \leq y \).

An asymmetric dual is given in [18], and in Schrijver[37, p. 95] gives the form

\[
\max \{ c^T x : Ax \leq b \} = \min \{ b^T y : A^T y = c, y \geq 0 \},
\]

if either exists. \hspace{1cm} (2.28)

Here we use the \( \geq \) form of the primal LP, proceeding as follows:

\[
\max \{ c^T x : Ax \geq b \} = \max \{ c^T x : -Ax \leq -b \} \hspace{1cm} (2.28)
\]

\[
= \min \{ (-b)^T y : (-A)^T y = c, y \geq 0 \}
\]

\[
= \min \{ b^T (-y) : A^T (-y) = c, -y \leq 0 \} = \min \{ b^T y : A^T y = c, y \leq 0 \}.
\]

So

\[
\max \{ c^T x : Ax \geq b \} = \min \{ b^T y : A^T y = c, y \leq 0 \},
\]

if either exists. \hspace{1cm} (2.29)

This equation should be compared with (2.28) of which it is the dual under the operation of exchanging \( \geq \) and \( \leq \).

We call the L.H.S. of this equation the original primal and the R.H.S of this equation the original dual; together they comprise the asymmetric dual pair, that is the original primal is given by problem 1.1 and the original dual is

\[
\text{find the vectors } y \text{ which minimize } b^T y, \text{ subject to } A^T y = c, y \leq 0. \hspace{1cm} (2.30)
\]

Let \( x_f \) be a feasible vector for the original primal and \( y_f \) be a feasible vector for the original dual, then

\[
Ax_f \geq b \hspace{1cm} (2.31)
\]

and

\[
A^T y_f = c. \hspace{1cm} (2.32)
\]
Since $y_f \leq 0$, multiplying Equation 2.31 on the left by $y_f^T$, yields

$$y_f^T Ax_f \leq y_f^T b;$$

(2.33)

Further, $A^T y_f = c$ implies

$$x_f^T A^T y_f = x_f^T c,$$

(2.34)

and from (2.33) and (2.34) it follows that $c^T x_f \leq b^T y_f$, or

$$c^T x_f - b^T y_f \leq 0$$

(2.35)

and, from (2.29) we have

$$c^T x_s - b^T y_s = 0$$

(2.36)

for solutions $x_s$ and $y_s$ to the original primal and asymmetric dual respectively.

### 2.6 Quasi-Boundedness

If Problem 1.1 is feasible then there exists a solution, say $x_f$ such that $Ax_f \geq b$. Now $Ax_f \geq b \Rightarrow A[x_f + \lambda(I - A^+ A)c] \geq b$, so $x_f + \lambda(I - A^+ A)c$ is also feasible and its objective value is $c^T [x_f + \lambda(I - A^+ A)c] = c^T x_f + c^T \lambda(I - A^+ A)c = c^T x_f + \lambda c^T (I - A^+ A)c = c^T x_f + \lambda \|I - A^+ A\| c^2$, which is unbounded unless $\|I - A^+ A\| c^2 = 0 \Rightarrow (I - A^+ A)c = 0 \Rightarrow A^+ Ac = c$, so we introduce the following definition:

*Problem 1.1 is said to be quasi-bounded if $A^+ Ac = c$, (2.37)*

so

**Lemma 2.6.1** Problem 1.1 is feasible bounded $\Rightarrow$ Problem 1.1 is quasi-bounded.

Note that usually $A$ is of full rank and $m \geq n$, so $A^+ A = I$, which implies $A^+ Ac = c$, and so such problem is quasi-bounded.

At this point we can introduce the original primal-dual form:

$$Ax \geq b$$

$$y \geq 0,$$

$$A^T y = c$$

$$c^T x - b^T y = 0$$

(2.38)

and note that if the original primal-dual form is feasible then the original problem is quasi-bounded.

**Lemma 2.6.2** Problem 1.1 is quasi-bounded $\iff A^T c = c$ $\iff$ there exists $y$ such that $A^T y = c$. 
2.7. EXERCISES

Proof: Problem 1.1 is quasi-bounded \( \Rightarrow A^T \mathbf{c} = (2.12\text{c}) \) \( A^T (A^T + c) = (A^T A^T + c) = (A^T A^T + c) = (A^+ A)^T c \)

\( (2.37) \)

\[ A^+ Ac = \mathbf{c} \Rightarrow A^T \mathbf{c} = \mathbf{c} \Rightarrow (A^T \mathbf{c} = \mathbf{c}) \land (A^+ AA^T \mathbf{c} = A^+ Ac) \Rightarrow (A^T \mathbf{c} = \mathbf{c}) \land (A^T \mathbf{c} = A^+ Ac) \Rightarrow \]

\( (2.37) \)

\[ A^+ Ac = \mathbf{c} \Rightarrow \text{Problem 1.1 is quasi-bounded, so Problem 1.1 is quasi-bounded } \Leftrightarrow A^T \mathbf{c} = \mathbf{c}. \]

Further, there exists \( y \) such that \( A^T y = \mathbf{c} \Rightarrow (A^T y = \mathbf{c}) \land (\mathfrak{A}y = AA^+ y = (AA^+)^T y = A^+ A^T A^T y = A^+ T c = A^T c = \mathbf{c}) \Rightarrow (A^T y = \mathbf{c}) \land (\mathfrak{A}y = \mathbf{c}) \Rightarrow (A^T y = \mathbf{c}) \land (A^T \mathfrak{A}y = A^T \mathbf{c}) \Rightarrow (A^T y = \mathbf{c}) \land (A^T y = A^T \mathbf{c}) \Rightarrow A^T \mathbf{c} = \mathbf{c}, \text{ and since the obverse holds, } A^T \mathbf{c} = \mathbf{c} \Leftrightarrow \text{ there exists } y \text{ such that } A^T y = \mathbf{c}. \]

From Lemmas 2.6.1 and 2.6.2 we have

**Lemma 2.6.3** Problem 1.1 is feasible bounded \( \Rightarrow A^T \mathbf{c} = \mathbf{c}. \]

Also,

**Lemma 2.6.4** \( A^T y = \mathbf{c} \Leftrightarrow \mathfrak{A}y = \mathbf{c}. \)

Proof: \( A^T y = \mathbf{c} \Rightarrow (A^+ T A^T y = A^+] \mathbf{c}) \land (A^T \mathbf{c} = \mathbf{c}) \Rightarrow ((A A^+) y = A^+ T c) \land (A^T \mathbf{c} = \mathbf{c}) \Rightarrow \)

\[ (2.12\text{c}) \)

\[ (A^+ y = \mathbf{c}) \land (A^T \mathbf{c} = \mathbf{c}) \Rightarrow (A^T \mathfrak{A} y = A^T \mathbf{c}) \land (A^T \mathbf{c} = \mathbf{c}) \Rightarrow (A^T y = A^T \mathbf{c}) \land (A^T \mathbf{c} = \mathbf{c}) \Rightarrow A^T y = \mathbf{c}. \]

2.7 Exercises

1. (a) Use the Scheme function `ginverse` to compute the pseudo-inverse of the matrix \( C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \).
   
   (b) Compute \( CC^T \). (Hint: Use the function `mat*mat`).
   
   (c) Compute \( C^T C \); check that this product is symmetric and idempotent.

2. (a) Compute the matrix \( \mathfrak{A} = AA^+ \), where \( A = \begin{bmatrix} 5 & 6 \\ 4 & -2 \\ 7 & -6 \end{bmatrix} \).
   
   (b) Compute the matrix \( I - \mathfrak{A} \). (Hint: Use the functions `idmat` and `mat-`).
   
   (c) Compute \( \mathfrak{A}(I - \mathfrak{A}), \mathfrak{A}^2 \), and \( (I - \mathfrak{A})^2 \).

3. (a) Check that the pseudo-inverse is correctly computed for matrix \( \mathfrak{A} \). (use nonconstructive)

   (b) Compute the inverse for the matrix \( B = \begin{bmatrix} 1.0 & 2.0 & 3.0 \\ 4.0 & 5.0 & 6.0 \end{bmatrix} \), and compare the result with the inverse of \( C \).
Chapter 3

Fixed-Points

Having cast the LP problem as a fixed-point problem we now consider the solution of this fixed-point problem, first from a general stand-point in this chapter, and then from a more specific point of view in the following Chapter 6.

In this chapter we introduce orthogonal projection matrices $P$ and $K$, and a swapping-matrix $S$. From these matrices we construct a matrix $U$ which is unitary in nature. We characterize the fixed-points of $U$ and show how they can be computed by a converging series approach, a regression approach, and by a vector lattice approach. In Chapter 6 the developed theory will be used in the specific context of matrix $\mathfrak{P}$ of Chapter 5, and matrices $\mathfrak{r}_z$ and $\mathfrak{G}$ which are introduced in Chapter 6. The specific matrices $\mathfrak{P}$, $\mathfrak{r}_z$ and $\mathfrak{G}$ correspond in their general properties with those of $P$, $K$ and $S$ respectively and relate directly to the solution of the fixed-point problem; thus the general results of this chapter apply in the later specific context of the following chapters.

3.1 Theory

3.1.1 Swapping Matrices

$S$ is called a *swapping-matrix* if $S$ is unitary Hermitian. We say that $S$ swaps $Q$ (or $Q$ is swapped by $S$) if $Q$ is an Hermitian idempotent (i.e. an orthogonal projection), and

$$S Q S = I - Q;$$  \hfill (3.1)

If $S$ is a swapping-matrix which swaps $Q$ then

$$Q^2 = Q \quad (a)$$
$$Q^T = Q \quad (b)$$
$$S^2 = I \quad (c)$$
$$S^T = S \quad (d)$$

(3.2)

since (a) and (b) follow from $Q$ being an symmetric idempotent, (c) follows since $S$ unitary implies $S S^T = I \Rightarrow S^2 = I$ since $S$ is symmetric, (d) follows since $S$ is required to be symmetric.
Lemma 3.1.1

Proof: By verifying the four conditions 2.11 for the pseudo-inverse: 1. $SXX^+SSX = SXX^+X$

(2.11 a) $S(I - Q)S = S^2 - SQS = I - (I - Q)$

(b) $S(QS) = I - Q \Rightarrow S^2QS = S(I - Q) \Rightarrow QS = S - SQ \Rightarrow QS + SQ = S$

(c) $Q^2 = Q \Rightarrow Q(I - Q) = 0 \Rightarrow QSQS = 0 \Rightarrow QS(QS)^2 = 0 \Rightarrow QS = 0$

(d) $(I - S)Q(I - S) = (Q - SQ)(I - S) = Q - QS - SQ + SQS = Q - (QS + SQ) + (I - Q) = Q - S + I - Q = I - S$

(e) $(I + S)Q(I + S) = Q + (SQ + QS) + SQS = Q + S + I - Q = I + S$

Lemma 3.1.2 $S^2 = I \Rightarrow (SX)^+ = X^+S$.

Proof: By verifying the four conditions 2.11 for the pseudo-inverse: 1. $SXX^+SSX = SXX^+X$

(2.11 a) $S(I - Q)S = S^2 - SQS = I - (I - Q)$

(b) $S(QS) = I - Q \Rightarrow S^2QS = S(I - Q) \Rightarrow QS = S - SQ \Rightarrow QS + SQ = S$

(c) $Q^2 = Q \Rightarrow Q(I - Q) = 0 \Rightarrow QSQS = 0 \Rightarrow QS(QS)^2 = 0 \Rightarrow QS = 0$

(d) $(I - S)Q(I - S) = (Q - SQ)(I - S) = Q - QS - SQ + SQS = Q - (QS + SQ) + (I - Q) = Q - S + I - Q = I - S$

(e) $(I + S)Q(I + S) = Q + (SQ + QS) + SQS = Q + S + I - Q = I + S$

Corollary 3.1.3 $S$ swaps $Q \Rightarrow (SQ)^+ = Q^+S = QS$.

3.1.2 Projections and Oblique Projectors

We need a few results relating to projection matrices:

Lemma 3.1.4 If $Q$ is an orthogonal projection then $\|x\|^2 = \|Qx\|^2 + \|x - Qx\|^2$.

Proof: $\|x\|^2 = \|Qx + (I - Q)x\|^2 = (Qx)^TQx + 2(Qx)^T(I - Q)x + [(I - Q)x]^T(I - Q)x$

$= \|Qx\|^2 + 2x^TQ^T(I - Q)x + \|(I - Q)x\|^2 = \|Qx\|^2 + \|(I - Q)x\|^2$. $\square$

Corollary 3.1.5 If $Q$ is an orthogonal projection then $\|Qx\|^2 \leq \|x\|^2$.

Proof: Obvious from Lemma 3.1.4. $\square$

Lemma 3.1.6 If $Q$ is an orthogonal projection then $\|Qx\|^2 = \|x\|^2 \iff Qx = x \iff x^TQx = x^Tx$.

Proof: Obvious from Lemma 3.1.4. $\square$

Define

$$\overline{Q} = Q(I - S)$$

(3.3)

then we have
3.1. THEORY

Lemma 3.1.7

\[ (a) \overline{Q}^2 = \overline{Q} \]
\[ (b) \overline{Q}^T \overline{Q} = I - S \]

Proof: (a) \(\overline{Q}^2 = (Q(I-S))^2 = Q(I-S)Q(I-S) = \overline{Q}\), (b) \(\overline{Q}^T \overline{Q} = (Q(I-S))^T Q(I-S) = (I-S)^T Q(I-S) = (I-S)QQ(I-S) = (I-S)Q(I-S) = I - S\). □

Note that \(\overline{Q}\) is what Afriat [3] calls an oblique projector.

3.1.3 Operations on Vectors

Notionally the vectors are considered to be column vectors on which conjugation is defined; for matrix \(X\) and vectors \(u\) and \(v\) we require

\[
Xu \cdot v = u \cdot X^Tv \quad (a) \\
v \cdot u = v \cdot u \quad (b)
\]

Lemma 3.1.8

If \(S\) swaps \(Q\) and \(Qz = z\), then \(z \perp Sz\), that is \(z^T Sz = 0\).

Proof: \(z^T Sz = (Qz)^T S(Qz) = z^T Q^T SQz = z^T QSQz = z^T 0z = 0\). □

Lemma 3.1.9

\(Qz = z \iff \overline{Q}z = z\).

Proof: \(Qz = z \Rightarrow (Qz - QSQz = z) \land (Qz = z) \Rightarrow (Q^2z - QSQz = z) \land (Qz = z) \Rightarrow (Q(I-S)Qz = z) \land (Qz = z) \Rightarrow Qz = z\), while \(\overline{Q}z = z \Rightarrow (Q(I-S)z = z) \land (\overline{Q}z = z) \Rightarrow (Q^2(I-S)z = Qz) \land (\overline{Q}z = z) \Rightarrow (Q(I-S)z = Qz) \land (\overline{Q}z = z) \Rightarrow (Q^2z - QSQz = z) \land (Qz = z) \Rightarrow Qz = z\). □

3.1.4 The General \(P\)-Unitary Matrix \(U\)

From this point we assume the existence of general idempotent symmetric \(K\) and \(P\), both swapped by \(S\). Referring forward, in Chapter 6 it will become apparent that the specific future rôle we have in mind for \(P\) is as the fixed-point matrix \(\Psi\) given by Equation 5.11b, while that for \(K\) is the matrix \(\mathfrak{K}_z\) given by Equation 6.3 which forces non-negativity and orthogonality, and thus eventually the complementary slackness condition of Chapter 4.3.2 to hold.

With matrices \(K\) and \(P\) swapped by \(S\), we have \(PKSP = P(KS + SK - SK)P = P(S - SK)P = PSP - PSKP = -PSKP\), that is:

\[ PKSP = -PSKP, \] (3.5)
so $PK(I + S)P = PKP + PKSP = PKP - PSKP = P(K - SK)P = P(I - S)KP$, Another similar result is obtained by transposition, so

$$PK(I + S)P = P(I - S)KP \quad \text{(a)}$$

$$P(I + S)KP = PK(I - S)P \quad \text{(b)} \quad (3.6)$$

Results 3.6' dual to Equation set 3.6, where $P$ and $K$ are exchanged, also obtain.

For idempotent $B$ define the $B$ component of vector $a$ or matrix $A$ to be $a^T Ba$ or $A^T BA$ respectively.

Now $z^TPKSPz = z^T(-PSKP)z = -z^TPSKPz = -z^T PKSPz, \text{ since } P, K \text{ and } S \text{ are Hermitian.}$

Thus $z^TPKSPz = -z^TPKSPz$, which implies

$$z^TPKSPz = 0, \text{ for arbitrary } z \quad (3.7)$$

and, of course, the dual result $(3.7')$: $z^TKPSKz = 0$, for arbitrary $z$. Thus

**Lemma 3.1.10** $z = Pz \Rightarrow z^T KSz = 0 \quad \blacksquare$

Consistent with Equation 3.3 a, we define the general oblique complementary slackness matrix

$$\overline{K} = K(I - S) \quad (3.8)$$

If $Pz = z$ then

$$(a) \quad (\overline{K}z)^T \overline{K}z = z^T z,$$

$$(b) \quad z^T \overline{K}z = z^T Kz .$$

**Lemma 3.1.11**

Proof: (a) $(\overline{K}z)^T \overline{K}z = z^T \overline{K}^T \overline{K}z \quad (L3.1.7c)$$

$$= z^T z - z^T Sz = z^T z - z^T P^T SPz = z^T z - z^T Pz = z^T z,$$

(b) $z^T \overline{K}z = z^T K(I - S)z = z^T Kz - z^T KSz = z^T Kz - z^T P^T KSPz = z^T Kz - z^T PKSPz \quad (L3.1.10)$

$$= z^T Kz - z^T PKSPz \Rightarrow z^T Kz. \quad \blacksquare$$

**Corollary 3.1.12** If $Pz = z$ then $z^T \overline{K}z \leq z^T z$.

Proof: $z^T \overline{K}z = z^T Kz \quad (L3.1.11b)$$

$$\leq z^T z \quad (L3.1.15) \Rightarrow z^T Kz \leq z^T z. \quad \blacksquare$

We now construct the matrix $U$ and show that it is $P$-unitary:

Define

$$U = P(I + S)K(I - S)P , \quad (3.9)$$

where $K, P$ and $S$ are as defined at the beginning of the section - that is $S$ swaps both $K$ and $P$. 
**Theorem 3.1.13** \( U \) is \( P \)-unitary.


\[
\begin{align*}
&= P(I - S)K(I + S)PP(I + S)K(I - S)P \\
&= P(I - S)K(I + S)P(I + S)K(I - S)P \\
&= P(I - S)K(I + S)K(I - S)P \\
&= P(I - S)K^2(I - S)P \\
&= P(I - S)P = P^2 = P.
\end{align*}
\]

Similarly \( UU^T = P \). □

In other words, with the definition \( R(P) = \{ Pz : z \in \mathbb{R}^m \} \), \( U \) when restricted to domain \( R(P) \), is a unitary function, so any vector \( z \) in the range of \( P \) (i.e. \( Pz = z \)) will be mapped by \( U \) to a vector with the same norm - that is \( Pz = z \Rightarrow \| Uz \|^2 = (Uz)^T(Uz) = z^T U^T U z = z^T Pz = z^T z = \| z \|^2 \).

**Lemma 3.1.14**

(a) \((Uz)^T S \omega = 0 \)

(b) \((Uz)^T(U \omega) = z^T \omega \)

Proof:

(a) \((Uz)^T S \omega = (P(I + S)K(I - S)Pz)^T S \omega = z^T P(I - S)K(I + S)PS \omega = 0 \).

(b) \((Uz)^T(U \omega) = z^T U^T U \omega = z^T P \omega = z^T \omega \). □

One final result is needed for the next section:

**Lemma 3.1.15** \( z = Uz \Leftrightarrow (z = Pz) \land (z^T U z = z^T z) \).

Proof: \((z = Pz) \land (z^T U z = z^T z) \)

\[
\begin{align*}
\Leftrightarrow (||z - Uz||^2 = z^T z - z^T U z - z^T U^T U z) \land (z = Pz) \land (z^T U z = z^T z) \\
\Leftrightarrow (||z - Uz||^2 = z^T z - 2z^T U z + z^T U^T U z) \land (z = Pz) \land (z^T U z = z^T z) \\
\Leftrightarrow (||z - Uz||^2 = z^T z - 2z^T z + z^T U^T U z) \land (z = Pz) \land (z^T U z = z^T z) \\
\Leftrightarrow (||z - Uz||^2 = z^T z - 2z^T z + z^T P z) \land (z = Pz) \land (z^T U z = z^T z) \\
\Leftrightarrow (||z - Uz||^2 = 0) \land (z = Pz) \land (z^T U z = z^T z) \\
\Leftrightarrow z = Uz .
\end{align*}
\]

### 3.1.5 Matrix Orbits

Here we develop some results on the \( P \)-unitary matrix \( U \) and give them a geometric interpretation.
Lemma 3.1.17

(b) The $K$ component of $U^n$ remains constant at $PKP$

(see (3.11), and the product of the idempotents $P$ and $K$.)

Proof: from Equation 3.11, $U(U^T + 2P + U) = 4UPKP$, $\Rightarrow P + 2U + U^2 = 4UPKP$, $\Rightarrow (P + U)^2 = 4UPKP$, $\Rightarrow (P + U)/2 = UPKP$, $\Rightarrow V^2 = UPKP$. Multiplication on the right by $U$ yields the other part of the lemma. 

Define the averaging matrix

$$V = (P + U)/2.$$ (3.10)


$$U^T + 2P + U = 4PKP$$ (3.11)

The following results connect the developed theory with the theory of reciprocal spaces [3], and the product of the idempotents $P$ and $K$. 

Lemma 3.1.17 $V^2 = UPKP = PKPU$. 

Proof: from Equation 3.11, $U(U^T + 2P + U) = 4UPKP$, $\Rightarrow P + 2U + U^2 = 4UPKP$, $\Rightarrow (P + U)^2 = 4UPKP$, $\Rightarrow (P + U)/2 = UPKP$, $\Rightarrow V^2 = UPKP$. Multiplication on the right by $U$ yields the other part of the lemma. 

Proof: (a) Obvious from the definition of $U$ and Theorem 3.1.13.
3.2. CHARACTERIZATION

**Corollary 3.1.18** \( VPKP = PKPV \)

Proof: Follows easily from the previous lemma. □

This leads to the interesting result:

**Corollary 3.1.19** \( V^{2i} = U^i(KP)^i = (PK)^iU^i \).

Proof: Exercise. □

**Lemma 3.1.20** \( V^TV = PKP \).


**Lemma 3.1.21** \( V^nTV^n = (PKP)^n = (PK)^nP = P(KP)^n \).

Proof: \( V^nTV^n = V^{(n-1)}TPKPV^{n-1} = V^{(n-1)}TV^{n-1}PKP = \ldots = (PKP)^n = (PK)^nP = P(KP)^n \). □

### 3.2 Characterization

In view of Lemma 3.1.9 we have

**Corollary 3.2.1** \( Kz = z \iff Kz = z \)

**Lemma 3.2.2** \( z \) is a fixed-point of \( U \) iff \( z \) is a fixed-point of \( V \).

Proof is obvious. □

**Lemma 3.2.3** \( z \) is a fixed-point of \( U \) iff \( z \) is a fixed-point of \( P \) and \( K \).

\[
(3.1.15) \quad Uz = z \iff (z^TUz = z^Tz) \land (Pz = z)
\]

\[
(3.9) \quad \iff (z^TP(I+S)K(I-S)Pz = z^Tz) \land (Pz = z)
\]

\[
\iff (z^TPKPz + z^TPSKPz - z^TPKSPz - z^TPSKPz = z^Tz) \land (Pz = z)
\]

\[
\iff (z^TPKPz - z^TPSKPz = z^Tz) \land (Pz = z)
\]

\[
\iff (z^T(Kz - z^TSPz = z^Tz) \land (Pz = z)
\]

\[
(3.1) \quad \iff (z^TKz - z^T(I-K)z = z^Tz) \land (Pz = z)
\]
Lemma 3.2.4 $z$ is a fixed-point of $V$ iff $\|z\| = \|Vz\|$.

Proof: $z = Vz \Rightarrow z = Uz$ \iff $z^Tz = z^TUz \iff 2z^Tz = z^Tu + z^Tu \iff z^Tz = (z^Tu + z^Tu)/2 \iff z^Tz = (2z^Tu + 2z^Tu)/4 \iff z^Tz = ((I + U)z + (I + U)z)/2$ \iff $\|z\| = \|Vz\|$.

Summing up Corollary 3.2.1 and the previous three lemmas we have:

- $z$ is a fixed-point of $U$ \iff $z$ is a fixed-point of $V$
- $z$ is a fixed-point of $P$ and $K$ \iff $z$ is a fixed-point of $P$ and $\bar{K}$
- $\|z\| = \|Vz\|$.

This theorem is illustrated by Figure 3.1.

3.3 Computational Methods

3.3.1 Averaging

Theorem 3.2.5 is the basis for this method.

With $1 = [1 \cdots 1]^T$ we consider the sequence $\{g_i\}$ where

\begin{align*}
    g_{i+1} &= Ug_i, \\
    g_1 &= P1,
\end{align*}

(3.12)
and the sequence \( \{v_i\} \) where

\[ v_{i+1} = Vv_i, \quad v_1 = P1. \]

Note that

\[ v_1 = g_1 \]

Since \( U \) is \( P \)-unitary it follows that \( \|g_n\| \) is constant; on the other hand we show that \( \{v_i\} \) converges, and to this end we have the following

**Lemma 3.3.1** \( \|z - Vz\|^2 = \|z\|^2 - \|Vz\|^2 \)

Proof: \( \|z - Vz\|^2 = \|z - (I + U)z/2\|^2 = \|(I - U)z/2\|^2 = \|(I - U)z\|^2/4 = [(I - U)z]^T(I - U)z/4 = z^T(I - U)Uz/4 = z^T(I - U - U^TUz)/4 = \|z\|^2 - z^T(I + U + U^TU)z/4 = \|z\|^2 - \|v_i\|^2 \)

Figure 3.2 and Pythagoras make the above lemma obvious.

---

**Figure 3.2: Distance Between Successive Vectors**

Now we are in a position to show that \( \{v_i\} \) is a Cauchy sequence and thus has a limit:

\[
\|v_i - v_{i+j}\| = \|[v_{i+1} - v_{i+k+1}]\| \leq \sum_{k=0}^{j-1} \|v_{i+k+1}\| \leq \sum_{k=0}^{j-1} \|v_{i+k+1}\| = \sum_{k=0}^{j-1} \|v_{i+k+1}\| \]

\[
= \sum_{k=0}^{j-1} \|V^k(v_i - v_{i+1})\| \leq \sum_{k=0}^{j-1} \|V^k(v_i - v_{i+1})\|
\]
\[ \leq \sum_{k=0}^{j-1} \sqrt{(v_i - v_{i+1})^T V^k T^k V^k (v_i - v_{i+1})} \]

where \( \lambda \) is the largest eigenvalue of \( PK \) other than unity (in Appendix A we show that the eigenvalues of \( PK \) lie in the interval \([-1, 1]\)).

\[ \leq \sum_{k=0}^{j-1} \sqrt{\|v_i - v_{i+1}\| \lambda^{2k} \|v_i - v_{i+1}\|} \]

That is \( \|v_i - v_{i+j}\| \leq \frac{1}{1-\lambda} \|v_i - v_{i+1}\| \), so

\[ \|v_i - v_{i+j}\|^2 \leq \frac{1}{(1-\lambda)^2} \|v_i - v_{i+1}\|^2 \]

and so, using Lemma 3.3.1,

\[ \|v_i - v_{i+j}\|^2 = \frac{1}{(1-\lambda)^2} (\|v_i\|^2 - \|v_{i+1}\|^2) \] (3.14)

and since \( \|v_i\|^2 \) is a Cauchy sequence, it follows that \( \{v_i\} \) is also a Cauchy sequence, which necessarily converges; we set

\[ v_\infty = \lim_{i \to \infty} v_i \]

Obviously

\[ U^i v_\infty = v_\infty \quad \forall i \geq 0 \]

### 3.3.2 Affine Regression

#### 3.3.2.1 General Approach

An affine sub-space of a vector space \( V \) is a set of the form \( x + S \) where \( S \) is a sub-space of \( V \). Alternatively we can define an affine space as a set which is closed under the binary operation \( +_\lambda \) for all \( \lambda \), where \( x +_\lambda y = \lambda x + (1-\lambda)y \).

An affine sub-space of an inner product space has a unique point of minimum norm, since if we choose an arbitrary point \( a \) and form an infinite sequence of vectors of strictly decreasing norm beginning with \( a \) they will all lie in \( A = \{x : \|x\| \leq \|a\|\} \) and since \( A \) is closed and bounded and hence compact this
sequence will have an accumulation point which, in view of the continuity of the norm, will have minimum norm, so there exists a point of minimum norm. Now suppose there are two vectors of minimum norm, say \( w_1 \) and \( w_2 \), then the point \( (w_1 + w_2)/2 \) which is also in the affine space has a smaller norm than both \( w_1 \) and \( w_2 \), contradicting their minimality; therefore there is a unique point of minimum norm.

Let \( g_1 \) be a vector in the range of \( P \), and \( p \) be a fixed-point of \( U \) and thus also in the range of \( P \) such that \( g_1 \cdot p > 0 \). (in Chapter 6.4 we construct a vector which satisfies this requirement for a non-negative fixed-point of the specific \( U \) introduced in Chapter 6.1.4). We consider the affine space

\[
A_j = \left\{ \sum_{i=1}^{j} \lambda_i g_i : \sum_{i=1}^{j} \lambda_i = 1 \right\}.
\]

Obviously \( A_1 \subseteq A_2 \subseteq A_3 \cdots \), moreover it can be shown that \( A_{i+1} = A_i \Rightarrow A_{i+2} = A_i \), so we define

\[
A = A_j \text{ where } j \text{ is such that } A_j = A_{j+1}.
\]

Take \( g_1 = p_1 + q_1 \), where \( p_1 = g_1 \triangleright p \) and \( q_1 = g_1 \triangleleft p \) which is a non-trivial orthogonal decomposition in view of Lemma 2.3.2, and since \( g_1 \cdot p > 0 \). So \( A \) is not equal to the range of \( P \) since each of its elements is of the form \( p_1 + r \), where \( r \) is orthogonal to \( p \), and thus a proper affine space. It follows that the vector of smallest norm in \( A \) (not necessarily \( p \)) is a fixed-point of \( U \); this vector is given by Equation 2.27.

Note that \( v_1 = g_1 \), so \( v_1 \in A \), also that \( v_i \in A \Rightarrow v_i \in A_j \Rightarrow v_i+1 \in A_{j+1} \Rightarrow v_i+1 \in A; \) since \( A \) is compact it follows that \( v_\infty \in A \).

Now \( (v_i - v_\infty)^T v_\infty = (v_1 - v_\infty)^T P v_\infty = (v_i - v_\infty)^T U^T U v_\infty = (v_i - v_\infty)^T U^T U v_\infty = (Uv_i - Uv_\infty)^T v_\infty = (Uv_1 - v_\infty)^T v_\infty \) that is

\[
(v_i - v_\infty)^T v_\infty = (Uv_i - v_\infty)^T v_\infty.
\]

Averaging this equation and the tautology \( (v_i - v_\infty)^T v_\infty = (v_1 - v_\infty)^T v_\infty \) yields

\[
(v_i - v_\infty)^T v_\infty = (v_{i+1} - v_\infty)^T v_\infty
\]

from which it follows that

\[
(v_i - v_\infty)^T v_\infty = 0 \quad \forall i \geq 1.
\]

Further, \( (g_i - v_\infty)^T v_\infty = (g_i - v_\infty)^T P v_\infty = (g_i - v_\infty)^T U^T U v_\infty = [U(g_i - v_\infty)]^T U v_\infty = (g_{i+1} - v_\infty)^T v_\infty \) that is

\[
(g_i - v_\infty)^T v_\infty = (g_{i+1} - v_\infty)^T v_\infty.
\]

Finally, since \( g_1 = v_1 \), it follows that \( (g_1 - v_\infty)^T v_\infty = (v_1 - v_\infty)^T v_\infty \) and so

\[
(g_i - v_\infty)^T v_\infty = 0 \quad \forall i \geq 1.
\]
Thus $v_\infty$ is the unique point of minimum norm in the affine space $\mathcal{A}$.

From these computations it also follows that $g_n = U^\tau g_1$ orbits $v_\infty$ at a fixed distance from $v_\infty$ in the affine space $\mathcal{A}$, as shown in Figure 3.3.

The above calculations form the basis for an algorithm for computing a fixed-point as follows.

Let

$$G_i = [g_1 \cdots g_i] \tag{3.15}$$

then we minimize $\|G_i x\|^2$ subject to $1_i^T x = 1$. The solution to this minimization problem was given in Chapter 2.4.2, yielding

$$\sigma_i = G_i x_s = \frac{G_i^+ G_i^T 1_i}{\|G_i^+ G_i^T 1_i\|^2} = \frac{G_i (G_i^T G_i)^+ 1_i}{1_i^T (G_i^T G_i)^+ 1_i} \tag{3.16}$$

Note that in the above formulae the pseudo-inverse is used, but the ordinary inverse obtains if $i \leq q$.

### 3.3.3 Ordered Spaces

Another method of solution of the general fixed-point problem involves using a vector lattice approach, which is detailed in this section.

A lattice\(^1\) is a triple $(L, \lor, \land)$ where $\lor : L^2 \to L$, and $\land : L^2 \to L$. These binary functions are normally written between their operands, and are required to satisfy

---

\(^1\)Some of the treatment here is an extension of that in [5].
3.3. COMPUTATIONAL METHODS

1. \( x \lor x = x; \ x \land x = x \), (nilpotency)

2. \( x_1 \lor x_2 = x_2 \lor x_1; \ x_1 \land x_2 = x_2 \land x_1 \), (commutativity)

3. \( x_1 \lor (x_2 \lor x_3) = (x_1 \lor x_2) \lor x_3; \ x_1 \land (x_2 \land x_3) = (x_1 \land x_2) \land x_3 \), (associativity)

4. \( x_1 \lor (x_1 \land x_2) = x_1; \ x_1 \land (x_1 \lor x_2) = x_1 \), (absorptivity)

If, in addition

\[
x_1 \lor (x_2 \land x_3) = (x_1 \lor x_2) \land (x_1 \lor x_3),
\]

or

\[
x_1 \land (x_2 \lor x_3) = (x_1 \land x_2) \lor (x_1 \land x_3),
\]

holds, then the other can be shown to hold and the lattice is said to be *distributive*.

We can associate an *order relation* with a lattice by

\[
x_1 \geq x_2 \text{ if } x_1 \lor x_2 = x_1.
\]

or, if we have an order relation and \( sup(x_1, x_2) \) and \( inf(x_1, x_2) \) both exist for all \( x_1 \) and \( x_2 \), we take \( x_1 \lor x_2 = sup(x_1, x_2) \) and \( x_1 \land x_2 = inf(x_1, x_2) \), and can show that these functions satisfy the above four defining properties for a lattice.

### 3.3.3.1 Vector Lattices

This sub-section requires a knowledge of basic lattice theory. Specifically the result that there is a natural one to one correspondence between the class of lattices and the class of ordered sets whose order admits an least upper bound and a greatest lower bound. Any basic text on lattice theory should suffice to follow the development.

We have an *ordered vector space* if

\[
x_1 \geq x_2 \Rightarrow x_1 + x_3 \geq x_2 + x_3, \ \forall x_1, x_2, x_3 \in L,
\]

and

\[
x_1 \geq x_2 \Rightarrow \alpha x_1 \geq \alpha x_2, \ \forall \alpha \in \mathbb{R}^+ \cup \{0\}, \ and \ \forall x_1, x_2 \in L.
\]

If the order on an ordered vector space defines a lattice then we have (as a definition) a *vector lattice*.

Note that we give priority to multiplication over \( \lor \) and \( \land \), and priority to \( \lor \) and \( \land \) over unary and binary + and -. Thus, for example \(-x_1 \lor x_2 = -(x_1 \lor x_2), \ x_1 + x_2 \land x_3 = x_1 + (x_2 \land x_3), \ and \ \alpha x_1 \lor x_2 = (\alpha x_1) \lor x_2, \)
For a vector lattice the following results hold

\[
\begin{align*}
x_1 \lor x_2 &= -(x_1) \land (x_2) \\
x_1 \lor x_2 + x_3 &= (x_1 + x_3) \lor (x_2 + x_3) \\
\alpha(x_1 \lor x_2) &= (\alpha x_1) \lor (\alpha x_2)
\end{align*}
\]

These results and a proof of (b) (which provides the flavour for proving the others) are given in [5]. Dual results where \(\lor\) is replaced by \(\land\), and vice versa hold and will be denoted by a \(\prime\) in the equation designator: Thus, for example,

\[
x_1 \land x_2 = -(x_1) \lor -(x_2).
\]

We define the positive and negative parts of a vector, and the absolute value of a vector as follows:

\[
\begin{align*}
x^\lor &= x \lor 0 \\
x^\land &= x \land 0 \text{ and } \\
|x| &= x^\lor - x^\land
\end{align*}
\]

(3.17)

**Lemma 3.3.2** \(x = x^\lor + x^\land\).

Proof: \(x - x^\lor = x - x \lor 0 = x - [-(x \lor 0)] = x + (-x) \land 0 = (x - x) \land (x + 0) = 0 \land x = x^\land. \)  
Thus \(x - x^\lor = x^\land\), which implies \(x = x^\lor + x^\land. \)  

**Lemma 3.3.3** \(x_1 + x_2 = x_1 \lor x_2 + x_1 \land x_2\).

Proof: From the previous lemma,

\[
x_1 - x_2 = (x_1 - x_2)^\lor + (x_1 - x_2)^\land = (x_1 - x_2) \lor 0 + (x_1 - x_2) \land 0,
\]

\[
= x_1 \lor x_2 - x_2 + x_1 \land x_2 - x_2 = x_1 \lor x_2 + x_1 \land x_2 - 2x_2.
\]

So \(x_1 - x_2 = x_1 \lor x_2 + x_1 \land x_2 - 2x_2, \)  \(\Rightarrow x_1 + x_2 = x_1 \lor x_2 + x_1 \land x_2. \)

3.3.3.2 Hilbert Lattices

Consider a vector lattice on which a norm is defined - that is a map \(\|\| : V \rightarrow \mathbb{R}; \) we require

\[
\|x\| = \|x\|
\]

in which case the norm is said to be a lattice norm, and the lattice is a normed vector lattice. if the lattice is complete with respect to the norm then it is said to be a Banach lattice.

Consider a normed vector lattice on which an inner product is defined and for which the norm is defined by \(\|x_1\| = \sqrt{x_1 \cdot x_1}\). if the vector space is complete with respect to the norm induced by the inner product then such a lattice is called a Hilbert lattice. The inner product of \(x_1\) and \(x_2\) will be denoted by
$x_1 \cdot x_2$, or where the vectors are cartesian by $x_1^T x_2$, or $x_1^* x_2$, depending on whether the vector space is over the Real or Complex field. If $x_1 \cdot x_2 = 0$ then we will say that $x_1$ is perpendicular to $x_2$, or $x_1 \perp x_2$.

Note: We require that

$$x_1, x_2 \geq 0 \Rightarrow x_1 \cdot x_2 \geq 0,$$  \hspace{1cm} (3.18)

and the existence of a swapping-matrix $S$ which is consonant with the lattice order, that is $x_1 \geq x_2 \Rightarrow Sx_1 \geq Sx_2$ from which it follows that

$$S(x_1 \lor x_2) = (Sx_1) \lor (Sx_2)\quad \text{(a)}$$
$$S(x_1 \land x_2) = (Sx_1) \land (Sx_2)\quad \text{(b)}$$

**Lemma 3.3.4** For a normed vector lattice $x^\lor \perp x^\land$.

Proof: We have $\|\|x\|=\|x\| \Rightarrow \|\|x\|^2 = \|x\|^2 \Rightarrow |x|^T |x| = x^T x$

$$\Rightarrow (x^\lor - x^\land) \cdot (x^\lor - x^\land) = (x^\lor + x^\land) \cdot (x^\lor + x^\land)$$

$$\Rightarrow x^\lor \cdot x^\lor - 2x^\lor \cdot x^\land + x^\land \cdot x^\land = x^\lor \cdot x^\lor + 2x^\lor \cdot x^\land + x^\land \cdot x^\land$$

$$\Rightarrow -2x^\lor \cdot x^\land = 2x^\lor \cdot x^\land \Rightarrow x^\lor \cdot x^\land = 0 \Rightarrow x^\lor \perp x^\land.$$

**Lemma 3.3.5** $x_1 \lor x_2 \cdot x_1 \land x_2 = x_1 \cdot x_2$

Proof: From Lemma 3.3.4 we have $(x_1 - x_2)^\lor \perp (x_1 - x_2)^\land$; further,

$$(x_1 - x_2)^\lor \perp (x_1 - x_2)^\land \Rightarrow (x_1 - x_2) \lor 0 \perp (x_1 - x_2) \land 0$$
$$\Rightarrow (x_1 - x_2) \lor 0 + x_2 - x_2 \perp (x_1 - x_2) \land 0 + x_2 - x_2$$
$$\Rightarrow x_1 \lor x_2 - x_2 \perp x_1 \land x_2 - x_2 \Rightarrow (x_1 \lor x_2 - x_2) \cdot (x_1 \land x_2 - x_2) = 0$$
$$\Rightarrow x_1 \lor x_2 \cdot x_1 \land x_2 - x_1 \lor x_2 \cdot x_2 - x_2 \cdot x_1 \land x_2 + x_2 \cdot x_2 = 0$$
$$\Rightarrow x_1 \lor x_2 \cdot x_1 \land x_2 - x_2 \cdot (x_1 \lor x_2 + x_1 \land x_2) + x_2 \cdot x_2 = 0$$
$$\Rightarrow x_1 \lor x_2 \cdot x_1 \land x_2 - x_2 \cdot (x_1 + x_2) + x_2 \cdot x_2 = 0$$
$$\Rightarrow x_1 \lor x_2 \cdot x_1 \land x_2 - x_2 \cdot x_1 = 0.$$

$$\Rightarrow x_1 \lor x_2 \cdot x_1 \land x_2 = x_1 \cdot x_2. \blacksquare$$

**Corollary 3.3.6** For a normed vector lattice $x_1 \lor x_2 \perp x_1 \land x_2 \Leftrightarrow x_1 \perp x_2$. 

3.3.3.3 Positive Fixed-Points

Let $S$ swap $P$ then, given $z = Pz$, from Lemma 3.1.8 $z \perp Sz$, so the decomposition $z + Sz = z \lor Sz$ is an orthogonal decomposition in view of Corollary 3.3.6. Multiplying each side of this equation by the idempotent $P$ we have the decomposition

$$z = P(z \lor Sz) + P(z \land Sz),$$

which is also an orthogonal decomposition in view of the following

**Lemma 3.3.7** $(z \lor Sz) \perp (z \land Sz) \Rightarrow P(z \lor Sz) \perp P(z \land Sz)$.

**Proof.** $SP(z \lor Sz) = SPS(z \lor Sz) = (I - P)(z \lor Sz) = (z \lor Sz) - P(z \lor Sz),$

hence

$$SP(z \lor Sz) = (z \lor Sz) - P(z \lor Sz).$$

Taking the inner product of this equation w.r.t. $z \land Sz$,

$$(z \land Sz) \cdot SP(z \lor Sz) = (z \land Sz) \cdot (z \lor Sz) - (z \land Sz) \cdot P(z \lor Sz)$$

$$\Rightarrow (z \land Sz) \cdot SP(z \lor Sz) = -(z \land Sz) \cdot P(z \lor Sz)$$

$$(3.2 \text{ d})$$

$$\Rightarrow (z \land Sz) \cdot S^TP(z \lor Sz) = -(z \land Sz) \cdot P(z \lor Sz)$$

$$(3.4 \text{ a})$$

$$\Rightarrow S(z \land Sz) \cdot P(z \lor Sz) = -(z \land Sz) \cdot P(z \lor Sz)$$

$$(3.3.3.2)$$

$$\Rightarrow (z \land Sz) \cdot P(z \lor Sz) = -(z \land Sz) \cdot P(z \lor Sz)$$

$$\Rightarrow (z \land Sz) \cdot P(z \lor Sz) = 0 \Rightarrow (z \land Sz) \cdot P^2(z \lor Sz) = 0$$

$$(3.4)$$

$$\Rightarrow (z \land Sz) \cdot P^TP(z \lor Sz) = 0 \Rightarrow P(z \land Sz) \cdot P(z \lor Sz) = 0$$

$$\Rightarrow P(z \lor Sz) \perp P(z \land Sz). \square$$
3.3. COMPUTATIONAL METHODS

Summing up:

\begin{align*}
\text{(L.3.1.8)} & \quad z = Pz \\
\text{(C.3.3.6)} & \quad \Rightarrow z \perp Sz \\
\text{(3.3.7)} & \quad \Rightarrow P(z \lor Sz) \perp P(z \land Sz)
\end{align*}

Note also that

\begin{align*}
P(z \lor Sz) \lor P(z \land Sz) & + P(z \lor Sz) \land P(z \land Sz) \\
& = P(z \lor Sz) + P(z \land Sz) = P(z \lor Sz + z \land Sz) \\
& = P(z + Sz) = z;
\end{align*}

so we have the orthogonal decomposition

\begin{align*}
z &= z_1 + z_2, \quad \text{where} \\
z_1 &= P(z \lor Sz) \lor P(z \land Sz), \quad \text{and} \\
z_2 &= P(z \lor Sz) \land P(z \land Sz).
\end{align*}

Since $z_1$ and $z_2$ are orthogonal,

\begin{align*}
\|z_1 - z_2\| &= \|z\|, \\
\|P(z_1 - z_2)\| &\leq \|z\|
\end{align*}

and so

\begin{align*}
\|Pz_1 - Pz_2\| &\leq \|z\|.
\end{align*}

Let $p$ be a non-negative fixed-point of $P$, that is $p \geq 0$ and $Pp = p$. We now show that the $p$-proportion of $Pz_1 - Pz_2$ is at least as great as that of $z$.

To this end note that, from Equation 3.19,

\begin{align*}
p \cdot (Pz_1 + Pz_2) = p \cdot P(z_1 + z_2) = (P^T p) \cdot (z_1 + z_2) \\
= (Pp) \cdot (z_1 + z_2) = p \cdot z; \quad \text{that is}
\end{align*}

\begin{align*}
p \cdot (Pz_1 + Pz_2) = p \cdot z.
\end{align*}
Also note that $p \cdot P(z_1) = p \cdot [P(z \lor Sz) \lor P(z \land Sz)] = (p \cdot P) \cdot [P(z \lor Sz) \lor P(z \land Sz)] = (Pp) \cdot [P(z \lor Sz) \lor P(z \land Sz)] = p \cdot (z \lor Sz) \geq p \cdot z$. and so

$$p \cdot (Pz_1) \geq p \cdot z ;$$  \hspace{1cm} (3.22)

From Equations 3.21 and 3.22,

$$p \cdot (Pz_1 - Pz_2) \geq p \cdot z ;$$ \hspace{1cm} (3.23)

In view of Equation 3.23, the $p$-quantity in $Pz_1 - Pz_2$ is at least equal to that of $z$, and from Equation 3.20 it follows that the $p$ proportion of $Pz_1 - Pz_2$ is strictly greater than that of $z$, unless $Pz_1 - Pz_2 = z_1 - z_2$; but $z_1 - z_2 \geq 0$, so this would mean we had reached a non-negative fixed-point. ⇒

Thus the recursion

$$z_1 = P1$$

$$z_{n+1} = P[P(z_n \lor Sz_n) \lor P(z_n \land Sz_n) - P(z_n \lor Sz_n) \land P(z_n \land Sz_n)]$$

converges to a positive fixed-point of $P$. 
Chapter 4
The Invariant Problems

We introduce the invariant problems then primal and dual function-pairs and invariant linear programming problems then we relate the invariant and original problems using these function-pairs.

Notation in this and subsequent chapters differs from previous chapters in that vectors are no longer necessarily denoted by boldface; the reader may have to discern the nature of symbols from definitions and context. In the following computations the superscript + denotes the Moore-Penrose pseudo-inverse which was defined in 2.2.5.

From the original problems we compute what we call invariant problems.

Substituting $A$ for $A$, $b$ for $b$ and $c$ for $c$ in Equation 2.29

$$\max \{c^T x : Ax \geq b\} = \min \{b^T y : \Delta y = c, \eta \leq 0\},$$
if either exists. \hfill (4.1)

Now $\{b^T y : \Delta y = c, \eta \leq 0\} = \{b^T y : (I-\Delta)y = c, \eta \leq 0\} = \{b^T y : \eta - \Delta y = c, \eta \leq 0\} = \{b^T y : \eta \geq 0, -\Delta y = c, \eta \geq 0\}$, that is

$$\{b^T y : \Delta y = c, \eta \leq 0\} = -\{b^T y : \Delta y - c = \eta \geq 0\}$$

so (4.1) can be written

$$\max \{c^T x : Ax \geq b\} = \min(-\{b^T y : \Delta y - c = \eta \geq 0\}) = -\max\{b^T y : \Delta y - c = \eta \geq 0\},$$
if either max or min exists. \hfill (4.3)

or

$$\max \{c^T x : Ax \geq b\} + \max\{b^T y : \Delta y - c = \eta \geq 0\} = 0$$
if either maximum exists. \hfill (4.4)

Lemma 4.0.8

(a) $\{c^T x : Ax - b \geq 0\} = \{c^T x : Ax \geq b\}$
(b) $\{b^T y : \Delta y - c = \eta \geq 0\} = \{b^T y : \Delta y \geq c\}$
CHAPTER 4. THE INVARIANT PROBLEMS

Proof: (a): \( \{c^T \mathbf{r} : \mathbf{A} \mathbf{r} - \mathbf{b} = \mathbf{r} \geq \mathbf{0}\} \subseteq \{c^T \mathbf{r} : \mathbf{A} \mathbf{r} - \mathbf{b} \geq \mathbf{0}\} = \{c^T \mathbf{r} : \mathbf{A}(\mathbf{A} \mathbf{r} - \mathbf{b}) - \mathbf{b} = \mathbf{A} \mathbf{r} - \mathbf{b} \geq \mathbf{0}\} = \{c^T(\mathbf{A} \mathbf{r} - \mathbf{b}) : \mathbf{A}(\mathbf{A} \mathbf{r} - \mathbf{b}) - \mathbf{b} = \mathbf{A} \mathbf{r} - \mathbf{b} \geq \mathbf{0}\} \subseteq \{c^T : \mathbf{A} \mathbf{r} - \mathbf{b} = \mathbf{r} \geq \mathbf{0}\}. (b): proof is analogous to proof of (a). □

From (4.4) and Lemma 4.0.8 (a)

**Lemma 4.0.9** \( \max\{c^T \mathbf{r} : \mathbf{A} \mathbf{r} - \mathbf{b} = \mathbf{r} \geq \mathbf{0}\} + \max\{b^T \mathbf{y} : \mathbf{D} \mathbf{y} - \mathbf{c} = \mathbf{y} \geq \mathbf{0}\} = 0 \), if either maximum exists.

and from (4.4) and Lemma 4.0.8 (b)

**Lemma 4.0.10** \( \max\{c^T \mathbf{r} : \mathbf{A} \mathbf{r} \geq \mathbf{b}\} + \max\{b^T \mathbf{y} : \mathbf{D} \mathbf{y} \geq \mathbf{c}\} = 0 \). if either maximum exists.

In view of these computations we define the *invariant primal* as

\[ \text{find the vectors } \mathbf{r} \text{ which maximize } c^T \mathbf{r}, \text{ subject to } \mathbf{A} \mathbf{r} \geq \mathbf{b}. \] (4.5)

and the *invariant dual* as

\[ \text{find the vectors } \mathbf{y} \text{ which maximize } b^T \mathbf{y}, \text{ subject to } \mathbf{D} \mathbf{y} \geq \mathbf{c}. \] (4.6)

Note that the invariant programs are always quasi-bounded as \( \mathbf{A}^+ \mathbf{A} = \mathbf{A}^2 = \mathbf{A} \mathbf{c} = \mathbf{c} \), and \( \mathbf{D}^+ \mathbf{D} = \mathbf{D}^2 \mathbf{b} = \mathbf{D} \mathbf{b} = \mathbf{b} \).

4.1 The Primal and Dual Function Pairs

The original and invariant problems are related, in the next section, using two function-pairs as follows:

the primal function-pair \( \{f_p, f_p\} \) where

\[
\begin{align*}
    f_p : \mathbb{R}^m &\to \mathbb{R}^m, \mathbf{x} \mapsto A \mathbf{x} - \mathbf{b}, \text{ and} \\
    f_p : \mathbb{R}^m &\to \mathbb{R}^m, \mathbf{x} \mapsto A^+(\mathbf{x} + \mathbf{b})
\end{align*}
\] (a) (b) (4.7)

and the dual function-pair \( \{f_d, f_d\} \) where

\[
\begin{align*}
    f_d : \mathbb{R}^m &\to \mathbb{R}^m, \mathbf{y} \mapsto -\mathbf{D} \mathbf{y} - \mathbf{c} \text{ and} \\
    f_d : \mathbb{R}^m &\to \mathbb{R}^m, \mathbf{y} \mapsto \mathbf{c} - \mathbf{D} \mathbf{y}
\end{align*}
\] (a) (b) (4.8)

We label the solution sets as

<table>
<thead>
<tr>
<th>Original</th>
<th>Invariant</th>
</tr>
</thead>
<tbody>
<tr>
<td>primal</td>
<td>( X = {x : A \mathbf{x} \geq \mathbf{0}} )</td>
</tr>
<tr>
<td>dual</td>
<td>( Y = {y : A^T \mathbf{y} = \mathbf{c}, y \leq 0} )</td>
</tr>
</tbody>
</table>

For the primal mappings we compute the compositions

\[
\begin{align*}
    x f_p f_p & = (A \mathbf{x} - \mathbf{b}) f_p = A^+((A \mathbf{x} - \mathbf{b}) + \mathbf{b}) = A^+ A \mathbf{x}, \text{ and } f_p f_p = (A^+(\mathbf{x} + \mathbf{b})) f_p = A(A^+(\mathbf{x} + \mathbf{b})) - \mathbf{b}
\end{align*}
\] (4.7 a) (4.7 b)
4.1. THE PRIMAL AND DUAL FUNCTION PAIRS

\[ 4.1. \quad \text{THE PRIMAL AND DUAL FUNCTION PAIRS} \]

\[ 51 \]

\[ A + (\mathfrak{A} + \mathfrak{B} - b) = \mathfrak{A} - b , \text{ that is} \]

\[ x f_p f_p = A^+ A x \quad \text{(4.9)} \]

\[ y f_d f_d = \mathfrak{D} y + c \quad \text{(4.11)} \]

\[ \eta f_d f_d = \mathfrak{D} \eta - c \quad \text{(4.12)} \]

The central feasible sets w.r.t. the function-pairs for the original and invariant problems are indicated by the blue boxes of Figure 4.3.

For the primal mappings we compute the triple compositions

\[ x f_p f_p = (A^+ A x) f_p = A(A^+ A x) - b = A x - b = x f_p , \quad \text{(4.7a)} \]

\[ y f_d f_d = \mathfrak{D} y + c \quad \text{(4.11)} \]

\[ \eta f_d f_d = \mathfrak{D} \eta - c \quad \text{(4.12)} \]

From (4.13) and (4.14) we see that the function-pair \( \{ f_p, f_p \} \) is regular, and from (4.15) and (4.16) that the function-pair \( \{ f_d, f_d \} \) is regular, so Lemma 2.1.1 applies giving Theorem 4.1.1

\[ \text{(a) The four central mappings } f_p, f_p, f_{d c}, f_{d c} \text{ are bijective.} \]

\[ \text{(b) The primal and dual central function-pairs } \{ f_p, f_{d c} \} \text{ and } \{ f_{d c}, f_p \} \text{ comprise mutually inverse functions.} \]

\[ \text{(c) The conditions } x = A^+ A x, x = \mathfrak{A} x - b, y = \mathfrak{D} y + c, \text{ and } \eta = \mathfrak{D} \eta - c \text{ define precisely the central elements of } X, \mathfrak{X}, Y, \text{ and } \mathfrak{Y} \text{ respectively.} \]

Proof: (a) and (b) follow since the function-pairs \( \{ f_p, f_p \} \) and \( \{ f_d, f_d \} \) satisfy the conditions for Lemma 2.1.1; (c) follows from 2.1.1(f) or (g), or Lemma 2.1.2, together with (4.9), (4.10), (4.11) and (4.12) respectively. □
\( \max \{ c^T x : Ax \geq b, x = A^+ A x \} \)

Figure 4.1: Original and Invariant Primal Solutions
4.2 Problem Comparison

We show that there are, effectively, one-to-one correspondences between solutions to the original and invariant problems. Under these correspondences original residuals are equated with invariant residuals, original feasible solutions are shown to correspond to invariant feasible solutions, original optimal feasible solutions to invariant optimal feasible solutions, and original objective values are shown to differ by a fixed quantity from invariant objective values (with a sign change for the dual problems). We conclude by showing that complementary slackness corresponds to optimality, in the context of the invariant primal.

The condition that the original problem be quasi-bounded is necessary in order that the one-to-one relationship exist between the original and invariant problems established in this section.

4.2.1 Equivalence of Residuals

For the primal and dual function-pairs the maps leave residuals unchanged, as can be seen from the following

**Lemma 4.2.1**

(a) \( \mathbf{A}(xf_p) - b = Ax - b \)

(b) \( A(\bar{r}_p) - b = \bar{A}x - b \)

(c) \( D(yf_d) - c = -y \)

(d) \( \eta_{f_d} = -(D\eta - c) \)

**Proof:**

(a) \( \mathbf{A}(xf_p) - b = xf_p(f_p) = x f_p = Ax - b \)

(b) \( A(\bar{r}_p) - b = (\bar{r}_p)f_p = r(f_p) = \bar{A}x - b \)

(c) \( D(yf_d) - c = D(-Dy - c) - c = Dy - c \)

\( = (AA^+)^T x y - y - c = A^T x A^Ty - y - c = -y + A^T c - c = -y \)

(d) \( \eta_{f_d} = c - D\eta = -(D\eta - c) \).

**Lemma 4.2.2**

(a) \( \mathbf{A}(\bar{x}f) - b = \bar{A}x - b \)

(b) \( D(D\eta - c) - c = D\eta - c \)

**Proof:**

(a) \( \mathbf{A}(\bar{x}f) - b = (\bar{x}f)f_p = \bar{x}f_p = \bar{A}x - b \)

(b) \( D(D\eta - c) - c = (D\eta - c)f_p = \eta_f f_p = D\eta - c \).

4.2.2 Feasible Vectors Correspond

From Lemma 4.2.1 we see that the function-pairs induce a correspondence between feasible vectors:
Corollary 4.2.3
\[
\begin{align*}
(a) & \quad Ax \geq b \iff \mathcal{X}(xf_p) \geq b \\
(b) & \quad Ax \geq b \iff A\mathcal{Y}(yf_d) \geq b \\
(c) & \quad D(yf_d) \geq \xi \iff y \leq 0 \quad \text{if } A^Ty = c \\
(d) & \quad (A^T\eta_d) = c \land \eta_d \leq 0 \iff D\eta \geq c \quad \text{if Problem 1.1 is quasi-bounded}
\end{align*}
\]

It follows, from Lemma 2.1.3, that feasible central points of the original and invariant problems are in one-to one correspondence.

### 4.2.3 Equivalence of Objective Values

Next we relate the objective values of elements and their images.

The objective values of vectors are related by the primal function-pair as follows:

\[
\begin{align*}
(a) & \quad c^T x f_p = c^T (Ax - b) = c^T Ax - c^T b \\
(b) & \quad c^T \mathcal{Y}(f) = c^T \mathcal{X} + b^T \xi \\
(c) & \quad b^T y f_d = b^T \xi - b^T y \\
(d) & \quad b^T \mathcal{Y} \eta_d = b^T \xi - b^T \eta
\end{align*}
\]

**Proof:**

\[
\begin{align*}
(a) & \quad c^T x f_p = c^T (Ax - b) = c^T Ax - c^T b \\
& \quad = (A^Tc)^T Ax - c^T b = c^T A^T Ax - c^T b = c^T x - b^T \xi \\
(b) & \quad c^T \mathcal{Y}(f) = c^T (A^T (x + b)) = (A^Tc)^T (x + b) \\
& \quad = (A^Tc)^T (x + b) = c^T (x + b) = c^T x + b^T \xi \\
(c) & \quad b^T y f_d = (Db)^T y f_d = b^T D(y f_d) = b^T (y + c) = b^T \xi - b^T y \\
(d) & \quad b^T \mathcal{Y} \eta_d = b^T (\xi - D\eta) = b^T \xi - b^T \eta.
\end{align*}
\]

We compare the objective values of the original and invariant problems, with emphasis on centrality, showing a very simple linear relationship between original and invariant objective values, and showing that the notion of complementary slackness applies to both types of problem.

The central mappings preserve objective values:

\[
\begin{align*}
(a) & \quad c^T x f_p = c^T x \\
(b) & \quad c^T \mathcal{Y} f_p = c^T \mathcal{Y} \\
(c) & \quad b^T y f_d = b^T y \\
(d) & \quad b^T \mathcal{Y} \eta_d = b^T \eta
\end{align*}
\]

**Proof:** Straightforward application of the primal and dual function-pairs.

The original objective functions and invariant objective functions are related as follows:

\[
\begin{align*}
(a) & \quad c^T x f_p + b^T y f_d = c^T x - b^T y \\
(b) & \quad c^T \mathcal{Y} f_p - b^T \mathcal{Y} \eta_d = c^T \mathcal{X} + b^T \eta
\end{align*}
\]

**Proof:**

(a) compute L 4.2.4 (a) + (c)

(b) compute L 4.2.4 (b) - (d).
4.3. SOLUTION CONDITIONS

4.2.4 Solutions Correspond

For the primal problems we have \( c^T x_1 = c^T (x_1 f_p) + c^T b \) and \( c^T x_2 = c^T (x_2 f_p) + c^T b \), so \( c^T x_1 \geq c^T x_2 \Rightarrow c^T (x_1 f_p) + c^T b \geq c^T (x_2 f_p) + c^T b \Rightarrow c^T (x_1 f_p) \geq c^T (x_2 f_p) \) and it follows, since \( f_p \) is onto, that if \( x_1 \) is maximal feasible then \( x_1 f_p \) is also, showing that solutions to the original primal are mapped to central solutions to the invariant primal. Similarly \( c^T y_1 \geq c^T y_2 \Rightarrow c^T (y_1 f) \geq c^T (y_2 f) \) which shows that solutions to the invariant primal are mapped to central solutions to the original primal. Central solutions to the original and invariant primals, of course, correspond one to one.

For the dual problems we have \( b^T y_1 = b^T c - b^T (y_1 f_d) \) and \( b^T y_2 = b^T c - b^T (y_2 f_d) \), so \( b^T y_1 \leq b^T y_2 \Rightarrow b^T c - b^T (y_1 f_d) \leq b^T c - b^T (y_2 f_d) \Rightarrow b^T (y_1 f_d) \geq b^T (y_2 f_d) \) and it follows, since \( f_d \) is onto, that if \( x_1 \) is minimal feasible then \( x_1 f_d \) is maximal feasible, showing that solutions to the original primal are mapped to central solutions to the invariant primal. Similarly \( c^T y_1 \geq c^T y_2 \Rightarrow c^T (y_1 f) \geq c^T (y_2 f) \) which shows that solutions to the invariant primal are mapped to central solutions to the original primal. Central solutions to the original and invariant primals, of course, correspond one to one.

The relationship between the original and invariant forms of the dual is summarised by Figure 4.2.

4.3 Solution Conditions

We derive conditions for the objective function of the original primal to be maximal, however the derivation is predicated on the existence of a maximal solution, so first we consider attainment.

4.3.1 Attainment

Assuming we have a feasible bounded original primal we know, from Lemma 4.2.4, that the invariant primal is feasible bounded; we now show that this maximum is attained.

Consider the invariant primal: \( \text{max } c^T \mathbf{r} \) such that \( \mathbf{A} \mathbf{r} \geq \mathbf{b} \), and note that if \( \mathbf{A} \mathbf{r} \geq \mathbf{b} \) then \( \mathbf{A} \mathbf{r} \geq \mathbf{0} \) and \( \mathbf{A}(\mathbf{A} \mathbf{r} \geq \mathbf{b}) = \mathbf{A} \mathbf{r} - \mathbf{A} \mathbf{b} = \mathbf{A} \mathbf{r} \geq \mathbf{b} \) and, further, \( c^T (\mathbf{A} \mathbf{r} - \mathbf{b}) = c^T \mathbf{A} \mathbf{r} - c^T \mathbf{b} = (c^T \mathbf{A}) \mathbf{r} = c^T \mathbf{r} \), so the problem is equivalent to \( \text{max } c^T \mathbf{r} \) such that \( \mathbf{A} \mathbf{r} \geq \mathbf{b}, \mathbf{r} \geq \mathbf{0} \).

Also consider the invariant dual: \( \text{max } b^T \eta \) such that \( \mathbf{D} \eta \geq \mathbf{c} \), and note that if \( \mathbf{D} \eta \geq \mathbf{c} \) then \( \mathbf{D} \eta \geq \mathbf{c} \) and \( \mathbf{D}(\mathbf{D} \eta \geq \mathbf{c}) = \mathbf{D}^2 \eta \geq \mathbf{c} \) and, further, \( b^T (\mathbf{D} \eta - \mathbf{c}) = b^T \mathbf{D} \eta - b^T \mathbf{c} = (b^T \mathbf{D}) \eta = b^T \eta \), so the problem is equivalent to \( \text{max } b^T \eta \) such that \( \mathbf{D} \eta \geq \mathbf{c}, \eta \geq \mathbf{0} \).

If \( \mathbf{f} \) is an arbitrary feasible solution to the invariant primal and \( \eta \) is an arbitrary feasible solution for the invariant dual then \( (\mathbf{A} \mathbf{f} - \mathbf{b})^T (\mathbf{D} \eta - \mathbf{c}) \geq 0 \Rightarrow \mathbf{f}^T \mathbf{A} \mathbf{D} \eta - \mathbf{f}^T \mathbf{A} \mathbf{c} - b^T \mathbf{D} \eta + b^T \mathbf{c} \geq 0 \Rightarrow -\mathbf{f}^T \mathbf{c} - b^T \eta \geq 0 \Rightarrow c^T \mathbf{f} + b^T \eta \leq 0 \)

which implies \( b^T \eta \leq -c^T \mathbf{f} \), that is the invariant dual is feasible bounded. Summing up we have:
CHAPTER 4. THE INVARIANT PROBLEMS

Figure 4.2: Original and Invariant Dual Problems
Lemma 4.3.1 \textit{The invariant primal is feasible bounded iff the invariant dual is feasible bounded.}

With the following lemma we take a more analytic approach than the usual constructive proof:

Lemma 4.3.2 \textit{If the invariant primal (dual) is feasible bounded then a feasible vector exists which attains this bound.}

Proof: Suppose $f = \max \{c^T x : Ax \geq b\}$ and consider the sequence of central solutions $\{x^{(1)}\}$ where $\lim_{i \to \infty} c^T x^{(1)}_i = f$; let $S\{x^{(1)}\} = \{ j : (x^{(1)}_i)_j = 0 \forall i \}$. Assume $\{x^{(1)}\}$ has no accumulation point, so $\{x^{(1)}\}$ has a sub-sequence $\{x^{(1)}_i\}$ with $\lim_{i \to \infty} c^T x^{(1)}_i = f$ and $||x^{(1)}_i|| \to \infty$.

Consider the sequence $\{q^{(2)}\}$ defined by $q^{(2)}_i = x^{(1)}_i / ||x^{(1)}_i||$, with accumulation point say $q_2$, with sub-sequence $\{q^{(2s)}\}$ defined by $q^{(2s)}_i = x^{(2s)}_i / ||x^{(2s)}_i||$ with $\lim_{s \to \infty} q^{(2s)}_i = q_2$ where $\{x^{(2s)}\}$ is a sub-sequence of $\{x^{(1)}\}$ and of course $||x^{(2s)}_i|| \to \infty$, then $2q^{(2s)}_i - b = x^{(2s)}_i - 2q^{(2s)}_i / ||x^{(2s)}_i|| \geq 0 \Rightarrow 2q^{(2s)}_i / ||x^{(2s)}_i|| = x^{(2s)}_i / ||x^{(2s)}_i|| \geq 0 \Rightarrow 2q^{(2s)}_i / ||x^{(2s)}_i|| = q^{(2s)}_i \geq 0 \Rightarrow 2q^{(2s)}_i = q_2 \geq 0$, where $||q_2|| = 1$, and $c^T q_2 = c^T \lim q^{(2s)}_i = \lim c^T x^{(2s)}_i / ||x^{(2s)}_i|| = \lim f / ||x^{(2s)}_i|| = 0$, that is $c^T q_2 = 0$.

Now we form the sequence $\{x^{(2t)}\}$ defined by $x^{(2t)}_i = x^{(2s)}_i - \lambda q_2$, choosing $\lambda_i$ so that $x^{(2t)}_i \geq 0$ and $(x^{(2t)}_i)_j = 0$ for some $j \notin S\{x_i\}$ (where $j$ may vary depending on $i$), so $S\{x^{(2t)}\} \supset S\{x\}$.

Further, $2x^{(2t)}_i - b = 2x^{(2s)}_i - 2\lambda q_2 - b = 2x^{(2s)}_i - b - \lambda_2 A q_2 = x^{(2s)}_i - \lambda_2 A q_2 = x^{(2t)}_i \geq 0$, that is $2x^{(2t)}_i - b = x^{(2t)}_i \geq 0$.

Next, by the pigeon-hole principle $\{x^{(2t)}\}$ has a sub-sequence say $\{x^{(2u)}\}$ which, for some index $k \notin S\{x\}$, has an infinite number of members with $(x^{(2u)}_i)_k = 0$. We form the subsequence $\{x^{(2v)}\}$ of $\{x^{(2u)}\}$ comprising precisely those members for which $(x^{(2u)}_i)_k = 0$.

We have constructed a sequence of central solutions $\{x^{(2v)}\}$ whose limiting objective value is maximal and moreover $S\{x^{(2v)}\} \supset S\{x^{(1)}\} \cup \{k\}$. In this manner we construct a series of sequences whose $S$’s strictly increase. As this cannot continue indefinitely, eventually we reach a sequence of central solution which in the limit attains the maximum and the sequence has an accumulation point which achieves this maximum. \hfill \Box

4.3.2 \textbf{Lagrange’s Method}

Note that this subsection is quite speculative in nature, however Lemma 4.3.3 may also be proved from result 2.28.

Given to vectors, $v_1$ and $v_2$ we define $v_1 v_2$ by $(v_1 v_2)_i = (v_1)_i \ast (v_2)_i$, and $v_1^T v_2 = \sum_i (v_1)_i \ast (v_2)_i$ (that is the usual inner product).

We form the Lagrangian

$$\mathcal{L} = x^T \lambda + \lambda^T (\lambda \lambda - \lambda^2) + \mu^T (\lambda - (\lambda \lambda - \lambda^2))$$
where \( \lambda, \mu \) and \( \mathbf{z} \) are vectors, and \( \mathbf{z}^2 = [z_1^2 \cdots z_m^2]^T \) is consistent with the definition above, then setting the Lagrangian’s first partial derivatives to zero, knowing from Lemma 4.3.2 that the problem not only is feasible bounded but also achieves its maximum for some value \( x_s \), and assuming the existence of a \( \lambda_s \) and a \( \mu_s \) we have

\[
\begin{align*}
(a) & \quad \partial L_s / \partial x = c + \mathbf{A}\lambda_s + \mathbf{D}\mu_s = 0 \\
(b) & \quad \partial L_s / \partial \mathbf{z} = -2\lambda_s \mathbf{z}_s = 0 \\
(c) & \quad \partial L_s / \partial \lambda = \mathbf{A}x_s - \mathbf{b} - \mathbf{z}_s^2 = 0 \\
(d) & \quad \partial L_s / \partial \mu = \mathbf{r}_s - (\mathbf{A}x_s - \mathbf{b}) = 0 \\
(e) & \quad \partial^2 L_s / \partial \mathbf{z}^2 = -2 \text{diag}(\lambda_s) \leq 0
\end{align*}
\]

where \( \partial \mathbf{z}^2 \) has the usual connotation.

Considering the last partial derivative, since the problem is one of maximization, the matrix \( \partial^2 L / \partial \mathbf{z}^2 \) must necessarily be negative semi-definite, so \(-2 \text{diag}(\lambda_s) \leq 0 \) and

\[
\lambda_s \geq 0 .
\]

(4.19)

From (4.18 a), \( c + \mathbf{A}\lambda_s + \mathbf{D}\mu_s = 0 \Rightarrow \mathbf{A}(c + \mathbf{A}\lambda_s + \mathbf{D}\mu_s) = 0 \Rightarrow c + \mathbf{A}\lambda_s = 0 \Rightarrow c + (I - \mathbf{D})\lambda_s = 0 \Rightarrow

\[
\mathbf{D}\lambda_s - \epsilon = \lambda_s
\]

(4.20)

Combining with (4.20):

\[
\mathbf{D}\lambda_s - \epsilon = \lambda_s \geq 0 .
\]

(4.21)

and we see that \( \lambda_s \) is a central solution to the invariant dual.

Further, from (4.18 b) it follows that

\[
\lambda_s \mathbf{z}_s^2 = 0 ,
\]

(4.22)

and from (4.18 c, d) that \( \mathbf{r}_s = \mathbf{z}_s^2 \), so we have the complementary slackness condition

\[
\lambda_s \mathbf{r}_s = 0
\]

(4.23)

noting that this means \( \lambda_{si} \mathbf{r}_s = 0 \) for \( i = 1, \cdots, m \).

From (4.18 d) we also have the predestined

\[
\mathbf{r}_s = \mathbf{A}\mathbf{r}_s - \mathbf{b}
\]

(4.24)

From (4.23) we have the weaker

\[
\lambda_s^T \mathbf{r}_s = 0
\]

(4.25)
4.4. SUMMARY

Finally, $\lambda^T_s r_s = 0 \Rightarrow (\mathcal{D} \lambda_s - c)^T (\mathcal{A} y_s - b) = 0 \Rightarrow$

\[ c^T r_s + b^T \lambda_s = 0. \]  \hspace{1cm} (4.26)

where $c^T r_s$ is maximal, which establishes that the upper bound of Lemma 4.0.9 is attained by a central solution $r_s$ to the invariant primal and a central feasible point $\lambda_s$ for the invariant dual, which feasible point may then be inferred to be a solution to the invariant dual.

The calculations have strengthened (4.17) to

\[ \text{If the original primal is feasible bounded then there exist central solutions} \]

**Lemma 4.3.3** $\xi_f$ and $\eta_f$ for the invariant primal and dual respectively such that

\[ c^T \xi_f + b^T \eta_f = 0. \]

Proof: follows from (4.17), Lemmas 4.3.1 and 4.3.2 and (4.26). \qed

We now see that Equation set 4.18 and the consequent 4.25 provide conditions which are not only necessary but also sufficient for a solution to the invariant primal. Summing up

**Theorem 4.3.4** $\xi_f$ and $\eta_f$ are respective central solutions for the invariant primal and dual

iff $\xi_f$ and $\eta_f$ are respective central feasible vectors for the invariant primal and dual and $c^T \xi_f + b^T \eta_f = r_f^T y_f = 0$.

Proof: $\xi_f$ and $\eta_f$ are respective central solutions for the invariant primal and dual

(L 4.3.3)

\[ \Leftrightarrow c^T \xi_f + b^T \eta_f = 0 \]

\[ \wedge \xi_f, \eta_f \text{ feasible central} \]

\[ \Leftrightarrow (\mathcal{A} y_f)^T (\mathcal{D} \eta_f) + b^T c - b^T (\mathcal{D} \eta_f) - (\mathcal{A} y_f)^T c = 0 \]

\[ \wedge \xi_f, \eta_f \text{ feasible central} \]

\[ \Leftrightarrow (\mathcal{A} y_f - b)^T (\mathcal{D} \eta_f - c) = 0 \]

\[ \wedge \xi_f, \eta_f \text{ feasible central} \]

\[ \Leftrightarrow c^T \xi_f + b^T \eta_f = r_f^T y_f = 0 \]

\[ \wedge \xi_f, \eta_f \text{ feasible central} \]

Definition: The respectively feasible vectors $\xi_f$ and $\eta_f$ satisfy the complementary slackness condition if

$(\mathcal{A} y_f - b)^T (\mathcal{D} \eta_f - c) = 0$; we see from the above proof that

**Theorem 4.3.5** $\xi_f$ and $\eta_f$ are respective central solutions for the invariant primal and dual

iff $\xi_f$ and $\eta_f$ are feasible and satisfy the complementary slackness condition.

Given quasi-boundedness, we can now map back to optimal solutions to the original problems, gaining the result

\[ \max c^T x \text{ s.t. } Ax \geq b = \min b^T y \text{ s.t. } A^T y = c, y \leq 0 \]

**Theorem 4.3.6** in case either exists, and in such case the bound is attained both for the primal and the dual.

4.4 Summary

The results of this chapter are summarised by Figure 4.3, which combines the information in Figures 4.1 and 4.2.
4.5 Examples

4.5.1 A Two Variable Problem

Consider the problem of maximizing $3x_1 + 4x_2$ subject to $x_1 + x_2 \leq 3$, $x_1 - x_2 \leq 1.5$, and $2x_1 + x_2 \geq 4$. The problem can be seen to be feasible and bounded by referring to the following diagram in which the set of feasible solutions is represented by the green triangular area, the constraints are represented by the lines forming the boundary of this area, and the objective function is represented by the indicated line which should be moved until it osculates the hatched area.

The problem can be written in matrix form as

$$
\text{maximize} \begin{bmatrix} 3 \\ 4 \end{bmatrix}^T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ subject to } \begin{bmatrix} -1 & -1 \\ -1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq \begin{bmatrix} -3 \\ -1.5 \\ 4 \end{bmatrix}.
$$

So $A = \begin{bmatrix} -1 & -1 \\ -1 & 1 \\ 2 & 1 \end{bmatrix}$, $b = \begin{bmatrix} -3 \\ -1.5 \\ 4 \end{bmatrix}$, and $c = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. 

Figure 4.3: Original and Invariant Problems
The Moore-Penrose pseudo-inverse can be computed as follows:

\[
A^+ = \left( \begin{bmatrix} -1 & -1 \\ -1 & 1 \\ 2 & 1 \end{bmatrix} \right)^T \left( \begin{bmatrix} -1 & -1 \\ -1 & 1 \\ 2 & 1 \end{bmatrix} \right)^{+} \left( \begin{bmatrix} -1 & -1 \\ -1 & 1 \\ 2 & 1 \end{bmatrix} \right)^T
\]

\[
= \left[ \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix} \right]^{-1} \left[ \begin{bmatrix} -1 & -1 \\ -1 & 1 \\ 2 & 1 \end{bmatrix} \right]^T = \left[ \begin{bmatrix} 3 & -2 \\ -2 & 6 \end{bmatrix} \right] \left[ \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \right] / 14
\]

\[
= \left[ \begin{bmatrix} -1 & -5 & 4 \\ -4 & 8 & 2 \end{bmatrix} \right] / 14,
\]

and

\[
AA^+ = \left[ \begin{bmatrix} -1 & -1 \\ -1 & 1 \\ 2 & 1 \end{bmatrix} \right] \left[ \begin{bmatrix} -1 & -5 & 4 \\ -4 & 8 & 2 \end{bmatrix} \right] / 14 = \left[ \begin{bmatrix} 5 & -3 & -6 \\ -3 & 13 & -2 \\ -6 & -2 & 10 \end{bmatrix} \right] / 14.
\]

\[
b = (I - AA^+)b = \left[ \begin{bmatrix} 9 & 3 & 6 \\ 3 & 1 & 2 \\ 6 & 2 & 4 \end{bmatrix} \right] \left[ \begin{bmatrix} -3 \\ -1.5 \\ 4 \end{bmatrix} \right] / 14 = \left[ \begin{bmatrix} -7.5 \\ -2.5 \\ -5.0 \end{bmatrix} \right] / 14.
\]

\[
c = A^Tc = \left[ \begin{bmatrix} -1 & -4 \\ -5 & 8 \\ -4 & 2 \end{bmatrix} \right] \left[ \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right] / 14 = \left[ \begin{bmatrix} -19 \\ 17 \\ 17 \end{bmatrix} \right] / 14.
\]
Thus the invariant primal is

\[
\text{maximize } \begin{bmatrix} -19 & 17 & 20 \end{bmatrix}^T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} / 14 \quad \text{s.t.} \quad \begin{bmatrix} 5 & -3 & -6 \\ -3 & 13 & -2 \\ -6 & -2 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} / 14 \geq \begin{bmatrix} -7.5 \\ -2.5 \\ -5.0 \end{bmatrix} / 14.
\]

These computations can be effected using the Scheme function `invprob`, provided in the document `solve_lp.drscm`. Note that `solve_lp.drscm` loads `linalg_1.drscm`.

### 4.5.2 The Transportation Problem

Here we take the transport problem, which was introduced in 1.3.2, and specialize the invariant form theory to it.

Recall that the transportation problem has the form

\[
\text{minimize } b^T w \text{ s.t. } A^T_x w = c, \ w \geq 0, \quad (a)
\]

where \( A^T_x = \begin{bmatrix} I_m \otimes 1_n^+ \\ 1_m^+ \otimes I_n \end{bmatrix} \), \quad (b)

and that this formulation has, in view of Equation 2.28, the dual LP formulation

\[
\text{max } c^T x \text{ s.t. } A_x x \leq b,
\]

which has the form given by Equation 1.2 (a).

Define \( I_{m,n} = I_m - m1_m^+1_n^+ \) and \( I_{n,m} = I_n - n1_n^+1_m^+ \),

then \( A^+_x = \begin{bmatrix} I_{m,n} \otimes 1_n^+ \\ 1_m^+ \otimes I_{n,m} \end{bmatrix} \),

so

\[
A_x A^+_x = \begin{bmatrix} I_m \otimes 1_n \otimes I_n \\ 1_m^+ \otimes I_{n,m} \end{bmatrix} \left[ \begin{bmatrix} I_{m,n} \otimes 1_n^+ \\ 1_m^+ \otimes I_{n,m} \end{bmatrix} \right]
\]

\[
= I_{m,n} \otimes 1_n^+1_n^+ + 1_m^+1_m^+ \otimes I_{n,m}
\]

\[
= I_m \otimes 1_n^+1_n^+ + 1_m^+1_m^+ \otimes I_n - 1_m^+1_m^+ \otimes 1_n^+1_n^+.
\]

Note that \( I - A_x A^+_x = I_m \otimes I_n - I_m \otimes 1_n^+1_n^+ - 1_m^+1_m^+ \otimes I_n + 1_m^+1_m^+ \otimes 1_n^+1_n^+ \)

\[
= (I_m - 1_m^+1_m^+ \otimes (I_n - 1_n^+1_n^+)) = [(I_m - 1_m^+1_m^+)I_m] \otimes [I_n(I_n - 1_n^+1_n^+)]
\]

\[
= [(I_m - 1_m^+1_m^+) \otimes I_n][I_m \otimes (I_n - 1_n^+1_n^+)],
\]

and so the invariant constraint vector is

\[
b = (I - A_x A^+_x)b = [(I_m - 1_m^+1_m^+) \otimes I_n][I_m \otimes (I_n - 1_n^+1_n^+)]b ;
\]
with \( \{b_{ij}\} \) written as a matrix, \( b \) is computed by row mean correction of \( b \) followed by column mean correction, or vice versa (since they commute).

The invariant objective vector is \( c = A^T x \)

\[
= \left[ I_{m,n} \otimes 1_n^+ \ 1_m^+ \otimes I_{n,m} \right] \begin{bmatrix} s \\ d \end{bmatrix} \\
= [I_{m,n} \otimes 1_n^+] s + [1_m^+ \otimes I_{n,m}] d \\
= [I_{m,n}s] \otimes 1_n^+ + 1_m^+ \otimes [I_{n,m}d].
\]

Nothing further can be done to simplify the calculation of \( b^T b \), however \( c^T c \) can be simplified as follows:

\[
c^T c = \left\{ s^T I_{m,n} \right\} \otimes 1_n^+ + 1_m^+ \otimes \left\{ d^T I_{n,m} \right\} \times \\
\left\{ [I_{m,n}s] \otimes 1_n^+ + 1_m^+ \otimes [I_{n,m}d] \right\}. \\
= s^T (I_m - 1_m 1_m^n) s/n + \frac{mn}{(m+n)^2} \left( \sigma + \bar{d} \right)^2 + d^T (I_n - 1_n 1_n^m) d/m.
\]

These computations form part of the basis for efficient calculation of an optimal solution to the standard transportation problem.

Finally note that the approach allows a good deal of parallel processing.

### 4.6 Exercises

Prove:

**Corollary 4.6.1** If the problem maximize \( c^T x \) subject to \( Ax \leq b \) is feasible bounded then \( \max \{c^T x : Ax \leq b\} = b^T c - \max \{b^T \eta : D \eta \leq c\} \).
Chapter 5

The Fixed-Point Problem

We represent the linear program 1.1 as the problem of finding the non-trivial non-negative fixed-points of an orthogonal projection matrix. Such representation and the subsequent gradient projection method of solution are the work of Rosen, Pyle, Cline, Bruni et al [36, 31, 32, 9]. The development given here, however, is given in a different LP context and the results are stated with slightly greater generality.

First we specify the fixed-point problem, then we establish a correspondence between solutions to the invariant primal and dual on the one hand and non-negative solutions to a fixed-point problem on the other. The matrix \( \mathcal{P} \) introduced here satisfies the properties of the general \( P \) of Chapter 3 and here we make use the general results of that chapter.

In Chapter 4 we showed that the solution of the original linear program 1.1 and its dual can, provided the original primal is quasi-bounded, be transformed to finding the solutions to the invariant primal and dual, and from the invariant primal and dual to the central invariant primal and dual - that is:

<table>
<thead>
<tr>
<th>Original</th>
<th>Invariant</th>
<th>Central Invariant</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Ax \geq b )</td>
<td>( A^T \mathbf{r} \geq b )</td>
<td>( A^T \mathbf{r} - b = \mathbf{r} \geq 0 )</td>
</tr>
<tr>
<td>( A^T \mathbf{y} = c, \mathbf{y} \leq 0 )</td>
<td>( D \mathbf{\eta} \geq c )</td>
<td>( D \mathbf{\eta} - c = \mathbf{\eta} \geq 0 )</td>
</tr>
<tr>
<td>( c^T x - b^T \mathbf{y} = 0 )</td>
<td>( c^T \mathbf{r} + b^T \mathbf{\eta} = 0 )</td>
<td>( c^T \mathbf{r} + b^T \mathbf{\eta} = 0 )</td>
</tr>
<tr>
<td>( A^+ Ac = c )</td>
<td>( A^+ Ac = c )</td>
<td>( A^+ Ac = c )</td>
</tr>
</tbody>
</table>

In this chapter we perform another step by transforming the invariant problems to a fixed-point problem.
5.1 Heuristic Development

5.1.1 Writing as a Matrix Primal-Dual System

We know, from Lemma 4.3.3, that for solutions, \( c^T x_s + b^T \eta_s = 0 \). It follows that the original linear program can be solved if the augmented primal-dual system

\[
\begin{bmatrix}
\mathbf{A} & 0 & -\mathbf{b} \\
0 & \mathbf{D} & -\mathbf{c} \\
\mu \mathbf{c}^T & \mu \mathbf{b}^T & 0
\end{bmatrix}
\begin{bmatrix}
x \\
\eta \\
1
\end{bmatrix} \geq 0, \quad \mu > 0,
\]

can be solved.

That this system can be solved implies in turn that there exist \( x, \eta \) and \( z \), with \( z \) positive such that the system

\[
\begin{bmatrix}
\mathbf{A} & 0 & -\mathbf{b} \\
0 & \mathbf{D} & -\mathbf{c} \\
\mu \mathbf{c}^T & \mu \mathbf{b}^T & 0
\end{bmatrix}
\begin{bmatrix}
x \\
\eta \\
z
\end{bmatrix} \geq 0, \quad \mu > 0
\]

can be solved.

We write this system as

\[
Qw_a \geq 0, \quad \text{where (a)}
\]

\[
Q = \begin{bmatrix}
\mathbf{A} & 0 & -\mathbf{b} \\
0 & \mathbf{D} & -\mathbf{c} \\
\mu \mathbf{c}^T & \mu \mathbf{b}^T & 0
\end{bmatrix}, \quad \text{and (b)}
\]

\[
w_a = \begin{bmatrix}
x \\
\eta \\
z
\end{bmatrix}. \quad \text{(c)}
\]

5.1.2 Transformation to a Projection Problem

\[
Qw_a \geq 0, \quad \text{where (a)}
\]

\[
QQ^+ Qw_a \geq 0, \quad \text{where (b)}
\]

\[
Q = \begin{bmatrix}
\mathbf{A} & 0 & -\mathbf{b} \\
0 & \mathbf{D} & -\mathbf{c} \\
\mu \mathbf{c}^T & \mu \mathbf{b}^T & 0
\end{bmatrix}, \quad \text{and (c)}
\]

\[
w_a = \begin{bmatrix}
x \\
\eta \\
z
\end{bmatrix}. \quad \text{(c)}
\]

Define

\[
\omega_a = Qw_a \quad \text{(5.3)}
\]
and

$$P_a = QQ^+, \quad (5.4)$$

which implies that the following system can be solved:

$$P_a\omega_a = \omega_a \geq 0. \quad (5.5)$$

So the LP problem has been transformed to a new problem - that of finding a non-negative fixed-point of an orthogonal projection.

Now, from Appendix 5.A,

$$Q^+ = \begin{bmatrix} \mathbf{A} - \frac{\mu^2 \mathbf{c}\mathbf{c}^T}{\eta} & -\frac{\mu^2 \mathbf{c}\mathbf{b}^T}{\nu} & \frac{\mu\mathbf{c}}{\nu} \\ \frac{\mu^2 \mathbf{b}\mathbf{c}^T}{\nu} & \mathbf{D} - \frac{\mu^2 \mathbf{b}\mathbf{b}^T}{\nu} & \frac{\mu\mathbf{b}}{\nu} \\ -\frac{\mathbf{b}^T}{\eta} & -\frac{\mathbf{c}^T}{\eta} & 0 \end{bmatrix}, \quad (5.6)$$

where

$$\eta = \mathbf{b}^T\mathbf{b} + \mathbf{c}^T\mathbf{c}, \quad (5.7)$$

and

$$\nu = 1 + \mu^2\eta. \quad (5.8)$$

From Equations 5.1 (b), 5.4 and 5.6,

$$P_a = \begin{bmatrix} \mathbf{A} + \frac{\mathbf{b}\mathbf{b}^T}{\eta} & \frac{\mathbf{b}\mathbf{c}^T}{\eta} & \frac{\mu\mathbf{c}}{\nu} \\ \frac{\mu^2 \mathbf{b}\mathbf{c}^T}{\nu} & \mathbf{D} + \frac{\mathbf{c}\mathbf{b}^T}{\nu} & \frac{\mu\mathbf{b}}{\nu} \\ \frac{\mu\mathbf{c}^T}{\nu} & \frac{\mu^2 \mathbf{c}^T}{\nu} & \frac{\mu^2 \eta}{\nu} \end{bmatrix}, \quad (5.9)$$

5.1.3 Simplification of the Projection Problem

We have, using calculations from Appendix 5.B the decomposition

$$P_a = \begin{bmatrix} \mathbf{A} & 0 & 0 \\ 0 & \mathbf{D} & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \mathbf{b} & \mathbf{b}^+ & \mathbf{c} & \mathbf{c}^+ \end{bmatrix} + \begin{bmatrix} \mathbf{b} \\ \mathbf{b} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{c} \\ \mathbf{b} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (5.10)$$

We continue by specializing Equation 5.9 by letting $\mu$ approach infinity, in which case

$$\lim_{\mu \to \infty} \begin{bmatrix} \mathbf{c} \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$
CHAPTER 5. THE FIXED-POINT PROBLEM

and

$$\lim_{\mu \to \infty} \mathcal{P}_\mu = \begin{bmatrix} A & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} b \\ c \\ 0 \end{bmatrix}^+ - \begin{bmatrix} c \\ b \\ 0 \end{bmatrix}^+ + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}^+.$$ 

The fourth term of this expression for $\mathcal{P}_\infty$ is redundant, so the problem can in essence be written as

Find $\omega$ s.t.

$$\mathcal{P}_\omega = \omega \geq 0,$$  \hspace{1cm} (a)

where

$$\mathcal{P} = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} + \begin{bmatrix} b \\ c \end{bmatrix}^+ - \begin{bmatrix} c \\ b \end{bmatrix}^+,$$  \hspace{1cm} (b)

and

$$\omega = \begin{bmatrix} \xi \\ \zeta \end{bmatrix}.$$  \hspace{1cm} (c)

Equation set (5.11) is called the heuristic fixed-point problem. In the following section the connection between this problem and the invariant problems is spelt out precisely.

5.2 Precise Development

A fixed-point problem is constructed from the invariant problems including a precise correspondence between its solutions and the solutions to the invariant problems. Specifically, from central solutions to the invariant problems we construct solutions to the fixed-point problem; we investigate the properties of these fixed-point solutions finding the necessary and sufficient conditions for such solution to yield solutions to the invariant problems.

5.2.1 Preliminaries

A point $\omega$ is said to be a fixed-point if $\mathcal{P}_\omega = \omega$. For partitioned points of the form $\omega = \begin{bmatrix} \xi \\ \zeta \end{bmatrix}$, where $\xi$ and $\zeta$ are $m$ dimensional vectors, we will say that $\omega$ is a feasible-point if $\xi$ and $\zeta$ are feasible for the invariant primal and dual respectively, we will say that $\omega$ is a central point if $A\xi - c = \xi$ and $D\zeta - b = \zeta$, that $\omega$ is a quasi-optimal point if $c^T \xi + b^T \zeta = 0$, and that $\omega$ is a solution-point if $\xi$ and $\zeta$ are respective solutions to the invariant primal and dual.

From Equation 4.4 and the definition of centrality we have

**Lemma 5.2.1** $\omega = \begin{bmatrix} \xi \\ \zeta \end{bmatrix}$ is a central solution-point iff

(a) $A\xi - b = \xi \geq 0$

(b) $D\zeta - c = \zeta \geq 0$

(c) $c^T \xi + b^T \zeta = 0$
5.2. PRECISE DEVELOPMENT

We define the following short forms:

\[
\begin{align*}
\Pi_\alpha &= \begin{bmatrix} \alpha & 0 \\ 0 & \mathbb{D} \end{bmatrix} \\
\beta &= \begin{bmatrix} b \\ c \end{bmatrix} \\
\gamma &= \begin{bmatrix} c \\ b \end{bmatrix} \\
\Pi_\beta &= \beta \beta^+ \\
\Pi_\gamma &= \gamma \gamma^+
\end{align*}
\]

(5.12)

then

\[
\begin{align*}
\Pi_\alpha^2 &= \Pi_\alpha \quad \text{(a)} \\
\Pi_\alpha \Pi_\beta &= 0 \quad \text{(b)} \\
\Pi_\alpha \Pi_\gamma &= \Pi_\gamma \quad \text{(c)} \\
\Pi_\beta^2 &= \Pi_\beta^T \quad \text{(d)} \\
\Pi_\beta &= \Pi_\beta \quad \text{(e)} \\
\Pi_\gamma^2 &= \Pi_\gamma \quad \text{(f)} \\
\Pi_\alpha \Pi_\beta &= 0 \quad \text{(g)} \\
\Pi_\alpha \Pi_\gamma &= 0 \quad \text{(h)} \\
\Pi_\beta \Pi_\gamma &= 0 \quad \text{(i)} \\
\Pi_\beta^2 &= \Pi_\beta \quad \text{(j)} \\
\Pi_\gamma^2 &= \Pi_\gamma \quad \text{(k)} \\
0 + \Pi_\beta - \Pi_\gamma &= 0 \quad \text{(L2.2.7)} \\
\gamma^T \omega &= 0 \quad \Leftrightarrow \quad \gamma^+ \omega = 0 \quad \Leftrightarrow \quad \gamma \gamma^+ \omega = 0 \quad \Leftrightarrow \quad \Pi_\gamma \omega = 0 \quad \Leftrightarrow \quad \omega \text{ is quasi-optimal.}
\end{align*}
\]

(5.17)
and that for any solution-point \( \hat{z} \), \( \Pi_{\gamma} \hat{z} = \gamma \gamma^+ \hat{z} = \gamma^T \gamma \hat{z} = \gamma^T 0 = 0 \), that is
\[
\Pi_{\gamma} \hat{z} = 0 \quad \text{for any solution-point } \hat{z} \tag{5.18}
\]

### 5.2.2 Solution Characterization

We show that fixed-points are quasi-optimal:

**Lemma 5.2.2** \( \omega \) is fixed \( \Rightarrow \) \( \gamma^T \omega = 0 \) \( \iff \) \( \Pi_{\gamma} \omega = 0 \) \( \iff \) \( \omega \) is quasi-optimal.

**Proof:**
\[
\begin{align*}
\omega &= \mathcal{P} \omega \Rightarrow \gamma^T \omega = \gamma^T \mathcal{P} \omega \quad \Rightarrow \quad \gamma^T \omega = 0^T \omega \Rightarrow \gamma^T \omega = 0 \\
&\iff \gamma^+ \omega = 0 \iff \Pi_{\gamma} \omega = 0 \iff \omega \text{ is quasi-optimal.}
\end{align*}
\]

Obviously

**Lemma 5.2.3** If \( \omega \) is a feasible point then \( \omega \) is non-negative.

and in particular we have

**Corollary 5.2.4** If \( \omega \) is a solution-point then \( \omega \) is non-negative.

We now consider the properties of central solution-points in the context of the fixed-point matrix \( \mathcal{P} \).

Let \( \hat{z} = \begin{bmatrix} f \\ \eta \end{bmatrix} \) be a central solution-point, then
\[
\mathcal{P} \hat{z} = (\Pi_{\alpha} + \Pi_{\beta} - \Pi_{\gamma}) \hat{z} = \Pi_{\alpha} \hat{z} + \Pi_{\beta} \hat{z} - \Pi_{\gamma} \hat{z} = \Pi_{\alpha} \hat{z} + \Pi_{\beta} \hat{z} = \hat{z} + (1 + \beta^+ \hat{z}) \beta
\]
that is
\[
\mathcal{P} \hat{z} = \hat{z} + (1 + \beta^+ \hat{z}) \beta \tag{5.19}
\]
and from this equation \( \mathcal{P}^2 \hat{z} = \mathcal{P}(\hat{z} + (1 + \beta^+ \hat{z}) \beta) = \mathcal{P} \hat{z} = \mathcal{P}(\hat{z} + (1 + \beta^+ \hat{z}) \mathcal{P} \beta \Rightarrow (1 + \beta^+ \hat{z}) \mathcal{P} \beta = 0 \) which implies, from Equation 5.15 a,
\[
(1 + \beta^+ \hat{z}) \beta = 0 \quad \text{for central solution-point } \hat{z}. \tag{5.20}
\]
Substituting (5.20) into (5.19) and bearing in mind Corollary 5.2.4 we have \( \mathcal{P} \hat{z} = \hat{z} \), that is
5.2. PRECISE DEVELOPMENT

Lemma 5.2.5 \( \exists \) is a central solution-point \( \Rightarrow \mathcal{P}_3 = \exists \geq 0 \) and \( (1 + \beta^+ \exists) \beta = 0 \).

Next we consider the converse of this lemma - that is

Lemma 5.2.6 \( \mathcal{P}_\omega = \omega \geq 0 \) and \( (1 + \beta^+ \omega) \beta = 0 \)
\( \Rightarrow \omega \) is a central solution-point.

With \( \omega = \begin{bmatrix} \xi \\ \zeta \end{bmatrix} \),
\( \mathcal{P}_\omega = \omega \geq 0 \) and \( (1 + \beta^+ \omega) \beta = 0 \Rightarrow \)
\( (\Pi_\omega + \Pi_\beta - \Pi_\gamma) \omega = \omega \geq 0 \) and \( (1 + \beta^+ \omega) \beta = 0 \Rightarrow \)
\( \Pi_\omega \omega + \Pi_\beta \omega - \Pi_\gamma \omega = \omega \geq 0 \) and \( (1 + \beta^+ \omega) \beta = 0 \Rightarrow \)
\( \Pi_\omega \omega + \Pi_\beta \omega - \Pi_\gamma \omega = \omega \geq 0 \) and \( (1 + \beta^+ \omega) \beta = 0 \Rightarrow \)
\( \begin{bmatrix} A \xi - b \\ D \zeta - c \end{bmatrix} = \begin{bmatrix} \xi \\ \zeta \end{bmatrix} \geq 0 \) and \( \begin{bmatrix} c \\ b \end{bmatrix}^T \begin{bmatrix} \xi \\ \zeta \end{bmatrix} = 0 \Rightarrow \)
\( \omega \) is a central solution-point.

So we have

Theorem 5.2.7 \( \exists \) is a central solution-point \( \iff \mathcal{P}_3 = \exists \geq 0 \) and \( (1 + \beta^+ \exists) \beta = 0 \). \( \square \)

To complete the characterization of the central solutions to the invariant problems in terms of fixed-points of \( \mathcal{P} \) we need to define the discriminant:

\[ d(\omega) = -\beta^+ \omega, \]  
(5.21)

then we have

Lemma 5.2.8 \( (1 + \beta^+ \omega) \beta = 0 \iff \)
\( (a) \beta \neq 0 \) and \( d(\omega) = 1, \) or
\( (b) \beta = 0 \) and \( d(\omega) = 0. \)

\( (a) \beta \neq 0 \iff 1 + \beta^+ \omega = 0 \iff -\beta^+ \omega = 1 \iff d(\omega) = 1 \)

\( (5.21) \)

Proof: \( (b) \beta = 0 \iff \beta^+ = 0 \Rightarrow d(\omega) = 0 \Rightarrow (d(\omega) = 0) \land (1 + \beta^+ \omega) \beta = 0 \)
\( \Rightarrow (d(\omega) = 0) \land (1 + \beta^+ \omega = 0 \lor \beta = 0) \Rightarrow (d(\omega) = 0) \land (\beta^+ \omega = -1 \lor \beta = 0) \)
\( \Rightarrow (d(\omega) = 0) \land (d(\omega) = 1 \lor \beta = 0) \Rightarrow \beta = 0. \) \( \square \)

Note that if \( \omega \) is a nonnegative fixed-point with \( d(\omega) > 0 \) we simply normalize it (or centralize it using the mapping \( \mathcal{F} : \omega = \begin{bmatrix} \xi \\ \zeta \end{bmatrix} \mapsto \begin{bmatrix} \xi F_p \\ \zeta F_d \end{bmatrix} \), where \( F_p = f_p f_p \) and \( F_d = f_d f_d \) ) to obtain a representative point.

Lemma 5.2.9 For \( \omega \) representative, \( d(\omega) = 0 \iff \beta = 0 \iff b = 0 \land c = 0. \)

If \( d(\omega) = 0 \) then \( -\beta^+ \omega = 0 \) and \( (1 + \beta^+ \omega) \beta = 0 \Rightarrow (1 + 0) \beta = 0 \Rightarrow \beta = 0. \)

Proof: Conversely, if \( \beta = 0 \) then \( d(\omega) = -\beta^+ \omega = -0^+ \omega = -0^T \omega = 0. \)

Obviously \( \beta = 0 \) is equivalent to \( b = 0 \land c = 0. \)
Lemma 5.2.10 For $\omega$ representative, $d(\omega) = 0 \Rightarrow$ all feasible solutions to the original problems are optimal.

Proof: Assuming Problem 1.1 is quasi-bounded, if $b = c = 0$ then $c = 0 \Rightarrow A^T \epsilon = 0$ \Rightarrow $c = 0$, and we have the problem of finding the set $\{x : Ax \geq b\}$, also $c = 0 \Rightarrow \max\{b^T y : A^T y = c, y \leq 0\} = \max\{b^T y : A^T y = 0, y \leq 0\} = \max\{b^T y : AA^+ y = 0, y \leq 0\} = \max\{b^T y : (I - AA^+) y = y, y \leq 0\} = \max\{0^T y : (I - AA^+) y = y, y \leq 0\} = \{y : AA^+ y = 0, y \leq 0\}$. Perhaps more obviously, if $b = c = 0$ then all feasible solutions to the invariant problems are optimal, and since the feasible and the optimal feasible solutions for the original and invariant problems are in on-to-one correspondence it follows that all feasible solutions to the original problems are optimal. \hfill $\square$

We sum up the relation between the fixed-point, invariant and original problems with Figure 5.1.

5.2.3 Special Cases

5.2.3.1 Ambivalence:

There are fixed-point problems which solve two LP problems: note that if $b$ is replaced by $-b$ and $c$ is replaced by $-c$ then $\Pi_\beta$ and $\Pi_\gamma$ are unchanged, so $\Psi$ is unchanged, and the fixed-point problem could be solving

$$\max(-c^T)x, \text{ subject to } Ax \geq -b.$$  \hfill (5.22)

which is equivalent to

$$\max c^T(-x), \text{ subject to } A(-x) \leq b.$$  \hfill (5.23)

which is equivalent to

$$\max c^T x, \text{ subject to } Ax \leq b.$$  \hfill (5.24)

Here we are really only interested in ambivalent situations where Problems (1.1) and (5.24) are both feasible bounded in which case there is the possibility that we will find a solution to Problem (5.24) and fail to realize that there also exists a solution to (1.1).\footnote{We are not interested in the situation where we have mis-apprehended a problem as being of type (1.1) when it purely of type (5.24).}

5.2.3.2 The Zero Solution

We define the zero solution as the central solution $\bar{z}_c = 0$. The zero solution only occurs under rather trivial conditions given by the following
Figure 5.1: Fixed-Point, Invariant and Original Problem Relationship
Lemma 5.2.11 A zero central solution point exists iff \( b = 0 \) and \( c = 0 \).

Proof: if \( \bar{z}_c \) is a zero central solution to the fixed-point problem then \( \bar{z}_c = 0 \) \( \Rightarrow \) \( d(\bar{z}_c) = 0 \) \( \Rightarrow \) \( \beta = 0 \Rightarrow b = c = 0 \); conversely if \( b = 0 \) and \( c = 0 \) then the invariant primal and dual can be written as

\[
\max 0 \quad \text{s.t.} \quad Ax \geq 0 \quad \text{and} \quad Dy \geq 0
\]

respectively, that is

\[
\max 0 \quad \text{s.t.} \quad Ax \geq 0 \quad \text{and} \quad \max 0 \quad \text{s.t.} \quad Dy \geq 0
\]

that is find \( x \) s.t. \( Ax \geq 0 \) and \( y \) s.t. \( Dy \geq 0 \). Since for \( x_s = 0 \) we have \( Ax_s = 0 \geq 0 \), and \( Ax_s - b = 0 - 0 = 0 = x_s \) it follows that \( x_s = 0 \) is a central solution to the invariant primal, while similarly we see that \( y_s = 0 \) is a central solution to the invariant dual, so there is a central solution point equal to zero. \( \square \)
5.A Appendix: Augmented Matrix Pseudo-Inverse

The pseudo-inverse of the augmented matrix
\[
\begin{bmatrix}
\mathbb{A} & 0 & -b \\
0 & \mathbb{D} & -c \\
\mu\epsilon^T & \mu\epsilon & 0
\end{bmatrix}
\]
is computed as follows: From Lemma 2.2.3
\[
\begin{bmatrix}
\mathbb{A} & 0 & -b \\
0 & \mathbb{D} & -c \\
\mu\epsilon^T & \mu\epsilon & 0
\end{bmatrix}^+ =
\begin{bmatrix}
\mathbb{A} & 0 & \mu\epsilon \\
0 & \mathbb{D} & \mu\epsilon \\
-b^T & -c^T & 0
\end{bmatrix}^+. 
\] (5.25)

With the partitioning following the scheme and notation of Albert [4, pp. 43-44], as given in Chapter 2.2.5.4,

\[
C_m = \begin{bmatrix}
\mathbb{A} & 0 \\
0 & \mathbb{D} \\
-b^T & -c^T
\end{bmatrix}
\] (5.26)

\[
c_{m+1} = \begin{bmatrix}
\mu\epsilon \\
\mu\epsilon \\
0
\end{bmatrix}
\] (5.27)

We see that with \( \eta \) given by Equation 5.7, that is \( \eta = \|b\|^2 + \|c\|^2 \),

\[
C_m^+ = \begin{bmatrix}
\mathbb{A} & 0 \\
0 & \mathbb{D} \\
-b^T & -c^T
\end{bmatrix}^+ = \begin{bmatrix}
\mathbb{A} & -b/\eta \\
0 & \mathbb{D} / \eta
\end{bmatrix},
\]
and thus

\[
C_mC_m^+ = \begin{bmatrix}
\mathbb{A} & 0 \\
0 & \mathbb{D} \\
0 & 0
\end{bmatrix}, \quad \Rightarrow I - C_mC_m^+ = \begin{bmatrix}
\mathbb{D} & 0 & 0 \\
0 & \mathbb{A} & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad (5.28)
\]

\[
\Rightarrow (I - C_mC_m^+)c_{m+1} = \begin{bmatrix}
\mathbb{D} & 0 & 0 \\
0 & \mathbb{A} & 0 \\
0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
\mu\epsilon \\
\mu\epsilon \\
0
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}, \quad (5.29)
\]

(which is the case covered by Theorem 2.2.12c). Further,

\[
C_m^{+}c_{m+1} = \begin{bmatrix}
\mathbb{A} & 0 & -b / \eta \\
0 & \mathbb{D} & -c / \eta
\end{bmatrix} \begin{bmatrix}
\mu\epsilon \\
\mu\epsilon \\
0
\end{bmatrix} = \begin{bmatrix}
\mu\epsilon \\
\mu\epsilon \\
0
\end{bmatrix}, \quad (5.30)
\]

and

\[
C_m^{+T}C_m^{+}c_{m+1} = \begin{bmatrix}
\mathbb{A} & 0 & 0 \\
0 & \mathbb{D} & 0 \\
-b^T / \eta & -c^T / \eta
\end{bmatrix} \begin{bmatrix}
\mu\epsilon \\
\mu\epsilon \\
0
\end{bmatrix} = \begin{bmatrix}
\mu\epsilon \\
\mu\epsilon \\
0
\end{bmatrix}. \quad (5.31)
\]
So, following Theorem 2.2.12c,

\[
k_{m+1} = \frac{C_m^+ C_m^+ c_{m+1}}{1 + \|C_m^+ c_{m+1}\|^2} = \begin{bmatrix} \mu c \\ \mu b \\ 0 \end{bmatrix} / (1 + \mu^2 \eta). \tag{5.32}
\]

and, following part (a) of the theorem,

\[
\begin{bmatrix} \mathcal{A} & 0 & -b/\eta \\ 0 & \mathcal{D} & -c/\eta \\ -b^T & -c^T & 0 \end{bmatrix}^+ = \begin{bmatrix} \mathcal{A} & 0 & -b/\eta \\ 0 & \mathcal{D} & -c/\eta \\ I - \begin{bmatrix} \mu c \\ \mu b \\ 0 \end{bmatrix} / \nu \end{bmatrix} = \begin{bmatrix} \mu c^T / \nu & \mu b^T / \nu & 0 \\ \mu c^T / \nu & \mu b^T / \nu & 0 \end{bmatrix},
\]

where \( \nu \) is given by Equation 5.8, that is \( \nu = 1 + \mu^2 \eta \).

The required pseudo-inverse is found by transposing the preceding equality, yielding

\[
\begin{bmatrix} \mathcal{A} & 0 & -b \\ 0 & \mathcal{D} & -c \\ \mu c & \mu b^T & 0 \end{bmatrix}^+ = \begin{bmatrix} \mathcal{A} & \mu c^T / \nu & -\mu c T / \nu \\ -\mu c / \nu & \mathcal{D} - \mu b b / \nu & -c / \nu \end{bmatrix}.
\]

Checking the four conditions for the pseudo-inverse to confirm this result is straightforward.

5. B Appendix: Projection Decomposition

\[
\begin{bmatrix} \mathcal{A} + b b / \eta \\ \mathcal{D} + c c / \eta \\ \mu c / \nu \end{bmatrix} = \begin{bmatrix} \mathcal{A} & b b / \eta & -\mu c c / \nu \\ -\mu c b / \nu & \mathcal{D} + \mu b / \nu & -c c / \nu \\ \mu c / \nu & \mu b / \nu & \mu^2 \eta / \nu \end{bmatrix} = \begin{bmatrix} b b / \eta & b c / \eta & 0 \\ c b / \eta & c c / \eta & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} -\mu c c T / \nu & -\mu c b T / \nu & \mu c / \nu \\ -\mu c b T / \nu & \mu b b T / \nu & \mu b / \nu \\ \mu c T / \nu & \mu b T / \nu & \mu^2 \eta / \nu \end{bmatrix}.
\]
\[
\begin{align*}
&= \begin{bmatrix}
A & 0 & 0 \\
0 & D & 0 \\
0 & 0 & 0
\end{bmatrix} + \begin{bmatrix}
b \\
c \\
0
\end{bmatrix} \begin{bmatrix}
\mu \\
\nu \\
0
\end{bmatrix} + \begin{bmatrix}
\frac{-\mu^2 c c^T}{\nu} & \frac{-\mu^2 b b^T}{\nu} & \frac{\mu c}{\nu} \\
\frac{-\mu^2 b c^T}{\nu} & \frac{\mu^2 b b^T}{\nu} & \frac{\mu b}{\nu} \\
\frac{\mu c^T}{\nu} & \frac{\mu b^T}{\nu} & \frac{\mu^2 \eta}{\nu}
\end{bmatrix} \\
&= \begin{bmatrix}
A & 0 & 0 \\
0 & D & 0 \\
0 & 0 & 0
\end{bmatrix} + \begin{bmatrix}
b \\
c \\
0
\end{bmatrix} \begin{bmatrix}
\mu \\
\nu \\
0
\end{bmatrix} + \begin{bmatrix}
\frac{-\mu^2 c c^T}{\nu} & \frac{-\mu^2 b b^T}{\nu} & \frac{\mu c}{\nu} \\
\frac{-\mu^2 b c^T}{\nu} & \frac{\mu^2 b b^T}{\nu} & \frac{\mu b}{\nu} \\
\frac{\mu c^T}{\nu} & \frac{\mu b^T}{\nu} & \frac{\mu^2 \eta}{\nu}
\end{bmatrix} \\
&= \begin{bmatrix}
A & 0 & 0 \\
0 & D & 0 \\
0 & 0 & 0
\end{bmatrix} + \begin{bmatrix}
b \\
c \\
0
\end{bmatrix} \begin{bmatrix}
\mu \\
\nu \\
0
\end{bmatrix} + \begin{bmatrix}
\frac{-\mu^2 c c^T}{\nu} & \frac{-\mu^2 b b^T}{\nu} & \frac{\mu c}{\nu} \\
\frac{-\mu^2 b c^T}{\nu} & \frac{\mu^2 b b^T}{\nu} & \frac{\mu b}{\nu} \\
\frac{\mu c^T}{\nu} & \frac{\mu b^T}{\nu} & \frac{\mu^2 \eta}{\nu}
\end{bmatrix} \\
&= \begin{bmatrix}
A & 0 & 0 \\
0 & D & 0 \\
0 & 0 & 0
\end{bmatrix} + \begin{bmatrix}
b \\
c \\
0
\end{bmatrix} \begin{bmatrix}
\mu \\
\nu \\
0
\end{bmatrix} + \begin{bmatrix}
\frac{-\mu^2 c c^T}{\nu} & \frac{-\mu^2 b b^T}{\nu} & \frac{\mu c}{\nu} \\
\frac{-\mu^2 b c^T}{\nu} & \frac{\mu^2 b b^T}{\nu} & \frac{\mu b}{\nu} \\
\frac{\mu c^T}{\nu} & \frac{\mu b^T}{\nu} & \frac{\mu^2 \eta}{\nu}
\end{bmatrix} \\
&= \begin{bmatrix}
A & 0 & 0 \\
0 & D & 0 \\
0 & 0 & 0
\end{bmatrix} + \begin{bmatrix}
b \\
c \\
0
\end{bmatrix} \begin{bmatrix}
\mu \\
\nu \\
0
\end{bmatrix} + \begin{bmatrix}
\frac{-\mu^2 c c^T}{\nu} & \frac{-\mu^2 b b^T}{\nu} & \frac{\mu c}{\nu} \\
\frac{-\mu^2 b c^T}{\nu} & \frac{\mu^2 b b^T}{\nu} & \frac{\mu b}{\nu} \\
\frac{\mu c^T}{\nu} & \frac{\mu b^T}{\nu} & \frac{\mu^2 \eta}{\nu}
\end{bmatrix}.
\end{align*}
\]
Chapter 6

Fixed-Point Problem Solution

In this chapter we develop an algorithm for solution of the fixed-point problem following the scheme of Chapter 3. We continue with $\mathcal{P}$ as specified in Chapter 5 and introduce a matrix $\mathcal{S}$ which swaps $\mathcal{P}$. We introduce the function $\mathcal{R}_{x}$ and show that $\mathcal{S}$ also swaps $\mathcal{R}_{x}$ and thus $\mathcal{R}_{x}$ serves as a specific example of the idempotent symmetric $K$. We then construct $\mathcal{U}_{x}$ and $\mathcal{V}_{x}$ in a manner analogous to Chapter 2.2.4, thus the results of Chapter 3 apply.

6.1 Construction of the Specific $\mathcal{P}$-Unitary Matrix $\mathcal{U}$

6.1.1 Specific $\mathcal{S}$

Define the $2m \times 2m$ matrix

$$
\mathcal{S} = \begin{bmatrix}
0 & I_m \\
I_m & 0
\end{bmatrix}
$$

then $\mathcal{S}$ is a swapping-matrix since $\mathcal{S}^2 = I_{2m}$, and $\mathcal{S}^T = \mathcal{S}$ imply $\mathcal{S}\mathcal{S}^T = \mathcal{S}^T\mathcal{S} = I$, so $\mathcal{S}$ is unitary Hermitian.

Note that

$$
\mathcal{S} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}
$$

(6.1)

and, in particular,

$$
\begin{bmatrix} x \\ y \end{bmatrix}^+ \mathcal{S} \overset{(L3.1.2)}{=} (\mathcal{S} \begin{bmatrix} x \\ y \end{bmatrix})^+ = \begin{bmatrix} y \\ x \end{bmatrix}^+
$$

(6.2)
6.1.2 Specific P

Our specific $P$ is given by Equation 5.11(b) - that is

$$P = [A 0 0 D] + [b c] [b c]^+ - [c b] [c b]^+.\)

From Equation 5.11(b), $S\Psi\mathcal{S} = \mathcal{S} \left\{ [A 0 0 D] + [b c] [b c]^+ - [c b] [c b]^+ \right\} \mathcal{S}$, which, by multiplication and using Equations 6.1 and 6.2, is equal to

$$\left[ \mathcal{D} 0 \mathcal{A} \right] + [c b] [c b]^+ - [b c] [b c]^+ = I - \Psi;$$

that is $S\Psi\mathcal{S} = I - \Psi$ and, since $\Psi$ is an Hermitian idempotent, $\mathcal{S}$ swaps $\Psi$ and thus (3.2), Lemmas 3.1.1 and 3.1.2, and Corollary 3.1.3 obtain.

**Lemma 6.1.1** If $\omega_r$ and $\omega_s$ are fixed-points of $\Psi$ then $\omega_r^T \mathcal{S}\omega_s = 0$.

Proof: $\omega_r^T \mathcal{S}\omega_s = (\Psi\omega_r)^T \mathcal{S}\Psi\omega_s = \omega_r^T \Psi^T \mathcal{S}\Psi\omega_s = \omega_r^T \Psi\mathcal{S}\Psi\omega_s = 0$. $\square$

**Lemma 6.1.2** $\xi_s^T \zeta_s = 0$

Proof: $\xi_s^T \zeta_s = \left[ \xi_s \ z_s \right]^T \mathcal{S} \left[ \xi_s \ z_s \right] / 2 \overset{\text{(L 6.1.1)}}{=} 0$. $\square$

6.1.3 Specific K

Here we continue within the context of Section 2.2.4, specifying $K$ as a diagonal matrix which forces the orthogonality (i.e. complementary slackness condition) to hold. From this specific $K$ we construct a specific unitary matrix and an averaging matrix in a manner exactly analogous to Equations 3.9 and 3.10 respectively.

Define

$$i' = \text{mod}_{2m}(i + m)$$

Given a vector $z$ of length $2m$ we define the Karush function as the $2m$ by $2m$ diagonal matrix $\mathcal{R}_z$

$$(\mathcal{R}_z)_{ii} = \begin{cases} 1 & \text{if } z_i > z_{i'} \\ 1 & \text{if } z_i = z_{i'} \text{ and } i < i' \\ 0 & \text{otherwise} \end{cases} \overset{(6.3)}{=}$$

Thus $\mathcal{R}_z$ is an Hermitian idempotent which has the value 1 at one of the indices $i, i'$, and the value 0 at the other index.
6.1. CONSTRUCTION OF THE SPECIFIC \( \mathfrak{P} \)-UNITARY MATRIX \( \mathfrak{U} \)

Lemma 6.1.3 \( \mathfrak{S} \) swaps \( \mathfrak{R}_z \).

Proof: From (6.3) we see that \( \mathfrak{R}_z \) is an Hermitian idempotent matrix; further, we can write \( \mathfrak{R}_z = \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix} \) where \( D_1 \) and \( D_2 \) are diagonal matrices and \( D_1 + D_2 = I \). Further,

\[
\mathfrak{S} \mathfrak{R}_z \mathfrak{S} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} = \begin{bmatrix} 0 & D_2 \\ D_1 & 0 \end{bmatrix} = \begin{bmatrix} I - D_2 \\ D_1 - D_2 \end{bmatrix} = I - \mathfrak{R}_z. \]

6.1.4 Specific \( \mathfrak{U} \)

Analogous to Equation 3.9, we define

\[
\mathfrak{U}_z = \mathfrak{P}(I + \mathfrak{S}) \mathfrak{R}_z (I - \mathfrak{S}) \mathfrak{P}, \tag{6.4}
\]

and analogous to Equation 3.10 the averaging matrix

\[
\mathfrak{V}_z = (\mathfrak{P} + \mathfrak{U}_z)/2. \tag{6.5}
\]

Note that \( \mathfrak{U}_z \) is \( \mathfrak{P} \)-unitary in view of Theorem 3.1.13.

Analogous to the definition of the general oblique Karush matrix \( \mathfrak{K} \) given by Equation 3.8 in Chapter 2.2.4, define

\[
\mathfrak{R}_z = \mathfrak{R}_z (I - \mathfrak{S}). \tag{6.6}
\]

Lemma 6.1.4 \( \mathfrak{R}_z z = [(I - \mathfrak{S}) z] \lor 0 \geq 0 \).

Proof: \( (\mathfrak{R}_z z)_i = (\mathfrak{R}_z (I - \mathfrak{S}) z)_i \)

\[
= \begin{cases} 
((I - \mathfrak{S}) z)_i & \text{if } z_i > z_{i'} \\
((I - \mathfrak{S}) z)_i & \text{if } z_i = z_{i'} \text{ and } i < i' \\
0 & \text{else}
\end{cases} = \begin{cases} 
z_i - z_{i'} & \text{if } z_i > z_{i'} \\
z_i - z_{i'} & \text{if } z_i = z_{i'} \text{ and } i < i' \\
0 & \text{else}
\end{cases} = \begin{cases} 
z_i - z_{i'} & \text{if } z_i > z_{i'} \\
0 & \text{else}
\end{cases} = |z_i - z_{i'}| \lor 0 = [(I - \mathfrak{S}) z]_i \lor 0 , \text{ so } \mathfrak{R}_z z = [(I - \mathfrak{S}) z] \lor 0 \geq 0. \]

So

\[
\mathfrak{R}_z z = \mathfrak{R}_z (I - \mathfrak{S}) z = [(I - \mathfrak{S}) z] \lor 0 = (z - \mathfrak{S} z)/2 + |z - \mathfrak{S} z|/2. \tag{6.7}
\]
Further, assuming $z = \mathcal{P}z$, \( \mathcal{U}z = \mathcal{P}(I + \mathcal{S})z = \mathcal{P}(I + \mathcal{S})z = \mathcal{P}(I + \mathcal{S})[z - \mathcal{S}z]/2 + |z - \mathcal{S}z|/2 = \mathcal{P}(I + \mathcal{S})|z - \mathcal{S}z|/2 = \mathcal{P}|z - \mathcal{S}z| \) , so we have the simple computational form

\[
\mathcal{U}z = \mathcal{P}|z - \mathcal{S}z| , \quad \text{for } z = \mathcal{P}z .
\] (6.8)

Note that Nguyen [28, p. 34], defines the function \( A(x) = (|x| + x)/2 \) which is shown to be the proximity map onto the non-negative cone \( \{ x \in \mathbb{R}^{2m} : x \geq 0 \} \). He shows that repeated application of \( \mathcal{PA} \) or of \( \mathcal{P}A \) leads to a non-negative fixed-point of \( \mathcal{P} \).

Define

\[
\mathcal{U} : z \mapsto \mathcal{U}z ,
\] (6.9)
and

\[
\mathcal{V} : z \mapsto \mathcal{V}z ,
\] (6.10)
then we see from Equation 6.8 that \( \mathcal{U} \) and \( \mathcal{V} \) are continuous functions. Note that \( \mathcal{U} \) and \( \mathcal{V} \) are not linear functions.

### 6.2 Proximality

We introduce the notion of proximality of vectors and show that, in the context of the LP fixed-point problem, proximality implies linear behaviour; this lays some of the groundwork for a solution method.

The vector \( p \) is said to be \textit{proximal} to \( q \) if \((I - \mathcal{S})q_i > 0 \Rightarrow (I - \mathcal{S})p_i \geq 0 \forall i \), while \( p \) is said to be proximal to \( q \) for component \( i \) if \((I - \mathcal{S})q_i > 0 \Rightarrow (I - \mathcal{S})p_i \geq 0 \). We define the \textit{i}th component-pair of a \( 2m \)-dimensional vector \( x \) to be the pair \((x_i, x_i')\); a \( 2m \) dimensional vector \( x \) is said to have a \textit{zero component-pair} at \( i \) if \( x_i = x_i' = 0 \).

Note that proximality is reflexive since a vector is obviously proximal to itself, and symmetric in view of

**Lemma 6.2.1** \( p \) is proximal to \( q \) \( \iff \) \( q \) is proximal to \( p \).

Proof: \( p \) is proximal to \( q \)
implies \((I - \mathcal{S})q_i > 0 \Rightarrow (I - \mathcal{S})p_i \geq 0 \forall i \)
implies \((I - \mathcal{S})p_i < 0 \Rightarrow (I - \mathcal{S})q_i \leq 0 \forall i \)
implies \((I - \mathcal{S})p_i' > 0 \Rightarrow (I - \mathcal{S})q_i' \geq 0 \forall i \)
implies \((I - \mathcal{S})p_i' > 0 \Rightarrow (I - \mathcal{S})q_i' \geq 0 \forall i' \)
implies \((I - \mathcal{S})p_i > 0 \Rightarrow (I - \mathcal{S})q_i \geq 0 \forall i \)
implies \( q \) is proximal to \( p \). \( \square \)
6.3. ORBITS

This means we may say “p and q are proximal” rather than p is proximal to q; proximality however is not transitive.

**Lemma 6.2.2** If x has a zero component-pair at i then for arbitrary q, x is proximal with q for component i.

Proof: Exercise.

**Lemma 6.2.3** If \( \omega_r \) and \( \omega_s \) are non-negative and \( \omega_r^T \mathcal{S} \omega_s = 0 \) then \( \omega_r \) and \( \omega_s \) are proximal.

Proof: \( \omega_r^T \mathcal{S} \omega_s = 0 \Rightarrow (\omega_r)_i (\omega_s)_{i'} = 0 \forall i \). So \( (\omega_r)_i - (\omega_r)_{i'} > 0 \forall i \Rightarrow (\omega_r)_i > (\omega_r)_{i'} \geq 0 \forall i \Rightarrow (\omega_r)_i > 0 \Rightarrow (\omega_s)_{i'} \geq 0 \Rightarrow (\omega_r)_i - (\omega_s)_{i'} \geq 0 \), that is \( \forall i : (\omega_r)_i - (\omega_r)_{i'} > 0 \Rightarrow (\omega_s)_i - (\omega_s)_{i'} \geq 0 \), that is \( \forall i : (I - \mathcal{S}) \omega_r \geq 0 \Rightarrow (I - \mathcal{S}) \omega_s \geq 0 \), so \( \omega_r \) and \( \omega_s \) are proximal.

**Lemma 6.2.4** If \( \omega_r \) and \( \omega_s \) are non-negative fixed-points of \( \Psi \) then \( \omega_r \) and \( \omega_s \) are proximal.

Proof: The lemma follows from Lemmas 6.1.1 and 6.2.3.

6.3 Orbits

We define a sequence of vectors \( \{ g_i \} \) which orbits the fixed-points of \( \Psi \) and a sequence of vectors \( \{ v_i \} \) which converges to a fixed-point of \( \Psi \). An algorithm which combines the averaging function \( \Psi \) and affine regression is proposed and it is shown that this algorithm terminates.

The first scheme we adopt is to form the sequence \( \{ v_i \} \) analogous in construction to the sequence \( \{ v_i \} \) of Sub-section 3.3.1. We know that this sequence converges, but usually the convergence is slow, so to accelerate the process we modify the scheme by using the affine regression method detailed in Chapter 3.3.2, however this is still not computationally satisfactory as we would need to compute \( n \) points and then regress without being sure that the points were proximal; to overcome this problem an incremental affine regression algorithm is developed in Appendix B.1.

If the regression does not yield a non-negative fixed-point then either (a) proximality has not been reached and we should proceed further with the series \( \{ v_i \} \) or (b) proximality has been reached but a zero component-pair exists. Although fixed-point solutions containing a zero component-pair can be constructed, to date no case has occurred when using the algorithm developed beginning in the following section.

6.4 Fixed-Point Analysis

Here we build on the fixed-point theory of Chapter 3, laying the basis for the affine regression algorithm of the following section.

---

1 By Lemma 6.2.4 the non-negative fixed-points of \( \Psi \) are mutually proximal.
Definitions: Let $p$ be a non-negative non-trivial fixed-point of $P$ then, with $\triangleright$ as defined by Equation 2.15a, the vector $p_z = z \triangleright p$ is the fixed-point component of $z$ w.r.t $p$. With $\prec$ as defined by Equation 2.15b, the residual component of $z$ w.r.t $p$ is defined as the vector $p^z = z \prec p$. The scalar $\|p_z\| / \|z\|$ is the fixed-point proportion of $p$ in $z$.

We start with a vector which necessarily contains a positive component of a non-negative fixed-point of $P$, if such fixed-point exists; a suitable choice is $v_1 = P1_{2m}$, where $1_{2m}$ is a $2m$-dimensional vector each of whose entries is unity, then we consider the sequences $\{g_i\}$ and $\{v_i\}$ where

$$u_{i+1} = Ug_i, \quad u_1 = P1$$

and

$$v_{i+1} = Vv_i, \quad v_1 = P1$$

and note that if a non-negative non-trivial fixed-point of $P$, say $p$, exists then $p^T u_1 = p^T P1 = p^T 1 > 0$, so $u_1$ contains a positive component of $p$.

The first result is that the fixed-points of $U$ are precisely the non-negative fixed-points of $P$.

**Theorem 6.4.1** $Uz = z \Leftrightarrow z$ is a non-negative fixed-point of $P$.

Proof: $Uz = z \Leftrightarrow U_z z = z \Leftrightarrow (Pz = z$ and $K_z z = z$) \hspace{1cm}$\Leftrightarrow (Pz = z$ and $K_z z = z$) $\Rightarrow Pz = z \geq 0$.

Conversely, suppose $Pz = z \geq 0$, then $K_z z = (z - \mathcal{G}z) \vee 0 \geq z - \mathcal{G}z$,

so $z^T K_z z \geq z^T (z - \mathcal{G}z) = z^T z \Rightarrow z^T K_z z = z^T z$,

so $(Pz = z) \wedge \|z - \overrightarrow{K_z z}\|^2 = z^T z - 2z^T \overrightarrow{K_z z} + (\overrightarrow{K_z z})^T \overrightarrow{K_z z}$ \hspace{1cm} $(C \ref{C.3.14})$

$\Rightarrow (Pz = z) \wedge z^T z - 2z^T \overrightarrow{K_z z} + z^T z = z^T z - 2z^T z + z^T z = 0$

$\Rightarrow (Pz = z) \wedge (\overrightarrow{K_z z} = z) \Rightarrow (Pz = z) \wedge (K z = z) \Rightarrow U_z z = z$. □

We next show that, for any non-negative fixed-point $p$ of $P$, the component of this fixed-point in the sequence $\{g_i\}$ is increasing:

**Lemma 6.4.2** If $p$ is a non-negative fixed-point of $P$ and $z$ is any fixed-point of $P$ then $p \cdot Uz \geq p \cdot z$; further, $p \cdot Uz = p \cdot z \Leftrightarrow z$ is proximal to $p$.
6.4. FIXED-POINT ANALYSIS

Proof: \( p \cdot Uz = p \cdot U_jz = p \cdot (j - Sz) \geq p \cdot (j - Sz) = p \cdot j - p \cdot Sz \)

\[ (6.8) \]

\[ (6.9) \]

Thus fixed-point content of \( \{g_i\} \) increases monotonically until proximality is reached, then it remains constant unless \( \{g_i\} \) becomes non-proximal again (which can, but eventually won’t, happen!).

**Lemma 6.4.3** For a non-negative fixed-point of \( \Psi \) and \( j \) any fixed-point of \( \Psi \), \( p \cdot Uz \geq p \cdot z \), moreover \( p \cdot Uz = p \cdot z \Leftrightarrow j \) is proximal to \( p \).

\[ (L.6.4.2) \]

Proof: \( p \cdot Uz = p \cdot (I + U)z/2 = p \cdot z/2 + p \cdot Uz/2 \geq p \cdot z/2 + p \cdot j/2 = p \cdot j \). The second part of the lemma also follows on applying the second part of Lemma 6.4.2 to the calculations.

**Lemma 6.4.4** Let \( p \) be a non-negative fixed-point of \( \Psi \) then \( p \cdot v_{i+1} \geq p \cdot v_i \), moreover \( p \cdot v_{i+1} = p \cdot v_i \Leftrightarrow v_i \) is proximal to \( p \).

\[ (L.6.4.3) \]

Proof: \( p \cdot v_{i+1} = p \cdot Uv_i \geq p \cdot v_i \).

Let \( v_\infty = \lim_{i \to \infty} \Psi^i \cdot 1_{2m} \), and \( z = \Psi^1 \cdot 1_{2m} \), then

\[ v_\infty \cdot \Psi^i \cdot 1_{2m} \geq v_\infty \cdot \Psi^1 \cdot 1_{2m} \]

\[ v_\infty \cdot \lim_{i \to \infty} \Psi^i \cdot 1_{2m} \geq v_\infty \cdot 1_{2m} \]

\[ \Rightarrow 1 \geq \frac{v_\infty \cdot 1_{2m}}{v_\infty \cdot 1_{2m}} \]

\[ \Rightarrow v_\infty \geq \frac{v_\infty \cdot 1_{2m}}{v_\infty \cdot 1_{2m}} \cdot v_\infty \]

\[ \Rightarrow v_\infty \geq 1_{2m} > v_\infty \]

It is possible for the sequence \( g_i \) to move from proximality to non-proximality however eventually it will end up permanently proximal to all the fixed-points. The situation is described by Figure 6.1.

Although just as for \( \{u_i\} \) if a non-trivial non-negative fixed-point exists then the fixed-point content of \( \{v_i\} \) only increases until proximality is attained, \( \{v_i\} \) decreases in norm and thus its fixed-point proportion continues to increase. By a slight extension of the argument of Chapter 3.3.1 we can prove that the sequence \( \{v_i\} \) converges to a non-negative fixed-point, and that this limit will be non-trivial.
6.5 The Algorithm

The algorithm relies on the process of averaging leading eventually to proximality.

We know that if we have a \( g_1 \) which is proximal then \( g_i = U_{g_1}^{i-1} g_1 \) orbits a fixed-point. We don’t worry whether or not \( g_1 \) to \( g_i \) are proximal - it is the proximality of the binding pattern which generates the sequence which matters: we simply use the binding pattern from \( g_1 \) to determine \( U \) and keep this same \( U \) as we generate \( g_2, \cdots, g_i \), even if the binding pattern of subsequent \( g \)'s changes, and apply affine regression. The regression is performed step by step in a manner similar to the conjugate gradient method (refer to 3.3.2) and at each step the resulting vector is checked to see if the negative proportion has decreased; if it fails to decrease, the previous vector together with its binding pattern becomes the starting point for a new affine regression. Note that the first step in affine regression is obviously the same as the averaging matrix.

6.5.1 Infeasibility Detection

We prove that if there is a solution to the fixed-point problem then the solution will have a component-pair containing a zero entry with the other entry greater than or equal to unity, it then follows the norm of a vector in the sequence \( \{ v_i \} \) lies between unity and \( \sqrt{m} \). If the norm falls below unity then the problem is infeasible or feasible unbounded.

From Lemma 6.4.4 for \( p \) a non-negative fixed-point of \( U \), \( p \cdot v_{\infty} \geq p \cdot v_1 \). Thus \( p \cdot v_{\infty} \geq p \cdot v_1 = p \cdot \Psi 1_{2m} = (\Psi^T p) \cdot 1_{2m} = (\Psi p) \cdot 1_{2m} = p \cdot 1_{2m} \), that is

\[
p \cdot v_{\infty} \geq p \cdot 1_{2m}
\]
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Since \( v_\infty \) is a non-negative fixed-point of \( \mathcal{U} \) we can set \( p = v_\infty \) in this equation arriving at

\[
\begin{align*}
v_\infty \cdot v_\infty &\geq v_\infty \cdot 1_{2m} \Rightarrow 1 \geq \frac{1_{2m} \cdot v_\infty}{v_\infty \cdot v_\infty} \Rightarrow v_\infty \geq \frac{1_{2m} \cdot v_\infty}{v_\infty \cdot v_\infty} v_\infty
\end{align*}
\]

which, applying Equation 2.15a,

\[
\Rightarrow v_\infty \geq 1_{2m} \triangleright v_\infty.
\]

This result is illustrated by Figure 6.2.

![Figure 6.2: The Unit and Limit Vectors](image)

Now consider the orthogonal projection \( z = 1_{2m} \triangleright v_\infty \) of the vector \( 1_{2m} \) (that is a vector comprising \( 2m \) 1's) onto the vector \( v_\infty \). We have

\[
(1_{2m} - z)^T z = (1_{2m} - \mathcal{P}1_{2m})^T \mathcal{P}1_{2m}
\]

\[
= ((I - \mathcal{P})1_{2m})^T \mathcal{P}1_{2m} = 1_{2m}^T (I - \mathcal{P})^T \mathcal{P}1_{2m} = 0
\]

\[
\Rightarrow \sum_{i=1}^{2m} (z_i - 1)z_i = 0 \Rightarrow \sum_{i=1}^{2m} z_i^2 = \sum_{i=1}^{2m} z_i \Rightarrow \sum_{i=1}^{2m} (z_i - 0.5 + 0.5)^2 = \sum_{i=1}^{2m} z_i
\]

\[
\Rightarrow \sum_{i=1}^{2m} (z_i - 0.5)^2 + \sum_{i=1}^{2m} (z_i - 0.5) + \sum_{i=1}^{2m} 0.25 = \sum_{i=1}^{2m} z_i \Rightarrow \sum_{i=1}^{2m} (z_i - 0.5)^2 = m/2
\]

\[
\Rightarrow \sum_{j=1}^{m} [(z_j - 0.5)^2 + (z_{j+m} - 0.5)^2] = m/2
\]
It follows that, for some \( j \in \{1, \cdots, m\} \),

\[
(z_j - 0.5)^2 + (z_{j+m} - 0.5)^2 \geq 0.5
\]  

(6.11)

Now, with reference to Equation 6.11, in Figure 6.3 the colour green is used for the case of equality and red for the > case; the solid part of each large circle represents the pointset satisfying the equation and non-negativity; in each case the small unfilled circle represents the notional point \( \left( \frac{z_j}{z_{1+m}} \right) \), while \( a \) and \( b \) represent the points of minimum norm of the variable \( \left( \frac{z_j}{z_{1+m}} \right) \) subject to the constraints of Equation 6.11 and non-negativity.

\[
\begin{align*}
(x_1 - 0.5)^2 + (x_{m+1} - 0.5)^2 > 0.5 \\
(x_1 - 0.5)^2 + (x_{m+1} - 0.5)^2 = 0.5
\end{align*}
\]

Figure 6.3: Minimum Norm Projections

It is apparent that the minimum norm solutions are given by the points represented by the small filled circles, and it follows that for a pair of entries of maximum norm of any minimum norm solution, at least one member of the pair is zero - in the present case it has to be \( z_j = 0 \) or \( z_{m+j} = 0 \) which when substituted into Equation 6.11 yields for example

\[
(z_j - 0.5)^2 + (0 - 0.5)^2 \geq 0.5 \Rightarrow (z_j - 0.5)^2 \geq 0.25
\]

\[
\Rightarrow (z_j - 0.5)^2 \geq 0.25 \Rightarrow z_j - 0.5 \geq 0.5 \text{ or } z_j - 0.5 \leq -0.5
\]

\[
\Rightarrow z_j \geq 1 \text{ or } z_j \leq 0 \Rightarrow z_j \geq 1 \text{ or } z = 0.
\]
6.6. CONCLUSION

So \((z_j \geq 1 \text{ and } z_{j+m} = 0)\) or \((z_j = 0 \text{ and } z_{m+j} \geq 1)\) or \(z = 0\); so the minimum possible norm of \(z\) is unity, otherwise \(z\) is zero.

Thus we have a test for infeasibility/feasible unboundedness, assuming a non-zero discriminant.

6.5.2 Difficulties

The problem is that the sequence \(\{g_i\}\) might converge to a point having a zero component-pair, and the binding pattern never stabilize. Although this is believed not to happen in practice, a modification to the algorithm is given later to cover this possibility.

6.6 Conclusion

In summary it has been shown that it is possible to solve the linear programming problem by transforming it to an invariant form, then to a fixed-point problem, and then solving the resulting fixed-point problem using linear and linear inequality transformations. The approach is a full solution to the LP problem, handling degenerate cases in a manner which is transparent from the linear algebraic point of view. A terminating algorithm has been described which handles even the most problematic linear programs. Moreover it will be shown in Chapter 8 that the approach generalizes to convex optimization in finite-dimensional spaces.
Chapter 7

Computations

In this chapter computer programming of the theory developed in Chapters 4 to 6 is used to check said theory.

Calculations have been carried out using DrScheme, so the Scheme interpreter DrScheme needs to be installed. To do this visit the DrScheme website. Note that the only change made to the selected “R5RS, with no frills” is that the box “Case sensitive” has been ticked. The program document we use is solve_LP.drscm, which contains a number of Scheme functions. Note that when the document is executed it loads another document called linalg_1.drscm which contains various linear algebra functions including ginverse, for computing the pseudo-inverse of a matrix.

7.1 The Invariant Form

Here we compute the invariant form for an LP and check some of its properties which were adduced in Chapter 4. We use function invprob contained in the document solve_LP.drscm. The function invprob takes as inputs the data A, b and c; it is invoked by typing (invprob A b c).

We test this function for the specific LP

\[
\begin{align*}
\text{maximize} & \quad 3x_1 + 4x_2 \\
\text{subject to} & \quad \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq \begin{bmatrix} 3 \\ 3/2 \\ -4 \end{bmatrix} 
\end{align*}
\]

with the specific test function invar_check.drscm, we obtain the following output:

Welcome to DrScheme, version 352.
Language: Standard (R5RS) custom.
loading linear algebra functions:
defining LP:
A = ((1 1) (1 -1) (-2 -1))
b = (3 1 1/2 -4)
c = (3 4)

computing the invariant LP:
(((5/14 -3/14 -3/7) (-3/14 13/14 -1/7) (-3/7 -1/7 5/7))
(15/28 5/28 5/14)
(1 5/14 -1 3/14 -1 3/7))

**********************************************************
checking that the computed invariant form fits the theory:
**********************************************************

A_inv =
(((5/14 -3/14 -3/7) (-3/14 13/14 -1/7) (-3/7 -1/7 5/7))
b_inv =
(15/28 5/28 5/14)
c_inv =
(1 5/14 -1 3/14 -1 3/7)
Checking that the problem is quasi-bounded, i.e. that $A^{-T}c_{inv} - c = 0$ (Equation 2.6.2):
(0 0)
Checking that $A_{inv} - A_{inv}^T = 0$, i.e. that $A_{inv}$ is symmetric (Lemma 2.13(a)):
((0 0 0) (0 0 0) (0 0 0))
and $A_{inv} - A_{inv}^{-2} = 0$, i.e. that $A_{inv}$ is idempotent (Equation 2.13(b)):
((0 0 0) (0 0 0) (0 0 0))
Checking that $A_{inv} b_{inv} = 0$ (Equation 2.13(e)):
(0 0 0)
Checking that $c_{inv} - A_{inv} c_{inv} = 0$, i.e. that $A_{inv} c_{inv} = c_{inv}$ (Equation 2.13(g)):
(0 0 0)
>

These results confirm some of the elementary theory for the invariant form of the specific original LP above.

7.2 Test Problems

Here, for specific linear programs, we check that repeated application of the averaging matrix $V$ defined by Equation 3.10 produces a sequence of vectors which converges to a fixed-point, yielding a solution to the respective linear program. Note that this recursion is not efficient (just as the theory suggests) and for this reason the number of applications has been set to a high value (100).

The function LP is contained in the document `solve_lp.drscm` and is run with the command

```
(LP A b c siter verbose variant).
```

where $A, b$ and $c$ are as previously defined, `siter` is the number of iterations, and `verbose` is a logical variable which determines the amount of output from the function LP, and `variant` is a logical variable - if `#f` then $e = (1 1 1 ...)$ is used as the starting vector for the recursion, and if `#t` then $-e$ is used.
Note that for some of the following problems the value variant = #t is not used as the result is basically the same as for variant = #f. Also most of the problems have been recast in the form of Equation 1.1.

### 7.2.1 Problem 1a

This is the problem which was introduced in Chapter 4.5. It is run with the document problem_1a.drscm. Output is

```
Welcome to DrScheme, version 300.
Language: Textual (MzScheme, includes R5RS) custom.

*******************************
******* averaging approach *******
*******************************
check for optimality
fc^Txi_t + fb^Tzeta_t = 2.220446049250313e-16
(should be zero)
********
discriminant (d) = 0.9999999999924527
Case Ia (>= form)
solution to invariant problem is
(4.04118759626134e-13 2.5000000000523386 -1.7341022646581653e-11
  5.000000000056121 6.128392683880913e-12 1.000000000019803)
solution to original problem is
(0.999999999976324 2.000000000027315)
Ax-b = (-3.638866985511413e-12 2.500000000050991 -2.0036861059224975e-11)

*******************************
******* lattice approach *******
*******************************
check for optimality
fc^Txi_t + fb^Tzeta_t = -7.771561172376096e-16
(should be zero)
********
discriminant (d) = 1.000000000001756
Case Ia (>= form)
solution to invariant problem is
(3.0637437031692065e-12 2.499999999978454 3.836645872648498e-12
  4.999999999968233 -7.66214869327377e-12 0.999999999829194)
solution to original problem is
(1.000000000008463 1.999999999875273)
Ax-b = (4.004352405218015e-12 2.499999999979059 4.4639847374128294e-12)
```

In both cases the solution to the invariant problem is very close to (0 2.5 0 5 0 1), and the solution to the original problem is very close to (1 2).

### 7.2.2 Problem 1b

This is the same problem 1a, however an extra constraint $0x_1 + 0x_2 \geq 0$ has been added. It is run with the document problem_1b.drscm. Output is

```
Welcome to DrScheme, version 300.
Language: Textual (MzScheme, includes R5RS) custom.
```
In both cases the solution to the original problem is very close to \((1,2)\), which is the same as Problem 1a. Note however that the averaging and lattice solutions to the fixed-point problem problem differ and this leads us to the construction of a zero component-pair solution as follows:

We see that there is the fixed-point

\[
p_1^T = \begin{pmatrix} 0 & 0 & 2.5 & 0 & 3.7247934543988825 & 5 & 0 & 1 \end{pmatrix}
\]

from the averaging method, and the fixed-point

\[
p_2^T = \begin{pmatrix} 0 & 0 & 2.5 & 0 & 3.483566542084479 & 5 & 0 & 1 \end{pmatrix}
\]

from the lattice method.

Taking a particular linear combination of \(p_1\) and \(p_2\) we obtain the fixed-point

\[
v^T = \begin{pmatrix} 0 & 0 & 2.5 & 0 & 0 & 5 & 0 & 1 \end{pmatrix}
\]

which has the zero component-pair \((v_1, v_5) = (0,0)\), so zero component-pairs do exist, but we haven’t managed to find one directly during the solution of an actual problem.
Note also that subtracting $p_2$ from $p_1$ yields the positive fixed-point $p_3 = (00001000)$, however this fixed-point has a zero discriminant so it doesn’t yield solutions.

### 7.2.3 Problem 2

The second test linear program is from Ecker & Kupferschmid [13, Exercise 2.3].

Minimize $x_1 - x_2$ subject to

\[ x_1 + 2x_2 \geq 4, \quad x_2 \leq 4, \quad 3x_1 - 2x_2 \leq 0, \quad x_2 \geq 0 \]

Refer to Figure 7.1 for a graphical solution.

![Graph of Problem 2](image)

Figure 7.1: **Problem 2**

The program is run using the document `problem_2.drs`. Output is:

```plaintext
Welcome to DrScheme, version 300.
Language: Textual (MzScheme, includes R5RS) custom.
*********************************************
******* averaging approach *********
*********************************************
check for optimality
$f_c^T x_t + f_b^T zeta_t = 6.19621743763556e-20$
(should be zero)
********
discriminant $(d) = 1.0$
Case Ia $(>= \text{ form})$
solution to invariant problem is
$(1.2534376289078367e-19 1.2083945054468448e-19 20.0 4.0$
$1.0 3.0 -1.6978445119504833e-20 -3.7073303033685564e-20)$
solution to original problem is
```
Ax-b = (0.0 0.0 20.0 4.0)

************ lattice approach ************

check for optimality
fc^Txi_t + fb^Tzeta_t = -9.447481464182589e-16
(should be zero)

********
discriminant (d) = 0.999999998255478

Case Ia (>= form)
solution to invariant problem is
(1.6822863359441972e-08 -1.1140389183115618e-08 20.000000042143576 4.000000011838199 1.0000000022327813 3.000000006731263 6.861095777049602e-10 7.190281589514207e-10)
solution to original problem is
(-4.000000006338486 4.000000011555752)

Again a correct solution has been found.

7.2.4 Problem 3

This problem is from Ravindran (op cit):

Minimize \(-3x_1 + x_2 + x_3\) subject to

\[
\begin{align*}
x_1 - 2x_2 + x_3 & \leq 11, \\
-4x_1 + x_2 + 2x_3 & \geq 3, \\
2x_1 - x_3 & = -1
\end{align*}
\]

\[
x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0
\]

We solve this problem by executing the document problem_3.drscm.

At first sight the problem appears unusual, but remember that the third and fourth constraints relate to an equality constraint, so we can regard the solution as having only three binding constraints. The computed results are:

Welcome to DrScheme, version 300.
Language: Textual (MzScheme, includes R5RS) custom.

************ averaging approach ************

check for optimality
fc^Txi_t + fb^Tzeta_t = 1.5201291195688791e-15
(should be zero)

********
discriminant (d) = 1.0000000102807172

Case Ia (>= form)
solution to invariant problem is
(-4.086822339208776e-09 -2.18214781572826e-08 -1.605498945821598e-08)
The lattice approach is seen not to converge as quickly as the averaging approach.

### 7.2.5 Problem 4a

Now a problem with multiple solutions:

Maximize \(2x + 2y + z\) subject to

\[
\begin{align*}
  x &\geq 0, \\
  y &\geq 0, \\
  z &\geq 0, \\
  x + y + z &\leq 1
\end{align*}
\]

So we maximize

\[
\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}^T \begin{bmatrix} x \\ y \\ z \end{bmatrix}
\]

subject to

\[
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}
\]

Note that the solution set for the original problem is given by

\[
(x, y, z) \in [(0, 0, 0), (0, 0, 1)]
\]

as can be seen by referring to Figure 7.2.

Further, the slack and binding constraints for the fixed-point problem are:
CHAPTER 7. COMPUTATIONS

Figure 7.2: Problem 4a

1. for \((x, y, z) = (1, 0, 0)\) the constraints are \((s, b, b, b)\).

2. for \((x, y, z) \in ((1, 0, 0), (0, 1, 0))\), the constraints are \((s, s, b, b)\).

3. for \((x, y, z) = (0, 1, 0)\) the constraints are \((b, s, b, b)\).

We solve this problem by executing the document `problem_4a.drscm`. Output is

Welcome to DrScheme, version 300.
Language: Textual (MzScheme, includes R5RS) custom.

**********************************
******* averaging approach *******
**********************************

check for optimality
\(fc^T x_i + fb^T zeta_t = -3.2723045615006252e-47\)
(should be zero)

********

discriminant \((d) = 1.0\)
Case Ia \((>= \text{ form})\)
solution to invariant problem is
\((0.5, 0.5, 1.0, 9.07681871668751e-47, 2.1815363743337502e-47, 5.4538409358343754e-47, 5.4538409358343754e-47, 1.0, 2.0)\)
solution to original problem is
\((0.5, 0.5, 0.0)\)
\(Ax - b = (0.5, 0.5, 0.0, 0.0)\)

**********************************
******* lattice approach *******
**********************************

check for optimality
\(fc^T x_i + fb^T zeta_t = -9.270948240292157e-17\)
(should be zero)

********
7.2. TEST PROBLEMS

discriminant \( d = 1.0000000064856773 \)

Case Ia (\( \geq \) form)

solution to invariant problem is
\[
\begin{pmatrix}
0.4999999958173318 & 0.4999999958173318 & 1.2219490454111951e-08 \\
-1.0339831438869336e-08 & 8.460172423626721e-09 & 8.460172423626721e-09 \\
1.0000000019744952 & 1.999999995488818 &
\end{pmatrix}
\]

solution to original problem is
\[
\begin{pmatrix}
0.4999999974387511 & 0.4999999974387511 & 1.3840909807161239e-08 \\
-8.718411992347797e-09 & &
\end{pmatrix}
\]

So in both cases a solution to the invariant problem is found at approximately \((-0.5 -0.5 0 0)\), and in both cases a solution to the original problem is approximately \((0.5 0.5 0)\).

7.2.6 Problem 4b

Now we try, with this problem, to manufacture a zero component-pair by shrinking the feasible set down to a single point.

Maximize \(2x + 2y + z\) subject to

\[
x \geq 0, \ y \geq 0, \ z \geq 0, \ x + y + z \leq 1, x \leq 0.5, y \leq 0.5
\]

So we maximize
\[
\begin{pmatrix}
2 \\
2 \\
1
\end{pmatrix}^T \begin{pmatrix}
x \\
y \\
z
\end{pmatrix} \text{ subject to } 
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
-1 & -1 & -1 \\
-1 & 0 & 0 \\
0 & -1 & 0
\end{bmatrix} \begin{pmatrix}
x \\
y \\
z
\end{pmatrix} \geq \begin{pmatrix}
0 \\
0 \\
0 \\
-1 \\
-0.5 \\
-0.5
\end{pmatrix}
\]

Note that the solution is given by
\[
(x \ y \ z) = (0.5 \ 0.5 \ 0)
\]

as can be seen by referring to Figure 7.3.

We solve this problem by executing the document `problem_4b.drscm`. Output is

Welcome to DrScheme, version 300.
Language: Textual (MzScheme, includes R5RS) custom.
**********************************
******* averaging approach *******
**********************************

check for optimality
\( fc^T x_i + fb^T \zeta_t = -5.551115123125783e-17 \)
(should be zero)

********

discriminant \( d = 1.0 \)

Case Ia (\( \geq \) form)

solution to invariant problem is
Figure 7.3: Problem 4b

Note that the algorithm precludes convergence to an extreme point of the optimal feasible region, as the starting vector of ones contains a positive component of every extreme point of the optimal feasible set, and this component does not decrease. Thus a major source of problematic zero component-pair (probably all of such pairs) creates no difficulties for the algorithm.
7.2. TEST PROBLEMS

7.2.7 Problem 5

An over-identified problem:

Maximize $x$ subject to

\[
x + y + z \leq 1, \quad x + y - z \leq 1, \quad x - y + z \leq 1, \quad x - y - z \leq 1
\]

So we maximize \[
\begin{bmatrix}
1 \\
0
\end{bmatrix}^T
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
\]
subject to \[
\begin{bmatrix}
-1 & -1 & -1 \\
-1 & 1 & 1 \\
-1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
\geq
\begin{bmatrix}
-1 \\
-1 \\
-1
\end{bmatrix}
\]

We solve this problem by executing the document `problem_5.drscm`. Output is:

Welcome to DrScheme, version 300.
Language: Textual (MzScheme, includes R5RS) custom.

*************************************************************
******* averaging approach ******
*************************************************************
check for optimality
fc^Txi_t + fb^Tzeta_t = 0.0
(should be zero)

****** discriminant (d) = 1.0
Case Ia (>= form)
solution to invariant problem is
(0.0 0.0 0.0 0.0 0.25 0.25 0.25 0.25)
solution to original problem is
(1.0 0.0 0.0)
Ax-b = (0.0 0.0 0.0 0.0)

*************************************************************
******* lattice approach ******
*************************************************************
check for optimality
fc^Txi_t + fb^Tzeta_t = 0.0
(should be zero)

****** discriminant (d) = 1.0
Case Ia (>= form)
solution to invariant problem is
(0.0 0.0 0.0 0.0 0.25 0.25 0.25 0.25)
solution to original problem is
(1.0 0.0 0.0)
Ax-b = (0.0 0.0 0.0 0.0)

*************************************************************
******* variant -**********
*************************************************************

check for optimality
fc^Txi_t + fb^Tzeta_t = 0.0
(should be zero)

******
discriminant (d) = 0.0
Case II (unusual form)
solution to invariant problem is
(0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0)
solution to original problem is
(1.0 0.0 0.0)
Ax-b = (0.0 0.0 0.0 0.0)

**********************************
******* lattice approach *******
**********************************
check for optimality
fc^Txi_t + fb^Tzeta_t = 0.0
(should be zero)
********
discriminant (d) = 1.0
Case Ia (>= form)
solution to invariant problem is
(0.0 0.0 0.0 0.0 0.25 0.25 0.25 0.25)
solution to original problem is
(1.0 0.0 0.0)
Ax-b = (0.0 0.0 0.0 0.0)

7.2.8 Problem 6

The following two problems of the unusual type described in Chapter 5.2.3:

(a) Maximize $x + 2y + z$ subject to

$$x + y + z \geq 1, \quad x - y + z \geq 1, \quad -x + 0y - z \geq -1, \quad x \geq 0, \quad x \geq 1$$

So we maximize

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix}^T \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

subject to

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \geq \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

We solve this problem by executing the document `problem_6av.drscm`. Output is

Welcome to DrScheme, version 300.
Language: Textual (MzScheme, includes R5RS) custom.

**********************************
******* averaging approach *******
**********************************
check for optimality
fc^Txi_t + fb^Tzeta_t = 1.232595164407831e-32
(should be zero)
********
discriminant (d) = -1.0
Case Ib (<= form)
solution to invariant problem is
(6.530723674265627e-17 3.0041328901621885e-16 -9.000000000000025 -7.000000000000016
 -8.000000000000025 -9.0000000000000015 -11.000000000000032 -5.224578939412501e-17
 5.224578939412501e-17)

So we maximize

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix}^T \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

subject to

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \geq \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

We solve this problem by executing the document `problem_6av.drscm`. Output is
solution to original problem is
(-8.000000000000025 -1.1102230246251565e-16 9.000000000000025)
Ax-b = (0.0 0.0 0.0 -8.000000000000025 -9.000000000000025)

********* lattice approach *********

check for optimality
fc^Txi_t + fb^Tzeta_t = 3.1554436208840472e-30
(should be zero)

********
discriminant (d) = -1.0000000000000002

Case Ib (<= form)
solution to invariant problem is
(-1.938317556187377e-14 -1.1307669524591808e-13 6.622993540389593e-14
 -8.00000000000000036 -9.00000000000000036 -7.000000000000000195
 -5.000000000000000002 -11.00000000000000043 2.7463584280148387e-14
 -2.7463584280148387e-14)
solution to original problem is
(-8.000000000000036 4.6851411639181606e-14 8.999999999999968)
Ax-b = (-2.042810365310288e-14 -1.1368683772161603e-13
 6.750155989720952e-14 -8.00000000000000036 -9.00000000000000036)

==========================================
= variant -=================================

==========================================
= averaging approach ======================

check for optimality
fc^Txi_t + fb^Tzeta_t = -2.2597578014143566e-32
(should be zero)

********
discriminant (d) = 1.0

Case Ia (>= form)
solution to invariant problem is
(-7.183796041692191e-17 -1.959217102279688e-17 4.5715065719859385e-17
 2.5816326530612225 1.5816326530612252 3.581632653061227
 5.581632653061227 10.163265306122454 -9.79608551139844e-17
 9.79608551139844e-17)
solution to original problem is
(2.5816326530612255 -5.551115123125783e-17 -1.5816326530612255)
Ax-b = (0.0 0.0 0.0 2.5816326530612225 1.5816326530612255)

********* lattice approach *********

check for optimality
fc^Txi_t + fb^Tzeta_t = 3.1554436208840472e-30
(should be zero)

********
discriminant (d) = -1.0000000000000002

Case Ib (<= form)
solution to invariant problem is
(-1.938317556187377e-14 -1.1307669524591808e-13 6.622993540389593e-14
 -8.00000000000000036 -9.00000000000000036 -7.000000000000000195
 -5.000000000000000002 -11.00000000000000043 2.7463584280148387e-14
 -2.7463584280148387e-14)
solution to original problem is
(-8.000000000000036 4.6851411639181606e-14 8.999999999999968)
Ax-b = (-2.042810365310288e-14 -1.1368683772161603e-13
 6.750155989720952e-14 -8.00000000000000036 -9.00000000000000036)
(b) Maximize \(-x - 2y - z\) subject to

\[
x + y + z \geq -1, \quad x - y + z \geq -1, \quad -x + 0y - z \geq 1, \quad x \geq 0 \quad x \geq -1
\]

So we maximize \[
\begin{bmatrix}
-1 \\
-2 \\
-1
\end{bmatrix}^T
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
\] subject to
\[
\begin{bmatrix}
1 & 1 & 1 \\
1 & -1 & 1 \\
1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
\geq
\begin{bmatrix}
-1 \\
1 \\
0
\end{bmatrix}.
\]

We solve this problem by executing the document `problem_6bv.drs`. Output is:

Welcome to DrScheme, version 300.
Language: Textual (MzScheme, includes R5RS) custom.
**********************************
******* averaging approach *******
**********************************
check for optimality
\[
f^T \xi_t + fb^T \zeta_t = -1.232595164407831e-32
\] (should be zero)
********

discriminant (d) = 1.0
Case Ia (>= form)
solution to invariant problem is
\[
\]
solution to original problem is
\[
(8.000000000000025 -9.000000000000025)
\]
Ax-b = \[
(0.0 0.0 8.000000000000025 9.000000000000025)
\]********

******** lattice approach *******
********

check for optimality
\[
f^T \xi_t + fb^T \zeta_t = -3.1554436208840472e-30
\] (should be zero)
********

discriminant (d) = 1.0
Case Ia (>= form)
solution to invariant problem is
\[
\]
solution to original problem is
\[
(8.000000000000036 -8.999999999999968)
\]
Ax-b = \[
(2.042810365310288e-14 -6.750155989720952e-14 8.000000000000036 9.000000000000036)
\]================================================
=============== variant ================
================================================
7.3. CONCLUSION

The theory for both the unitary and lattice approaches works in practice.
Chapter 8

Convex Optimization

Our aim is to develop operational conditions for solving convex optimization problems.

We consider the general problem

\[
\begin{align*}
\text{maximize} & \quad f(x), \ x \in \mathbb{R}^n \\
\text{subject to} & \quad g(x) \geq 0, \ e \ \mathbb{R}^m.
\end{align*}
\] (8.1)

Here \(f\) is a scalar function of the unknown \(n\)-dimensional vector \(x\), and \(g\) is an \(m\)-dimensional vector function of \(x\). The objective function \(f\) and the inequality constraint \(g\) are required to be continuously differentiable concave\(^1\) functions.

8.1 Convexity

Given a column vector \(x\) and a scalar function, \(y\), we introduce the vector derivative or gradient of \(y\) as the column vector function \(y_{x} = \partial y/\partial x = \nabla_x y\) defined by

\[
(y_{x})_i = \partial y/\partial x_i .
\] (8.2)

Define \(\hat{f} = f(\hat{x})\) and \(\hat{\nabla} f = f_x|\hat{x}\), i.e. \(\hat{\nabla} f\) is the \(n\) dimensional vector whose \(j^{th}\) term is \(\partial f(x)/\partial x_j | \hat{x}\).

A differentiable function \(f\) on a convex domain is convex \(\Leftrightarrow f(x) \geq f(\hat{x}) + \hat{\nabla} f^T (x - \hat{x})\), while a differentiable function \(f\) on a convex domain is concave \(\Leftrightarrow -f(x) \geq -f(\hat{x}) + \nabla f(-f)^T(x - \hat{x}) \Leftrightarrow -f(x) \geq -f(\hat{x}) + \nabla f(-f)^T(x - \hat{x}) \Leftrightarrow f(x) \leq f(\hat{x}) + \nabla f^T(x - \hat{x})\), so

**Lemma 8.1.1** \(\text{Differentiable function } f \text{ on a convex domain is concave} \Leftrightarrow f(x) \leq f(\hat{x}) + \hat{\nabla} f^T (x - \hat{x}) \Leftrightarrow f(x) \leq (f(\hat{x}) - \hat{\nabla} f^T \hat{x}) + \hat{\nabla} f^T x\)

so the objective function has now been represented as a “constant” component, \(f(\hat{x}) - \hat{\nabla} f^T \hat{x}\), plus a component \(\hat{\nabla} f^T x\) linear in \(\hat{x}\), whereby \(\hat{\nabla} f\) is our \(\mathbf{c}\).

\(^1\)The reason for this is that the usual formulation is a minimization problem with \(g(x) \leq 0\).
8.2 Types of Problem

Typically the nature of the objective function is such that we do not need to restrict the solutions so that they satisfy 8.1(b) (refer to Figure 8.1). However it is also possible that linearizing a problem results in a solution set which is an affine space while the non-linear problem’s solution set is a proper subset of this affine space (refer to Figure 8.2); in this case condition 8.1(b) must be forced to obtain, usually by Newton’s method (refer to Figure 8.4). Finally there is the aberrant case of an empty linearized solution set for the reason that the linearized problem’s feasible set is unbounded in a direction for which the non-linear problem feasible set is bounded (refer to Figure 8.3); such problems might best be recast within the affine space containing the linearized problem’s feasible set.

8.2.1 The Typical Case

A typical bounded problem is illustrated by Figure 8.1; in this case the linearized problem yields a solution directly, using the fixed-point method.
8.2.2 The Atypical Case

An atypical bounded problem is illustrated by Figure 8.2; in this case the solutions to the linearized problem comprise an affine space; the solutions to the non-linear problem comprise a single point in this space. In this case it is necessary to combine the fixed-point method with Newton’s method.

Figure 8.2: The Atypical Case

8.2.3 The Aberrant Case

An aberrant bounded problem is given in Figure 8.3; in this case the set of feasible points of the linearized problem is an unbounded affine space; the solutions to the non-linear problem comprise a single point in this space. Usually in this case only Newton’s method need be applied to reach a solution. However there is the unusual aberrant case where there are multiple feasible points of the non-linear problem, lying in the affine space of feasible points of the linearized problem.

8.3 Solution by Linearization

If \( \mathbf{x} \) and \( \mathbf{y} \) are both column vectors, then we form the Jacobian matrix \( \mathbf{y}_x \) defined by

\[
(y(x))_{ij} = \frac{\partial y_i}{\partial x_j}.
\] (8.3)

Given an estimate, \( \hat{x} \), of the solution, write

\[
\hat{g} = g(\hat{x}), \quad \text{and} \quad \hat{G} = g_x|\hat{x},
\]
We approximate $g$ by $g = \hat{g} + \hat{G}(x - \hat{x})$, where $\hat{G}$ is the $m \times n$ Jacobian matrix whose $(i,j)$ term is $\partial g(x)_i/\partial x_j$. Then, for concave $g$, since $g \leq \hat{g} + \hat{G}(x - \hat{x})$, we have $g \geq 0 \Rightarrow \hat{g} + \hat{G}(x - \hat{x}) \geq 0$, which implies $\hat{G}x \geq \hat{G}\hat{x} - \hat{g}$. So feasibility of the convex non-linear program 8.1 implies feasibility of its linearization. Thus, given an initial estimate $\hat{x}$ of a solution to the problem, notionally we have

$$
\hat{A} = \hat{G} \quad \text{(a)} \\
\hat{b} = \hat{G}\hat{x} - \hat{g} \quad \text{(b)} \\
\hat{c} = \hat{\nabla}f \quad \text{(c)}
$$

We have now linearized the problem to:

$$
\text{maximize } \hat{c}^T x \text{ subject to } \hat{A}x \geq \hat{b} .
$$

where $\hat{A}$, $\hat{b}$ and $\hat{c}$ are given by Equations 8.4 (a), (b) and (c) respectively.

We also require 8.1(b) as there is no guarantee that the solution set for the problem has not been increased by linearization; to ensure this we periodically apply Newton’s method using the current binding constraints. (refer to Figure 8.4).
### 8.4 Solution Conditions

Analogous to Equation 2.37, we impose

\[
\hat{G}^+ \hat{G} \triangledown f = \triangledown f .
\]  

(8.5)

As with LP’s case we can construct invariant primal and dual LP’s, and fixed-point problem

\[
\text{find } \begin{bmatrix} \xi \\ \eta \end{bmatrix} \text{ satisfying } \begin{bmatrix} \Psi \\ \Phi \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} \geq 0
\]

(8.6)

where, in the present context, we have

\[
\begin{align*}
\hat{A} & = \hat{G} \hat{G}^+ & \text{(a)} \\
\hat{b} & = -(I - \hat{G} \hat{G}^+) \hat{g} & \text{(b)} \\
\hat{c} & = \hat{G}^T \hat{\triangledown} f & \text{(c)} \\
\hat{D} & = I - \hat{G} \hat{G}^+ & \text{(d)} \\
\end{align*}
\]

Collecting relevant equations and properties, the conditions are as for Lemma 5.2.1, with the additional \( \hat{g} \geq 0 \) to override the increase in the solution set due to linearization.

\[
\begin{align*}
\hat{\xi} & = \hat{A} \hat{\xi} - \hat{b} \geq 0 & \text{(a)} \\
\hat{\eta} & = \hat{D} \hat{\eta} - \hat{c} \geq 0 & \text{(b)} \\
\hat{c}^T \hat{\xi} + \hat{b}^T \hat{\eta} & = 0 & \text{(c)} \\
\hat{g} & \geq 0 & \text{(d)} \\
\hat{\xi} & = \hat{A} \hat{\xi} - \hat{b} & \text{(e)} \\
\hat{x} & = \hat{A}^+ (\hat{x} + \hat{b}) & \text{(f)}
\end{align*}
\]

(8.8)

Note: \((e) \wedge (f) \Rightarrow \hat{\xi} = \hat{A} \hat{\xi} - \hat{b} .

Note that these conditions are

1. necessary and sufficient if the problem is one of convex optimization and is non-aberrant; if the problem is aberrant then we delineate the affine space of feasible points of the linearized problem and recast the problem within this lower dimensional context, eventually arriving at a context within which the problem is non-aberrant.

2. operational since \( \hat{\xi} \) and \( \hat{\eta} \) can be constructed using the fixed-point approach,

3. free of any constraint qualification such as quasi-normality,

4. distinct from the Karush-Kuhn-Tucker conditions as condition 8.8(c) is scalar in nature while the corresponding KKT stationarity condition is a vector condition.
8.5 The Algorithm

We adopt the following recursive scheme for the non-aberrant case:

1. Choose an initial \( \hat{x} \) (try \( \hat{x} = 0 \)) as a solution to the primal,

2. Compute \( \hat{A}, \hat{b} \) and \( \hat{c} \), using Equation 8.4.

3. Compute \( \hat{A}, \hat{b}, \hat{c} \) and \( \hat{D} \) using Equation 8.7. (consistent with Equation 2.12); optionally, \( \hat{b} \) and \( \hat{c} \) may be normalized. In detail, compute:
   
   (a) \( \hat{G}^+ \) either by direct inversion or by updating using the iterative method of Ben-Israel and Cohen:
   
   \[ \hat{G}^+_{i+1} = 2\hat{G}^+_{i} - \hat{G}^+_{i}\hat{G}^+_{i} \]
   
   where \( \hat{G}^+_{0} \) is the MPGI of the previous \( \hat{G} \),

   (b) \( \mathfrak{A} = \hat{G}\hat{G}^+ \), \( \mathfrak{b} = -(I - \hat{G}\hat{G}^+)\hat{g} \), \( \mathfrak{c} = \hat{G}^{T+}\hat{\nabla}f \) and \( \mathfrak{D} = I - \mathfrak{A} \),

4. Compute \( \mathfrak{P} \) using Equation 5.11 (b).

5. Fixed-point estimate: initial \( \hat{z} = \mathfrak{P}1 \); update with
   
   (a) \( \hat{z}_{new} = \mathfrak{P}|\hat{z}_{old} - \mathfrak{S}|\hat{z}_{old} \) , or

   (b) when binding pattern becomes stable
   
   (i) perform affine regression (refer to Chapter 3.3.2), or
   
   (ii) apply Newton’s Method using the binding constraints (to overcome the difficulty described in Subsection 8.6).

6. Compute the discriminant, \( d \), using Equation 5.21 and with fixed-point estimate \( \hat{z} = \begin{bmatrix} \hat{\xi} \\ \hat{\eta} \end{bmatrix} \), using \( \hat{\xi}_r = \hat{\xi}/d \)

   - that is \( \hat{\xi}_r \) is the primal component of the representative fixed-point solution \( \hat{z}_r = \hat{z}/d \).

7. Consistent with Equation 4.7 (b), compute \( \hat{x} = \hat{A}^+(\hat{\xi} + \hat{b}) = \hat{G}^+(\hat{\xi} - (I - \hat{G}\hat{G}^+)\hat{g}) = \hat{G}^+\hat{\xi}, \) as an estimated solution for the original problem.

8. Stop if conditions 8.8 are met, otherwise go to step 2.

If the scheme for the non-aberrant case fails to converge set \( c = 0 \) to determine feasible points, determine the affine space in which the feasible points lie, and recast the problem within this affine space.

8.6 Application

Here we apply the theory for the non-linear case to solving two problems. The algorithm works well for the first case but fails for the second, suggesting that the approach requires further refinement involving Step 5b above.
8.6.1 Non-Linear Problem I

Maximize $x + y + z$ subject to $x^2 + y^2 + z^2 \leq 1$, $x \geq -0.5$, $x \leq 0.5$.

With reference to Figure 8.4, when the objective function $x + y + z$ is restricted to the value 1.7247 we get a hyperplane of solutions some of which are depicted by the triangular blue area which osculates the feasible set (which is coloured dark green, green and red) at the unique solution $x = 0.5$, $y = 0.61235$, $z = 0.61235$ (indicated by a red dot).

The recursive linearization algorithm when applied to this problem gives this result, however this is more by luck than good management as there are in fact multiple solutions to the linearized problem, as indicated in Figure 8.4 by the line of solutions to the linearized problem; this is an atypical problem.

8.6.2 Non-Linear Problem II

We apply the approach to the following problem (refer to [13, p.301]):

$$\begin{align*}
\text{maximize} & \quad 3x_1 - \frac{x_2^2}{2} \\
\text{subject to} & \quad x_1^2 + x_2^2 \leq 1 \\
 & \quad x_1 \geq 0 \\
 & \quad x_2 \geq 0
\end{align*}$$

What we find is that when using the recursive linearization algorithm the solution doesn’t converge properly. We can see why this happens by referring to Figure 8.5. The figure shows the feasible and
optimal feasible solutions to the limiting linearized problem. The optimal feasible solutions to the limiting linearized problem form the set \( \{(x_1, x_2) : x_1 = 1; x_2 \geq 0\} \), so there is no guarantee that the algorithm will converge to \((x_1, x_2) = (1, 0)\). To overcome this difficulty we could apply the Newton-Raphson algorithm once the linearized form has converged to a solution, using the binding constraints, after Step 7 of the algorithm; this problem is also atypical.

![Figure 8.5: Non-Linear Problem II](image)

### 8.7 Conclusion

Both problems are atypical. The solution to the first problem was correct due to the symmetry of the starting guess. The solution reached for the second problem was only to the linearized problem, however the algorithm also successfully identified the slack and binding constraints, without incorporation of Newton’s Method. So, generally speaking, in order to solve atypical non-linear optimization problems Newton’s Method needs to be incorporated into the solution algorithm.

Having covered the typical and atypical non-linear optimization cases, the aberrant case needs to be considered. This case can be analysed by first delineating the affine set of feasible points of the linearized problem, and then recasting the non-linear problem in this context.
Appendices
Appendix A

Convergence

Here we consider the problem of convergence in greater depth. Specifically, the convergence of the averaging matrix $V$ is investigated.

Note Corollary 3.1.19, which states that $V^{2i} = U^i(KP)^i = (PK)^iU^i$. This implies that we can analyse the spectral properties of say $PK$ in order to gain knowledge of the convergence properties of $V$.

Afriat’s [2] theory of reciprocal spaces is used to investigate the spectral structure of $PK$ and $KP$, leading to a better understanding of the $P$-unitary matrix $U$, through the development of some associated algebra. It is shown that the eigenvalues of $PK$ (excluding those associated with fixed-points) are all in $(0, 1)$. Note that if the greatest of these is near unity then, after a number of iterations, neither $(PK)^n h$ nor $V^n h$ will converge quickly to a fixed-point of both $P$ and $K$, which suggests that the affine regression approach is required.

A.1 Reciprocal Vectors and Spaces

Following Afriat [2], spaces $A$ and $B$ are said to be reciprocal spaces in spaces $P, K$ if they are orthogonal projections of each other in $P, K$; thus if $P$ and $K$ are the symmetric idempotents defining the orthogonal projections onto $P$ and $K$ respectively, then $A = PB, B = KA$. Vectors which span reciprocal rays give rise to a reciprocal pair of vectors in the relevant spaces; if the vectors are of the same length we say they are a balanced pair. The coefficient of inclination between the reciprocal spaces $P$ and $K$ is $R = \text{trace}(PK)$. (Afriat uses $e$ and $f$ to denote a pair of symmetric idempotents - this notation is more common when analyzing more general structures.)

An eigenvalue in $(0, 1)$ is said to be proper; an eigenvector having a proper eigenvalue is also said to be proper; a reciprocal vector associated with a proper eigenvalue is said to be proper, and the subspace generated by the set of all proper eigenvalues of $PK$ and $KP$ is called the proper subspace and is denoted by $\mathcal{H}$. Note also that if $x \in P$ then $Kx = 0 \Rightarrow (I - K)x = x, \Rightarrow SKSx = x, \Rightarrow KSx = Sx$.

Bearing in mind the possibility that $\rho$ may be a complex number, let $PKx_P = \rho^2 x_P$. If $\rho \neq 0$ define
Theorem A.1.3

Summing up we have the balanced reciprocal scheme

\begin{align*}
P K x_p &= \rho^2 x_p, \\
K P x_K &= \rho^2 x_K, \\
K x_p &= \rho x_K, \quad \text{and} \\
P x_K &= \rho x_p.
\end{align*}

Note that, from Equation A.1 (c), \( x_p^T K x_p = \rho x_p^T x_K \Rightarrow x_p^T x_p = \rho x_p^T x_K \), (since \( K x_K = x_K \)). Similarly \( x_p^T x_K = \rho x_p^T x_p \); it follows that \( \| x_p \| = \| x_K \| \), and \( \cos(x_p, x_K) = x_p^T x_K / (\| x_p \| \| x_K \|) = x_p^T x_K / \| x_p \|^2 \); thus given the eigenvector \( x_p \) of \( PK \), with eigenvalue \( \rho^2 \), take \( x_K = K x_p / \rho \), then \( x_p, x_K \) is a pair of reciprocals in \( \mathcal{P}, \mathcal{K} \). Moreover, since they are of the same length we say they are a balanced pair.

Lemma A.1.1

An eigenvector of \( PK \) and an eigenvector of \( KP \) with distinct eigenvalues are orthogonal.

Proof: Let \( x_p \) be an eigenvector of \( PK \) with eigenvalue \( \rho^2 \), and \( x'_K \) be an eigenvector of \( KP \) with eigenvalue \( \rho^2 \), distinct from \( \rho^2 \), and assume without loss of generality that \( \rho^2 \neq 0 \), then

\[ \rho^2 x_K x_p = (K P x_K)^T x_p = (x_K^T P K) x_p = \rho^2 x_K^T x_p. \]

Thus \( (\rho^2 - \rho^2)x_K^T x_p = 0 \Rightarrow x_K^T x_p = 0 \square \)

Lemma A.1.2

Two eigenvectors of \( PK \) with distinct eigenvalues are orthogonal.

Proof: Let \( x_p \) and \( x'_K \) be eigenvectors of \( PK \) with eigenvalues \( \rho^2 \) and \( \rho^2 \) respectively. Assume without loss of generality that \( \rho^2 \neq 0 \). Now

\[ x_p^T x_p = (P K P x_p / \rho^2)^T x_p = x_p^T K P x_p / \rho^2 = x_p^T P K x_p / \rho^2 = (K x_p')^T x_p / \rho^2 \]

\[ = (\rho^2 x_K')^T x_p / \rho^2 = x_K^T x_p / \rho^2 = 0 \quad (\text{by Lemma A.1.1}). \square \]

Summing up we have

Theorem A.1.3

Eigenvectors of \( PK \) and \( KP \) with distinct eigenvalues are orthogonal.

The angle between \( x_p \) and \( x_K \) is given by \( \cos(x_p, x_K) = \rho \); the coefficient of inclination is \( \rho^2 \). If the scheme has \( \rho \) positive then it is said to be an acute reciprocal scheme, and if it has \( \rho \) negative then it is said to be obtuse. Let \( (x_p|x_K) \) denote a reciprocal pair \( x_p, x_K \) in \( \mathcal{P}, \mathcal{K} \); let \( [x_p|x_K] \) denote a balanced reciprocal pair \( x_p, x_K \) in \( \mathcal{P}, \mathcal{K} \), and let \( \{x_p|x_K\} \) denote a normalized balanced reciprocal pair \( x_p, x_K \) in \( \mathcal{P}, \mathcal{K} \); define \( \alpha(x_p|x_K) = (\alpha x_p|\alpha x_K) \) and regard two pairs \( (x_p|x_K) \) and \( (x'_p|x'_K) \) as equivalent if \( (x_p|x_K) = \beta(x'_p|x'_K) \) for some \( \beta > 0 \).

Define the “sum” vector \( s = x_p + x_K \) and the “difference” vector \( d = x_p - x_K \); note that \( s \perp d \). Now consider the symmetric positive definite matrix \( W = (P + K) / 2 \); it can be shown that \( W s = \frac{1+\rho}{2} s \), and
A.1. RECIPROCAL VECTORS AND SPACES

\[ Wd = \frac{1}{\rho^2}d. \] Thus a pair of eigenvalues and vectors has been delineated for \( W \); since this is a symmetric positive definite matrix it follows that \( \rho \) is real; further, since \( P \) and \( K \) are symmetric idempotents, \( \rho \in [-1, 1] \). It is now apparent that the eigenvalues, \( \rho^2 \) of \( PK \) and of \( KP \) are all positive real numbers in the closed interval \([0, 1]\).

The above discussion can be tightened to the following

**Lemma A.1.4** \( \rho^2 \) is an eigenvalue of \( PK \) if and only if \((1 \pm \rho)/2 \) is an eigenvalue of \( W = (P + K)/2 \).

**Proof:** The forward implication has already been covered. For the reverse implication let \( x \) be an eigenvector of \( W \) with eigenvalue \((1 + \rho)/2\); this implies \((P + K)x = (1 \pm \rho)x\), \( \Rightarrow P(P + K)x = (1 \pm \rho)Px\), \( \Rightarrow Px + PKx = (1 \pm \rho)Px\), \( \Rightarrow PKx = \pm \rho Px\),

\[ KPKx = \pm \rho KPx, \tag{A.2} \]

and, similarly,

\[ PKPx = \pm \rho PKx. \tag{A.3} \]

Thus from equations A.2 and A.3, \( PKPKx = \pm \rho PKPx = \rho^2 PKx \); similarly \( KPKx = \rho^2 KPx \). So \( \rho^2 \) is an eigenvalue of \( PK \) and of \( KP \). \( \square \)

It follows that the eigenvalues of \( PK \) all lie in \([0, 1]\).

Define \( \eta = \sqrt{1 - \rho^2} \), \( \tau = \eta^{-1} \), and

\[
\begin{align*}
\xi_P &= \tau PSx_K, \text{ and} \\
\xi_K &= \tau KSx_P, \tag{a} (b)
\end{align*}
\]

Note that

\[ \eta^2 = 1 - \rho^2. \tag{A.5} \]

The map \( \mathcal{B} : \{x_K|x_K\} \mapsto [\xi_P|\xi_K] = \tau(PSx_K|KSx_P) \) is well-defined, \( \mathcal{B}^2 \) is the identity map, and \( \mathcal{B} \) is one to one and onto for the set of all reciprocal pairs, and when restricted to the set of normalized reciprocals in \( \mathcal{P}, \mathcal{K} \).

**Lemma A.1.5**

**Proof:** Now \( \xi_P = \tau PSx_K \), and \( \xi_K = \tau KSx_P \), so

\[
PK\xi_P = \tau PKPSx_K = \tau PKPSx_K = -\tau PKSPKx_K = \tau PSKPKx_K = \tau PSKPKx_K = \rho^2 \tau PSx_K
\]

\[ = \rho^2 \xi_P. \]

Similarly \( KP\xi_K = \rho^2 \xi_K \).

Also \( K\xi_P = \tau KPSx_K = \tau KPSx_K = -\tau KSPKx_K = -\tau KSPx_K = -\rho KSPx_P = -\rho \xi_K \); similarly \( P\xi_K = -\rho \xi_P \).
In summary

\[ PK\xi_P = \rho^2 \xi_P, \quad \text{(a)} \]
\[ KP\xi_K = \rho^2 \xi_K, \quad \text{(b)} \]
\[ K\xi_P = -\rho \xi_K, \quad \text{and (c)} \]
\[ P\xi_K = -\rho \xi_P. \quad \text{(d)} \]

Thus we have a balanced reciprocal pair and may write \([\xi_P|\xi_K]\) and so \(B\) is well-defined. Since \(\|\xi_P\|^2 = \|\tau PSx_K\| = \tau^2 x_K^T S P S x_K = \tau^2 (I - P)x_K = \tau^2 (1 - \rho^2)\|x_K\|^2 = \|x_K\|^2\), we have \(\{x_P|x_K\} \Rightarrow \{\xi_P|\xi_K\}\). Further, for any pair \([x_P|x_K]\),

\[ B^2[x_P|x_K] = \tau B[PSx_K|KSx_P] \]
\[ = \tau^2[PSKSx_P|KSx_K] = \tau^2[P(I - K)x_P|K(I - P)x_K] \]
\[ = \tau^2[x_P - PKx_K - KPx_K] = \tau^2(1 - \rho^2)[x_P|x_K] \]
\[ = [x_P|x_K]; \]

that is \(B^2\) is the identity map, from which it follows that \(B\) is one to one and onto.

Thus a new pair, \([\xi_P|\xi_K]\), of balanced reciprocals in \(P, K\) has been constructed, corresponding to the same eigenvalue, \(\rho^2\), of \(PK\) and \(KP\). However \([x_P|x_K]\) and \([\xi_P|\xi_K]\) have opposite angularity. If the original pair is normalized then the new pair will also be.

Let \(G\) stand for \(P\) or \(K\); let \(\bar{G}\) stand for \(P\) or \(K\); and let \(\bar{P} = K, \bar{K} = P\), then the first of the following equations is a direct consequence of Equation A.4, while the later is the result of a short computation:

\[ GSx_G = \eta \xi_G, \quad \text{(a)} \]
\[ GS\xi_G = \eta x_G. \quad \text{(b)} \]

**Lemma A.1.6**

\[ (a) \quad \bar{x}_G^T \xi_G = \bar{x}_G^T(\tau GSx_G) = \tau \bar{x}_G^T GSGx_G/\rho = \tau \bar{x}_G^T GSGGx_G/\rho = 0 \quad \text{or} \quad 3.7 \quad 3.7' \]
\[ (b) \quad \bar{x}_G^T S \xi_G = \bar{x}_G^T G S \xi_G = \bar{x}_G^T (\eta x_G) = \eta \bar{x}_G^T x_G = \eta. \]

**A.2 Spectra of Products of Idempotents**

Spectral decomposition is an important source of articulation in analyzing multivariate problems: Using the method of reciprocals we establish a near orthogonality of bases for \(PK\) and \(KP\). Essentially we show that only eigenvectors associated with the same eigenvalue are oblique - the rest are orthogonal.
Let \([x_P|x_K]\) and \([x'_P|x'_K]\) be balanced reciprocal pairs; if \(x_P \perp x'_P, x_P \perp x'_K, x_K \perp x'_P,\) and \(x_K \perp x'_K,\) then write \([x_P|x_K] \perp [x'_P|x'_K]\); we say that the reciprocal pairs are orthogonal. If \(a \perp x_P\) and \(a \perp x_K\) write \(a \perp [x_P|x_K].\)

**Lemma A.2.1** If \([x_P|x_K], g \in G\), and \(g \perp x_G\) then \(g \perp [x_P|x_K].\)

Proof: \(g \perp x_G \Rightarrow g^T x_G = 0 \Rightarrow g^T G \hat{G} S x_G = 0 \Rightarrow g^T (I - \hat{G}) x_G = 0 \Rightarrow g^T \hat{G} x_G = 0 \Rightarrow g^T (\rho x_G) = 0 \Rightarrow g \perp x_G.\) Thus \(g \perp [x_P|x_K].\)

We specialize this result to what will be called the *Separation Lemma*:

**Lemma A.2.2** \(x'_G \perp x_G \Leftrightarrow x'_G \perp x_G.\)

Proof: \(x'_G \perp x_G \Leftrightarrow x'_G^T x_G = 0, \Leftrightarrow x'_G^T \rho x_G = 0 \Leftrightarrow x'_G^T G x_G = 0 \Leftrightarrow x'_G^T x_G = 0 \Leftrightarrow x'_G \perp x_G = 0.\)

**Corollary A.2.3** \(x'_G \perp x_G \Leftrightarrow x'_G \perp x_G.\)

Proof: Swap \(x\) with \(x'\) in the lemma.

**Corollary A.2.4** \(x'_G \perp x_G \Leftrightarrow x'_G \perp x_G.\)

Proof: Set \(G\) equal to \(\hat{G}\) in Corollary A.2.3.

Thus, from Corollaries A.2.3 and A.2.4 we have the (*Four Musketeers Lemma*):

**Lemma A.2.5** \(x'_G \perp x_G \Leftrightarrow x'_G \perp x_G \Leftrightarrow x'_G \perp x_G \Leftrightarrow x'_G \perp x_G.\)

Continuing, there is the *Transposition Lemma*:

**Lemma A.2.6** \(x'_G \perp x_G \Leftrightarrow \xi'_G \perp \xi_G.\)

Proof: \(x'_G \perp x_G \Leftrightarrow x'_G^T x_G = 0, \Leftrightarrow x'_G^T \hat{G} S \xi_G = 0 \Leftrightarrow x'_G^T S \hat{G} \xi_G = 0 \Leftrightarrow \xi'_G^T \xi_G = 0 \Leftrightarrow \xi'_G \perp \xi_G.\)

From these results it is straightforward to establish the following *Four Musketeers Theorem*:

**Theorem A.2.7** The spaces \(<x_P,x_K,\xi_P,\xi_K>\) and \(<x'_P,x'_K,\xi'_P,\xi'_K>\) are orthogonal if and only if \(x'_P \perp <x_P,\xi_P>\).

This theorem shows that mutually orthogonal pairs of balanced reciprocal pairs can be constructed—that is we can construct

\[
\{[x_P^{(1)}|x_K^{(1)}], [\xi_P^{(1)}|\xi_K^{(1)}]\}, \{[x_P^{(2)}|x_K^{(2)}], [\xi_P^{(2)}|\xi_K^{(2)}]\}, \ldots,
\]

with

\[
\{[x_P^{(i)}|x_K^{(i)}], [\xi_P^{(i)}|\xi_K^{(i)}]\}, \{[x_P^{(j)}|x_K^{(j)}], [\xi_P^{(j)}|\xi_K^{(j)}]\} \forall i, j \neq j.
\]
as long as we can find an eigenvector (with eigenvalue in $(0,1)$) which is perpendicular to those already delineated.

Proof: The above lemmas show that repeated roots do not cause problems in the construction of such pairs; furthermore note that the complete set of delineated eigenvectors of $P+K$ is a mutually orthogonal set, as $P+K$ is positive definite, so eigenvectors of $PK$ (or $KP$) associated with different eigenvalues must be orthogonal. □

A.3 Conclusion

By spectral decomposition of the matrix $PK$ we have shown that the convergence of the sequence $(PK)^nx$ is geometric, however no upper bound other than unity has been found for the coefficient of convergence.
Appendix B

Efficient Computation

In this appendix we consider a few problems in relation to efficient computation.

B.1 Incremental Affine Regression

Equation 3.16 is computational theory and naturally we wish to take advantage of Theorem 2.2.12 to lighten the computational load. It is apparent from Chapter 6 that we are searching for the correct slack and binding conditions in a context where extra structure obtains; an algorithm which takes advantage of this extra structure is is developed here since it is not practical to perform an affine regression each time proximality (defined in Chapter 6.2) may have been reached.

We develop a method of regression which is incremental, along the lines of the conjugate gradient method; it is similar to the conjugate gradient method, but we first need some theory.

Write \( \gamma_i = G_i^+ T \), then the affine regression solution given by Equation 3.16 is \( \gamma_i/\|\gamma_i\|^2 \). Further, from Theorem 2.2.12 we see that with \( G_i = [g_1, \ldots, g_i] \),

\[
G_{i+1}^+ T_{i+1} = [I - k_{i+1} g_i^T] G_i^+ T_i + k_{i+1},
\]

where \( k_{i+1} \)

\[
\begin{align*}
&= \frac{(I - G_i G_i^+)}{\left\| (I - G_i G_i^+) g_{i+1} \right\|^2} g_{i+1}, & \text{if } (I - G_i G_i^+) g_{i+1} \neq 0, \\
&= \frac{G_i^+ T G_i^+ g_{i+1}}{1 + \left\| G_i^+ g_{i+1} \right\|^2}, & \text{otherwise.} \\
\end{align*}
\]

So we have, from Equation B.1a the recursive scheme

\[
\gamma_{i+1} = \gamma_i + (1 - g_i^T \gamma_i) k_{i+1}.
\]
This leaves us with the problem of computing \( k_i \) and for the special case of Equation 3.12; we proceed as follows.

Consider \( g_1, g_2, \ldots \) generated by successive application of the \( P \)-unitary matrix \( U \); in this case, using the Gram-Schmidt function defined in Chapter 2.3 we define, for \( i < j \),

\[
\begin{align*}
  y_{i,j} &= g_i \prec \{g_{i+1}, \ldots, g_j\} \\
  z_{j,i} &= g_j \prec \{g_{j-1}, \ldots, g_i\}
\end{align*}
\]

(B.2)

then, in view of Theorem 2.3.5, Equations B.2 can be re-written as

\[
\begin{align*}
  y_{1,i} &= g_1 \prec \{g_{i-1}, \ldots, g_1\} = (g_{i-1} \prec \{g_{i-2}, \ldots, g_1\}) \prec (g_{i-2} \prec \{g_{i-3}, \ldots, g_1\}) = y_{1,i-1} \prec z_{i,2} \\
  z_{i,1} &= g_i \prec \{g_{i-1}, \ldots, g_1\} = (g_{i-1} \prec \{g_{i-2}, \ldots, g_2\}) \prec (g_{i-2} \prec \{g_{i-3}, \ldots, g_1\}) = z_{i,2} \prec y_{1,i-1}
\end{align*}
\]

(B.3)

that is, with \( i \geq 3 \)

\[
\begin{align*}
  y_{1,2} &= g_1 \prec g_2 \\
  z_{2,1} &= g_2 \prec g_1 \\
  y_{1,i} &= y_{1,i-1} \prec z_{i,2} \\
  z_{i,1} &= z_{i,2} \prec y_{1,i-1}
\end{align*}
\]

(B.4)

Now note that \( (I - G_{i-1}G_{i-1}^+)g_i = g_i \prec \{g_{i-1}, \ldots, g_1\} = z_{i,1} \) and since

\[
z_{i,2} = g_i \prec \{g_{i-1}, \ldots, g_2\} = U(g_{i-1} \prec \{g_{i-2}, \ldots, g_1\}) = U g_{i-1,1} = U z_{i-1,1}
\]

so with \( U \) taking precedence over \( \prec \) (i.e. \( Ux \prec y = (Ux) \prec y \)) we may write Equations B.4 as

\[
\begin{align*}
  y_{1,2} &= g_1 \prec g_2 \\
  z_{2,1} &= g_2 \prec g_1 \\
  y_{1,i} &= y_{1,i-1} \prec U z_{i-1,1} \\
  z_{i,1} &= U z_{i-1,1} \prec y_{1,i-1}
\end{align*}
\]

(B.5)

Thus we have the recursion

\[
\begin{align*}
  y_{1,i} &= g_1 \prec g_{i-1,1} \\
  z_{i,1} &= g_2 \prec g_{i-1,1} \\
  k_i &= g_{i,1}/\|g_{i,1}\|^2
\end{align*}
\]

(B.6)

This recursion reduces the computational burden associated with the affine regression method for computing a fixed-point and may well make the method computationally effective.

**B.2 Equality Constraints Problem Solution**

A linear program may contain equality constraints, so rather than

\[
\text{maximize } c^T x \text{ subject to } Ax \geq b,
\]
the problem may be, as in Chapter 2.2.7.2,

\[
\text{maximize } c^T x \quad \text{subject to} \quad A_1 x \geq b_1 \text{ and } A_2 x = b_2.
\]

There are a number of ways of handling this problem, which are detailed below.

As detailed in Chapter 2.2.7.2 one solution is to purge \( A_2 x \geq b_2 \) of any “correlation” with \( A_1 \) by forming the inequality problem

\[
\text{maximize } c^T (I - A_2^+ A_2) x \quad \text{subject to} \quad A_1 (I - A_2^+ A_2) x \geq b_1 - A_1 A_2^+ b.
\]

Solution of this purged problem however may require care to maintain computational stability.

Another possibility which is probably more robust is to write each equality as a pair of inequalities, thus if the \( i^{th} \) row of the LP were an equality, say \( p_i x = b_i \), it would be written as

\[
\begin{bmatrix}
p_i \\
-p_i
\end{bmatrix} x \geq \begin{bmatrix} b_i \\
-b_i
\end{bmatrix}.
\]

This would appear to increase the number of rows of the matrix \( A \) at the same time introducing ill-conditioning, however the effects of these problems can be avoided:

Assume for simplicity that there is only one equality and it is in the first row and write

\[
A_{a} = \begin{bmatrix}
a_1/\sqrt{2} \\
-a_1/\sqrt{2} \\
a_2 \\
\vdots \\
a_m
\end{bmatrix}, \quad b_{a} = \begin{bmatrix} b_1/\sqrt{2} \\
-b_1/\sqrt{2} \\
b_2 \\
\vdots \\
b_m
\end{bmatrix}.
\]

then the LP becomes

\[
\text{maximize } c^T x \quad \text{subject to } A_{a} x \geq b_{a}.
\]

Note that \( A_{a} = I_{a} A \), and \( b_{a} = I_{a} b \), where

\[
I_{a} = \begin{bmatrix} 1/\sqrt{2} & 0^T \\
-1/\sqrt{2} & 0^T \\
0 & I
\end{bmatrix}
\]

Further, \( I_{a}^T I_{a} = I \), so \( I_{a}^+ = I_{a}^T \).

Let \( A^+ = Q = [q_1 q_2 \cdots q_n] \); note that \( A_{a}^T A_{a} = A^T A \) and, further,

\[
A_{a}^+ = (A_{a}^T A_{a})^+ A_{a}^T = (A^T A)^+ A_{a}^T = [q_1/\sqrt{2} - q_1/\sqrt{2} q_2 \cdots q_n] = A^+ I_{a}^T,
\]

that is

\[
A_{a}^+ = A^+ I_{a}^T,
\]
which in any case holds in view of Exercise 4.14 (a) of Albert [4].

Define \( \mathfrak{A}_a = A_a A_a^+ \), then \( \mathfrak{A}_a = A_a A_a^+ = I_a A A^+ I_a^T = I_a \mathfrak{A}_a I_a^T \). The term \( b_a = (I - A_a A_a^+) b_a = (I - I_a A A^+ I_a^T) I_a b = I_a b - I_a A A^+ b = I_a (I - AA^+) b = I_a b \), so

\[
  b_a = I_a b.
\]

Similarly, \( c = A_a^+ c = I_a A_a^+ c = I_a c \), that is

\[
  c_a = I_a c.
\]

Therefore

\[
  P_a = \begin{bmatrix} \mathfrak{A}_a & 0 \\ 0 & I - \mathfrak{A}_a \end{bmatrix} + \begin{bmatrix} b_a \\ c_a \end{bmatrix} + \begin{bmatrix} b_a \\ c_a \end{bmatrix}^+ - \begin{bmatrix} b_a \\ c_a \end{bmatrix}^+,
\]

and

\[
  P_a = \mathcal{T}_a P \mathcal{T}_a^T,
\]

where

\[
  \mathcal{T}_a = \begin{bmatrix} I_a & 0 \\ 0 & I_a \end{bmatrix}.
\]

Thus left multiplication by \( P_a \) can be written as

\[
  P_a x = I_a P \mathcal{T}_a^T x = I_a (P \mathcal{T}_a x) \,.
\]

Hence we have the three steps:

1. Form \( h_i^{(n)} = (h_{ai1}^{(n)} - h_{ai2}^{(n)})/\sqrt{2} \) if \( i \) corresponds to an equality constraint, and if not leave \( h_i \) unchanged,

2. left multiplication by \( P \) to form \( h^{(n+1)} \),

3. splitting the rows of \( h^{(n+1)} \) corresponding to equality constraints thus:

\[
  h_i^{(n+1)} \rightarrow h_{ai}^{(n+1)} = \begin{bmatrix} h_i^{(n+1)}/\sqrt{2} \\ -h_i^{(n+1)}/\sqrt{2} \end{bmatrix}.
\]

At this stage we apply the Karush matrix \( M \) again.
B.3 Scaling

Scaling the rows has no effect on the solution, while scaling the columns can be described by the transformation \( x = Dx' \), so the problem becomes: Maximize \( c^T Dx' \) subject to \( ADx' \geq b \).

This means that it is necessary to scale \( c \) together with the columns of \( A \). Note that \( x \) is reconstructed using the transformation \( x = Dx' \).

The columns and the rows can be scaled alternately a number of times, accumulating the factors, then applying the accumulated factors to \( A, b, \) and \( c \).

B.4 Invariant Form Computations

Here we consider methods for computing \( A, b \) and \( c \).

B.4.1 Computing Invariant \( c \)

Two methods of and two main approaches to computing \( c \) are given. They are:

1. the conjugate gradient method, (Program CONJGRAD)
2. the Jacobi method (Program JACOBI)

and there are two options which can be used with either of these methods:

1. the ATA option (solving \( A^T Ax = c \))
2. the AAT option (solving \( AA^T x = Ac \))

B.4.1.1 The Conjugate Gradient Method

The following discussion relies on Section B.6 of Chapter 3. The reader may also refer to Dew, [12, p. 178-181, 183-4, and Appendix 2].

The conjugate gradient method solves the equation \( Mu = v \) for \( u \), and the solution \( u_s = M^+ v \) is obtained, provided \( M \) is symmetric and the recursion begins with \( u_0 \) in the row space of \( M \).

For the first option, where the equation of the form \( A^T Ax = c \) is solved, we obtain as the result of the recursion \( x = (A^TA)^+ c = A^+ A^T c \). We then form \( Ax = AA^T Ax = A^T c = c \). Multiplication by \( A \) has the additional effect of virtually eliminating drift into the null space of \( A \).

The conjugate gradient method would normally be followed by “cleaning up” using the Jacobi method.
B.4.1.2 The Jacobi Method

We first describe the ATA option in detail, AAT being similar:

**The ATA option:** Write $A^T A \mu = c$, then this equation can be solved recursively using Jacobi’s method as follows: Write $Q = A^T A$, then $Q \mu = c$, that is

$$q_{11} \mu_1 + q_{12} \mu_2 + \cdots + q_{1m} \mu_m = c_1$$

$$
\vdots
$$

$$q_{m1} \mu_1 + q_{m2} \mu_2 + \cdots + q_{mm} \mu_m = c_m$$

with the recursive solution

$$\mu_{k+1}^i = \left\{ c_i - \sum_{j \neq i} q_{ij} \mu_j^k \right\}/q_{ii}$$

$$= \left\{ c_i - \sum_{j=1}^m q_{ij} \mu_j^k \right\}/q_{ii} + \mu_i^k$$

In vector form we have the Jacobi recursion

$$\mu_{k+1} = \mu_k + \mathcal{D}^{-1}_Q (c - Q \mu_k)$$  \hfill (B.7)

with this recursion $\mu_k \to (A^T A)^+ c + (I - A^+ A) z$, for some $z$ so $A \mu_k \to A (A^+ A^T c + (I - A^+ A) z) = A^+ T c = \epsilon$, and if $\mu_e = \lim_{k \to \infty} \mu_k$ then $A \mu_e = c$.

**The AAT Option:** This involves the equation $A A^T c = A A^T A^+ T c = A (A^+ A)^T c = A A^+ A c = A c$. That is $A A^T \epsilon = A c$.

Thus the equation $A A^T \mu = A c$ has $\epsilon$ as a solution.

This equation can also be solved recursively using Jacobi’s method as follows: Write $V = A A^T$, and $d = A c$, then $V \epsilon = d$, yielding in a similar manner to above, the recursive solution

$$\mu_{k+1} = \mu_k + \mathcal{D}^{-1}_V (d - V \mu_k).$$

If the rows of $A$ are normalized so that $q_{ii} = 1 \forall i$, then $\mathcal{D}_V = I$ and we can write

$$\mu_{k+1} = \mu_k + d - V \mu_k.$$

In this case $\mu_k$ remains in the column space of $A$ and therefore approaches $\epsilon$.

B.4.2 Computing Invariant $b$

The computation of $b$ follows that of $c$ quite closely. Write $A^T A \mu = A^T b$, that is $Q \mu = A^T b$, then either the conjugate gradient, or the Jacobi recursion for this equation, which is similar to step (1) above, converges to

$$\mu_s = A^+ b + (I - AA^+) z, \exists z.$$

Then $A \mu_s = AA^+ b$ and $b$ can be computed as $b - A \mu_s$. 
B.4.3 Effect of Invariant A on a Vector

The matrix $\mathcal{A}$ can be regarded as the orthogonal projection onto the column space of $\mathcal{A}$. This means we can use the residual function defined by 2.15 as follows:

Writing $A = [p_1, p_2, \ldots, p_m]$, $A_b = \begin{bmatrix} A, x \end{bmatrix}$, we form

$A^{(1)} = [p_1 \triangleleft p_2, \ldots, p_1 \triangleleft p_m, p_1 \triangleleft x] = \begin{bmatrix} p_2^{(1)}, \ldots, p_m^{(1)}, x^{(1)} \end{bmatrix}$

$A^{(2)} = \begin{bmatrix} p_2^{(1)} \triangleleft p_3^{(1)}, \ldots, p_2^{(1)} \triangleleft p_m^{(1)}, p_2^{(1)} \triangleleft x^{(1)} \end{bmatrix}$

then the sequence $\{x^{(i)}\}$ will terminate with $(I - \mathcal{A})x$, enabling $\mathcal{A}x$ to be computed.

Note

1. that before using the Jacobi method, the columns and rows are successively normalized, ending with row normalization, so that $\mathcal{D}V = I$.
2. great accuracy in the first step is not necessary as the aim is only to have a reasonably good starting value for the second step so it does not go on for too long and drift into the subspace perpendicular to $\mathcal{R}(P)$.
3. in Equation B.7 $Q\mu_k = A^T A\mu_k$ is computed as $A^T (A\mu_k)$.
4. in the Jacobi step successive over-relaxation can be applied. (refer to [8]).

B.5 Fixed-Point Computation

We wish to compute $U h_n$ but $Uh = P(I + S)K(I - S)h = P\{(I + S)K(I - S)h\} = P\begin{bmatrix} w \\ w \end{bmatrix}$, where

$\begin{bmatrix} w \\ w \end{bmatrix} = (I + S)K(I - S)h$.

Thus

$U h = \left\{ \begin{bmatrix} A & 0 \\ 0 & I - A \end{bmatrix} + \begin{bmatrix} b \\ c \end{bmatrix} \begin{bmatrix} b \\ c \end{bmatrix}^T - \begin{bmatrix} c \\ b \end{bmatrix} \begin{bmatrix} c \\ b \end{bmatrix}^T \right\} \begin{bmatrix} w \\ w \end{bmatrix}$

$= \begin{bmatrix} Aw + [(b + c)^T w/d][b - c] \\ w - \{Aw + [(b + c)^T u/d][b - c]\} \end{bmatrix} = \begin{bmatrix} w \\ w - v \end{bmatrix}$,

where

$v = Aw + [(b + c)^T w/d][b - c]$. 
B.6 A Conjugate Gradient Algorithm

The conjugate gradient method was suggested by M.R. Stiefel [16] in 1952; it can be regarded as a modified steepest descent algorithm, with the step direction chosen to assure mutual conjugacy of steps. It gives rise to a number of specific algorithms, and here we give Fletcher and Reeves [14] version along the lines of the development in Aoki [7].

We start with the problem of solving, or finding an approximate solution to the matrix equation

\[ Mu = v, \]  
(B.8)

where \( M \) is an \( m \times m \) positive semi-definite symmetric matrix and \( u, v \in \mathbb{R}^m \). The assumption of semi-definiteness is weaker than usual, moreover we do not require that the equation have a solution; this makes the present development more general than is normally presented.

We begin with an initial guess \( u_0 \) for a solution to the equation, and improve on it by making successive steps in directions \( d_0, d_1, \cdots \) which are required to be mutually conjugate that is

\[ d_i^T M d_j = 0 \quad \forall \ i, j, \ i \neq j. \]  
(B.9)

For a matrix of rank \( p \leq m \), the \( p^{th} \) step yields an exact solution.

We use the error function

\[ h_k = \epsilon_k^T M^+ \epsilon_k \]  
(B.10)

where

\[ \epsilon_k = v - Mu_k, \]  
(B.11)

thus

\[ h_k = (v - Mu_k)^T M^+ (v - Mu_k), \]

\[ = v^T M^+ v - 2 v^T M^+ Mu_k + u_k^T M u_k. \]  
(B.12)

Note that \( M^+ M v - Mu_k = M^+ M v - v + v - Mu_k = v - Mu_k - (I - M^+ M)v \) (B.11)

\[ = \epsilon_k - (I - M^+ M)v, \]  
that is

\[ M^+ M v - Mu_k = \epsilon'_k, \]

where \( \epsilon'_k = \epsilon_k - (I - M^+ M)v \) (a)

We assume that the next iterate is

\[ u_{k+1} = u_k + \alpha_k d_k, \]

where \( d_k \in \mathcal{R}(M) \) is known; (a)

\[ \alpha_k \]  
(b)
we then attempt to minimize $h_{k+1}$:

Now

$$h_{k+1} = v^T M^+ v - 2v^T M^+ M u_{k+1} + u_{k+1}^T M u_{k+1},$$

$$= v^T M^+ v - 2v^T M^+ M (u_k + \alpha_k d_k)$$

$$+ (u_k + \alpha_k d_k)^T M (u_k + \alpha_k d_k),$$

and, setting the partial derivative of $h_{k+1}$ w.r.t. $\alpha_k$ equal to zero,

$$\frac{\partial h_{k+1}}{\partial \alpha_k} = -2v^T M^+ M d_k + 2d_k^T M (u_k + \alpha_k d_k) = 0.$$  \hspace{1cm} (B.16)

Solving this equation for $\alpha_k$ gives the best step factor in the direction of steepest descent. To this end

$$- v^T M^+ M d_k + d_k^T M u_k + \alpha_k d_k^T M d_k = 0,$$  \hspace{1cm} (B.17)

$$\Rightarrow$$

$$\alpha_k = \frac{d_k^T M^+ M v - d_k^T M u_k}{d_k^T M d_k} = \frac{d_k^T M^+ M (M^+ M v - M u_k)}{d_k^T M d_k},$$  \hspace{1cm} (B.18)

and it follows that

$$\alpha_k = \frac{d_k^T M^+ M [\epsilon_k - (I - M^+ M) v]}{d_k^T M d_k} = \frac{d_k^T M^+ M \epsilon_k}{d_k^T M d_k}.$$  \hspace{1cm} (B.19)

In view of assumption B.14 (b) we have

$$\alpha_k = \frac{d_k^T \epsilon_k}{d_k^T M d_k}.$$  \hspace{1cm} (B.20)

The direction of steepest ascent is given by

$$\frac{\partial h_k}{\partial u_k} = -2M^+ M v + 2M u_k = -2(M^+ M v - M u_k) = -2\epsilon_k.$$  \hspace{1cm} (B.13)

Thus the direction of steepest descent is $\epsilon_k$, which we cannot compute, however we show in the next paragraph that the step direction $\epsilon_k$, although longer, is equally effective; moreover it is computable and thus we use it. The step factor is not changed - it remains at $\alpha_k$.

≪ We use a † to denote values based on the direction of steepest descent $\epsilon_k$. Values without a prime are used to denote results based on the direction $\epsilon_k$. We assume the same step factor of $\lambda$ in each case. We have $u_{k+1}^\dagger = u_k + \lambda \epsilon_k$, so $\epsilon_{k+1}^\dagger = v - M u_{k+1}^\dagger = v - M \{u_k + \lambda \epsilon_k\} = v - M \{u_k + \lambda \epsilon_k - (I - M^+ M) v\} = v - M \{u_k + \lambda \epsilon_k\}$, and we have $u_{k+1} = u_k + \lambda \epsilon_k$. So $\epsilon_{k+1} = v - M u_{k+1} = v - M (u_k + \lambda \epsilon_k)$, hence $\epsilon_{k+1} = \epsilon_{k+1}$, and $h_{k+1}^\dagger = h_{k+1}$; in other words the error functions for the two step directions are the same, and therefore the step direction $\epsilon_k$ is equally effective as the direction of steepest descent $\epsilon_k$. ≫
The direction $\epsilon_k$ is not however generally conjugate to the previous step, so it is modified to achieve conjugacy by choosing

$$d_k = \epsilon_k - \beta_k d_{k-1},$$

where the scalar $\beta_k$ is chosen so that $d_k$ is conjugate to $d_{k-1}$, that is

$$d^T_{k-1} Md_k = 0.$$  \hspace{1cm} (B.22)

Substituting Equation B.21 into Equation B.22,

$$d^T_{k-1} M(\epsilon_k - \beta_k d_{k-1}) = 0,$$

$$\Rightarrow \beta_k = \frac{d^T_{k-1} \epsilon_k}{d^T_{k-1} p_{k-1} d_{k-1}},$$

where

$$p_k = M d_k.$$  

We avoid computing one matrix product by observing that

$$\epsilon_{k+1} = v - M u_k = v - M(u_k + \alpha_k d_k) = v - M u_k - \alpha_k M d_k = \epsilon_k - \alpha_k p_k,$$

that is

$$\epsilon_{k+1} = \epsilon_k - \alpha_k p_k.$$  \hspace{1cm} (B.23)  

Summarizing:

First, initialize with:

$$u_0 := \text{initial solution estimate}$$  \hspace{1cm} (a)

$$d_0 := v - M u_0$$  \hspace{1cm} (b)

$$\epsilon_0 := d_0$$  \hspace{1cm} (c)

$$k := 0,$$  \hspace{1cm} (d)

then apply the recursion:

$$p_k := M d_k$$  \hspace{1cm} (a)

$$\alpha_k := \frac{d^T_k \epsilon_k}{d^T_k p_k}$$  \hspace{1cm} (b)

$$u_{k+1} := u_k + \alpha_k d_k$$  \hspace{1cm} (c)

if $\alpha_k = 0$ then $result := u_k$ \hspace{1cm} (d)

$\epsilon_{k+1} := \epsilon_k - \alpha_k p_k$  \hspace{1cm} (e)

$\beta_{k+1} := \frac{p^T_k \epsilon_{k+1}}{p^T_k d_k}$  \hspace{1cm} (f)

$d_{k+1} := \epsilon_{k+1} - \beta_{k+1} d_k$  \hspace{1cm} (g)

$k := k + 1$  \hspace{1cm} (h)

if $\|\epsilon_{k+1}\| > tol$ go to (a) \hspace{1cm} (i)

otherwise $result := u_{k+1}$. \hspace{1cm} (j)

Note that this recursive system ensures that $d_k$ lies in the column space of $M$, thus satisfying Equation B.14 (b). Further note that, since $M$ is symmetric positive semi-definite, $M^+$ is also, and thus
h_k ≥ 0. Setting \( u' = M^+v, \ e' = v - Mu' = v - MM^+v = (I - MM^+)v \). So \( h' = e'^TM^+e' = v^T(I - MM^+)M^+(I - MM^+)v = 0 \). Thus \( h \) assumes its minimum when \( u' = M^+v \). It can be shown that \( u' \) is the only vector in the range of \( M \) for which \( h \) assumes its minimum\(^1\), and so the nature of the algorithm is such that the finite sequence \( Mu_k \) must necessarily stop at \( MM^+v \).

We establish, along the lines of Aoki [7, pp 120-122], the mutual M-conjugacy of the sequence \( \{d_i\} \) and the mutual orthogonality of the error sequence \( \{e_i\} \) by induction:

First note that
\[
\epsilon_0^T \epsilon_1 = \epsilon_0^T (\epsilon_0 - \alpha_0 p_0) = \|\epsilon_0\|^2 - \alpha_0 \epsilon_0^T p_0 = \|\epsilon_0\|^2 - \alpha_0 \epsilon_0^T d_0 = \|\epsilon_0\|^2 - (\alpha_0 \epsilon_0^T d_0) d_0^T p_0 = \|\epsilon_0\|^2 - d_0^T \epsilon_0 = \|\epsilon_0\|^2 - \|\epsilon_0\|^2 = 0.
\]
That is
\[
\epsilon_0^T \epsilon_1 = 0. \tag{B.26}
\]

Also
\[
\epsilon_1^T d_0 = \epsilon_1^T \epsilon_0 = 0. \tag{B.27}
\]

Assume that \( \{\epsilon_0, \cdots, \epsilon_{k-1}\} \) are mutually orthogonal and that \( \{d_0, \cdots, d_{k-1}\} \) are mutually M-conjugate, then
\[
\epsilon_k^T d_{k-2} = (\epsilon_{k-1} - \alpha_{k-1} p_{k-1})^T d_{k-2} = \epsilon_{k-1}^T d_{k-2} - \alpha_{k-1} p_{k-1}^T d_{k-2}
\]
\[
= 0 - \alpha_{k-1} p_{k-1}^T d_{k-2} = -\alpha_{k-1}(M d_{k-1})^T d_{k-2} = -\alpha_{k-1} d_{k-1}^T M^T d_{k-2}
\]
\[
= -\alpha_{k-1} d_{k-2} d_{k-1} = 0.
\]
Thus
\[
\epsilon_k^T d_{k-2} = 0. \tag{B.28}
\]
These results are needed to start the induction.

Continuing with computations to establish the inductive argument, for \( k ≥ 2 \),
\[
\epsilon_k^T d_{k-1} = (\epsilon_{k-1} - \alpha_{k-1} p_{k-1})^T d_{k-1} = \epsilon_{k-1}^T d_{k-1} - \alpha_{k-1} p_{k-1}^T d_{k-1} \]
\[
= \epsilon_{k-1}^T d_{k-1} - (d_{k-1}^T \epsilon_{k-1} / d_{k-1}^T p_{k-1}) p_{k-1}^T d_{k-1} = 0,
\]
\[
\Rightarrow \epsilon_k^T d_{k-1} = 0, \Rightarrow \epsilon_k^T (\epsilon_{k-1} - \beta_{k-1} d_{k-2}) = 0,
\]
\[
\Rightarrow \epsilon_k^T \epsilon_{k-1} - \beta_{k-1} \epsilon_k^T d_{k-2} = 0,
\]
\(^1\)\( M \) is symmetric positive semi-definite \( \Rightarrow M^+ \) is symmetric positive semi-definite, thus \( x^T M^+ x ≥ 0 \forall x \). We then show that \( x \mapsto x^T M^+ x \) is convex and that if \( a^T M^+ a = b^T M^+ b = 0 \), then \( M(a - b) = 0 \), which suffices.
\[ \Rightarrow \epsilon_k^T \epsilon_{k-1} = 0, \quad \text{in view of Equation B.28 (b)}. \]

Now for \( i \in \{0, \cdots, k-2\} \),

\[ \epsilon_i^T \epsilon_k = \epsilon_i^T (\epsilon_{k-1} - \alpha_{k-1} \beta_{k-1}) = -\alpha_{k-1} \epsilon_i^T \beta_{k-1} \]

\[ = -\alpha_{k-1} \epsilon_i^T M d_{k-1} \quad \text{(B.25g)} \]

\[ = -\alpha_{k-1} (d_i + \beta_i d_{i-1})^T M d_{k-1} = 0; \]

thus \( \epsilon_0, \cdots, \epsilon_k \) are mutually orthogonal.

Turning to \( d_k \), we know \( d_{k-1}^T M d_k = 0 \); we now establish that \( d_i^T M d_k = 0 \) for \( i \in \{0, \cdots, k-2\} \):

\[ d_i^T M d_k = d_i^T M (\epsilon_k - \beta_k d_{k-1}) = d_i^T M \epsilon_k = \epsilon_i^T M d_i = \epsilon_i^T M p_i = \epsilon_i^T (\epsilon_i - \epsilon_{i+1})/\alpha_i = 0. \]

(In view of Equation B.25 (d), the divisor \( \alpha_i \) cannot be zero if the recursion has reached \( k \).)

This completes the calculations: To start the induction we have shown that \( \{d_0, d_1\} \) is a mutually M-conjugate set of elements, and that \( \{\epsilon_0, \epsilon_1\} \) is a set of mutually orthogonal elements. We have shown that if the elements of \( \{d_0, \cdots, d_{k-1}\} \) are mutually M-conjugate, and the elements of \( \{\epsilon_0, \cdots, \epsilon_{k-1}\} \) are mutually orthogonal then the elements of \( \{d_0, \cdots, d_k\} \) are mutually M-conjugate, and \( \{\epsilon_0, \cdots, \epsilon_k\} \) is a set of mutually orthogonal elements.  \( \Rightarrow \)

### B.7 Exercises: Conjugate Gradient Method

1. Solve the equation \( Mx = v \), where \( M = \begin{bmatrix} 1.0 & 2.0 & 3.0 \\ 2.0 & 5.0 & 6.0 \\ 3.0 & 6.0 & 7.0 \end{bmatrix} \), and \( v = \begin{bmatrix} 8.0 \\ 9.0 \\ 10.0 \end{bmatrix} \), using the conjugate gradient method.

(Use the function \texttt{conjgrad}.)

2. Is the answer to the previous exercise correct?

3. Pose and solve a problem without an exact solution; confirm that the solution satisfies the comments at the beginning of \texttt{conjgrad}. \( \gg \)
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