An exact algorithm for solving the ring star problem

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Abstract

This paper deals with the ring star problem that consists in designing a ring that pass through a central depot, and then assigning each non visited customer to a node of the ring. The objective is to minimize the total routing and assignment costs. A new chain based formulation is proposed. Valid inequalities are proposed to strengthen the linear programming relaxation and are used as cutting planes in a branch-and-cut approach. A large set of instances are tested and show the effectiveness of the method that outperforms the results previously obtained on the problem.

Keywords: mixed integer programming, polyhedral analysis, branch and cut, network design.

1 Introduction

This paper addresses a telecommunication network problem called the ring star problem. It consists in finding a simple cycle through a subset of vertices of a graph which minimizes the cost of the cycle and the assignment cost of the vertices not in the cycle to their closest vertex on the cycle. The problem has several applications in telecommunication network design. Indeed, the ring topology is chosen in many fiber optic communication networks to guarantee continuous communication service to the customers (terminals). They are connected to the concentrators of the ring by point-to-point links which results in a star topology. In other words, the problem consists of selecting a subset of user locations where concentrators will be installed, interconnect them by a ring network and assign the other customer locations to the concentrators.

The Ring Star Problem (RSP) was firstly introduced by Labbé *et al.* [7] who derive a branch-and-cut method based on some polyhedral properties of the problem. A closely related problem called the median cycle problem [8] was recently studied. It consists of finding a simple cycle that minimizes the routing cost subject to an upper bound on the total assignment cost of the non-visited nodes. Moreno Pérez *et al.* [5] have proposed a variable neighborhood tabu search method for solving this location-allocation problem. Several authors have considered problems where instead of a cycle, a structure such as a undirected path or a tree must be identified and the nodes not on this structure must be assigned to the structure. A classification of these problems can be found in [6]. Recently, Baldacci *et al.* [3] present and discuss integer programs for a more general problem that is the capacitated $m-$ring star

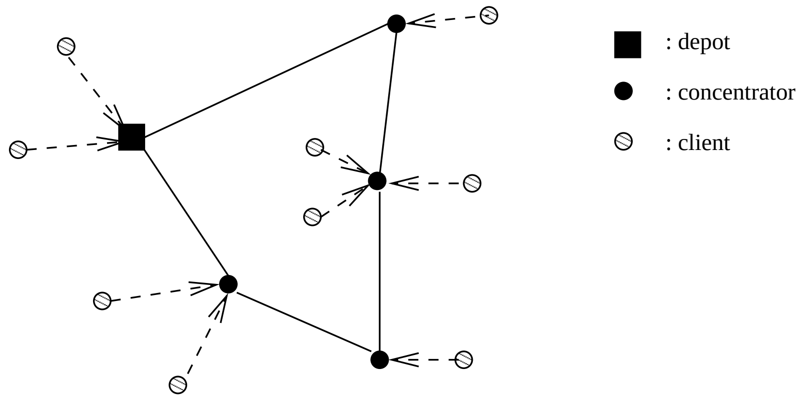

Figure 1: A solution of the ring star problem.

problem to design m node-disjoint simple rings that pass through the central depot and through some other nodes and a set of direct connections from each non-visited customer to a node of a ring. The total number of customer nodes is bounded by an upper limit which represents the ring capacity. The objective is to minimize the cost of the rings and the cost of the customer connections.

The ring-star problem can be formally stated as an optimization problem in graph theory. We consider a mixed graph $G = (V, E \cup A)$, where $V = \{v_1, v_2, \cdots, v_n\}$ is the vertex set, $E = \{v_iv_j : v_i, v_j \in V\}$ is the edge set and $A = \{(v_i, v_j) : v_i, v_j \in V\}$ is the arc set. Vertex v_1 is referred to as a root (or depot). Edges in E refer to the undirected concentrators links, and arcs in A refer to the directed assignment between customers and concentrators. A nonnegative ring cost c_{ij} is associated with each edge v_iv_j and a nonnegative assignment cost d_{ij} is associated with each arc (v_i, v_j). A solution of the ring star problem is a simple cycle through a subset C of V including v_1. The objective is to determine a solution for which the sum of the ring and the assignment costs is minimized. The ring cost of a solution is the sum of the ring costs of all edges on the cycle. The assignment cost is defined as $\sum_{v_i \in V \setminus C} \min_{v_j \in C} d_{ij}$.

The problem is NP-hard since the special case in which the assignment costs are very large compared to the ring cost is the classical traveling salesman problem.

The purpose of this paper is to propose a branch and cut exact algorithm for the ring star problem. We propose a new mixed integer linear formulation based on the *st-chains* which are the undirected simple paths between two specific nodes s and t. New facet-defining inequalities are given and are used as cutting planes in a branch-and-cut approach. Experimental results are reported on a large family of instances and show the effectiveness of our approach comparing to previous works.

The paper is organized as follows: In Section 2, we recall the cycle based formulation and describe the new chain based formulation. In Section 3, we analyze the convex hull of all the feasible solutions of the chain based formulation and derive new facet-defining inequalities. In particular, in Section 3.2 we show that every facet-defining inequality for the cycle based formulation can be transformed to a facet-defining inequality for the chain based formulation. In Section 3.3 we establish a link between the *st*-chain polyhedron and the convex hull of the feasible solutions of the chain based formulation. Thus, we derive a new subclass of facet-defining inequalities which is not issued from cycle based formulation. In Section 4, we describe briefly a branch-and-cut algorithm based on the new formulation and compare the computational results to those reported in [7].

2 Mathematical formulations

The ring-star problem can be formulated as a mixed integer linear programming (MIP) problem. We present first the MIP formulation introduced by Labbé *et al.* [7] and then a new formulation based on undirected simple paths rather than cycles.

2.1 Cycle based formulation

In the cycle based formulation, we define for each edge $v_iv_j \in E$, a binary variable x_{ij} equals to 1 if and only if edge v_iv_j appears in the cycle and y_{ij} a binary variable equals to 1 if and only if vertex v_i is assigned to vertex v_j of the cycle. Moreover, if a vertex v_i belongs to the cycle then $y_{ii} = 1$, that is, vertex v_i is assigned to itself. We define for $S \subset V$, $E(S) = \{v_iv_j \in E : v_i, v_j \in S\}$ and $\delta(S) = \{v_iv_j \in E : v_i \in S, v_j \notin S\}$. For each subset F of E, we define $x(F) = \sum_{v_iv_j \in F} x_{ij}$. If $S = \{v_i\}$, $\delta(\{v_i\})$ will be denoted $\delta(i)$. Thus, we have:

$$\min \sum_{v_iv_j \in E} c_{ij}x_{ij} + \sum_{(v_i,v_j) \in A} d_{ij}y_{ij} \tag{1}$$

Subject to:

$$x(\delta(i)) = 2y_{ii}, \qquad \forall v_i \in V \tag{2}$$

$$\sum_{v_j \in V} y_{ij} = 1, \qquad \forall v_i \in V \setminus \{v_1\} \tag{3}$$

$$x(\delta(S)) \geq 2 \sum_{v_j \in S} y_{ij}, \qquad \forall S \subset V : v_1 \notin S, v_i \in S \tag{4}$$

$$x_{ij} \in \{0, 1\}, \qquad v_iv_j \in E \tag{5}$$

$$y_{ij} \geq 0, \qquad (v_i, v_j) \in A \tag{6}$$

$$y_{11} = 1, \tag{7}$$

$$y_{1j} = 0, \qquad \forall v_j \in V \setminus \{v_1\} \tag{8}$$

$$y_{jj} \in \mathbb{N}, \qquad \forall v_j \in V \setminus \{v_1\} \tag{9}$$

The objective function (1) minimizes the total cycle and assignment costs. Constraints (2) ensure that the degree of a vertex v_i is equal to 2 if and only if $y_{ii} = 1$, that means that it belongs to the cycle. Constraints (3) mean that either v_i is a vertex of the cycle, in this case $y_{ii} = 1$, or v_i is assigned to a vertex v_j of the cycle and in this case $y_{ii} = 0$ and $y_{ij} = 1$. Constraints (4) represent the connectivity constraints. Indeed, they state that S must be connected to its complement by at least two edges of the cycle whenever at least one vertex $v_i \in S$ is assigned to $v_j \in S$. They guarantee that the solution will not include sub-cycles. Constraints (2), (5), (7) and (8) ensure that the solution will contain at least one cycle including the depot. Constraints (2), (3), (4) and (6) state that every vertex not belonging to the cycle is assigned to a vertex of the cycle. Constraints (9) are the integrity constraints for the cycle variables. We can notice that the integrality constraints on the y_{ij} variables are unnecessary since in an optimal solution of the formulation without constraints (9), a node v_i is always assigned totally to the nearest concentrator v_j on the cycle, i.e. $y_{ij} = 1$.

2.2 New chain based formulation

To strengthen the cycle based formulation, Labbé *et al.* [7] have studied the convex hull of all the feasible solutions of RSP. They have notably derived facet-defining inequalities from those of the circuit polytope [4] which is the convex hull of all the cycles. But as in a solution of RSP the cycle is not any cycle and must go through the node v_1, a facet-defining inequality for the circuit polytope is rather only a valid inequality and not a facet-defining inequality for RSP. Hence, some facet-defining inequalities for RSP may not be derived from the circuit polytope. To overcome this disadvantage, we present below a chain based formulation of the RSP problem in which a solution except the assignments is a *st*-chain. We shall derive facet-defining inequalities for RSP from the *st*-chain polyhedron noted PP_{st} which is the upper hull of all the *st*-chains. The one-one correspondence of the solutions of RSP and the *st*-chains will give us new facet-defining inequalities as we shall see in Section 3.3.
In the sequel of the paper, we set $s = v_1$ and add a dummy node which is a clone t of s in G to obtain a new graph $G' = (V', E' \cup A')$ where $V' = V \cup \{t\}$, $E' = E \cup \{st\} \cup \{tu|$ where $su \in E\}$ and $A' = A$. Let $n' = |V'|$, $m'_e = |E'|$, $m'_a = |A'|$ and $m' = m'_e + m'_a$.

Remark 1. *A solution of RSP consists of a st-chain in G' and an assignment of each remaining nodes to a node in the chain except t.*

The mixed integer linear formulation regarding the graph G' is nearly the same as the cycle based one. Particularly, the objective function is unchanged and the constraints related to the root vertex are now expressed for s and t. We obtain:

$$\min \sum_{v_iv_j \in E} c_{ij}x_{ij} + \sum_{(v_i,v_j) \in A} d_{ij}y_{ij} \tag{10}$$

Subject to:

$$x(\delta(v_i)) = 2y_{ii} \qquad \forall v_i \in V' \setminus \{s,t\} \tag{11}$$

$$\sum_{v_i \in V'} y_{ji} = 1 \qquad \forall v_j \in V' \setminus \{s,t\} \tag{12}$$

$$x(\delta(S)) \geq 2\sum_{j \in S} y_{ij} \qquad \forall S \subset V' \setminus \{s,t\}, \quad \forall v_i \in S \tag{13}$$

$$x_{ij} \in \{0,1\} \qquad v_iv_j \in E' \tag{14}$$

$$y_{ij} \geq 0 \qquad (v_i,v_j) \in A' \tag{15}$$

$$y_{ss} = 1, y_{tt} = 1, y_{si} = 0 \qquad \forall v_i \in V' \setminus \{s,t\} \tag{16}$$

$$x(\delta(s)) = 1, x(\delta(t)) = 1 \tag{17}$$

We can observe that there are $n' - 2$ equalities (11), $n' - 2$ equalities (12), n' equalities (16) and 2 equalities (17). Thus, the system (11), (12), (16) and (17) has $3n' - 2$ equalities.

3 Polyhedral analysis

Let (x, y) be any solution of RSP in G', let $\chi_{(x,y)}$ denote its incidence vector in $\mathbb{R}^{m'}$. We define

$$\mathcal{P}_{rs} = \text{ conv}(\{\chi_{(x,y)} | \text{ where } (x, y) \text{ is a solution of RSP}\})$$

This section is devoted to the study of facial structure of $\mathcal{P}_{rs}$. We begin to show that the system defined by (11), (12) (16) and (17) characterize the affine hull $\mathcal{P}_{rs}$. We establish then the relationship between $\mathcal{P}_{rs}$ and the st-chain polyhedron and derive new facet-defining inequalities for $\mathcal{P}_{rs}$. At the end of the section, we study the projection of $\mathcal{P}_{rs}$ on the variables y.

3.1 Affine hull and dimension of $\mathcal{P}_{rs}$

Theorem 1. *The affine hull of $\mathcal{P}_{rs}$ is defined by* (11)*,* (12) (16) *and* (17) *and its dimension, $dim(\mathcal{P}_{rs}) = m' - 3n' + 2$.*

Proof. It is obvious that the rows of this system defined by (11), (12), (16) and (17) are linearly independent, thus its rank is $3n' - 2$.
Let us assume now that there is another equality:

$$\alpha^t x + \beta^t y = \gamma \tag{18}$$

such that (18) is different from the equalities in (11), (12) (16) and (17) and the solutions of RSP satisfy (18). It is obvious that $\alpha \in \mathbb{R}^{m'_e}$ and $\beta \in \mathbb{R}^{m'_a}$. Let α_{ij} denote the component of α corresponding to the edge $v_i v_j$. Let β_{ij} and β_{ji} denote respectively the components corresponding to the arcs (v_i, v_j) and (v_j, v_i).
From (16), we can fix $\beta_{ss} = \beta_{tt} = 0$ and $\beta_{si} = 0$ for all $v_i \in V' \setminus \{s, t\}$. From (11), we can fix $\beta_{ii} = 0$ for all $v_i \in V' \setminus \{s, t\}$. At last, from (12), we can fix $\beta_{is} = 0$ for all $v_i \in V' \setminus \{s, t\}$.
For any of the nodes v_i, v_j, v_k such that $v_i, v_j \in V' \setminus \{s, t\}$ and $v_k \in V'$, let us consider a solution of RSP in which the st-chain contains the edge $v_i v_k$ and does not contain v_j. In this solution, every node that does not belong the st-chain is assigned to s. If we add v_j between v_i and v_k in the st-chain, we obtain another solution of RSP (see Figure 2).

Since both solutions satisfy (18), we derive $\alpha_{ik} = \alpha_{ij} + \alpha_{jk}$. This equation applied for all triplets v_i, v_j, v_k such that $v_i, v_j \in V' \setminus \{s, t\}$ and $v_k \in V'$ implies that $\alpha_{ij} = 0$ for all $v_i, v_j \in V' \setminus \{s, t\}$, $\alpha_{si} = \alpha_{sj}$ and $\alpha_{ti} = \alpha_{tj}$. From (17), we can fix some α_{si} and α_{ti} to 0 and thus all the components of α are equal to 0, i.e. $\alpha = \mathbf{0} \in \mathbb{R}^{m'_e}$. Now, we modify the first solution built above by assigning v_j to v_i. The equations (18) corresponding to this new solution and the old one allow us to deduce that $\beta_{ji} = 0$. Applying this deduction to all pairs v_i, v_j,

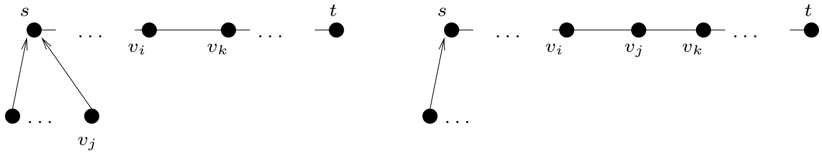

Figure 2: Two solutions of RSP.

we can conclude that the vector β is also the zero vector in $\mathbb{R}^{m'_a}$. Thus $\gamma = 0$ and then the affine hull of $\mathcal{P}_{rs}$ is described completely by constraints (11), (12), (16) and (17). Consequently, $dim(\mathcal{P}_{rs}) = m' - (3n' - 2) = m' - 3n' + 2$. □

3.2 Links between cycle based and chain based formulations

A general technique described in the appendix can be used to prove that a given valid inequality defines a facet for $\mathcal{P}_{rs}$. Let $a^t x + b^t y \geq c$ be a facet-defining inequality I of RSP for the cycle based formulation. We transform I to an inequality $a'^t x + b'^t y \geq c'$ denoted by I' for the chain based formulation as follows:

$$a'_{ij} = a_{ij} \;\; \forall i, j \in V' \setminus \{t\},$$
$$a'_{ti} = a_{si} \;\; \forall i, j \in V' \setminus \{s, t\},$$
$$b' = b, \;\; c' = c, \;\; a'_{st} = c' - \sum_{i \in V' \setminus \{s,t\}} b_{is}$$

Lemma 1. *The inequality I' defines a facet for $\mathcal{P}_{rs}$.*

Proof. We can see that the solutions of RSP satisfying respectively I and I' at equality are the same. Let $\mathcal{A}$ and $\mathcal{A}'$ be the matrices whose rows are the incidence vectors of these solutions respectively in the cycle based formulation and in the chain based formulation. We proceed to eliminate columns in $\mathcal{A}$ as described in the general framework. After that, the matrix $\mathcal{A}$ has $m - 3n + 1$ columns. Let $\mathcal{A}_1$ be the submatrix of $\mathcal{A}$ containing the columns corresponding to the variables x_{si} for all $i \in V \setminus \{s\}$. Let $\mathcal{A}_2$ be the submatrix containing the columns which are not in $\mathcal{A}_1$. The same procedure is applied for I' and in addition we use the row corresponding to the solution that consists of the edge st and the assignment of the remaining nodes to s to eliminate the column corresponding to the variable x_{st}. It remains $m' - 3n' + 1$ columns in $\mathcal{A}'$. We set $\mathcal{A}'_1$ the submatrix of $\mathcal{A}'$ containing the columns corresponding to the variables x_{si} and x_{ti} for all $i \in V \setminus \{s, t\}$ and $\mathcal{A}'_2$ the submatrix containing the columns which are not in $\mathcal{A}'_1$. We can see that $\mathcal{A}_2 = \mathcal{A}'_2$. As I defines a facet for the cycle based formulation, i.e. $\mathcal{A}$ is full-column rank and $rank(\mathcal{A}) \leq rank(\mathcal{A}_1) + rank(\mathcal{A}_2)$, $\mathcal{A}_1$ and $\mathcal{A}_2$ are full-column rank, i.e. $rank(\mathcal{A}_1) = n - 2$ and $rank(\mathcal{A}_2) = m - 4n + 3$. Thus, $rank(\mathcal{A}'_2) = m - 4n + 3 = m' - n' - 4(n' - 1) + 3 = m' - 5n' + 7$. The proof is complete if we can show that $\mathcal{A}'_1$ is full-column rank, i.e. $rank(\mathcal{A}'_1) = 2n' - 6$. This can be shown by considering $2n' - 6 = 2(n + 1) - 6 = 2n - 4$, $rank(\mathcal{A}_1) = n - 2$, and each tight solution with respect to I corresponds to two tight solutions with respect to I' since each neighbour of s in the cycle can be either a neighbour of s or a neighbour of t in the $st-$chain. □

Corollary 1. *The inequalities* (13) *define facets for* $\mathcal{P}_{rs}$.

Proof. The inequalities (13) can be obtained by applying the transformation defined above to the inequalities (4) in the cycle based formulation. Since constraints (4) define facets for the cycle based formulation described in section 2.1, thus the inequalities (13) define facets for $\mathcal{P}_{rs}$ according to Lemma 1. $\square$

As mentioned in [7], when $|S| = 2$ and $S = \{v_i, v_j\}$ with $v_i, v_j \in V' \setminus \{s, t\}$ and $v_i \neq v_j$, using equalities (11), we can rewrite the inequalities (13) as follows:

$$x_{ij} + y_{ji} \leq y_{jj} \text{ for all } v_i, v_j \in V' \setminus \{s, t\} \text{ and } v_i \neq v_j. \quad (19)$$

Corollary 2. *Let us consider*

$$y_{ij} + y_{jk} + y_{ki} \leq 1 \textit{ for all different nodes } v_i, v_j, v_k \in V' \setminus \{s, t\}$$

These inequalities along with the inequalities (14) *and* (15) *define facets for* $\mathcal{P}_{rs}$.

Proof. These inequalities define facets for the cycle based formulation [7] and remain unchanged under the transformation to the chain based formulation described in the proof of Lemma 1. $\square$

3.3 Facet-defining inequalities from the st-chain polyhedron PP_{st}

We establish the links between the facets of $\mathcal{P}_{rs}$ and the ones of its projection on x-space PP_{st} by the following properties. All the proofs are provided in the appendix.

Lemma 2. *Let $a^t x \geq b$ be a facet defining inequality I of PP_{st}. Then I also defines a facet of $\mathcal{P}_{rs}$ if for all pairs of vertices u, $v \notin \{s, t\}$, there are some st-chains satisfying I at equality containing u but not v, and conversely there are some st-chains satisfying I at equality containing v but not u.*

A *T-join* with $T = \{s, t\}$ is composed by a st-chain and eventually some additional cycles. It is well-known that the T-join polytope, i.e. the convex hull of the incidence vectors of all the T-joins is completely characterized by trivial inequalities and the *blossom* inequalities. A blossom inequality is associated to a vertex subset $U \subset V$ and an edge subset $F \subset \delta(U)$ such that $|U \cap \{s, t\}| + |F|$ is odd. Such a blossom inequality is defined as follows:

$$x(\delta(U) \setminus F) - x(F) \geq 1 - |F| \quad (20)$$

To fit to the st-chains, we make a slight change to the inequality (20) and we obtain a new inequality called st-chain-blossom. The set F could not contain the edge st. The new inequality st-chain-blossom is defined as follows:

$$x(\delta(U) \setminus F) - x(F) + (1 - |F|)x_{st} \geq 1 - |F| \;\; \forall U \subset V', F \subset \delta(U), |U \cap \{s, t\}| + |F| \text{ is odd} \quad (21)$$

Proposition 1. *The st-chain-blossom inequalities (21) define a facet for PP_{st} if and only if F is a matching.*

Proof. First, we show the validity of the st-chain-blossom inequalities. It is easy to see that all st-chains that contain at most $|F|-1$ edges in F satisfy (21). For those containing all the edges in F, the arrangement of s and t make that they contain at least one edge in $\delta(U)\setminus F$, hence satisfy (21).
Let us assume that F is not a matching, i.e. there exists two edges $uv_1, uv_2 \in F$ both incident to some vertex u. As we argue above, the st-chains satisfying (21) at equality contain at least $|F|-1$ edges in F. Hence, they should go through the vertex u and then satisfy also $x(\delta(u)) = 2$. Therefore (21) can not define facets when F is not a matching.

If F is a matching, let us assume that there is another inequality $\alpha^T x \geq \beta$ such that all the st-chains satisfying (21) satisfy also $\alpha^T x = \beta$. We can notice

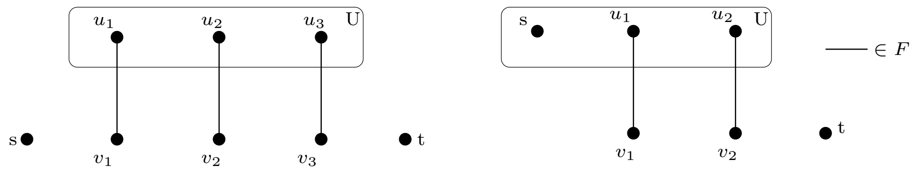

Figure 3: Two forms of the set U.

that when the set U contains either s or t then $|F|$ must be even, otherwise when U contains both s, t or none of them then $|F|$ must be odd. As s and t are symmetric as well as U and $V \setminus U$, we will only consider two cases :

- none of the nodes s and t are in U,
- only $s \in U$ and $t \notin U$

as depicted in Figure 3. The two cases shown in Figure 3 denote the smallest forms for st-chain-blossom inequalities, the proof will be done only for these forms. These properties can be easily extended to all the other cases.

- Proof for the first form when $s, t \notin U$: the chains $sv_1u_1u_2v_2t$ and $sv_1u_1u_3u_2v_2t$ satisfy (21) at equality, thus they satisfy also $\alpha^T x = \beta$. From the corresponding equations $\alpha^T x = \beta$ of those two chains, we deduce that $\alpha_{u_1u_2} = \alpha_{u_1u_3} + \alpha_{u_2u_3}$. By the perfect symmetry of nodes u_1, u_2 and u_3, we obtain $\alpha_{u_2u_3} = \alpha_{u_1u_2} + \alpha_{u_1u_3}$ and $\alpha_{u_1u_3} = \alpha_{u_1u_2} + \alpha_{u_2u_3}$. This implies that $\alpha_{u_1u_2} = \alpha_{u_1u_3} = \alpha_{u_2u_3} = 0$.
  The chains $sv_1u_1u_2v_2t$ and $sv_1u_1u_3v_3t$ are tight. From the two corresponding equations $\alpha^T x = \beta$, we deduce that $\alpha_{v_2t} = \alpha_{v_3t}$. By symmetry, $\alpha_{v_1t} = \alpha_{v_2t} = \alpha_{v_3t}$ and $\alpha_{v_1s} = \alpha_{v_2s} = \alpha_{v_3s}$. We fix these coefficients and β to zero.
  By considering the chains $sv_1u_1u_2v_2t$ and $sv_1u_1u_2v_2v_3t$, we have $\alpha_{v_2v_3} = 0$. Symmetrically, $\alpha_{v_1v_2} = \alpha_{v_2v_3} = \alpha_{v_3v_1} = 0$.
  In the tight chains containing edges u_iv_i, u_jv_j with $1 \leq i \neq j \leq 3$, $\alpha_{u_iv_i}$ and $\alpha_{u_jv_j}$ are the unique not null coefficient. By considering all the chains, we can derive that $\alpha_{u_1v_1} = \alpha_{u_2v_2} = \alpha_{u_3v_3} = 0$.
  We can see that the tight chains containing all the edges in F, i.e. the edges u_1v_1, u_2v_2 and u_3v_3, contain only one other edge in $\delta(U)$. Regarding these chains, we can deduce that $\alpha_e = 0$ for all $e \in \delta(U)\setminus F$. At last,

the tight chain which is the edge st gives α_{st}=0. Finally, we obtain $\alpha = \mathbf{0}$ and $\beta = 0$.

- Proof for the second form when $s \in U$ and $t \notin U$: From the two tight chains $su_1u_2v_2t$ and su_2v_2t, we derive $\alpha_{su_2} = \alpha_{u_1u_2} + \alpha_{su_1}$. Similarly, we also have $\alpha_{su_1} = \alpha_{u_1u_2} + \alpha_{su_2}$. This implies that $\alpha_{u_1u_2} = 0$ and $\alpha_{su_1} = \alpha_{su_2}$. Symmetrically, we deduce that $\alpha_{v_1v_2} = 0$ and $\alpha_{tv_1} = \alpha_{tv_2}$. Let us fix $\alpha_{su_1} = \alpha_{su_2} = 0$ and $\alpha_{tv_1} = \alpha_{tv_2} = 0$.
From the tight chain su_1v_1t, we deduce $\alpha_{u_1v_1} = 0$. Similarly, we have $\alpha_{u_2v_2} = 0$. Considering the tight chain $sv_1u_1u_2v_2t$, we derive $\alpha_{sv_1} = 0$. By symmetry, we have $\alpha_{sv_1} = \alpha_{sv_2} = 0$ and $\alpha_{tu_1} = \alpha_{tu_2} = 0$. Finally, the tight chain $su_1v_1u_2v_2t$ gives $\alpha_{v_1u_2} = \beta$. Again, by the same argument, we have $\alpha_{v_1u_2} = \alpha_{u_2v_1} = \beta$. Let us fix $\beta = 0$, this implies that all the components of the vector α are equal to 0. □

Corollary 3. *If F is a matching of the subgraph of G' induced by the edge set E', the st-chain-blossom inequalities define a facet for $\mathcal{P}_{rs}$.*

The proof is based on the fact that if F is a matching, the st-chain-blossom inequalities satisfy the conditions of Lemma 2. This proves the property. □

Labbé *et al.* [7] show that the following inequalities called 2-matching define facets for the cycle based formulation for all $U \subset V$ such that $|U| \geq 6$ and $F \subset \delta(U)$ such that F is a matching and $|F| \geq 3$ and odd:

$$x(E(U)) + x(F) \leq \sum_{v_i \in U} y_{ii} - \lfloor \frac{|F|-1}{2} \rfloor \tag{22}$$

We will show in the following that the 2-matching inequalities is a subcase of the st-chain-blossom inequalities when both $s, t \in U$ or both $s, t \notin U$ and $|F|$ is odd for the chain based formulation.
We can see that $x(\delta(U) \setminus F) - x(F) = x(\delta(U)) - 2x(F)$. The equalities (11) and (17) will give $x(\delta(U)) = \sum_{v_i \in U} y_{ii} - 2x(E'(U))$. Two cases are considered:

- if both $s, t \in U$ then the st-chain-blossom inequality becomes

$$x(E(U)) + x(F) \leq y_{ss} + y_{tt} + 2 \sum_{v_i \in U \setminus \{s,t\}} y_{ii} - \lfloor \frac{|F|-1}{2} \rfloor$$

$$x(E(U)) + x(F) \leq 2 + 2 \sum_{v_i \in V \setminus \{s,t\}} y_{ii} - \lfloor \frac{|F|-1}{2} \rfloor$$

  This form is the transformation of the 2-matching inequalities when $s \in U$ for the chain based formulation.

- if both $s, t \notin U$ then the st-chain-blossom inequality becomes

$$x(E(U)) + x(F) \leq 2 \sum_{v_i \in U} y_{ii} - \lfloor \frac{|F|-1}{2} \rfloor$$

  This form is the transformation of the 2-matching inequalities when $s \notin U$ for the chain based formulation.

- if either $s \in U, t \notin U$ or $t \in U, s \notin U$ then $|F|$ is even and then the st-chain-blossom inequality becomes

$$x(E(U)) + x(F) \leq 2 \sum_{v_i \in U \setminus \{s,t\}} y_{ii} - \frac{|F|}{2}$$

This form can not be obtained by the transformation of some facets known for the cycle based formulation, thus this defines a new facet for the chain based formulation $\mathcal{P}_{rs}$. Let us call this subcase of the st-chain-blossom inequalities, the *st-chain-blossom pairs.*

We can conclude by the following property:

Corollary 4. *The cycle based formulation described in Section 2.1 contains strictly the chain based formulation.*

Proof. Indeed, all the facet-defining inequalities given in [7] for the cycle based formulation are facet-defining inequalities for the chain based formulation in an equivalent form as described in Section 3.2. Moreover, there are facet-defining inequalities for the chain based formulation which are not derived from the cycle based formulation. □

4 Computational results

4.1 A branch-and-cut algorithm for the RSP

The basic steps of the branch-and-cut algorithm used to solve the RSP are similar to the algorithm described in [7]. The main difference is that we introduce the st-chain blossom pair inequalities as cuts to be added when they are violated. All the other inequalities used as cuts or included in the initial linear program are the ones used in [7] under the transformation described above. For a detailed description, we refer the reader to [7]. Moreover, we did not implement a primal heuristic while in [7], the authors applied a primal heuristic every 5 iterations. The other ingredients of a branch-and-cut algorithm such as initial heuristic, node selection policy, branching strategy are identical to the ones used in [7]. The branch-and-cut approach is outlined in what follows.

STEP 1 : Computing an upper bound

The principle of the heuristic that computes an upper bound $\bar{z}$ is similar to the one proposed by Labbé *al.* [7]. We start with a path $C = \{v_1, v_1'\}$ and successively insert a vertex $v_i \notin C$ to minimize $L(i, \lambda) = \lambda inc(C, i) + (1 - \lambda)(-dec(C, i))$, where $inc(C, i)$ is the minimum increment produced in the cost of the $st-$chain C when inserting v_i, and $dec(C, i)$ is the decrease produced in the assignment cost. This operation is repeated as long as $L(i, \lambda) < 0$ with a new insertion. This is done for $\lambda \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$ and the best solution is selected. The values of λ correspond to different trade off between the ring cost and the assignment cost. Moreover, the initial linear subproblem is then defined in what follows, the subproblem is solved and inserted in a list $\mathcal{L}$.

$$
\begin{aligned}
&\min \sum_{v_iv_j \in E} c_{ij}x_{ij} + \sum_{(v_i,v_j)\in A} d_{ij}y_{ij} \\
&\text{Subject to:} \\
&x(\delta(v_i)) = 2y_{ii} && \forall v_i \in V' \setminus \{s,t\} \\
&\sum_{v_i \in V'} y_{ji} = 1 && \forall v_j \in V' \setminus \{s,t\} \\
&x_{ij} \in \{0,1\} && v_iv_j \in E' \\
&y_{ij} \geq 0 && (v_i,v_j) \in A' \\
&y_{ss} = 1, y_{tt} = 1, y_{si} = 0 && \forall v_i \in V' \setminus \{s,t\} \\
&x(\delta(s)) = 1, x(\delta(t)) = 1
\end{aligned}
$$

STEP 2: Termination check and subproblem selection

If the list $\mathcal{L}$ is empty, stop. Otherwise, select a subproblem from the list according to a best-first policy, that is, select the subproblem having the lowest objective function value.

STEP 3: Subproblem solution

Set $t = t+1$. Let z be the objective function value of the current solution. If $z \geq \bar{z}$, go to Step 2. Otherwise, if the solution is feasible for the RSP, set $\bar{z} = z$, go to Step 2.

STEP 4: Constraint separation and generation

Introduce violated connectivity constraints (13) and blossom constraints (21). If no constraint can be generated, go to Step 6. Otherwise, go to Step 3.

STEP 5: Branching

Create two subproblems by branching on a fractional y_{ii} or x_{ij} variable. The first branching strategy consists in finding a y_{ii} variable. To this end, we applied the strong branching rule within the five y_{ii} variables with fractional value closest to 0.5. If all these variables are integer, select a x_{ij} variable using the same criterion. Insert both subproblems in $\mathcal{L}$ and go to Step 2.

We have developed the following separation procedures for Step 4. Let us denote by $G^* = (V^*, E^* \cup A^*)$ the support graph associated with a given (fractional) solution (x^*, y^*), that is, $V^* = \{v_i \in V' \ : \ 0 < y^*_{ii} < 1\}$, $E^* := \{v_iv_j \in E' \ : \ 0 < x_{ij} < 1\}$ and $A^* := \{(v_i,v_j) \in A' : \ i \neq j, \ 0 < y^*_{ij} < 1\}$.

Separation of the connectivity constraints (13):
We use the algorithm described in [7] to solve the separation problem of this constraint which can be reduced to maximum flow problems in G^*.

Separation of the st-chain blossom constraints (21):
The separation problem of the blossom constraint has been known to be polynomial since Padberg and Rao's seminal paper [9]. Recently, Letchford, Reinelt and Theis's [1, 2] proposed a new exact algorithm improving the complexity of Padberg and Rao's algorithm. We adapt this algorithm for the *st*-chain blossom inequalities.

4.2 Branch-and-cut framework and computational environment

The branch-and-cut algorithm is implemented in the C++ programming language. We have used the BCP a branch-and-cut framework of COIN-OR Foundation (http://www.coin-or.org). We have also used the linear solver CLP of COIN-OR to solve the linear program at each node of the search tree. Both BCP and CLP are open source softwares and downloadable from the COIN-OR website. The experiments are conducted on an IBM PC with a Intel Dual Core 3MGHz processor running on Linux operating system.

4.3 Numerical experiments

In order to assess the efficiency of the branch-and-cut algorithm based on the chain based formulation (ChB-BC), we drive a comparison with the same approach reported in Labbé *et al.* [7] for the cycle based formulation (CyB-BC). We choose the test instances in Class I described in [7], the other Classes II et III in [7] were generated randomly. The Class I is based on TSP instances from TSPLIB 2.1 involving between 50 and 200 vertices (problems eil51 to kroB200). We recall the main characteristics of these instances. The root was always chosen as the first vertex. If l_{ij} denotes the distance between vertices v_i and v_j given in the TSP files. To obtain optimal solutions visiting approximately 100, 75, 50, and 25% of the total number of vertices in the instances, the settings are $c_{ij} = \alpha \lceil l_{ij} \rceil$, $d_{ij} = \lceil (10-\alpha) l_{ij} \rceil$, for $\alpha \in \{3, 5, 7, 9\}$, and $d_{ii} = 0$ for all $v_i \in V$. To show that the linear relaxation of our formulation is stronger than the one given in [7], we compare the lower bound obtained at the root node of the branch-and-bound tree which represent the value of the objective function at the root node just before the branching, i.e. when no violated cuts have been found. As in [7], the author reported the percentage of this value over the value of the best solution found, we also report this percentage. Table 1 below shows the comparison of the percentage obtained respectively by the ChB-BC algorithm and the CyB-BC algorithm. The first column is the name of the instance, the second is the value of α. The third column is the percentage obtained by CyB-BC, the fourth column is the percentage obtained by ChB-BC. The last column report the number of *st*-chain blossom pair inequalities generated at the root node of the branch-and-bound tree. We only report the instances where optimal solution is not found at the root node. We can notice that for every instance for which the CyB-BC algorithm finds an optimal value at the root node, the ChB-BC algorithm finds it as well.

We can see in Table 1 that the chain based formulation has the same performance as the cycle based formulation when no *st*-chain-blossom pair inequalities

Name	α	CyB-BC	ChB-BC	st-chain blossom pair
st70	3	99.80	99.80	0
st70	5	99.84	99.84	0
pr76	3	98.59	98.593	68
pr76	5	99.66	99.68	61
pr76	7	99.80	99.80	0
rat99	3	99.89	99.89	0
kroA100	3	99.80	99.92	2
kroA100	5	99.78	99.78	0
kroB100	3	99.50	99.89	5
kroC100	3	99.81	99.81	0
kroD100	3	99.88	99.88	0
kroE100	3	99.28	99.57	39
eil101	3	99.84	99.84	0
pr124	3	98.82	98.82	0
pr124	3	99.79	99.79	2
bier127	3	99.84	99.87	18
ch130	3	99.85	99.85	2
ch130	3	99.64	99.64	138
pr136	3	99.45	99.5	70
pr144	3	99.83	99.83	10
pr144	5	99.55	99.55	0
kroA150	3	99.49	99.49	26
kroB150	3	99.51	99.57	15
pr152	3	99.51	99.55	3
pr152	5	96.05	96.11	155
pr152	7	96.66	96.76	308
rat195	3	99.68	99.74	99
kroA200	3	93.59	96.59	12
kroA200	9	97.15	98.51	151
kroB200	3	99.81	99.86	2
kroB200	9	95.13	97.52	66

Table 1: Percentage of the lower bound given at the root node over the best value of the problem given by CyB-BC.

are generated except for the instance ch130 with $\alpha = 3$. For the other instances, the addition of the st-chain-blossom pair inequalities induces the improvement of the lower bound. This shows that the chain based formulation is effectively stronger than the cycle based formulation.
We also compare the size of the branch-and-bound tree, i.e. the number of nodes, for some hard instances of the Class I. This way of comparison is less objective than the previous comparison on the lower bound at the root node. This can be biased because we did not implement the primal heuristic that in some cases can reduce the size of the branch-and-bound tree. Moreover, non-algorithmic parameters of the solver can somehow affect this size such as the numerical precision from which a solution is considered to be integer, the num-

ber of iterations a cut is considered ineffective before it is deleted by the solver can also affect this size. Nevertheless, we report some experiments in Table 2. We focused on the instances with $\alpha = 3$ since for other values of α the size is generally 1. Beyond the objectivity of the comparison, we can see that most of

Name	α	CyB-BC	ChB-BC	st-chain blossom pair
eil51	3	3	3	0
pr76	3	727	501	355
kroA100	3	11	11	2
kroB100	3	39	15	7
kroD100	3	5	9	3
kroE100	3	41	47	50
eil101	3	19	15	5
pr124	3	195	171	88
pr124	5	17	15	0
ch130	3	21	9	7
pr144	3	21	15	38
kroA150	3	115	93	225
rat195	3	61	59	88
d198	3	65	71	70
ch150	3	35	15	1
kroA200*	3	129	91	438

Table 2: The size in number of nodes of the branch-and-bound tree.

the time, ChB-BC method has a smaller search tree than CyB-BC. Note that we could solve to optimality the instance *kroA200* with $\alpha = 3$ with a search tree of 91 nodes whereas it is not solved completely within 2 hours and with a search tree of 129 nodes in [7]. We did not report the execution time since our computer is much more powerful than the one used in [7]. Moreover, it would be not accurate to quantify the difference between the linear solvers (Coin-Clp and Cplex 6.0). We could say nevertheless that by average our program is 15 times faster and in particular we have solved the instance *kroA200* with $\alpha = 3$ in 6 minutes.

5 Conclusion

We proposed a new formulation for the RSP based on chains. This formulation allowed to derive new facet-defining inequalities issued from the st-chain polyhedron. Computational results show that the new inequalities improve the linear relaxation in comparison to the one of the cycle based formulation.
The st-chain-blossom inequalities characterize the up hull of the st-chain polytope which is the convex hull of all the incidence vectors of the st-chains. These inequalities also define facets for the st-chain polytope. They correspond to the "easy" instances of the shortest st-chain problem, i.e. given a cost function which do not induce negative circuits, we can minimize it over the st-chain polytope using only the st-chain-blossom inequalities. It is thus interesting for a future work to study the facet-defining inequalities corresponding to the

“hard” instances of the shortest st-chain problem and using Lemma 2 convert them to a facet of the chain based formulation. Further works should also be invested to explore the location aspect of the problem, i.e. how to locate the concentrator among the nodes. This will allow to find new facet-defining inequalities involving the variables y’s. Several properties such as at most one assignment among variables y_{ij}, y_{jk} and y_{ki} is allowed have been discussed in [7], but as the authors have reported, their corresponding inequalities did not show efficiency in the branch-and-cut algorithm. Perhaps, it is more interesting to find mixed properties involving at the same time the variables x’s and y’s variables.

Appendix

General framework for facet-defining proof

In the following, we describe a general technique to prove that a given valid inequality defines a facet for $\mathcal{P}_{rs}$.
Let $a^t x + b^t y \geq c$ be a valid inequality denoted I for $\mathcal{P}_{rs}$. Let us call *tight solution with respect to* I a solution of RSP satisfying I at equality. We want to prove that I defines a facet for $\mathcal{P}_{rs}$. To do this, we will assume that every tight solution of RSP with respect to I satisfies also a valid inequality $\alpha^t x + \beta^t y \geq \gamma$ at equality. We will show that $\alpha = \delta a$, $\beta = \delta b$ and $\gamma = \delta c$ for some real δ. This is equivalent to fix γ to 0 and show that $\alpha = \beta = \mathbf{0}$.
Let us call $\mathcal{A}$ the matrix that rows are the incidence vectors of all the tight solutions with respect to I. The equations (11), (12) and (16) allow us to eliminate columns of $\mathcal{A}$ corresponding to some specific variables, equivalently the coefficients α or β corresponding to these variables are fixed to 0. From (16), we fix $\beta_{ss} = \beta_{tt} = 0$ and $\beta_{si} = 0$ for all $i \in V' \setminus \{s,t\}$. From (12), we fix $\beta_{is} = 0$ for all $i \in V' \setminus \{s,t\}$. From (11), we fix $\beta_{ii} = 0$ for all $i \in V' \setminus \{s,t\}$. At this stage, we have eliminated $n' + (n'-2) + (n'-2) = 3n' - 4$ columns of $\mathcal{A}$ and it remains $m' - 3n' + 4$ columns in $\mathcal{A}$.
Now considering (17), we can fix $\alpha_{is} = 0$ for some $i \in V' \setminus \{s,t\}$ and $\alpha_{jt} = 0$ for some $j \in V' \setminus \{s,t\}$. We thus eliminate two columns from $\mathcal{A}$. Hence, $\mathcal{A}$ has now $m' - 3n' + 2$ columns. Let α' and β' be respectively the vectors containing the components of α et β corresponding to these columns, i.e. without the components fixed to 0. The last step consists of proving that the system

$$\mathcal{A} \begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = \mathbf{0}$$

has a unique trivial solution, i.e. $\alpha' = 0$ and $\beta' = 0$.

Proof of the Lemma 2

Proof. As I is valid for PP_{st}, it is also valid for $\mathcal{P}_{rs}$.
To complete the proof, we will exhibit $m' - 3n' + 2$ affinely independent solutions of RSP satisfying I at equality. As I defines a facet for PP_{st} there are $m'_e - 2$ st-chains affinely independent satisfying I at equality. We transform them to $m'_e - 2$ affinely independent solutions of RSP by assigning the vertices not

included in the st-chains to s. It is clear that all these solutions satisfying I at equality.
Now for each pair of vertices u, $v \notin \{s,t\}$, we take any two st-chains satisfying I at equality such that one contains u and not v and the other contains v and not u. Let us call the two chains respectively chain-(uv) and chain-(vu). Thus we have:

$$m'_a - n' \text{ (the } n' \; y_{ii}) \; - (n'-2) \text{ (the } y_{si}) \; - (n-2) \text{ (the } y_{is}) \; = m'_a - 3n' + 4$$

st-chains which are not necessarily distinct. Each $uv-$chain corresponds to a solution of RSP by assigning v to u and the other vertices not included in the st-chain to s. This solution satisfies I at equality and considering them all we obtain $m'_a - 3n' + 4$ solutions of RSP satisfying I at equality. We can see that these solutions are affinely independent and they are also affinely independent to the above $m'_e - 2$ affinely independent solutions of RSP. Therefore, we have displayed $m'_a - 3n' + 4 + m'_e - 2 = m' - 3n' + 2$ affinely independent solutions of RSP satisfying I at equality $\Rightarrow$ I defines a facet for $\mathcal{P}_{rs}$. □

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