Representation of nonnegative convex polynomials

Jean B. Lasserre

Abstract. We provide a specific representation of convex polynomials nonnegative on a convex (not necessarily compact) basic closed semi-algebraic set $\mathbf{K} \subset \mathbb{R}^n$. Namely, they belong to a specific subset of the quadratic module generated by the concave polynomials that define $\mathbf{K}$.

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1. Introduction

An important research area of real algebraic geometry is concerned with representations of polynomials positive on a basic semi-algebraic set

$$\mathbf{K} := \{x \in \mathbb{R}^n \;:\; g_j(x) \geq 0, \quad j = 1, \ldots, m\} \subset \mathbb{R}^n \tag{1.1}$$

where $g_j \in \mathbb{R}[X]$, $j = 1, \ldots, m$.

An important result in this vein is Schmüdgen's Positivstellensatz [6] which states that if $\mathbf{K}$ is compact and $f \in \mathbb{R}[X]$ is positive on $\mathbf{K}$ then f belongs to the preordering $P(g)$ generated by the g_j's; bounds on the degrees in the representation are even provided in Schweighofer [7]. Under a rather weak additional assumption on the g_j's, Putinar's refinement [4] states that f even belongs to the quadratic module $Q(g)$ generated by the g_j's. The above mentioned representation results do not specialize when f is convex and the g_j's are concave (so that $\mathbf{K}$ is convex) a highly important case, particularly in optimization. Also, as soon as $\mathbf{K}$ is not compact any more then negative results, notably by Scheiderer [5], exclude to represent *any* f positive on $\mathbf{K}$ as an element of $P(g)$ or $Q(g)$ (except perhaps in

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low-dimensional cases). For more details, the interested reader is referred to the nice survey [5].

However, inspired and motivated by some classical results from convex optimization, we show that specialized representation results are possible when f is convex and the g_j's are concave, in which case $\mathbf{K} \subset \mathbb{R}^n$ is a closed (not necessarily compact) convex basic semi-algebraic set. Namely, a specific subset $Q_c(g)$ of the quadratic module $Q(g)$ is such that $Q_c(g) \cap F$ is *dense* (for the l_1-norm of coefficients) in the convex cone F of convex polynomials, nonnegative on $\mathbf{K}$.

2. Convex polynomials on a convex semi-algebraic set

2.1. Notation and Preliminaries

Let $\mathbb{R}[X]$ be the ring of real polynomials in the variables $X = (X_1, \ldots, X_n)$, and let $\Sigma^2 \subset \mathbb{R}[X]$ be the subset of sums of squares (sos) polynomials. If $f \in \mathbb{R}[X]$, write $f(X) = \sum_{\alpha \in \mathbb{N}^n} f_\alpha X^\alpha$, and denote its l_1-norm by $\|f\|_1$ $(= \sum_{\alpha \in \mathbb{N}^n} |f_\alpha|)$.

Let $Q(g) \subset \mathbb{R}[X]$ be the *quadratic module* generated by a set of polynomials $g = (g_j)_{j=1}^m \subset \mathbb{R}[X]$, that is,

$$Q(g) := \left\{ \sigma_0 + \sum_{j=1}^m \sigma_j \, g_j \; : \quad \sigma_j \in \Sigma^2, \; j = 0, \ldots, m \right\}. \tag{2.1}$$

Throughout the paper we make the following assumption.

Assumption 2.1. *$\mathbf{K} \subset \mathbb{R}^n$ is defined in (1.1) and is such that:*

(a) *g_j is concave for every $j = 1, \ldots, m$.*

(b) *There exists $z \in \mathbf{K}$ such that $g_j(z) > 0$ for every $j = 1, \ldots, m$.*

Assumption 2.1(b), known as Slater condition, is an important regularity condition for the celebrated Karush-Kuhn-Tucker optimality conditions.

Proposition 2.2. *Let Assumption 2.1 hold and let $f \in \mathbb{R}[X]$ be convex and such that $f^* := \inf_x \{f(x) \, : \, x \in \mathbf{K}\} = f(x^*)$ for some $x^* \in \mathbf{K}$.*

Then there exists $\lambda \in \mathbb{R}^m_+$ such that

$$\nabla f(x^*) - \sum_{j=1}^m \lambda_j \nabla g_j(x^*) \; = \; 0 \, ; \quad \lambda_j \, g_j(x^*) \; = \; 0, \quad j = 1, \ldots, m. \tag{2.2}$$

In other words, the Lagrangian $L_f \in \mathbb{R}[X]$ defined by

$$X \mapsto L_f(X) \; := \; f(X) - f^* - \sum_{j=1}^m \lambda_j \, g_j(X), \qquad X \in \mathbb{R}^n, \tag{2.3}$$

is a nonnegative polynomial which satisfies

$$L_f(x^*) \; = \; 0 \, ; \quad \nabla L_f(x^*) = 0. \tag{2.4}$$

See e.g. Polyak [3].

2.2. Convex Positivstellensatz

If one is interested in representation of polynomials nonnegative on $\mathbf{K}$, the first polynomial to consider is of course $f - f^*$ where $0 \leq f^* = \inf_{x \in \mathbf{K}} f(x)$. Indeed, any other positive polynomial is just $(f - f^*) + f^*$ with $f^* \geq 0$. And so, if $f - f^*$ belongs to some preordering or some quadratic module, then so does f. From Proposition 2.2 it is easy to establish the following result.

Corollary 2.3. *Let Assumption* 2.1 *hold and let $f \in \mathbb{R}[X]$ be convex and such that $f^* := \inf_x \{f(x) : x \in \mathbf{K}\} = f(x^*)$ for some $x^* \in \mathbf{K}$. If the nonnegative polynomial L_f of (*2.3*) is sos then*

$$f - f^* = \sigma + \sum_{j=1}^{n} \lambda_j g_j \tag{2.5}$$

*for some convex sos polynomial $\sigma \in \Sigma^2$ and some nonnegative scalars λ_j, $j = 1, \ldots, m$. That is, $f - f^* \in Q(g)$, with $Q(g)$ as in (*2.1*). In addition, the sos weights associated with the g_j's are just nonnegative constants, and σ is convex.*

Proof. Follows from the definition (2.3) of L_f, and the fact that L_f is sos. □

Hence in view of Corollary 2.3, an interesting issue is to provide sufficient conditions for L_f to be sos. For instance, consider the following definition from Helton and Nie [1]

Definition 2.4 (Helton and Nie [1]**).** A polynomial $f \in \mathbb{R}[X]$ is sos-convex if its Hessian $\nabla^2 f$ is a sum of squares (sos), that is, there is some integer p and some matrix polynomial $F \in \mathbb{R}[X]^{p \times n}$ such that

$$\nabla^2 f(X) := \left(\frac{\partial^2 f(X)}{\partial X_i \partial X_j} \right)_{ij} = F(X)^T F(X). \tag{2.6}$$

Corollary 2.5. *Let Assumption* 2.1 *hold, and let $f \in \mathbb{R}[X]$ be convex and such that $f^* := \inf_x \{f(x) : x \in \mathbf{K}\} = f(x^*)$ for some $x^* \in \mathbf{K}$.*

*If f is sos-convex and $-g_j$ is sos-convex for every $j = 1, \ldots, m$, then $f - f^* \in Q(g)$. More precisely, (*2.5*) holds for some convex sos polynomial $\sigma \in \Sigma^2$ and some nonnegative scalars λ_j, $j = 1, \ldots, m$.*

Proof. From Proposition 2.2, let L_f be as in (2.3). As f and $-g_j$ are sos convex, write

$$\nabla^2 f(X) = F(X)^T F(X); \quad -\nabla^2 g_j(X) = G_j(X)^T G_j(X), \quad j = 1, \ldots, m,$$

for some $F \in \mathbb{R}[X]^{p \times n}$ and some $G_j \in \mathbb{R}[X]^{p_j \times n}$, $j = 1, \ldots, m$. Hence,

$$\nabla^2 L_f = \nabla^2 f - \sum_{j=1}^{m} \lambda_j \nabla^2 g_j = F^T F + \sum_{j=1}^{m} \lambda_j G_j^T G_j = H^T H,$$

with $H^T := \left[F^T \,|\, \sqrt{\lambda_1}\, G_1^T | \cdots | \sqrt{\lambda_m}\, G_m^T \right]$, and so L_f is sos-convex. As (2.4) holds, by Lemma 3.2 in Helton and Nie [1], the polynomial L_f is sos, and so, by Corollary 2.3, the desired result (2.5) holds. □

Next, consider the subset $Q_c(g) \subset Q(g)$ defined by:

$$Q_c(g) := \left\{ \sigma + \sum_{j=1}^{m} \lambda_j \, g_j \;:\quad \lambda \in \mathbb{R}^m_+ \,;\, \sigma \in \Sigma^2,\, \sigma \text{ convex.} \right\} \subset Q(g). \tag{2.7}$$

The set $Q_c(g)$ is a specialization of $Q(g)$ to the convex case, in that the weights asociated with the g_j's are nonnegative scalars, i.e., sos polynomials of degree 0, and the sos polynomial σ is convex.

Theorem 2.6. *Let Assumption* 2.1 *hold, and let $Q_c(g)$ be as in* (2.7)*. Let $F \subset \mathbb{R}[X]$ be the convex cone of convex polynomials nonnegative on $\mathbf{K}$.*

Then $Q_c(g) \cap F$ is dense in F for the l_1-norm $\|\cdot\|_1$. In particular, if $\mathbf{K} = \mathbb{R}^n$ (so that F is now the set of nonnegative convex polynomials), then $\Sigma^2 \cap F$ is dense in F.

Proof. Let $f \in F$ and let $r_0 := \lfloor (\deg f)/2 \rfloor + 1$. Given $r \in \mathbb{N}$, let $\Theta_r \in \mathbb{R}[X]$ be the polynomial

$$X \mapsto \Theta_r(X) \,:=\, 1 + \sum_{i=1}^{n} X_i^{2r}. \tag{2.8}$$

For every $\epsilon > 0$, the polynomial $f_{\epsilon 0}(X) := f(X) + \epsilon \, \Theta_{r_0}(X)$ is convex and nonnegative on $\mathbf{K}$, i.e., $f_{\epsilon 0} \in F$. In addition,

$$0 \,\leq\, f^* := \inf_{x \in \mathbf{K}} f(x) \,\leq\, \inf_{x \in \mathbf{K}} f_{\epsilon 0}(x) \,=\, f_{\epsilon 0}(x^*_\epsilon) \,=:\, f^*_\epsilon,$$

for some $x^*_\epsilon \in \mathbf{K}$. Indeed, the level set $\{x \in \mathbf{K} \;:\; f_{\epsilon 0}(x) \leq \alpha\}$ is compact for every $\alpha \in \mathbb{R}$, and so, $f_{\epsilon 0}$ attains its minimum on $\mathbf{K}$. Obviously, we also have $\|f_{\epsilon 0} - f\|_1 \to 0$ as $\epsilon \downarrow 0$. Next, let $L_{f_{\epsilon 0}}$ be as in (2.3), i.e.,

$$L_{f_{\epsilon 0}} \,=\, f + \epsilon \, \Theta_{r_0} - f^*_\epsilon - \sum_{j=1}^{n} \lambda^\epsilon_j \, g_j,$$

for some nonnegative vector $\lambda^\epsilon_j \in \mathbb{R}^m_+$. As $L_{f_{\epsilon 0}} \geq 0$ on $\mathbb{R}^n$, by Corollary 3.3 in Lasserre and Netzer [2], there exists $r_\epsilon \in \mathbb{N}$ such that for every $r \geq r_\epsilon$, $L_{f_{\epsilon 0}} + \epsilon \, \Theta_r$ is sos. That is, $\sigma := L_{f_{\epsilon 0}} + \epsilon \, \Theta_r \in \Sigma^2$ and so

$$f_\epsilon \,:=\, f + \epsilon \, (\Theta_{r_0} + \Theta_r) \,=\, \sigma + f^*_\epsilon + \sum_{j=1}^{n} \lambda^\epsilon_j \, g_j.$$

Notice that by definition, $\sigma \in \Sigma^2$ is convex. Next, as $f^*_\epsilon \geq 0$, $\sigma + f^*_\epsilon \in \Sigma^2$, and so, equivalently, $f_\epsilon \in Q_c(g)$.

In addition, $f_\epsilon \in F$ because f_ϵ is convex (as $f_\epsilon = f + \epsilon(\Theta_{r_0} + \Theta_r)$) and nonnegative on $\mathbf{K}$ (as $f_\epsilon \geq f$), and so, $f_\epsilon \in Q_c(g) \cap F$. Finally, $\|f - f_\epsilon\|_1 = \epsilon \|\Theta_{r_0} + \Theta_r\|_1 \to 0$ as $\epsilon \downarrow 0$.

Finally, if $\mathbf{K} = \mathbb{R}^n$ (so that F is now the set of nonnegative convex polynomials), one obtains $Q_c(g) = \Sigma^2$. □

One may also replace Θ_r in (2.8) with the new perturbation

$$X \mapsto \theta_r(X) := \sum_{k=0}^{r} \sum_{j=1}^{n} \frac{X_j^{2k}}{k!}.$$

This perturbation also preserves convexity. In addition, not only $\|f - f_\epsilon\|_1 \to 0$ as $\epsilon \downarrow 0$, but the convergence $f_\epsilon \to f$ is also *uniform* on compact sets!

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Jean B. Lasserre
LAAS-CNRS and Institute of Mathematics
University of Toulouse
LAAS, 7 avenue du Colonel Roche
31077 Toulouse Cédex 4
France
e-mail: `lasserre@laas.fr`