An Infeasible Interior-Point Algorithm with full-Newton Step for Linear Optimization

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Abstract

In this paper we present an infeasible interior-point algorithm for solving linear optimization problems. This algorithm is obtained by modifying the search direction in the algorithm [8]. The analysis of our algorithm is much simpler than that of the algorithm [8] at some places. The iteration bound of the algorithm is as good as the best known iteration bound $O\left(n \log \frac{1}{\varepsilon}\right)$ for IIPMs.

Keywords: Linear optimization, infeasible interior-point method, primal-dual method, polynomial complexity.

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1 Introduction

Interior-point Methods (IPMs) are now among the most effective methods for solving linear optimization (LO) problems. For a survey we refer to recent books on the subject [9, 11, 13]. One may distinguish between IPMs according to wether they are feasible IPMs or infeasible IPMs (IIPMs). Feasible IPMs start with a strictly feasible interior point and maintain feasibility during the solution process. It is not trivial to find an initial feasible interior point. One method to overcome this problem is to use the homogeneous embedding model by introducing artificial variables. Such a homogenous self-dual was presented first by Ye et al.[14] for LO, and further developed by Andersen and Ye, etc. in [1, 9, 12].

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IIPMs start with an arbitrary positive point and feasibility is reached as optimality is approached. The choice of the starting point in IIPMs is crucial for the performance. Lustig [3] and Tanabe [10] were the first to present IIPMs for LO. The first theoretical result on primal-dual IIPMs was obtained by Kojima, Meggido and Mizuno [2]. They showed that an infeasible-interior-point variant of the primal-dual feasible IPM studied in [6] is globally convergent. The first polynomial-complexity result was obtained by Zhang [15] who proved that, with proper initialization, an IIPM has $O\left(n^2 \log \frac{1}{\varepsilon}\right)$-iteration complexity. Shortly after that, Mizuno [5] proved that the Kojima-Meggido-Mizuno algorithm also has $O\left(n^2 \log \frac{1}{\varepsilon}\right)$-iteration complexity. Mizuno [5] and Potra [7] presented two primal-dual IIPMs with $O\left(n \log \frac{1}{\varepsilon}\right)$-iteration complexity which is the best known iteration bound for IIPMs. Roos [8] presented the first primal-dual IIPM that uses full-Newton steps for solving the LO problem. He also proved that the complexity of his algorithm coincides with the best known iteration bound for IIPMs.

In this paper we consider primal-dual LO problems in the following the standard form:

$$(P) \qquad \min \left\{c^T x : Ax = b, \quad x \geq 0\right\},$$

and the dual problem is given by

$$(D) \qquad \max \left\{b^T y : A^T y + s = c, \quad s \geq 0\right\},$$

where $A \in \mathbf{R}^{m \times n}, b, y \in \mathbf{R}^m$ and $c, x, s \in \mathbf{R}^n$ and w.l.o.g rank $(A) = m$. The vectors x, y and s are the vectors of variables. As usual for IIPMs we assumed that the initial iterates $\left(x^0, \, y^0, \, s^0\right)$ are as follows:

$$x^0 = s^0 = \zeta\, e, \quad y^0 = 0, \quad \mu^0 = \zeta^2, \tag{1}$$

where μ^0 is the initial parameter and $\zeta > 0$ is such that

$$\|x^* + s^*\|_\infty \leq \zeta, \tag{2}$$

for some optimal solution (x^*, y^*, s^*) of (P) and (D). In the rest of this paper we use some notations like r_b^0 and r_c^0 which defined in [4, 8] as the initial residual vectors:

$$r_b^0 = b - Ax^0 = b - \zeta A e \tag{3}$$

$$r_c^0 = c - A^T y^0 - s^0 = c - \zeta e. \tag{4}$$

Using $\left(x^0\right)^T s^0 = n\zeta^2$, the total number of iterations in the algorithm of [8] is bounded above by

$$24\, n \, \log \frac{\max\left\{n\zeta^2, \, \left\|r_b^0\right\|, \, \left\|r_c^0\right\|\right\}}{\varepsilon}, \tag{5}$$

Up to a constant factor, the iteration bound (5) was first obtained by Mizuno [5] and it is still the best known iteration bound for IIPMs.

To describe the motivation and contribution of this paper we need to recall the main ideas underlying the algorithm in [8]. For any ν with $0 < \nu \leq 1$ we consider the perturbed problem (P_ν), defined by

$$(P_\nu) \qquad \min \left\{ \left(c - \nu r_c^0\right)^T x : Ax = b - \nu r_b^0, \quad x \geq 0 \right\},$$

and its dual problem (D_ν), which is given by

$$(D_\nu) \qquad \max\left\{\left(b-\nu r_b^0\right)^T y : A^T y + s = c - \nu r_c^0, \quad s \geq 0\right\}.$$

Note that if $\nu = 1$ then $x = x^0$ yields a strictly feasible solution of (P_ν), and $(y, s) = (y^0, s^0)$ a strictly feasible solution of (D_ν). Due to the choice of the initial iterates we may conclude that if $\nu = 1$ then (P_ν) and (D_ν) each have a strictly feasible solution, which means that both perturbed problems then satisfy the well known interior-point condition (IPC). More generally one has the following lemma (see also [8, Lemma 3.1]).

Lemma 1.1 (Theorem 5.13 in [13]) *The perturbed problems (P_ν) and (D_ν) satisfy the IPC for each $\nu \in (0, 1]$, if and only if the original problems (P) and (D) are feasible.*

We assume that problems (P) and (D) are feasible. By this assumption, Lemma 1.1 implies that the perturbed problem pair (P_ν) and (D_ν) satisfy the IPC, for each $\nu \in (0, 1]$. This guarantees that the following system

$$b - Ax = \nu r_b^0, \qquad x \geq 0 \tag{6}$$

$$c - A^T y - s = \nu r_c^0, \qquad s \geq 0 \tag{7}$$

$$xs = \mu e. \tag{8}$$

has a unique solution, for every $\mu > 0$. If $\nu \in (0, 1]$ and $\mu = \nu\zeta^2$ we denote this unique solution in the sequel as $(x(\nu),\, y(\nu),\, s(\nu))$. As a consequence, $x(\nu)$ is the μ-center of (P_ν) and $(y(\nu),\, s(\nu))$ the μ-center of (D_ν). Due to this notation we have, by taking $\nu = 1$, $(x(1),\, y(1),\, s(1)) = (x^0, y^0, s^0) = (\zeta e, 0, \zeta e)$.

Like [4, 8] we need to measure proximity of iterates (x, y, s) to the μ-center of the perturbed problems (P_ν) and (D_ν). To this end we use $\delta(x, s; \mu)$ as the quantity to measure closeness to μ-centers, which is defined as follows.

$$\delta(x, s; \mu) := \delta(v) := \frac{1}{2}\left\|v - v^{-1}\right\| \qquad \text{where} \qquad v := \sqrt{\frac{xs}{\mu}}. \tag{9}$$

Initially we have $x = s = \zeta e$ and $\mu = \zeta^2$, whence $\delta(x, s; \mu) = 0$. In the sequel we assume that at the start of each iteration, $\delta(x, s; \mu)$ is smaller than or equal to a (small) threshold value $\tau > 0$. So this is certainly true at the start of the first iteration.

Now we describe one iteration of our algorithm. Suppose that for some $\nu \in (0, 1]$ we have x, y and s satisfying the feasibility conditions (6) and (7) and such that

$$x^T s = n\mu \quad \text{and} \quad \delta(x, s; \mu) \leq \tau, \tag{10}$$

where $\mu = \nu\zeta^2$. First we reduce ν to $\nu^+ = (1-\theta)\nu$, with $\theta \in (0, 1)$, and find new iterates x^f, y^f and s^f that satisfy (6) and (7), with ν replaced by ν^+. As we will see, by taking θ small enough this can be realized by one so-called *feasibility step*, to be described below soon. So, as a result of the feasibility step we obtain iterates that are feasible for (P_{ν^+}) and (D_{ν^+}). Then we apply a limited number of centering steps with respect to the μ^+-centers of (P_{ν^+}) and (D_{ν^+}). The centering steps keep the iterates feasible for (P_{ν^+}) and (D_{ν^+}); their purpose is to get iterates x^+, y^+ and s^+ such that $(x^+)^T s^+ = n\mu^+$, where $\mu^+ = \nu^+\zeta^2$ and $\delta(x^+, s^+; \mu^+) \leq \tau$. This process is repeated until the duality gap and the norms of the residual vectors are less than some prescribed accuracy parameter ε.

Primal-Dual Infeasible IPM

Input:
Accuracy parameter $\varepsilon > 0$;
barrier update parameter θ, $0 < \theta < 1$
threshold parameter $\tau > 0$
parameter $\zeta > 0$.

begin
$x := \zeta e;\ y := 0;\ s := \zeta e;\ \nu = 1;$
while $\max\left(x^T s,\ \|b - Ax\|,\ \left\|c - A^T y - s\right\|\right) \geq \varepsilon$ **do**
begin
feasibility step: $(x,\ y,\ s) := (x,\ y,\ s) + (\Delta^f x,\ \Delta^f y,\ \Delta^f s);$
μ-update: $\mu := (1 - \theta)\mu;$
centering steps:
while $\delta(x, s; \mu) \geq \tau$ **do**
$(x,\ y,\ s) := (x,\ y,\ s) + (\Delta x,\ \Delta y,\ \Delta s);$
endwhile
end
end

Figure 1: Algorithm

Before describing the search directions used in the feasibility step and the centering step we give a more formal description of the algorithm in Figure 1. For the feasibility step in [8] search directions $\Delta^f x$, $\Delta^f y$ and $\Delta^f s$ are (uniquely) defined by the system

$$A\Delta^f x = \theta\nu r_b^0 \tag{11}$$

$$A^T\Delta^f y + \Delta^f s = \theta\nu r_c^0 \tag{12}$$

$$s\Delta^f x + x\Delta^f s = \mu e - xs. \tag{13}$$

It can easily be understood that if (x, y, s) is feasible for the perturbed problems (P_ν) and (D_ν) then after the feasibility step the iterates satisfy the feasibility conditions for (P_{ν^+}) and (D_{ν^+}), provided that they satisfy the nonnegativity conditions. Assuming that before the step $\delta(x, s; \mu) \leq \tau$ holds, and by taking θ small enough, it can be guaranteed that after the feasibility step the iterates x^f, y^f and s^f are nonnegative and moreover $\delta(x^f, s^f; \mu^+) \leq 1/\sqrt{2}$, where $\mu^+ = (1 - \theta)\mu$. So, after the μ-update the iterates are feasible for (P_{ν^+}) and (D_{ν^+}) and μ is such that $\delta(x, s; \mu) \leq 1/\sqrt{2}$.

In a centering step the search directions $\Delta x, \Delta y$ and Δs are the usual primal-dual Newton directions, (uniquely) defined by

$$\begin{aligned} A\Delta x &= 0, & (14)\\ A^T\Delta y + \Delta s &= 0, & (15)\\ s\Delta x + x\Delta s &= \mu e - xs. & (16) \end{aligned}$$

Denoting the iterates after a centering step as x^+, y^+ and s^+, we recall from [9] the following result.

Lemma 1.2 *If $\delta := \delta(x,s;\mu) \leq 1$, then the primal-dual Newton step is feasible, i.e., x^+ and s^+ are nonnegative, and $(x^+)^T s^+ = n\mu$. Moreover, if $\delta := \delta(x,s;\mu) \leq \frac{1}{\sqrt{2}}$, then $\delta(x^+,s^+;\mu) \leq \delta^2$.*

As discussed in [4, 8], by using centring steps we get iterates that satisfy $x^T s = n\mu$ and $\delta(x,s;\mu) \leq \tau$, where τ is (much) smaller than $1/\sqrt{2}$. By using Lemma 1.2, the required number of centering steps can easily be obtained. Because after the μ-update we have $\delta = \delta(x,s;\mu) \leq 1/\sqrt{2}$, and hence after k centering steps the iterates (x,y,s) satisfy

$$\delta(x,s;\mu) \leq \left(\frac{1}{\sqrt{2}}\right)^{2^k}.$$

This implies that at most

$$\log_2\left(\log_2 \frac{1}{\tau^2}\right) = \log_2\left(\log_2 64\right) \leq 3. \tag{17}$$

centering steps are needed.

In this paper, we modify the feasibility step by replacing the equation (13) by the equation

$$s\Delta_x^f + x\Delta_s^f = (1-\theta)\,\mu e - xs. \tag{18}$$

This modification makes the analysis new and much simpler than the analysis of the algorithm in [4, 8]. The iteration bound is as good as that in [4, 8] which is essentially the same as the best iteration bound for IIPMs.

To conclude this section we briefly describe how the paper is organized. Section 2 is devoted to the analysis of the feasibility step, which is the main part of the paper. The analysis presented in this section differs from the analysis in [4, 8]. The final iteration bound is derived in Section 3. Some concluding remarks can be found in Section 4.

Some notations used throughout the paper are as follows. $\|\cdot\|$ denotes the 2-norm of a vector. For any $x = (x_1;\ x_2;\ \cdots;\ x_n) \in \mathbf{R}^n$, $x_{\min}$ denotes the smallest and $x_{\max}$ the largest value of the components of x. Furthermore, e denotes the all-one vector of length n. We write $f(x) = O(g(x))$ if $f(x) \leq \gamma\, g(x)$ for some positive constant γ.

2 Analysis of the feasibility step

Let x, y and s denote the iterates at the start of an iteration, and assume $\delta(x,s;\mu) \leq \tau$. Recall that in the first iteration we have $\delta(x,s;\mu) = 0$.

2.1 Effect of the feasibility step; choice of θ

As established in Section 1, the feasibility step generates new iterates x^f, y^f and s^f that are feasible for the new perturbed problem pair (P_{ν^+}) and (D_{ν^+}). A crucial element in the analysis is to show that after the feasibility step $\delta(x^f, s^f; \mu^+) \leq 1/\sqrt{2}$, i.e., that the new iterates are within the region where the Newton process targeting at the μ^+-centers of (P_{ν^+}) and (D_{ν^+}) is quadratically convergent.

Defining

$$d_x^f := \frac{v\Delta^f x}{x}, \qquad d_s^f := \frac{v\Delta^f s}{s}, \tag{19}$$

with v as defined in (9). Now using (18) and $xs = \mu v^2$ we may write

$$x^f s^f = xs + \left(s\Delta^f x + x\Delta^f s\right) + \Delta^f x \Delta^f s = \mu^+ e + \Delta^f x \Delta^f s = \mu\left((1-\theta)e + d_x^f d_s^f\right). \tag{20}$$

Lemma 2.1 *The new iterates are certainly strictly feasible if* $(1-\theta)\,e + d_x^f d_s^f > 0$.

Proof: Note that if x^f and s^f are positive then (20) makes clear that $(1-\theta)\,e + d_x^f d_s^f > 0$. In the same way as Lemma 4.1 in [8] the converse can be proved. Thus we have that x^f and s^f are positive if and only if $(1-\theta)\,e + d_x^f d_s^f > 0$. Thus the lemma follows. □

Corollary 2.2 *The iterates* $\left(x^f, y^f, s^f\right)$ *are certainly strictly feasible if*

$$\left\|d_x^f d_s^f\right\|_\infty < (1-\theta)\,.$$

Using (19) we may also write

$$x^f = x + \Delta^f x = x + \frac{x d_x^f}{v} = \frac{x}{v}(v + d_x^f) \tag{21}$$

$$s^f = s + \Delta^f s = s + \frac{s d_s^f}{v} = \frac{s}{v}(v + d_s^f). \tag{22}$$

To simplify the presentation we will denote $\delta(x, s; \mu)$ below simply as δ. Recall that we assume that before the feasibility step one has $\delta \leq \tau$. In the sequel we denote

$$\omega(v) = \frac{1}{2}\sqrt{\|d_x^f\|^2 + \|d_s^f\|^2}. \tag{23}$$

This implies $\|d_x^f\| \leq 2\,\omega(v)$ and $\|d_s^f\| \leq 2\,\omega(v)$, and moreover,

$$\left(d_x^f\right)^T d_s^f \leq \left\|d_x^f\right\| \left\|d_s^f\right\| \leq \frac{1}{2}\left(\left\|d_x^f\right\|^2 + \left\|d_s^f\right\|^2\right) \leq 2\omega(v)^2 \tag{24}$$

$$\left\|d_x^f d_s^f\right\|_\infty \leq \left\|d_x^f\right\| \left\|d_s^f\right\| \leq 2\omega(v)^2. \tag{25}$$

Lemma 2.3 *Let* $\theta = \frac{\alpha}{\sqrt{2n}}$, $\alpha \leq 1$ *for* $n \geq 2$. *The iterates* $\left(x^f, y^f, s^f\right)$ *are strictly feasible if* $\omega(v) \leq \frac{1}{2}$.

Proof: Let $\omega < \frac{1}{2}$ and $\theta = \frac{\alpha}{\sqrt{2n}}$, $\alpha \leq 1$ for $n \geq 2$. Then (25) implies that $\left\|d_x^f d_s^f\right\|_\infty \leq \frac{1}{2} \leq 1-\theta$. By Corollary 2.2 this implies that the iterates $\left(x^f, y^f, s^f\right)$ are strictly feasible. □

Lemma 2.4 *One has*

$$\delta(v^f)^2 \le \frac{\omega(v)^4}{(1-\theta)\,(1-\theta-2\omega(v)^2)} \tag{26}$$

Proof: By definition (9),

$$\delta(x^f, s^f; \mu^+) = \delta(v^f) = \frac{1}{2}\left\| v^f - \frac{e}{v^f} \right\|, \;\text{ where } v^f = \sqrt{\frac{x^f s^f}{\mu^+}}.$$

After division of both sides in (20) by μ^+ we get

$$\left(v^f\right)^2 = \frac{\mu\left((1-\theta)\,e + d_x^f d_s^f\right)}{\mu^+} = e + \frac{d_x^f d_s^f}{1-\theta}. \tag{27}$$

By using the definition of the $\delta\left(v^f\right)$ we have

$$\delta(v^f)^2 = \frac{1}{4}\left\| v^f - \left(v^f\right)^{-1} \right\|^2 = \frac{1}{4}\left\| \left(v^f\right)^{-1}\left(e - \left(v^f\right)^2\right) \right\|^2 \le \frac{1}{4}\left\| (v^f)^{-1} \right\|_\infty^2 \left\| e - (v^f)^2 \right\|^2.$$

We proceed by deriving bounds for the last two norms. First we consider the second norm:

$$\begin{aligned} \left\| e - \left(v^f\right)^2 \right\| &= \left\| \frac{d_x^f d_s^f}{1-\theta} \right\| \\ &\le \frac{1}{1-\theta}\left\| d_x^f \right\| \left\| d_s^f \right\| \\ &\le \frac{2\omega(v)^2}{1-\theta}, \end{aligned}$$

where we used (27) for equality and (24) for the second inequality. For estimate of $\left\| \left(v^f\right)^{-1} \right\|_\infty$ we may write,

$$\begin{aligned} \left(v_i^f\right)^2 &= 1 + \frac{\left(d_x^f\right)_i \left(d_s^f\right)_i}{1-\theta} \\ &\ge 1 - \frac{2\omega(v)^2}{1-\theta}, \end{aligned}$$

where we used (25) for inequality. We therefore have, using the last inequality,

$$\left(v_i^f\right)^{-2} \le \frac{1-\theta}{1-\theta-2\omega\,(v)^2}.$$

Hence,

$$\left\| \left(v_i^f\right)^{-1} \right\|_\infty^2 \le \frac{1-\theta}{1-\theta-2\omega\,(v)^2}$$

which completes the proof. □

Since we need to have $\delta\left(v^f\right) \leq \frac{1}{\sqrt{2}}$, it follows from Lemma 2.4 that it suffices if

$$\frac{\omega(v)^4}{(1-\theta)\,(1-\theta-2\omega(v)^2)} \leq \frac{1}{2}.$$

Due to Lemma 2.3 we decide to choose

$$\theta = \frac{\alpha}{\sqrt{2n}}, \qquad \alpha \leq 1. \tag{28}$$

Then, for $n \geq 5$, one may easily verify that

$$\omega(v) \leq \frac{1}{2} \quad \Rightarrow \quad \delta(v^f) \leq \frac{1}{\sqrt{2}}. \tag{29}$$

We proceed by considering the vectors d_x^f and d_s^f more in detail.

2.2 An Upper bound for $\omega\,(v)$

One may easily check that the system (11)-(13), which defines the search directions $\Delta^f x$, $\Delta^f y$ and $\Delta^f s$, can be expressed in terms of the scaled search directions d_x^f and d_s^f as follows.

$$\bar{A} d_x^f = \theta\nu r_b^0, \tag{30}$$

$$\bar{A}^T \frac{\Delta^f y}{\mu} + d_s^f = \theta\nu v s^{-1} r_c^0, \tag{31}$$

$$d_x^f + d_s^f = (1-\theta)v^{-1} - v, \tag{32}$$

where

$$\bar{A} = AV^{-1}X,\ V = \text{diag}\ (v)\ \text{ and } X = \text{diag}\ (x)\,. \tag{33}$$

Let us denote the null space of the matrix $\bar{A}$ as $\mathcal{L}$. So,

$$\mathcal{L} := \left\{\xi \in \mathbf{R}^n : \bar{A}\xi = 0\right\}.$$

Obviously, the affine space $\left\{\xi \in \mathbf{R}^n : \bar{A}\xi = \theta\nu r_b^0\right\}$ equals $d_x^f + \mathcal{L}$. Note that due to a well-known result from linear algebra the row space of $\bar{A}$ equals the orthogonal complement $\mathcal{L}^\perp$ of $\mathcal{L}$. Therefore, (31) shows that the affine space $\left\{\theta\nu v s^{-1} r_c^0 + \bar{A}^T\xi \,:\, \xi \in \mathbf{R}^m\right\}$ equals $d_s^f + \mathcal{L}^\perp$. Since $\mathcal{L} \cap \mathcal{L}^\perp = \{0\}$, it follows that the affine spaces $d_x^f + \mathcal{L}$ and $d_x^f + \mathcal{L}^\perp$ meet in a unique point. This point is denoted below by q. We now recall a lemma from [8] which gives an upper bound for $\omega\,(v)$.

Lemma 2.5 (lemma 4.4 in [8]) *Let q be the (unique) point in the intersection of the affine spaces $d_x^f + \mathcal{L}$ and $d_s^f + \mathcal{L}$. Then*

$$\omega(v) \leq \sqrt{\|q\|^2 + (\|q\| + 2\delta(v))^2}.$$

From (29) we know that in order to have $\delta\left(v^f\right) \leq \frac{1}{\sqrt{2}}$, we should have $\omega\,(v) \leq \frac{1}{2}$. So, due to Lemma 2.5 this will hold if $\|q\|$ satisfies

$$\|q\| + (\|q\| + 2\,\delta\,(v))^2 \leq \frac{1}{4}. \tag{34}$$

2.3 Upper bound for $\|q\|$

From Lemma 2.5 we know that q is the (unique) solution of the system

$$\begin{aligned} \bar{A}q &= \theta\nu r_b^0, \\ \bar{A}^T\xi + q &= \theta\nu v s^{-1} r_c^0. \end{aligned}$$

We proceed to derive an upper bound for $\|q\|$. Before doing this we choose the initial point in the usual way as defined in (1) and (2).

Lemma 2.6 *Let $\left(x^0,\, y^0,\, s^0\right)$ be an initial point as defined in (1) and (2), we have*

$$\|q\| \leq \frac{\theta}{\zeta\, v_{\min}} \left(\|x\|_1 + \|s\|_1\right) \tag{35}$$

Proof: By using similar arguments as in Lemma 4.7 in [8] we obtain the following result:

$$\sqrt{\mu}\,\|q\| \leq \theta\nu\sqrt{\|D\left(\bar{s}-s^0\right)\|^2 + \|D^{-1}\left(\bar{x}-x^0\right)\|^2}, \tag{36}$$

where $\bar{x}$, $\bar{y}$ and $\bar{s}$ satisfy

$$\begin{aligned} A\bar{x} &= b, \\ A^T\bar{y} + \bar{s} &= c, \end{aligned} \tag{37}$$

and

$$D = \operatorname{diag}\left(\frac{xv^{-1}}{\sqrt{\mu}}\right). \tag{38}$$

We are still free to choose $\bar{x}$ and $\bar{s}$ such that they satisfy in system (37). We use $\bar{x} = x^*$ and $\bar{s} = s^*$ with x^* and s^* as defined in (2). Then we have

$$0 \leq x^0 - \bar{x} = x^0 - x^* \leq \zeta e, \quad 0 \leq s^0 - \bar{s} \leq \zeta e.$$

It follows that

$$\begin{aligned} \left\|D\left(\bar{s}-s^0\right)\right\|^2 &\leq \zeta^2\,\|De\|^2 \\ &\leq \zeta^2\left\|\frac{xv^{-1}}{\sqrt{\mu}}\right\|^2 = \frac{\zeta^2}{\mu}\left\|\frac{x}{v}\right\|^2 \\ &\leq \frac{\zeta^2}{\mu}\left\|\frac{x}{v_{\min}}\right\|^2 = \frac{\zeta^2}{\mu\, v_{\min}^2}\,\|x\|^2. \end{aligned} \tag{39}$$

where we used matrix D as defined in (38). In the same way it follows that

$$\left\|D^{-1}\left(\bar{x}-x^0\right)\right\|^2 \leq \frac{\zeta^2}{\mu\, v_{\min}^2}\,\|s\|^2. \tag{40}$$

Substitution (39), (40) and $\mu = \nu\mu^0 = \nu\zeta^2$ into (36) implies that

$$\|q\| \leq \frac{\theta}{\zeta\, v_{\min}}\sqrt{\|x\|^2 + \|s\|^2}.$$

Using $\|x\|^2 + \|s\|^2 \leq \left(\|x\|_1 + \|s\|_1\right)^2$ in the last inequality we have

$$\|q\| \leq \frac{\theta}{\zeta\, v_{\min}}\left(\|x\|_1 + \|s\|_1\right),$$

proving the lemma. □

2.4 Some bounds for $\|x\|_1$ and $\|s\|_1$ and $v_{\min}$; choice of α and τ

Let x and $(y,\, s)$ be feasible for (P_ν) and (D_ν), respectively. We need to find an upper bound for $\|x\|_1 + \|s\|_1$ and lower bound for smallest component, named $v_{\min}$, of vector v as defined in (9). For finding an lower bound on $v_{\min}$ we recall Lemma II.60 from [9] without further proof.

Lemma 2.7 (Cf. Lemma II.60 in [9]) *Let $\delta = \delta(v)$ be given by (9). Then*

$$\frac{1}{\rho(\delta)} \leq v_i \leq \rho(\delta), \tag{41}$$

where

$$\rho(\delta) := \delta + \sqrt{1+\delta^2}. \tag{42}$$

Lemma 2.8 *Let x and $(y,\, s)$ be feasible for the perturbed problems (P_ν) and (D_ν) respectively and $\left(x^0,\, y^0,\, s^0\right)$ as defined in (1). Then for any primal-dual optimization solution $(x^*,\, y^*,\, s^*)$, we have*

$$\begin{aligned} \nu\left(x^T s^0 + s^T x^0\right) &= s^T x + \nu^2 \left(s^0\right)^T x^0 \\ &+ \nu(1-\nu)\left(\left(s^0\right)^T x^* + \left(x^0\right)^T s^*\right) - (1-\nu)\left(s^T x^* + x^T s^*\right). \end{aligned} \tag{43}$$

Proof: Let

$$\begin{aligned} x' &= x - \nu x^0 - (1-\nu)\, x^*, \\ y' &= y - \nu y^0 - (1-\nu)\, y^*, \\ s' &= s - \nu s^0 - (1-\nu)\, s^*. \end{aligned}$$

From (3), (4) and definition of the perturbed problems (P_ν) and (D_ν), we can see easily that

$$\begin{aligned} Ax' &= 0 \\ A^T y' + s' &= 0, \end{aligned}$$

which shows that x' belongs in the null-space and s' is in row-space of matrix A which implies that x' and s' are orthogonal, i.e.,

$$\left(x'\right)^T s' = \left(x - \nu x^0 - (1-\nu)\, x^*\right)^T \left(s - \nu s^0 - (1-\nu)\, s^*\right) = 0.$$

By expanding the last equality and using the fact $(x^*)^T s^* = 0$ we obtain the desired result. □

Lemma 2.9 *Let x and $(y,\, s)$ be feasible for the perturbed problems (P_ν) and (D_ν) respectively and $\delta(v)$ be given as (9) and $x^0 = s^0 = \zeta e$, where $\zeta > 0$ is a constant such that $\|x^* + s^*\|_\infty \leq \zeta$ for some primal-dual optimal solution $(x^*,\, y^*,\, s^*)$. Then we have*

$$\|x\|_1 + \|s\|_1 \leq \left(\rho(\delta)^2 + 1\right) n\zeta, \tag{44}$$

where $\rho(\delta)$ is as defined in (42).

Proof: Since x, s, x^* and s^* are nonnegative, Lemma 2.8 implies that

$$x^T s^0 + s^T x^0 \leq \frac{s^T x}{\nu} + \nu \left(s^0\right)^T x^0 + (1-\nu)\left(\left(s^0\right)^T x^* + \left(x^0\right)^T s^*\right). \tag{45}$$

Since $x^0 = s^0 = \zeta e$ and $\|x^* + s^*\|_\infty \leq \zeta$, we have

$$\left(x^0\right)^T s^* + \left(s^0\right)^T x^* = \zeta e^T \left(x^* + s^*\right) \leq \zeta e^T \left(\|x^* + s^*\|_\infty e\right) = \zeta \|x^* + s^*\|_\infty \left(e^T e\right) \leq n\zeta^2.$$

Also by using $\left(x^0\right)^T s^0 = n\zeta^2$ in (45) we get

$$x^T s^0 + s^T x^0 \leq \frac{s^T x}{\nu} + \nu\zeta^2 = \frac{\mu\left(e^T v^2\right)}{\nu} + n\zeta^2 = \zeta^2\left(e^T v^2\right) + n\zeta^2,$$

where for the last equality we used $\nu = \frac{\mu}{\mu^0}$ and $\mu^0 = \zeta^2$. By using Lemma 2.7 in the last inequality we obtain

$$x^T s^0 + s^T x^0 \leq \left(\rho\left(\delta\right)^2 + 1\right) n\zeta^2.$$

Since $x^0 = s^0 = \zeta e$ we have

$$x^T s^0 + s^T x^0 = \zeta\left(e^T x + e^T s\right) = \zeta\left(\|x\|_1 + \|s\|_1\right).$$

Hence it follows that

$$\|x\|_1 + \|s\|_1 \leq \left(\rho\left(\delta\right)^2 + 1\right) n\zeta,$$

which proves the lemma. □

Substituting (41) and (44) into (35) we obtain

$$\|q\| \leq n\,\theta\,\rho\left(\delta\right)\left(1 + \rho\left(\delta\right)^2\right).$$

Now we choose

$$\tau = \frac{1}{8}. \tag{46}$$

Since $\delta \leq \tau = \frac{1}{8}$ and $\rho\left(\delta\right)$ is monotonically increasing with respect to δ, we have

$$\|q\| \leq n\,\theta\,\rho\left(\delta\right)\left(1 + \rho\left(\delta\right)^2\right) \leq n\,\theta\,\rho\left(\frac{1}{8}\right)\left(1 + \rho\left(\frac{1}{8}\right)\right) = 2.586\,n\,\theta.$$

Using $\theta = \frac{\alpha}{\sqrt{2}\,n}$ in the last inequality we obtain

$$\|q\| \leq \frac{2.586\,n\,\alpha}{\sqrt{2}\,n} = \frac{2.586\sqrt{2n}\,\alpha}{2}.$$

In order to have $\delta\left(v^f\right) \leq \frac{1}{\sqrt{2}}$, by (34) we should have $\|q\|^2 + \left(\|q\| + 2\delta\left(v\right)\right)^2 \leq \frac{1}{4}$. Therefore, since $\delta\left(v\right) \leq \tau = \frac{1}{8}$, it suffices if q satisfies $\|q\|^2 + \left(\|q\| + \frac{1}{4}\right)^2 \leq \frac{1}{4}$. So we have $\delta\left(v^f\right) \leq \frac{1}{\sqrt{2}}$ if $\|q\| \leq 0.455$. Since $\|q\| \leq \frac{2.586\,n\,\alpha}{\sqrt{2}\,n}$, the latter inequality is satisfied if we take

$$\alpha = \frac{1}{5\sqrt{n}}, \tag{47}$$

because

$$\frac{0.91}{3.6571} \geq \frac{1}{5}.$$

According to (28) this gives the following value for θ:

$$\theta = \frac{1}{5\sqrt{2}\,n}. \tag{48}$$

3 Iteration bound

In the previous sections we have found that if at the start of an iteration the iterates satisfy $\delta(x, s; \mu) \leq \tau$, with τ and θ as defined in (46) and (48), then after the feasibility step and the μ-update the iterates satisfy $\delta(x, s; \mu^+) \leq 1/\sqrt{2}$.

According to (17), at most

$$\log_2\left(\log_2 \frac{1}{\tau^2}\right) = \log_2\left(\log_2 64\right) = 3$$

centering steps suffice to get iterates that satisfy $\delta(x, s; \mu^+) \leq \tau$. So each iteration consists of one feasibility step and 3 centering steps. In each iteration both the duality gap and the norms of the residual vectors are reduced by the factor $1 - \theta$. Hence, using $\left(x^0\right)^T s^0 = n\zeta^2$, the total number of iterations is bounded above by

$$\frac{1}{\theta} \log \frac{\max\left\{n\zeta^2,\, \left\|r_b^0\right\|,\, \left\|r_c^0\right\|\right\}}{\varepsilon}.$$

Since

$$\theta = \frac{1}{5\sqrt{2}\,n},$$

the total number of inner iterations is bounded above by

$$20\sqrt{2}\,n \log \frac{\max\left\{n\zeta^2,\, \left\|r_b^0\right\|,\, \left\|r_c^0\right\|\right\}}{\varepsilon}.$$

Note that the order of this bound is exactly the same as the bound in [4, 8]. In the following we state without further proof our main result.

Theorem 3.1 *If (P) and (D) have optimal solutions x^* and $(y^*,\, s^*)$ such that $\|x^* + s^*\|_\infty \leq \zeta$, then after at most*

$$20\sqrt{2}\,n \log \frac{\max\left\{n\zeta^2,\, \left\|r_b^0\right\|,\, \left\|r_c^0\right\|\right\}}{\varepsilon},$$

iterations the algorithm finds an ε-solution of (P) and (D).

Due the theorem above we know that if there exist x^* and $(y^*,\, s^*)$ such that satisfy (2), the algorithm finds an ε-solution. One might ask what if this condition is not satisfied. From Lemma 2.6 we have that under assumptions (1) and (2) during the course of the algorithm $\|q\| \leq 0.455$. So, if during the excursion of the algorithm $\|q\| > 0.455$, then we may conclude that there exist no optimal solutions $(x^*,\, y^*,\, s^*)$ such that satisfy

$$\|x^* + s^*\|_\infty \leq \zeta.$$

4 Concluding remarks

We analyzed an algorithm with full-Newton steps for LO which differs from the algorithm presented in [4, 8] in the definition of the feasibility step. In the system of the feasibility step the equation (13) is replaced with:

$$s\Delta^f x + x\Delta^f s = (1-\theta)\mu e - xs,$$

whereas the feasibility step in [8] was determined by

$$s\Delta^f x + x\Delta^f s = \mu e - xs,$$

and in [4] was

$$s\Delta^f x + x\Delta^f s = 0.$$

The analysis for the feasibility step presented in Section 2 differs from the analysis in [4, 8]. The iteration bound of the algorithm is as good as the best known iteration bound for IIPMs.

Another topic for further research is the extension of the algorithm presented in this paper to symmetric cone optimization.

References

[1] E. D. Andersen and Y. Ye. A computational study of the homogeneous algorithm for large-scale convex optimization. *Computational Applications and Optimization*, 10 (1998) 243–269.

[2] M. Kojima, N. Megiddo, and S. Mizuno. A primal-dual infeasible-interior-point algorithm for linear programming. *Mathematical Programming*, 61 (1993) 263–280.

[3] I. J. Lustig. Feasibility issues in aprimal-dual interior point method for linear programming. *Mathematical Programming*, 49 (1990/91) 145–162.

[4] H. Mansouri and C. Roos. Simplified $O(n)$ infeasible interior-point algorithm for linear optimization using full-Newton step. *Optimization Methods and Software*, 22(3) (2007) 519–530.

[5] S. Mizuno. Polynomiality of infeasible interior point algorithms for linear programming. *Mathematical Programming*, 67 (1994) 109–119.

[6] S. Mizuno, M. J. Todd, and Y. Ye. On adaptive-step primal-dual interior-point algorithms for linear programming. *Mathematics of Operations Research*, 18 (1993) 964–981.

[7] F. A. Potra. An infeasible-interior-point predictor-corrector algorithm for linear programming. *SIAM Journal on Optimization*, 6(1) (1996) 19–32.

[8] C. Roos. A full-Newton step $O(n)$ infeasible interior-point algorithm for linear optimization. *SIAM Journal on Optimization*, 16(4) (2006) 1110–1136.

[9] C. Roos, T. Terlaky, and J.-Ph. Vial. *Theory and Algorithms for Linear Optimization. An Interior-Point Approach.* John Wiley & Sons, Chichester, UK, 1997 (2nd Edition, Springer, 2006).

[10] K. Tanabe. Centered Newton method for linear programming: Interior and 'exterior' point method (in Janpanese). *In: New Methods for Linear Programming. K. Tone (Ed.) 3*(1990) 98–100.

[11] S. J. Wright. *Primal-dual Interior-Point Methods*. SIAM, Philadelphia, 1996.

[12] F. Wu, S. Wu, and Y. Ye. On quadratic convergence of the $O(\sqrt{n}L)$-iteration homogeneous and self-dual linear programming algorithm. *Annals of Operations Research*, 87 (1999) 393–406.

[13] Y. Ye. *Interior Point Algorithms, Theory and Analysis.* John Wiley & Sons, Chichester, UK, 1997.

[14] Y. Ye, M. J. Todd, and S. Mizonu. An $O(\sqrt{n}L)$-iteration homogenous and self-dual linear programming algorithm. *Mathematical of Operations Research 19*(1994) 53–67.

[15] Y. Zhang. On the convergence of a class of infeasible-interior-point methods for the horizantal linear complementarity problem. *SIAM Journal on Optimization*, 4 (1994) 208–227.