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Strange Behaviors of Interior-point Methods for Solving Semidefinite Programming Problems in Polynomial Optimization

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Abstract

We observe that in a simple one-dimensional polynomial optimization problem (POP), the 'optimal' values of semidefinite programming (SDP) relaxation problems reported by the standard SDP solvers converge to the optimal value of the POP, while the true optimal values of SDP relaxation problems are strictly and significantly less than that value. Some pieces of circumstantial evidences for the strange behaviours of SDP solvers are given. This result gives a warning to users of SDP relaxation method for POP to be careful in believing the results of the SDP solvers. We also demonstrate how SDPA-GMP, a multiple precision SDP solver developed by one of the authors, can deal with this situation correctly.

Key words: Polynomial Optimization, Semidefinite Programming, Numerical Stability.

1. Introduction

Proposed by Lasserre [6] and Parrilo [13], the semidefinite programming (SDP) relaxation method for polynomial optimization problems (POPs) has been extensively studied. The latest results on this method are summarized in the nice survey [9] by Laurent. Several softwares of the SDP relaxation method are released and widely used, e.g., GloptiPoly [3], SOSTOOLS [14], and SparsePOP [18]. All the above softwares use the existing SDP solvers such as SDPA [1] or SeDuMi [16].

The theory of the SDP relaxation method for POP (e.g., Theorem 4.2 of [6]) requires to raise the number called *relaxation order* which determines the size of SDP relaxation problem infinitely large to obtain the optimal value of a POP. On the other hand, size of SDP relaxation problem explodes as the relaxation order increases; an SDP relaxation problem of a POP having n variables with the relaxation order r contains a $\binom{n+r}{r}$ by $\binom{n+r}{r}$ matrix called the *moment matrix* in its constraint. Even for moderate-sized POPs and relaxation orders, solving the corresponding SDP relaxation problem is impossible for the state-of-the-art SDP solvers just because of its size.

Fortunately, it is often observed in numerical experiments that very small relaxation orders, typically 2 or 3, are enough to obtain optimal values of POPs (See [6, 17]). This observation makes the SDP relaxation method for POP more practical, because sizes of SDP relaxation problems are not huge with such small relaxation orders.

One of the purpose of this paper is to give a warning to this optimistic view; we will pose a question whether we have successfully solved the SDP problems or not in the SDP relaxation method for POPs. Specifically, we observe that in a simple one-dimensional POP, the 'optimal' values reported by the SDP solvers converge to the optimal value of the POP, while the true optimal values of SDP relaxation problems are strictly and significantly less than that value. Namely, the convergence of 'optimal' values of SDP relaxation problems to the optimal value of the POP is an illusion. We put a stress on the fact that the wrong value is, if we provide a sufficiently large relaxation order, the optimal value of POP itself which we want to compute, and that the true optimal values of SDP relaxation problems never converge to this value.

Another purpose of this paper is to demonstrate that a multi-precition calculation is effective to remove this silent numerical problem. We use SDPA-GMP [1, 10] which employs GMP Library (GNU Multiple Precision Library [2]) in the interior-point method for SDP, and show that this software computes the optimal value of the SDP problems in an arbitrary precision if we designate an enough calculation precision when the software starts. In [10], they applied SDPA-GMP to highly degenerate SDP problems which we cannot solve with the standard SDP solvers arising from quantum physics. Maho Nakata, one of the authors, is the principal programmer of SDPA-GMP in the SDPA project.

Henrion and Lasserre [4] reported that in minimizing Motzkin polynomial over 2-dimensional Euclidean space, SeDuMi returns the optimal value of the original POP as the optimal value of the SDP relaxation problem, while the true optimal value of the SDP is $-\infty$ because Motzkin polynomial is a special polynomial which is nonnegative but not an SOS. In this paper, we report that this discrepancy can be observed in more natural situations. Our example is a constrained POP constructed by simple linear and quadratic polynomials, and an exact SOS solution exists; the corresponding SDP problem has an optimal solution. But it still has such discrepancy between the true and computed optimal values of the SDP relaxation problem.

Lasserre [7] proves that any nonnegative polynomial can be decomposed into an SOS by adding an SOS with small coefficients. His result implies that SeDuMi returns the optimal value of the POP for Motzkin polynomial due to the numerical error in the computation. His result is extended to a POP with a bounded feasible region in [8]. Our explanation on our case has a similar flavor to, but is essentially different from, the result of [8]. In contrast to using preordering in [8], we use quadratic modules which leads us the standard SDP relaxation for POP. However, our result is specific to our example.

The rest of this paper is organized as follows. In Section 2, we introduce a simple unbounded POP and its SDP relaxation, and prove that the optimal value of SDP relaxation is 0 while that of POP is 1. In Section 3, we prove two properties on the SDP relaxation problem for the POP to explain why the SDP solvers return the wrong optimal values which converge to the optimal value of the POP. In Section 4, we demonstrate that SDPA-GMP calculates the true optimal values of the SDP relaxation problems.

Section 5 deals with a bounded POP for which we know theoretically that the optimal values of the SDP relaxation problems converge to that of POP. We observe a similar phenomena to Section 2 for this case. Section 6 contains some concluding remarks.

2. An Unbounded Example

We consider the following one-dimensional POP:

$$\inf \left\{ x \;\middle|\; x \geq 0,\ x^2 \geq 1 \right\} \tag{1}$$

whose optimal value is 1. The SDP relaxation problem to (1) with the relaxation order $r(\geq 1)$ is as follows (See, e.g. [6]):

$$p_r^* = \inf \{ y_1 \mid (y_1, \ldots, y_{2r}) \in \mathcal{P}_r \} \tag{2}$$

where $\mathcal{P}_r \subseteq \mathbb{R}^{2r}$ is the set satisfying the folowing three positive semidefinite constraints:

$$\begin{bmatrix} 1 & y_1 & \cdots & y_r \\ y_1 & y_2 & \cdots & y_{r+1} \\ \vdots & \vdots & \ddots & \vdots \\ y_r & y_{r+1} & \cdots & y_{2r} \end{bmatrix} \succeq O. \tag{3}$$

$$\begin{bmatrix} y_1 & y_2 & \cdots & y_r \\ y_2 & y_3 & \cdots & y_{r+1} \\ \vdots & \vdots & \ddots & \vdots \\ y_r & y_{r+1} & \cdots & y_{2r-1} \end{bmatrix} \succeq O. \tag{4}$$

$$\begin{bmatrix} y_2 - 1 & y_3 - y_1 & \cdots & y_{r+1} - y_{r-1} \\ y_3 - y_1 & y_4 - y_2 & \cdots & y_{r+2} - y_r \\ \vdots & \vdots & \ddots & \vdots \\ y_{r+1} - y_{r-1} & y_{r+2} - y_r & \cdots & y_{2r} - y_{2r-2} \end{bmatrix} \succeq O. \tag{5}$$

Here, $A \succeq O$ implies that A is positive semidefinite.

The dual of (2) is equivalent to the following sums of squares (SOS) problem:

$$q_r^* = \sup \left\{ p \;\middle|\; x - p = s_0(x) + x s_1(x) + (x^2 - 1) s_2(x),\ s_0 \in \mathrm{SOS}_r, s_1, s_2 \in \mathrm{SOS}_{r-1} \right\}, \tag{6}$$

where SOS_r is the set of sums of squares in x whose degree is less than or equal to $2r$. Because $s \in \mathrm{SOS}_r$ is equivalent with the existence of positive semidefinite matrix X for which $s(x) = u_r(x)^T X u_r(x)$ where $u_r(x)^T = (1, x, \ldots, x^r)$, we can write the constraint of (6) as

$$\left.\begin{array}{l} x - p = u_r(x)^T X u_r(x) + x u_{r-1}(x)^T Y u_{r-1} + (x^2 - 1) u_{r-1}(x)^T Z u_{r-1}(x)\ (\forall x) \\ X \succeq O,\ Y \succeq O,\ Z \succeq O. \end{array}\right\} \tag{7}$$

Note that X is an $r+1$ by $r+1$ matrix, while Y and Z are r by r. From Theorem 4.2 of [6], we see that SDP (2) has an interior (i.e., positive definite) feasible solution because the feasible region of POP (1) has an interior (i.e., the values of x and $x^2 - 1$ are strictly positive) feasible solution . Moreover, as we will prove in Theorem 1, we see that SDP (6) has a feasible solution. Therefore, it follows from the duality theorem of SDP that we have $p_r^* = q_r^*$ and that q_r^* is attanined.

Tables 1 and 2 show the results on this problem with various relaxation orders by the standard latest SDP solvers SeDuMi 1.2 [16] and SDPA 7.1.1 [1], respectively. In each solver, we use the default parameters. The last column of Table 2 shows the status returned by SDPA at the termination of the interior-point method. The return value "pdOPT" implies that SDPA successfully obtained an approximate optimal solution, and "pdFEAS" that the computed solution is feasible but probably not optimal. From these tables, we observe the following:

(a) The 'optimal' values reported by the solvers increase as the relaxation order is raised, and converge to the optimal value of the POP (1) at the relaxation order 5.
(b) For $r = 5$ and 6, SeDuMi and SDPA successfully obtained approximate optimal solutions without any numerical difficulty. In particular, SDPA returns pdOPT in these cases.
(c) Looking at Table 1 closely, one notices that the duality gaps of the final solutions are negative, although they are very small in $r = 5$ and 6. This means that the final solutions could be slightly infeasible. In contrast, the solutions derived by SDPA have positive duality gaps.

Table 1. The result by SeDuMi 1.2

r	p_r^*	q_r^*	Feasibility of (2)	Feasibility of (6)	Duality gap
1	1.10e-13	8.91e-14	0.0e+00	3.3e-13	-9.8e-14
2	2.64e-03	2.82e-03	0.0e+00	9.8e-10	-1.8e-04
3	1.29e-01	1.34e-01	0.0e+00	1.2e-09	-0.4e-02
4	7.36e-01	7.46e-01	0.0e+00	1.1e-09	-1.0e-02
5	1.0e+00	1.0e+00	5.4e-11	3.8e-10	-2.0e-10
6	1.0e+00	1.0e+00	1.1e-10	9.7e-10	-6.0e-10

Table 2. The result by SDPA 7.1.1

r	p_r^*	q_r^*	Feasibility of (2)	Feasibility of (6)	Duality gap	Status
1	3.39e-09	-2.58e-08	7.3e-12	1.1e-12	2.9e-08	pdOPT
2	1.23e-02	1.52e-02	1.5e-11	9.4e-08	2.9e-03	pdFEAS
3	3.31e-01	3.74e-01	5.7e-14	8.9e-08	4.3e-02	pdFEAS
4	9.97e-01	9.98e-01	1.1e-14	5.3e-08	8.2e-04	pdFEAS
5	1.00e+00	1.00e+00	1.5e-11	7.6e-12	1.1e-16	pdOPT
6	1.00e+00	1.00e+00	7.3e-12	6.2e-11	4.4e-08	pdOPT

Note that the problem (1) has an unbounded feasible region and does not satisfy the condition of Theorem 4.2 of [6]. Consequently, we cannot ensure that the optimal value p_r^* of SDP (2) converges to the optimal value of POP (1). Even for such POPs, in practice, we often observe convergence to their optimal values.

However, we have the following.

Theorem 1. *The optimal value of* (2) *is* 0 *for any* $r \geq 1$.

Proof : We prove it by induction. For $r = 1$, note that the equality constraint is

$$x - p = X_{00} + 2X_{10}x + X_{11}x^2 + xY_{00} + (x^2 - 1)Z_{00} \ (\forall x).$$

Then the positive semidefiniteness of X and Z imply $X_{00} = Z_{00} = 0$, from which $X_{10} = 0$ follows. Now we have $-p = X_{00}$, thus $q_1^* = 0$.

Next we assume that $q_r^* = 0$ for $r \leq \bar{r}$, and consider q_{r+1}^*. Comparing the terms of $x^{2(\bar{r}+1)}$ in (6), we obtain the equality:

$$0 = X_{\bar{r}+1,\bar{r}+1} + Z_{\bar{r},\bar{r}}.$$

Because $X_{\bar{r}+1,\bar{r}+1} \geq 0$ and $Z_{\bar{r},\bar{r}} \geq 0$ by the positive semidefiniteness of X and Z, we have $X_{\bar{r}+1,\bar{r}+1} = 0$ and $Z_{\bar{r},\bar{r}} = 0$. Then, again due to the positive semidefiniteness of X and Z, we conclude that $X_{i,\bar{r}+1} = 0$ and $Z_{i,\bar{r}} = 0$ for any $0 \leq i \leq \bar{r}$. Now we compare the tems of $x^{2\bar{r}+1}$ in (6) to obtain:

$$0 = X_{\bar{r}+1,\bar{r}} + Y_{\bar{r},\bar{r}} + Z_{\bar{r}-1,\bar{r}}.$$

This means that $Y_{\bar{r},\bar{r}} = 0$ and thus $Y_{i,\bar{r}} = 0$ for any $1 \leq i \leq \bar{r}$. We have shown that the terms of $x^{2\bar{r}+1}$ and $x^{2\bar{r}+2}$ do not give any influence on the constraints of (6), thus we have $q_{\bar{r}+1}^* = q_{\bar{r}}^*$. □

By Theorem 1, we know that the optimal value of the SDP relaxation problem, 0, is strictly smaller than the optimal value of the POP (1) which is 1. However, the SDP solvers give the correct optimal value of POP (1) with relaxation order more than 5.

Moreover, from the proof, we see that SDP (6) does not have any interior feasible solutions. Indeed, for SDP (6) with relaxation order r, the feasible solution (X, Y, Z) forms

$$X = \begin{pmatrix} -p & 0_r^T \\ 0_r & O_{r,r} \end{pmatrix}, Y = \begin{pmatrix} 1 & 0_{r-1}^T \\ 0_{r-1} & O_{r-1,r-1} \end{pmatrix}, Z = O_{r,r},$$

where p is nonpositive, 0_r is the r-dimensional zero vector and $O_{r,r}$ is the $r \times r$ zero matrix. Since all matrices are not positive definite, all feasible solutions of SDP (6) are not interior feasible points. Therefore even if one applies the primal-dual interior-point method into SDP relaxation problems (2) and (6) constructed from POP (1), it may not converge to the optimal solutions for any given tolerance in polynomially many iterations.

3. More on the Problem

In this section, we prove two propositions which explains the strange behavior of the interior-point methods we have seen in the previous section.

Proposition 2. *If* $(y_1, \dots, y_{2r})$ *be a feasible solution of* (2) *with* $y_1 > 0$*, then*

$$y_k \geq \frac{y_2^{k-1}}{y_1^{k-2}}$$

for any $2 \leq k \leq 2r$*.*

Proof : We prove the proposition by induction. When $k = 2$, the inequlity $y_2 \geq y_2$ obviously holds. Suppose that for $2 \leq i \leq k$, the inequality holds. We divide the proof into two cases.

If k is even, then from (4),

$$\begin{pmatrix} y_1 & y_{k/2+1} \\ y_{k/2+1} & y_{k+1} \end{pmatrix} \succeq O$$

must hold, and thus we have

$$y_1 y_{k+1} \geq y_{k/2+1}^2.$$

By the induction assumption, we obtain

$$y_{k+1} \geq y_{k/2+1}^2 / y_1 \geq (y_2^{k/2} / y_1^{k/2-1})^2 / y_1 = y_2^k / y_1^{k-1},$$

which implies that the proposition holds for $k+1$.

If k is odd, then from (3),

$$\begin{pmatrix} y_2 & y_{(k+3)/2} \\ y_{(k+3)/2} & y_{k+1} \end{pmatrix} \succeq O$$

must hold, and thus we have

$$y_2 y_{k+1} \geq y_{(k+3)/2}^2.$$

The induction assumption and the fact that $y_2 \geq 1$ due to (5) imply

$$y_{k+1} \geq y_{(k+3)/2}^2 / y_2 \geq (y_2^{(k+1)/2} / y_1^{(k-1)/2})^2 / y_2 = y_2^k / y_1^{k-1}.$$

which means that the assersion holds for $k+1$. This completes the proof. □

Proposition 2 shows that it is very hard to compute an approximate optimal solution of (2), because the objective function y_1 is close to 0.

Proposition 3. *For any* $\epsilon > 0$*, there exists* $r_0 \in \mathbb{N}$ *such that for all* $r \geq r_0$ *we can find sum of squares* s_0, s_1, s_2 *satisfying the following identity:*

$$\epsilon x^{2r} + x - 1 = s_0 + x s_1 + (x^2 - 1) s_2. \tag{8}$$

Proof : Putting $s_1(x) = (x-1)^2$ and $s_2(x) = 2$, respectively, we define f_ϵ by

$$f_\epsilon(x) = \epsilon x^{2r} + x - 1 - 2(x^2 - 1) - x(x-1)^2 = \epsilon x^{2r} - x^3 + 1.$$

If we can prove that f_ϵ is a sum of squares, then we obtain the desired result. Noting that, in one-dimensional case, any nonnegative polynomial is also a sum of squares, we will prove that $f_\epsilon(x) \geq 0$ for all x in the following.

Because

$$f'(x) = 2r\epsilon x^{2r-1} - 3x^2,$$

the critical points of f satisfy:

$$x^2(2r\epsilon x^{2r-3} - 3) = 0,$$

and hence they are 0 or $\pm(3/2r\epsilon)^{1/(2r-3)}$. Since at these critical points, it holds that

$$f(x) = x^3 \left(\frac{3}{2r} - 1 \right) + 1,$$

parameter	value	short description
`maxIteration`	10000	maximum number of iterations
`epsilonStar`	ϵ	optimality tolerance
`lambdaStar`	1.0E4	initial point parameter 1
`omegaStar`	2.0	initial point parameter 2
`lowerBound`	-1.0E5	lower bound of the objective value of (2)
`upperBound`	1.0E5	upper bound of the objective value of (6)
`betaStar`	5.00e-01	direction parameter 1
`betaBar`	5.00e-01	direction parameter 2
`gammaStar`	5.00e-01	direction parameter 3
`epsilonDash`	ϵ	feasibility tolerance
`precision`	P	precision (significant bits) used in calculation

Table 3. Parameters of SDPA-GMP

the minimum is taken at $\bar{x} = (3/2r\epsilon)^{1/(2r-3)}$ if $r \geq 2$, and its value is

$$f(\bar{x}) = \left(\frac{3}{2r\epsilon}\right)^{3/(2r-3)} \left(\frac{3}{2r} - 1\right) + 1.$$

A straightforward calculation produces that $f(\bar{x}) \geq 0$ is equivalent with

$$\epsilon \geq \frac{3}{2r}\left(1 - \frac{3}{2r}\right)^{\frac{2r-3}{3}}.$$

Therefore, if we take r_0 as the minimum integer r satisfying $\epsilon \geq 3/2r_0$, then for $r \geq r_0$ we have

$$\epsilon \geq \frac{3}{2r_0} \geq \frac{3}{2r} \geq \frac{3}{2r}\left(1 - \frac{3}{2r}\right)^{\frac{2r-3}{3}}.$$

This completes the proof. □

Proposition 3 implies that if the objective function of the POP (1) is slightly perturbed with $\epsilon > 0$, then the corresponding SOS problem (6) has the feasible solution with $p = 1$. Arbitrarily small ϵ can be chosen if we allow r to be large. Although we do not add any kind of perturbation explicitly, there must be some very small numerical errors in the practical computation of the interior-point methods, which may cause a similar effect.

4. SDPA-GMP: A Remedy

In Section 2, we have observed that the 'optimal' values reported by the SDP solvers are significantly different from the real optimal value. In this section, we show that this phenomena is improved by using a multiple-precision calculation in solving the SDP. SDPA-GMP [1, 10] is exploiting GNU GMP Library [2] which allows us a multiple precision calculation.

To solve (2) and (6), we need to carefully adjust parameters of SDPA-GMP as shown in Table 3. See the manual of SDPA-GMP [1] for the details of these parameters. In this section, we use SDPA-GMP 7.1.1 and GMP 4.2.2, and solve (2) and (6) with relaxation order $r = 5$ and $r = 6$ for various tolerance ϵ and precision $P = 1000, 3000$. In this case, SDPA-GMP calculates floating point numbers with approximately 300 and 900 significant digits, respectively. Tables 4 and 5 show the optimal values of (2) by SDPA-GMP, the cpu times in seconds, the numbers of iterations and the status at the final solution by SDPA-GMP for each ϵ and $P = 1000, 3000$. Table 6 shows the solution of (2) with relaxation order 6 by SDPA-GMP with tolerance ϵ = 1.0e-50 and precision $P = 3000$. In Table 5, "dFEAS" means that SDPA-GMP found the feasible solution of SDP (6) which does not satisfy the optimality.

From these tables, we observe the following.

(a) SDPA-GMP with precision 3000 returns the positive values p_5^* and p_6^* less than the tolerances, which can be regarded as reasonable approximate optimal values.

(b) In $r = 5$ and 6, the interior-point method of SDPA-GMP with precision 3000 needs about 400 and 530 more iterations, respectively, to increase accuracy 10 digits more. It seems that the interior-point method of SDPA-GMP may not have superlinear convergence property.

Table 4. Results of SDPA-GMP with $r = 5$

P	1000				3000			
ϵ	p_5^*	cpu[sec]	#iter.	Status	p_5^*	cpu[sec]	#iter.	Status
1.0e-10	3.5e-09	6.94	400	pdOPT	3.5e-09	35.29	400	pdOPT
1.0e-20	3.8e-19	13.92	811	pdOPT	3.8e-19	71.10	811	pdOPT
1.0e-30	3.9e-29	20.93	1216	pdOPT	3.9e-29	106.19	1216	pdOPT
1.0e-40	4.2e-41	27.97	1630	pdOPT	3.9e-39	141.00	1619	pdOPT
1.0e-50	4.1e-41	27.68	1614	dFEAS	3.5e-49	176.56	2025	pdOPT

Table 5. Results of SDPA-GMP with $r = 6$

P	1000				3000			
ϵ	p_6^*	cpu[sec]	#iter.	Status	p_6^*	cpu[sec]	#iter.	Status
1.0e-10	6.8e-09	13.85	517	pdOPT	6.8e-09	72.17	517	pdOPT
1.0e-20	6.7e-19	28.10	1055	pdOPT	6.7e-19	146.39	1055	pdOPT
1.0e-30	1.4e-29	42.84	1609	dFEAS	6.6e-29	220.14	1588	pdOPT
1.0e-40	4.5e-30	41.90	1578	dFEAS	7.0e-39	293.62	2121	pdOPT
1.0e-50	3.8e-31	42.65	1600	dFEAS	6.8e-49	366.59	2653	pdOPT

Table 6. A solution of SDP (2) with relaxation order 6

y_1	+6.794272266270e-49	y_7	+1.872607837957e+249
y_2	+6.425924576302e+00	y_8	+1.256821138513e+299
y_3	+1.954800989153e+50	y_9	+9.047053385671e+348
y_4	+9.232905715919e+99	y_{10}	+7.170626187075e+398
y_5	+5.020811934847e+149	y_{11}	+6.750469860340e+448
y_6	+2.968009283344e+199	y_{12}	+1.125124049811e+499

(c) For $r = 5$ with tolerance more than 1.0e-30 and $r = 6$ with tolerance more than 1.0e-20, the number of iterations of precision 1000 are the same as that of precision 3000. This implies that the behaviors of the interior-point method of precision 1000 and 3000 are almost identical and the approximate optimal solutions derived by precision 1000 are reliable. For these relaxation orders and tolerances, we can expect that the SDPA-GMP finds the value and solution at the same number of iterations even though one increases precision by more than 1000.

(d) We consider the case that one wants to find a solution whose objective value $\tilde{p}_r^*$ is less than a given tolerance $\delta > 0$. Then Proposition 2 provides a lower bound of a necessary precision to obtain the accurate value and solution. Indeed, to obtain an approximate solution whose optimal value $\tilde{p}_r^* = y_1 < \delta$, y_{2r} must be greater than at least δ^{-2r+2} due to Proposition 2. In the case of $r = 6$ and δ =1.0e-48, we obtain the inequality $y_{12} \geq$ 1.0e+480. We see from Table 6 that y_{12} of the optimal solution by SDPA-GMP with tolerance ϵ =1.0e-50 and precision P =3000 satisfies this inequality. The interior-point method must deal with numbers having difference of more than 400 digits, and to accurately compute such numbers, we should set the precision P greater than at least 1200. (Recall that precision should be set aporoximately three times larger than digits needed.) In fact, SDPA-GMP with precision 3000 can compute approximate optimal solutions accurately, but precision 1000 cannot.

5. A bounded Example

Now we consider the following bounded POP:

$$\inf \left\{ x \;\middle|\; x \geq 0,\ x^2 \geq 1,\ x \leq a \right\} \tag{9}$$

where $a > 1$. The optimal value of (9) is 1. The SDP relaxation problem to (9) is:

$$\tilde{p}_r^* = \inf \left\{ y_1 \;\middle|\; (y_1, \dots, r_{2r}) \in \tilde{\mathcal{P}}_r \right\} \tag{10}$$

where $\tilde{\mathcal{P}}_r$ is the set satisfying (3), (4), (5), and

$$\begin{bmatrix} a-y_1 & ay_1-y_2 & \cdots & ay_{r-1}-y_r \\ ay_1-y_2 & ay_2-y_3 & \cdots & ay_r-y_{r+1} \\ \vdots & \vdots & \ddots & \vdots \\ ay_{r-1}-y_r & ay_r-y_{r+1} & \cdots & ay_{2r-2}-y_{2r-1} \end{bmatrix} \succeq O. \tag{11}$$

The dual SOS problem is:

$$\tilde{q}_r^* = \sup\left\{ p \;\middle|\; x-p = s_0(x)+xs_1(x)+(x^2-1)s_2(x)+(a-x)s_3(x),\ s_0\in \mathrm{SOS}_r, s_1,s_2,s_3\in\mathrm{SOS}_{r-1} \right\}. \tag{12}$$

Because the feasible region is bounded in (9), it is known that theoretically the optimal values of the SDP relaxation problems converge to 1, the optimal value of (9). See Theorem 2.1 of [5] and Theorem 4.2 of [6] for the details.

r	SeDuMi	SDPA-GMP
2	1.0047660e-02	1.0000000000000000e-02
3	9.3617005e-02	8.9832284318256281e-02
4	4.8495080e-01	2.4671417555058437e-01
5	1.0000000e+00	4.6654884910150963e-01
6	1.0000000e+00	7.1101762251342480e-01
7	1.0000000e+00	9.1325740792312672e-01

Table 7. SeDuMi vs. SDPA-GMP: $a = 100$

In all numerical experiments of this section, we set precision to be 1000 and tolerance to be 1.0e-50 for SDPA-GMP. This means that SDPA-GMP calculates floating point numbers with approximately 300 significant digits. The other parameters for SDPA-GMP is the same as the parameters in the previous section.

Table 7 compares SeDuMi and SDPA-GMP with $a = 100$ and various values of r. We see that that there exist some differences between the reported values of SeDuMi and SDPA-GMP. By SeDuMi, the optimal value of POP 1 is found at relaxation order 5, while SDPA-GMP returns strictly smaller values even for relaxation order 7.

Next we compare SeDuMi and SDPA-GMP with various a and the fixed relaxation order 3. Before showing the result, we derive the explicit formula for the optimal value of (12).

Proposition 4. *Let $a^* = 2+2\sqrt{2}$. The optimal value of* (10) *with relaxation order* 3 *is*

$$\tilde{p}_3^* = \begin{cases} 1 & \text{if } 1 < a \le a^*, \\ p(a) := \dfrac{(3a^2-4)^2}{a(a^4+16a^2-16)} & \text{if } a^* < a. \end{cases}$$

Proof : It follows from Theorem 4.2 of [6] or Corollary 4.7 of [15] that $\tilde{p}_r^* = \tilde{q}_r^*$ for every $r \ge 1$ and (12) is attained (i.e., an optimal solution exists) because (9) has an interior feasible point. In particular, we have $\tilde{p}_3^* = \tilde{q}_3^*$.

With $r = 3$, we have an SOS expression:

$$\begin{aligned} x - p(a) &= x\left(1-\frac{x}{a}\right)^2\left(\frac{a^2-4}{\sqrt{a^4+16a^2-16}} - \frac{4ax}{\sqrt{a^4+16a^2-16}}\right)^2 \\ &\quad +(x^2-1)\frac{1}{a}\left(\frac{4-3a^2}{\sqrt{a^4+16a^2-16}} + \frac{4ax}{\sqrt{a^4+16a^2-16}}\right)^2 \\ &\quad +(a-x)\left(\frac{x}{a}\right)^2\left(\frac{a^2-4}{\sqrt{a^4+16a^2-16}} - \frac{4ax}{\sqrt{a^4+16a^2-16}}\right)^2. \end{aligned} \tag{13}$$

This expression implies that $\tilde{q}_3^* \ge p(a)$ for all $a \ge 1$.

We observe that the optimal value of (12) is an nonincreasing function of a. Indeed, suppose that the SOS relation

$$x - p = \bar{s}_0 + x\bar{s}_1 + (x^2-1)\bar{s}_2 + (a-x)\bar{s}_3 \tag{14}$$

holds for optimal value p corresponding to a, where $\bar{s}_0$, $\bar{s}_1$, $\bar{s}_2$, and $\bar{s}_3$ are optimal SOSs. Then for $a' < a$, we have

$$x - p = \bar{s}_0 + (a - a')\bar{s}_3 + x\bar{s}_1 + (x^2 - 1)\bar{s}_2 + (a' - x)\bar{s}_3. \tag{15}$$

Now we see that p is feasible for (12) with $s_0 = \bar{s}_0 + (a - a')\bar{s}_3$, $s_1 = \bar{s}_1$, $s_2 = \bar{s}_2$, and $s_3 = \bar{s}_3$, thus p is a lower bound of the optimal value corresponding to a'.

Because $p(a^*) = 1$ and $\tilde{p}_3^* \leq 1$ due to the weak duality theorem, we have $p(a) = 1$ for $1 < a \leq a^*$.

The case where $a > a^*$ is proved in Appendix. □

Table 8 shows the values reported by SeDuMi and SDPA-GMP together with $\tilde{p}_3^*$. In the previous experiments, we could not identify which is wrong between SeDuMi or SDPA-GMP. However, now we can easily confirm that SDPA-GMP is reporting the correct optimal values on this problem, while SeDuMi is not.

a	SeDuMi	SDPA-GMP	$\tilde{p}_3^*$
2	1.0000000	1.000000000000000	1
3	1.0000000	1.000000000000000	1
4	1.0000000	1.000000000000000	1
5	0.99921172	0.999207135777998	0.999207135777998
6	0.97130304	0.971264367816092	0.971264367816092
7	0.92192366	0.921832033539197	0.921832033539197
⋮	⋮	⋮	⋮
10	0.75654357	0.756353591160221	0.756353591160221
100	0.093617005	0.089832284318256	0.089832284318256

Table 8. SeDuMi and SDPA-GMP vs. $\tilde{p}_3^*$: $r = 3$

6. Concluding Remarks

For simple one-dimensional POPs (1) and (9), we have investigated the correct optimal values of SDP relaxation problems and observed that SeDuMi and SDPA return the wrong values, while SDPA-GMP returns the correct optimal values. Moreover, we have seen that the wrong values coincide with the optimal value of (1) for a small relaxation order. Proposition 3 implies that the numerical error in the computation causes the convergence to the wrong value. From these observations, we conclude that for some SDP problems arising from POPs, users should be more careful of the results reported by the SDP solvers, and that the interior point method employing a multiple precision calculation is effective for SDP problems which is sensitive to the numerical errors.

This paper also makes us notice that an appropriate description of the feasible region of the given POP may provide a better lower bound or the exact value for the optimal value of the given POP. Indeed, the feasible region is $[1, +\infty)$. If we describe it as $\{x \in \mathbb{R} \mid x \geq 1\}$, then we can reformulate (1) into the following POP:

$$\inf \{ x \mid x \geq 1 \}. \tag{16}$$

This POP is equivalent with (1) in the sense that (16) has the same objective function and feasible region as (1). For POP (16), we can verify easily that the optimal values of SDP relaxation problems with all relaxation order $r \geq 1$ are 1 and they coincide with the optimal value of this POP. On the other hand, if we describe the feasible region as $\{x \in \mathbb{R} \mid x \geq 0, x^2 - 1 \geq 0\}$, then the optimal values of all SDP relaxation problems arising from (1) is not definitely 1, as we have already mentioned in Theorem 1.

From the above observation, one might think that the strange behavior arises only when the modeling to express the feasible region and/or the objective function is not appropriate, and such a case is pathological. However, we have another two-dimensional example:

$$\inf \left\{ x_2 \;\middle|\; x_1^4 + x_2^4 \geq 1,\ x_2 \geq x_1^2 \right\}.$$

In fact, we found the strange behavior for the first time on this problem appeared in [12]. For this problem, we see that SeDuMi reports positive optimal values of SDPs converging to the optimal value of POP, but can prove that the true optimal value of SDP is 0. The proof is combersome, and we omit it.

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Appendix

The purpose of Appendix is to prove the case of $a > a^*$ of Theorem 4. Because we have already known $\tilde{p}_3^* = \tilde{q}_3^* \geq p(a)$ for all $1 \leq a$ from (13), it is sufficient to prove the inequality $\tilde{p}_3^* \leq p(a)$ for all $a^* \leq a$. For this, we introduce an auxiliary SDP problem and its dual problem:

$$\zeta_3^* := \inf\left\{ y_1 \;\middle|\; \begin{array}{l} \boldsymbol{M}_1 := \begin{bmatrix} y_1 & y_2 & y_3 \\ y_2 & y_3 & y_4 \\ y_3 & y_4 & y_5 \end{bmatrix} \succeq O, \boldsymbol{M}_2 := \begin{bmatrix} y_2 - 1 & y_3 - y_1 \\ y_3 - y_1 & y_4 - y_2 \end{bmatrix} \succeq O, \\ \boldsymbol{M}_3 := \begin{bmatrix} a - y_1 & ay_1 - y_2 & ay_2 - y_3 \\ ay_1 - y_2 & ay_2 - y_3 & ay_3 - y_4 \\ ay_2 - y_3 & ay_3 - y_4 & ay_4 - y_5 \end{bmatrix} \succeq O. \end{array} \right\}. \tag{17}$$

$$\eta_3^* := \sup\left\{ p \;\middle|\; x - p = xs_1(x) + (x^2 - 1)s_2(x) + (a - x)s_3(x), s_1, s_3 \in \mathrm{SOS}_2, s_2 \in \mathrm{SOS}_1. \right\}. \tag{18}$$

Lemma 5. *We have $\tilde{p}_3^* = \tilde{q}_3^* = \zeta_3^* = \eta_3^* \geq p(a)$ for all $a > a^*$.*

Proof : We prove that $\tilde{p}_3^* = \tilde{q}_3^* \geq \zeta_3^* \geq \eta_3^* \geq p(a)$ for all $a > a^*$. We have already mentioned that $\tilde{p}_3^* = \tilde{q}_3^*$ at the proof of Proposition 4. Since (18) is the dual problem of (17), we have $\eta_3^* \leq \zeta_3^*$ from the weak duality theorem. We can prove $\zeta_3^* \leq \tilde{p}_3^*$ directly since $\boldsymbol{M}_2$ of (17) is the principal submatrix of (4) for (10) with relaxation order 3. In addition, from (13), we can see that $\eta_3^* \geq p(a)$.

We prove $\tilde{q}_3^* \leq \eta_3^*$. Indeed, for the identity of (12), comparing the coefficient of the monomial x^6, we have the equality $0 = s_{0,6} + s_{2,4}$, where $s_{0,6}$ and $s_{2,4}$ are coefficients of monomial x^6 in s_0 and monomial x^4 in s_2, respectively. The coefficients must be nonnegative because they are coefficients of the highest monomials in s_0 and s_2. This implies that we have $\deg(s_0) = 4$ and $\deg(s_2) = 2$ for each feasible solution (p, s_0, s_1, s_2, s_3) of (12) with relaxation order 3. Moreover, for each feasible solution (p, s_0, s_1, s_2, s_3) of (12) with relaxation order 3, $(p, (s_1 + s_0/a), s_2, (s_3 + s_0/a))$ is feasible for (18) with the same objective value because we have the identity

$$s_0 + xs_1 + (a - x)s_3 = x(s_1 + s_0/a) + (a - x)(s_3 + s_0/a)$$

and $\deg(s_1 + s_0/a) \leq 2, \deg(s_3 + s_0/a) \leq 2$. This implies $\tilde{q}_3^* \leq \eta_3^*$, and thus it follows from that $\tilde{p}_3^* = \tilde{q}_3^* = \zeta_3^* = \eta_3^* \geq p(a)$. □

From Lemma 5, the reminder of our task is to prove that $\zeta_3^* \leq p(a)$ for all $a > a^*$. For this, it is sufficient to make a feasible solution $(y_1, y_2, y_3, y_4, y_5) \in \mathbb{R}^5$ for SDP (17) with $y_1 = p(a)$. Indeed, the following solution is feasible for (17) with $y_1 = p(a)$:

$$\left\{\begin{array}{lcl} y_1 &=& p(a), \\ y_2 &=& \dfrac{(3a^2-4)(a^4-10a^2+8)}{a^2(a^4+16a^2-16)}, \\ y_3 &=& \dfrac{(3a^2-4)(a^6-19a^4+32a^2-16)}{2a^3(a^4+16a^2-16)}, \\ y_4 &=& \dfrac{(3a^2-4)(3a^8-71a^6+140a^4-144a^2+64)}{8a^4(a^4+16a^2-16)}, \\ y_5 &=& \dfrac{(3a^2-4)(11a^{10}-275a^8+552a^6-704a^4+640a^2-256)}{32a^5(a^4+16a^2-16)}. \end{array}\right. \tag{19}$$

In order to find the solution (19), we construct a polynomial system on $y_1, \ldots, y_5$ from some of the necessary conditions for the optimal solutions of (17) and (18), and solve the system by Symbolic Math Toolbox of MATLAB [11].

To check whether the solution (19) is feasible for (17), we prove that all principal minors for $\boldsymbol{M}_1, \boldsymbol{M}_2, \boldsymbol{M}_3$ are nonnegative. Let $\boldsymbol{M}_{i,I}$ be the principal submatrix for $I \subset \{1, 2, 3\}$ for $i = 1, 3$ or $I \subset \{1, 2\}$ for $i = 2$. Firstly, we focus on the matrix $\boldsymbol{M}_1$. We write 1×1 principal minors for $\boldsymbol{M}_1$:

$$\begin{array}{rcl} \det(\boldsymbol{M}_{1,\{1\}}) &=& \dfrac{(3a^2-4)^2}{a(a^4+16a^2-16)}, \\ \det(\boldsymbol{M}_{1,\{2\}}) &=& \dfrac{(3a^2-4)(a^6-19a^4+32a^2-16)}{2a^3(a^4+16a^2-16)} \\ &=& \dfrac{(3a^2-4)\left\{(a^2-4a-4)(a^4+4a^3+a^2+20a+116)+544a+448\right\}}{2a^3(a^4+16a^2-16)}, \\ \det(\boldsymbol{M}_{1,\{3\}}) &=& \dfrac{(3a^2-4)(11a^{10}-275a^8+552a^6-704a^4+640a^2-256)}{32a^5(a^4+16a^2-16)}. \end{array}$$

For $a > a^* = 2 + 2\sqrt{2}$, we have $3a^2 - 4 \geq 0$, $a^2 - 4a - 4 \geq 0$ and $a^4 + 16a^2 - 16 \geq 0$, and thus $\det(\boldsymbol{M}_{1,\{2\}})$ is nonnegative. In addition, we can decompose a part of the numerator of $\det(\boldsymbol{M}_{1,\{3\}})$ as follows:

$$\begin{array}{rcl} 11a^{10}-275a^8+552a^6-704a^4+640a^2-256 &=& (a^2-4a-4)(11a^8+44a^7-55a^6-44a^5 \\ && +156a^4+448a^3+1712a^2+8640a+42048) \\ && +202752a+167936 \\ &=& (a^2-4a-4)(11a^5((a-2)(a^2+6a+7)+10) \\ && +156a^4+448a^3+1712a^2+8640a+42048) \\ && +202752a+167936. \end{array}$$

From this decomposition and $a > 0$, we can see the nonnegativity of $\det(\boldsymbol{M}_{1,\{3\}})$. Next, we see the 2×2 principal minors.

$$\begin{aligned}
\det(\boldsymbol{M}_{1,\{1,2\}}) &= \frac{(3a^2-4)^2(a^2-1)(a^2+4)(a^2-4a-4)(a^2+4a-4)}{2a^4(a^4+16a^2-16)^2}, \\
\det(\boldsymbol{M}_{1,\{1,3\}}) &= \frac{(3a^2-4)^2(a^2-1)(a^2+4)(a^2-4a-4)(a^2+4a-4)(5a^2-4)^2}{32a^6(a^4+16a^2-16)^2}, \\
\det(\boldsymbol{M}_{1,\{2,3\}}) &= \frac{(3a^2-4)^2(a^2-1)(a^2+4)(a^2-4a-4)(a^2+4a-4)(a^2-4)^2}{32a^4(a^4+16a^2-16)^2}.
\end{aligned}$$

From $a > a^*$, the nonnegativity of all 2×2 principal minors easily follows. 3×3 principal minor $\det(\boldsymbol{M}_{1,\{1,2,3\}})$ is 0. Therefore $\boldsymbol{M}_1$ is positive semidefinite.

Next, we check the principal minors of $\boldsymbol{M}_2$. We have

$$\begin{aligned}
\det(\boldsymbol{M}_{2,\{1\}}) &= \frac{2(a^2-1)(a^2-4a-4)(a^2+4a-4)}{a^2(a^4+16a^2-16)}, \\
\det(\boldsymbol{M}_{2,\{2\}}) &= \frac{(3a^2-4)^2(a^2-1)(a^2-4a-4)(a^2+4a-4)}{8a^4(a^4+16a^2-16)}, \\
\det(\boldsymbol{M}_{2,\{1,2\}}) &= 0.
\end{aligned}$$

$\boldsymbol{M}_2$ is positive semidefinite because all principal minors are nonnegative provided $a > a^*$.

Finally, we check the principal minors of $\boldsymbol{M}_3$. We have

$$\begin{aligned}
\det(\boldsymbol{M}_{3,\{1\}}) &= \frac{(a^2-1)(a^2+4)^2}{a(a^4+16a^2-16)}, \\
\det(\boldsymbol{M}_{3,\{2\}}) &= \frac{(3a^2-4)(a^2-1)(a^2-4)(a^2+4)}{2a^3(a^4+16a^2-16)}, \\
\det(\boldsymbol{M}_{3,\{3\}}) &= \frac{(3a^2-4)(a^2-1)(a^2+4)(a^2-4)^3}{32a^5(a^4+16a^2-16)}, \\
\det(\boldsymbol{M}_{3,\{1,2\}}) &= \frac{(a^2-1)^2(a^2+4)^2(3a^2-4)(a^2-4a-4)(a^2+4a-4)}{2a^4(a^4+16a^2-16)^2}, \\
\det(\boldsymbol{M}_{3,\{1,3\}}) &= \frac{(a^2-1)^2(a^2+4)^2(3a^2-4)(a^2-4)^2(a^2-4a-4)(a^2+4a-4)}{32a^6(a^4+16a^2-16)^2}, \\
\det(\boldsymbol{M}_{3,\{2,3\}}) &= 0, \\
\det(\boldsymbol{M}_{3,\{1,2,3\}}) &= 0.
\end{aligned}$$

Aagin we can easily see the nonnegativity of all principal minors of $\boldsymbol{M}_3$ provided $a > a^*$, and thus $\boldsymbol{M}_3$ is positive semidefinite. We have completed proving that the solution (19) is feasible for (18), and that $\tilde{p}_3^* = \zeta_3^* = p(a)$ for all $a > a^*$.