

# Homogeneous Cone Complementarity Problems and $P$ Properties \*

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(April 9, 2009)

## Abstract

We consider existence and uniqueness properties of a solution to homogeneous cone complementarity problem (HCCP). Employing the  $T$ -algebraic characterization of homogeneous cones, we generalize the  $P, P_0, R_0$  properties for a nonlinear function associated with the standard nonlinear complementarity problem to the setting of HCCP. We prove that if a continuous function has either the order- $P_0$  and  $R_0$ , or the  $P_0$  and  $R_0$  properties then all the associated HCCPs have solutions. In particular, if a continuous function has the trace- $P$  property then the associated HCCP has a unique solution (if any); if it has the uniform-trace- $P$  property then the associated HCCP has the global uniqueness (of the solution) property (GUS). We present a necessary condition for a nonlinear transformation to have the GUS property. Moreover, we establish a global error bound for the HCCP with the uniform-trace- $P$  property. Finally, we study the HCCP with the relaxation transformation on a  $T$ -algebra and automorphism invariant properties for homogeneous cone linear complementarity problem.

**Keywords:** Homogeneous cone complementarity problem,  $P$  property, existence of a solution, globally uniquely solvability property, error bound.

**AMS Subject Classification:** 26B05, 65K05, 90C33

## 1 Introduction

In this paper, we are interested in the *homogeneous cone complementarity problem* (HCCP( $F, q$ ) for short) which is to find a vector  $x \in K$  such that

$$x \in K, y \in K^*, \langle x, y \rangle = 0, y = F(x) + q, \quad (1)$$

where  $K$  is a homogeneous cone (the automorphism group of the cone acts transitively on the interior of the cone, see Section 2 for details) in a finite-dimensional inner product space  $\mathbb{H}$  over  $\mathbb{R}$  with its dual  $K^*$  given by  $K^* := \{y \in \mathbb{H} : \langle x, y \rangle \geq 0, \forall x \in K\}$ ,  $F : \mathbb{H} \rightarrow \mathbb{H}$  is a continuous function and  $q \in \mathbb{H}$ . If  $F(x) = L(x)$  is linear, we call problem (1) the *homogeneous cone*

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\*The work was supported in part by the National Natural Science Foundation of China (10831006, 10671010), and a Discovery Grant from NSERC.

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*linear complementarity problem* (HCLCP( $L, q$ )). When  $K$  is a symmetric cone in a Euclidean Jordan algebra, it is the *symmetric cone complementarity problem* (SCCP), which includes the so-called *nonlinear complementarity problem* (NCP, where  $\mathbb{H} = \mathbb{R}^n$ , the space of  $n$ -dimensional real column vectors, and  $K = \mathbb{R}_+^n$ , the nonnegative orthant) and *semidefinite complementarity problem* (SDCP, where  $\mathbb{H} = \mathbb{S}^n$ , the space of  $n \times n$  real symmetric matrices, and  $K = \mathbb{S}_+^n$ , the cone of positive semidefinite symmetric matrices) as special cases.

In the NCP context, a continuous function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is said to be a *P function* (has the *P property*) if the following implication holds

$$(x - y) \circ [f(x) - f(y)] \leq 0 \quad \Rightarrow \quad x = y,$$

where “ $\circ$ ” denotes the Hadamard (componentwise) product and  $z \leq 0$  means that all components of  $z$  are nonpositive. There are many applications of *P* functions in engineering, economics, management science, and other fields, see, e.g., [7, 25]. We say  $f$  is a  $P_0$  function if  $f + \varepsilon I$  is a *P* function for any  $\varepsilon > 0$  where  $I$  is the identity transformation. This is a generalization of *P*-matrices. It is known that if  $f$  is a  $P_0$  function and satisfies the so-called  $R_0$  condition then the NCP( $f, q$ ) has a solution for every  $q \in \mathbb{R}^n$ . (Here, we only consider the  $R_0$  condition as described in Definition 3.2 below.) In the setting of a Cartesian product of sets in  $\mathbb{R}^n$ , Facchinei and Pang [7] gave *P* and  $P_0$  functions and studied some of their properties. In the setting of a Euclidean Jordan algebra, Gowda, Sznajder and Tao [14] studied some *P* and  $P_0$  properties for linear transformations; Tao and Gowda [26] introduced *P* and  $P_0$  functions and established the existence result for SCCP. Moreover, Gowda and Sznajder [13] studied the automorphism invariance of *P* and *globally uniquely solvability* (GUS) properties for linear transformations on Euclidean Jordan algebras. Symmetric cones are *homogeneous* and *self-dual*, see [8, 21]. A natural next step in generalization is to drop the requirement that  $K$  is self-dual. While there is a finite number of non-isomorphic symmetric cones of each dimension, the number is uncountable for homogeneous cones when the dimension  $n > 11$ , see [29]. There has been increasing interest and activity in the area of homogeneous cones and optimization problems over homogeneous cones, see, e.g., [1, 3, 4, 5, 6, 9, 15, 16, 20, 23, 24, 27, 28, 29, 30]. These papers deal with either certain theoretical properties of homogeneous cones, primal-dual interior-point methods for linear programming over homogeneous cones or their applications. In this paper, we work on the *P* properties for nonlinear transformations in the setting of HCCP. The aim of our work is to establish the existence and uniqueness results of a solution to HCCP.

With the help of the  $T$ -algebraic characterization of homogeneous cones, we first study the *metric projection* onto homogeneous cone  $K$  and its properties related to HCCP. Based on them, we introduce *P*, order-*P*, trace-*P*, uniform-trace-*P*, trace- $P_0$ , order- $P_0$ ,  $P_0$  and  $R_0$  properties for a function  $F : \mathbb{H} \rightarrow \mathbb{H}$  in the setting of HCCP. Then, we show that if  $F$  has either the order- $P_0$  and  $R_0$ , or the  $P_0$  and  $R_0$  properties then the HCCP( $F, q$ ) has a solution for every  $q \in \mathbb{H}$  by applying the degree theory; if  $F$  has the trace-*P* property then the associated HCCP( $F, q$ ) has a unique solution (if not empty); if  $F$  has the uniform-trace-*P* property then the associated HCCP( $F, q$ ) has GUS property. Moreover, we establish a global error bound for the HCCP with  $F$  having the uniform-trace-*P* property. We also present a necessary condition for a transformation to have the GUS property. Finally, we apply our results to the HCCP with  $F$  specified by the *relaxation transformation*  $R_f$  on a  $T$ -algebra that is induced by a vector valued function  $f$ ; in particular, we show the equivalent relationships between *P* ( $P_0$ , respectively) property of  $f$  and various *P* ( $P_0$ , respectively) properties of  $R_f$ . We apply our main results to the associated HCCP( $R_f, q$ ) and show that it has a bounded solution set if  $f$  has the  $P_0$  and  $R_0$  properties. Furthermore, in the context of HCLCP we show that  $R_0, Q$ , GUS and Lipschitzian GUS, ultra-*P* and ultra-GUS properties are automorphism invariant.

This paper is organized as follows. In Section 2, we briefly review some basic concepts and results on  $T$ -algebras, and describe some fundamental results on metric projection onto homogenous cones. In Section 3, we introduce the various  $P$  properties and show our existence result for HCCP. In Section 4, we study the GUS property and give an error bound for HCCP. In Section 5, as applications, firstly, we introduce the relaxation transformation, and study its monotonicity, various  $P$  and  $P_0$  properties; secondly, we consider the automorphism invariant properties for HCLCP. In Section 6, we include some concluding remarks. While most of our focus is on Homogeneous cones, many of our results apply more generally (in the setting of arbitrary convex cones).

## 2 Preliminaries

We first briefly review some basic concepts and results on homogeneous cones and  $T$ -algebras from [4, 29, 30], and then provide some fundamental results on metric projection onto homogeneous cones.

### 2.1 Homogeneous cones and $T$ -algebras

**Definition 2.1** *A closed, convex cone  $K$  with nonempty interior is homogeneous if the group of automorphisms of  $K$  acts transitively on the interior of  $K$ .*

Note that a cone  $K$  is homogeneous then so is its dual  $K^*$ . Vinberg [29] introduced a constructive way to build homogeneous cones by employing the so-called  $T$ -algebra which connects homogeneous cones to abstract matrices whose elements are vectors. We first review the following concept of *matrix algebra*.

**Definition 2.2** *A matrix algebra  $\mathcal{A}$  is a bi-graded algebra  $\bigoplus_{i,j=1}^r \mathcal{A}_{ij}$  over the reals with a bilinear product of  $a_{ij} \in \mathcal{A}_{ij}$  and  $a_{kl} \in \mathcal{A}_{kl}$  ( $1 \leq i, j, k, l \leq r$ ) satisfying*

$$a_{ij}a_{kl} \in \begin{cases} \mathcal{A}_{il} & \text{if } j = k, \\ 0 & \text{if } j \neq k, \end{cases}$$

where  $\mathcal{A}_{ij}$  is a  $n_{ij}$ -dimensional vector space. The positive integer  $r$  is called the rank of the matrix algebra  $\mathcal{A}$ .

Every element  $a \in \mathcal{A}$  is a generalized matrix with its component in  $\mathcal{A}_{ij}$  being an  $n_{ij}$ -dimensional generalized element of the matrix  $a_{ij}$ , i.e.,  $a_{ij}$  is the projection of  $a$  onto  $\mathcal{A}_{ij}$ . The multiplication of two elements  $a, b \in \mathcal{A}$  is analogous to the multiplication of matrices,

$$(ab)_{ij} = \sum_{k=1}^r a_{ik}b_{kj}.$$

Assume that for every  $i$ ,  $\mathcal{A}_{ii}$  is isomorphic to  $\mathbb{R}$ , and let  $\rho_i$  be the isomorphism and let  $e_i$  denote the representation of the *unit element* of  $\mathcal{A}_{ii}$  in  $\mathcal{A}$ . We define the *trace* of an element  $a$  as

$$\text{Tr}(a) = \sum_{i=1}^r \rho_i(a_{ii}).$$

The following notion generalizes the classical (conjugate) transpose. An *involution*  $*$  of the matrix algebra  $\mathcal{A}$  of rank  $r$  is a linear automorphism on  $\mathcal{A}$  that satisfies

- (i)  $(a^*)^* = a$  for all  $a \in \mathcal{A}$  (involutory).
- (ii)  $(ab)^* = b^*a^*$  for all  $a, b \in \mathcal{A}$  (anti-homomorphic).
- (iii)  $(a^*)_{ij} = (a_{ij})^*$  for  $1 \leq i, j \leq r$ .
- (iv)  $\mathcal{A}_{ij}^* = \mathcal{A}_{ji}$  for  $1 \leq i, j \leq r$ .

We are ready to state the following definition of  $T$ -algebra, which was originally introduced by Vinberg [29].

**Definition 2.3** *A  $T$ -algebra of rank  $r$  is a matrix algebra  $\mathcal{A}$  of rank  $r$  with involution  $(*)$  satisfying the following axioms:*

- I. For each  $1 \leq i \leq r$ , the subalgebra  $\mathcal{A}_{ii}$  is isomorphic to the reals.
- II. For each  $a \in \mathcal{A}$  and each  $1 \leq i, j \leq r$ ,

$$a_{ji}e_i = a_{ji} \quad \text{and} \quad e_i a_{ij} = a_{ij}.$$

- III. For each  $a, b \in \mathcal{A}$ ,  $\text{Tr}(ab) = \text{Tr}(ba)$ .
- IV. For each  $a, b, c \in \mathcal{A}$ ,  $\text{Tr}((ab)c) = \text{Tr}(a(bc))$ .
- V. For each  $a \in \mathcal{A}$ ,  $\text{Tr}(a^*a) \geq 0$ , with equality if and only if  $a = 0$ .
- VI. For each  $a, b, c \in \mathcal{A}$  and each  $i, j, k, l \in \{1, 2, \dots, r\}$  with  $i \leq j \leq k \leq l$ ,

$$a_{ij}(b_{jk}c_{kl}) = (a_{ij}b_{jk})c_{kl}.$$

- VII. For each  $a, b \in \mathcal{A}$  and each  $i, j, k, l \in \{1, 2, \dots, r\}$  with  $i \leq j \leq k$  and  $l \leq k$ ,

$$a_{ij}(b_{jk}b_{lk}^*) = (a_{ij}b_{jk})b_{lk}^*.$$

Thus, the cone associated with a  $T$ -algebra  $\mathcal{A}$  of rank  $r$ , denoted by  $\text{int}(K(\mathcal{A}))$ , is given by

$$\text{int}(K(\mathcal{A})) := \{tt^* : t \in \mathcal{A}, t_{ij} = 0 \forall 1 \leq j < i \leq r, \text{ and } t_{ii} > 0 \forall 1 \leq i \leq r\}.$$

It is easy to see from Axiom II that  $e := \sum_{i=1}^r e_i$  is the unit element in  $\mathcal{A}$ , i.e.,  $ea = ae$  for all  $a \in \mathcal{A}$ . It is necessary to note that multiplication in a  $T$ -algebra is neither commutative nor associative. Define the subalgebra of upper triangular elements of  $\mathcal{A}$  and the subspace of ‘‘Hermitian’’ elements, respectively, as

$$\mathcal{T} := \bigoplus_{i \leq j, i, j=1}^r \mathcal{A}_{ij}, \quad \mathcal{H} := \{a \in \mathcal{A} : a = a^*\}.$$

Clearly, Axiom VI is equivalent to  $t(uw) = (tu)w$  for all  $t, u, w \in \mathcal{T}$ . Taking involution, we get another equivalent statement:  $t(uw) = (tu)w$  for all  $t, u, w \in \mathcal{T}^*$ . Similarly, Axiom VII is equivalent to  $t(uu^*) = (tu)u^*$  for all  $t, u \in \mathcal{T}$ ; or equivalently,  $(u^*u)t = u^*(ut)$  for all  $t, u \in \mathcal{T}^*$ .

Let  $\mathcal{T}_*$  (resp.,  $\mathcal{T}_+$  and  $\mathcal{T}_{++}$ ) denote the set of elements of  $\mathcal{T}$  with nonzero (resp., nonnegative and positive) diagonal components. It is easy to see that  $\text{int}(K(\mathcal{A})) = \{tt^* : t \in \mathcal{T}_{++}\}$  and  $\mathcal{H}$  is the linear span of  $\text{int}(K(\mathcal{A}))$ .

We recall below a fundamental characterization of homogeneous cones established by Vinberg [29].

**Theorem 2.4** *( $T$ -algebraic representation of homogeneous cones) A cone  $K$  is homogeneous if and only if  $\text{int}(K)$  is isomorphic to the cone  $\text{int}(K(\mathcal{A}))$  associated with some  $T$ -algebra  $\mathcal{A}$ . Moreover, given  $\text{int}(K(\mathcal{A}))$ , the representation of an element from  $\text{int}(K)$  in the form  $tt^*$  is unique. Finally, the dual cone  $\text{int}(K^*)$  can be represented as  $\{t^*t : t \in \mathcal{T}_{++}\}$ .*

**Remark 2.1** This theorem is analogous to the representation of a symmetric cone as the set of squares over a Jordan algebra. However, for  $T$ -algebra and its associated cone it is not true that for a given  $a \in \mathcal{A}$  we have  $aa^* \in \text{int}(K(\mathcal{A}))$ . The positivity of the diagonal elements  $t \in \mathcal{T}_{++}$  is only required to ensure uniqueness. In general, the closures of the cone  $\text{int}(K(\mathcal{A}))$  and its dual are specified by

$$K(\mathcal{A}) = \{tt^* : t \in \mathcal{T}_+\}, \quad K^*(\mathcal{A}) = \{t^*t : t \in \mathcal{T}_+\}. \quad (2)$$

Thus,  $K(\mathcal{A})$  is a closed convex cone in the inner product space  $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ , where  $\langle \cdot, \cdot \rangle$  is given by  $\langle a, b \rangle = \text{Tr}(a^*b) = \text{Tr}(ab)$  for all  $a, b \in \mathbb{H}$ . It is worth noting that for all  $a, b, c \in \mathcal{A}$ ,

$$\langle ab, c \rangle = \langle b^*a^*, c^* \rangle = \langle a, cb^* \rangle = \langle b, a^*c \rangle. \quad (3)$$

We define the *norm* induced by the inner product as  $\|a\| := \sqrt{\langle a, a \rangle}$ . In what follows, we may simply write  $K$  and  $\mathbb{H}$  for  $K(\mathcal{A})$  and  $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ , respectively.

**Example 2.1** Consider the following five-dimensional closed convex cone with nonempty interior (Vinberg [29]):

$$K := \left\{ x \in \mathbb{R}^5 : \begin{pmatrix} x_1 & x_2 & x_4 \\ x_2 & x_3 & 0 \\ x_4 & 0 & x_5 \end{pmatrix} \in \mathbb{S}_+^3 \right\}.$$

This cone is homogeneous; but, it is not a symmetric cone since there does not exist any inner product on  $\mathbb{R}^5$  under which  $K = K^*$ . Let us choose the inner product implied by the trace inner product on  $\mathbb{S}^3$ ; that is,  $\forall x, y \in \mathbb{R}^5$ ,

$$\langle x, y \rangle := x_1y_1 + x_3y_3 + x_5y_5 + 2x_2y_2 + 2x_4y_4.$$

Then,

$$K^* = \left\{ y \in \mathbb{R}^5 : \begin{pmatrix} y_1 & y_2 \\ y_2 & y_3 \end{pmatrix} \in \mathbb{S}_+^2, \begin{pmatrix} y_1 & y_4 \\ y_4 & y_5 \end{pmatrix} \in \mathbb{S}_+^2 \right\}.$$

With this (natural) choice of the inner product,  $K \subseteq K^*$ . Moreover, it is straightforward to verify that  $a^2 \in K + K^*$  for all  $a \in \mathbb{H}$ .

**Remark 2.2** Note that every  $z \in \mathbb{H}$  may be rewritten as  $z = t + t^*$  with  $t \in \mathcal{T}$  and  $\langle z^2, e_i \rangle \geq 0$  for all  $i \in \{1, 2, \dots, r\}$ . Then  $z^2 = tt^* + t^*t + t^2 + (t^*)^2$ . Motivated by the fact that a symmetric cone is the set of squares over a Jordan algebra, we propose the following question:

for every homogeneous cone  $K$ , does there exist  
an inner product on  $\mathbb{H}$  such that  $z^2 \in K + K^*, \forall z \in \mathbb{H}$ ?

As we stated in the Introduction, we are generalizing the underlying theorems and in many cases their existing proofs from the symmetric cone setting to the more general, homogeneous cone setting. The existing results for the symmetric cone setting essentially fix an inner product under which  $K = K^*$  and treat both  $K$  and  $K^*$  in the same finite dimensional Euclidean space. This allows operations like  $x + y$  for  $x \in K, y \in K^*$  and the related metric projections onto the cones  $K, K^*$  to be treated in the same space. Since much of the related theory is based on metric projections, one is required to fix an inner product in our more general setting as well. We remind the reader that the choice of the inner product is up to the goals of the user of the theory (for example, to recover the existing results for the special case of symmetric cones, we would pick the inner product so that  $K = K^*$ ).

## 2.2 Metric Projection

Let  $\Pi_K(x)$  denote the *metric projection* of  $x$  onto  $K$ , i.e.,

$$\Pi_K(x) := \operatorname{argmin} \left\{ \frac{1}{2} \|x - z\|^2 : z \in K \right\}.$$

In other words,  $y = \Pi_K(x)$  if and only if  $y \in K$  and

$$\|x - y\| \leq \|x - z\|, \quad \forall z \in K,$$

or equivalently, the so-called obtuse angle property (or the Kolmogorov criterion) holds:

$$\langle z - \Pi_K(x), x - \Pi_K(x) \rangle \leq 0, \quad \forall z \in K.$$

It is well-known [31] that the metric projector  $\Pi_K$  is unique and contractive, i.e.,

$$\|\Pi_K(x) - \Pi_K(y)\| \leq \|x - y\|, \quad \forall x, y \in \mathbb{H}.$$

Utilizing the Moreau decomposition, any  $x \in \mathbb{H}$  can be written as

$$x = \Pi_K(x) - \Pi_{K^*}(-x) \quad \text{with} \quad \langle \Pi_K(x), \Pi_{K^*}(-x) \rangle = 0. \quad (4)$$

Based on the metric projection operator, we define the following operations for any  $x, y \in \mathbb{H}$ ,

$$x \wedge_K y := x - \Pi_K(x - y), \quad x \vee_K y := y + \Pi_K(x - y). \quad (5)$$

Then, by direct calculation, we obtain

$$x \wedge_K y = y \wedge_{K^*} x, \quad (-x) \wedge_K (-y) = -(x \vee_{K^*} y). \quad (6)$$

Summarizing the above arguments, we have the following proposition.

**Proposition 2.5** *Let  $K$  be a closed convex cone in  $\mathbb{H}$  with its dual  $K^*$ . Then the following statements hold for all  $x, y \in \mathbb{H}$ .*

(a) *We have  $x = \Pi_K(x) - \Pi_{K^*}(-x)$  with  $\langle \Pi_K(x), \Pi_{K^*}(-x) \rangle = 0$ . This decomposition is unique in the sense that if  $x = x_1 - x_2$  with  $x_1 \in K, x_2 \in K^*$  and  $\langle x_1, x_2 \rangle = 0$  then  $x_1 = \Pi_K(x)$  and  $x_2 = \Pi_{K^*}(-x)$ .*

(b)  *$x \wedge_K y = y \wedge_{K^*} x, \quad x \vee_K y = y \vee_{K^*} x$ .*

(c)  *$(-x) \wedge_K (-y) = -(x \vee_{K^*} y)$  and  $(-x) \wedge_{K^*} (-y) = -(x \vee_K y)$ .*

*In particular,  $\Pi_K(x) \wedge_K \Pi_{K^*}(-x) = 0$ , and  $\Pi_K(x) \vee_K \Pi_{K^*}(-x) = \Pi_K(x) + \Pi_{K^*}(-x)$ .*

Considering the characterization of homogeneous cones, from (4) and (2), we obtain that any  $x \in \mathbb{H}$  can be expressed as

$$x = uu^* - v^*v \quad \text{with} \quad \langle uu^*, v^*v \rangle = 0,$$

where  $u, v \in \mathcal{T}_+$ . Observe that

$$\begin{aligned} \langle uu^*, v^*v \rangle &= \langle (uu^*)v^*, v^* \rangle \\ &= \langle u(u^*v^*), v^* \rangle \\ &= \langle (vu)^*, u^*v^* \rangle \\ &= \langle (vu)^*, (vu)^* \rangle \\ &= \|vu\|^2, \end{aligned}$$

where the first equality holds by (3), the second equality holds by Axiom VII, the third holds by (3) and the fact that  $*$  is anti-homomorphic. Then, Axiom V implies that  $\langle uu^*, v^*v \rangle = 0$  if and only if  $vu = 0$ . We have actually proved the following.

**Theorem 2.6** *Let  $K$  be a homogeneous cone in  $\mathbb{H}$  with its dual  $K^*$ . Every  $x \in \mathbb{H}$  can be uniquely expressed as*

$$x = uu^* - v^*v \quad \text{with} \quad vu = 0, \quad u, v \in \mathcal{T}_+. \quad (7)$$

Moreover, we have  $\Pi_K(x) = uu^*$ ,  $\Pi_{K^*}(-x) = v^*v$ .

Applying Proposition 2.5 and the above theorem, we obtain the the following equivalent statements related to HCCP (1).

**Proposition 2.7** *Let  $K$  be a homogeneous cone in  $\mathbb{H}$  with its dual  $K^*$ . Then the following statements are equivalent.*

- (a)  $x \wedge_K y = 0$ .
- (b)  $y \wedge_{K^*} x = 0$ .
- (c)  $x \in K$ ,  $y \in K^*$ ,  $\langle x, y \rangle = 0$ .
- (d)  $x \in K$ ,  $y \in K^*$ ,  $\langle xy, e_i \rangle = \langle yx, e_i \rangle = 0$ ,  $\forall i \in \{1, 2, \dots, r\}$ .
- (e) There exist  $u, v \in \mathcal{T}_+$  such that  $x = uu^*$ ,  $y = v^*v$ ,  $vu = 0$ .
- (f)  $x \in K$ ,  $y \in K^*$ ,  $(xy)_{lj} = 0$ ,  $\forall l, j \in \{1, 2, \dots, r\}$  such that  $l \geq j$ .

In particular, if  $xy = yx$ , then (f) becomes

$$(f') \quad x \in K, \quad y \in K^*, \quad xy = 0.$$

**Proof.** By Proposition 2.5 and Theorem 2.6, (a)  $\Leftrightarrow$  (b)  $\Leftrightarrow$  (c)  $\Leftrightarrow$  (e). Clearly, (f)  $\Rightarrow$  (d)  $\Rightarrow$  (c). Therefore, we need only to show that (e)  $\Rightarrow$  (f). Choose any  $w^* \in \mathcal{T}^*$ . Applying arguments similar to those before Theorem 2.6, we obtain

$$\begin{aligned} \langle xy, w^* \rangle &= \langle (uu^*)(v^*v), w^* \rangle \\ &= \langle uu^*, w^*(v^*v) \rangle \\ &= \langle uu^*, (w^*v^*)v \rangle \\ &= \langle (uu^*)v^*, w^*v^* \rangle \\ &= \langle u(vu)^*, w^*v^* \rangle. \end{aligned}$$

Then, the desired conclusion follows. □

As a direct application of Theorem 2.6, we obtain the following lemma which is useful in the subsequent analysis.

**Lemma 2.8** *For every  $i, j \in \{1, 2, \dots, r\}$  such that  $i \neq j$ , let  $a_{ij} \in \mathcal{A}_{ij}$  and  $a_{ji} \in \mathcal{A}_{ji}$  be given with  $a_{ij} + a_{ji} \in \mathbb{H}$ . Then, for every large positive scalar  $\lambda$ ,*

$$e_i + \lambda e_j + a_{ij} + a_{ji} \in K \cap K^*.$$

**Proof.** We first show that  $e_i + \lambda e_j + a_{ij} + a_{ji} \in K^*$ . Choose any  $x \in K$ . By (2), we may rewrite  $x = uu^*$  with  $u \in \mathcal{T}_+$ . Let  $u = \sum_{k \leq l} u_{kl}$ . Thus,

$$\langle x, e_i + \lambda e_j + a_{ij} + a_{ji} \rangle = \langle uu^*, e_i \rangle + \lambda \langle uu^*, e_j \rangle + \langle uu^*, a_{ij} \rangle + \langle uu^*, a_{ji} \rangle. \quad (8)$$

Note that

$$\begin{aligned} \langle uu^*, e_j \rangle &= \langle u, e_j u \rangle \\ &= \left\langle \sum_{k \leq l} u_{kl}, e_j \sum_{k \leq l} u_{kl} \right\rangle \\ &= \left\langle \sum_{k \leq l} u_{kl}, \sum_{k=1}^j u_{kj} + \sum_{l=j+1}^r u_{kj} \right\rangle \\ &= \sum_{k=1}^j \|u_{kj}\|^2 + \sum_{l=j+1}^r \|u_{kj}\|^2 \geq 0, \end{aligned} \quad (9)$$

where the first equality follows from  $\langle ab, c \rangle = \langle a, cb^* \rangle$  for  $a, b, c \in \mathcal{A}$ , the second one follows from the definition of  $T$ -algebra. Similarly,

$$\langle uu^*, e_i \rangle \geq 0. \quad (10)$$

Thus, if  $\langle uu^*, e_j \rangle \neq 0$ , we have  $\langle uu^*, e_j \rangle > 0$  and the desired conclusion holds by (8). If  $\langle uu^*, e_j \rangle = 0$ , then  $\sum_{k=1}^j u_{kj} + \sum_{l=j+1}^r u_{kj} = 0$ . Therefore, direct calculation yields

$$\langle uu^*, a_{ij} \rangle = \langle u, a_{ij} u \rangle = \left\langle \sum_{k \leq l} u_{kl}, a_{ij} \sum_{k \leq l} u_{kl} \right\rangle = \left\langle \sum_{l=i}^r u_{il}, a_{ij} \sum_{l=j}^r u_{jl} \right\rangle = 0,$$

and

$$\langle uu^*, a_{ji} \rangle = \langle u, a_{ji} u \rangle = \left\langle \sum_{k \leq l} u_{kl}, a_{ji} \sum_{k \leq l} u_{kl} \right\rangle = \left\langle \sum_{l=j}^r u_{jl}, a_{ji} \sum_{l=i}^r u_{il} \right\rangle = 0.$$

These together with (10) lead to  $\langle x, e_i + \lambda e_j + a_{ij} + a_{ji} \rangle = \langle uu^*, e_i \rangle \geq 0$ . Hence, we proved that for every  $\lambda > 0$  large enough,  $\langle x, e_i + \lambda e_j + a_{ij} + a_{ji} \rangle \geq 0$  for every  $x \in K$ . That is, for every  $\lambda > 0$  large enough,

$$e_i + \lambda e_j + a_{ij} + a_{ji} \in K^*. \quad (11)$$

We next show that  $e_i + \lambda e_j + a_{ij} + a_{ji} \in K$ . Take any  $x \in K^*$ . By (2),  $x = v^*v$  with  $v \in \mathcal{T}_+$ . Let  $v = \sum_{k \leq l} v_{kl}$ . Applying the same arguments as above, we conclude that for every  $\lambda > 0$  large enough,  $\langle x, e_i + \lambda e_j + a_{ij} + a_{ji} \rangle \geq 0$  for every  $x \in K^*$ . The desired conclusion follows.  $\square$

We next address the following result which will be used to establish the connection among various  $P$ -properties in this paper.

**Proposition 2.9** *Let  $K$  be a homogeneous cone in  $\mathbb{H}$  with its dual  $K^*$ . For every  $x, y \in \mathbb{H}$ , the following statements hold:*

- (i) if  $x \in K$ ,  $y \in K^*$ , then  $\langle xy, e_i \rangle \geq 0$ ,  $\forall i \in \{1, 2, \dots, r\}$ ;

$$(ii) \langle (x \wedge_K y)(x \vee_K y), e_i \rangle = \langle xy, e_i \rangle, \quad \forall i \in \{1, 2, \dots, r\}.$$

Moreover, if  $x \wedge_K y \in -(K \cap K^*)$ ,  $x \vee_K y \in K + K^*$ , then  $\langle xy, e_i \rangle \leq 0$ ,  $\forall i \in \{1, 2, \dots, r\}$ . Furthermore, in this case,  $\langle x, y \rangle = \sum_{i=1}^r \langle xy, e_i \rangle \leq 0$ .

**Proof.** Note that  $e_i^2 = e_i$  and  $e_i x = x e_i$  for all  $x \in \mathcal{A}$  and  $i \in \{1, 2, \dots, r\}$ . We first prove (i). By Theorem 2.6, if  $x \in K$ ,  $y \in K^*$ , then there exist  $u, v \in \mathcal{T}_+$  such that  $x = uu^*$ ,  $y = v^*v$ . Thus, by (3) and the similar arguments before Theorem 2.6,

$$\begin{aligned} \langle xy, e_i \rangle &= \langle (uu^*)(v^*v), e_i \rangle \\ &= \langle uu^*, e_i(v^*v) \rangle \\ &= \langle uu^*, (v^*v)e_i \rangle \\ &= \langle uu^*, v^*(ve_i) \rangle \\ &= \langle v(uu^*), ve_i \rangle \\ &= \langle (vu)u^*, e_i v \rangle \\ &= \langle vu, (e_i v)u \rangle \\ &= \langle vu, e_i(vu) \rangle \\ &= \|e_i(vu)\|^2 \geq 0. \end{aligned}$$

Therefore, the desired conclusion (i) holds.

For part (ii), direct calculation yields

$$\begin{aligned} \langle (x \wedge_K y)(x \vee_K y), e_i \rangle &= \langle [x - \Pi_K(x - y)][y + \Pi_K(x - y)], e_i \rangle \\ &= \langle xy, e_i \rangle + \langle x\Pi_K(x - y), e_i \rangle - \langle \Pi_K(x - y)y, e_i \rangle - \langle \Pi_K(x - y)\Pi_K(x - y), e_i \rangle \\ &= \langle xy, e_i \rangle + \langle x, \Pi_K(x - y)e_i \rangle - \langle y, \Pi_K(x - y)e_i \rangle - \langle \Pi_K(x - y), \Pi_K(x - y)e_i \rangle \\ &= \langle xy, e_i \rangle + \langle (x - y) - \Pi_K(x - y), \Pi_K(x - y)e_i \rangle \\ &= \langle xy, e_i \rangle, \end{aligned}$$

where the last equality follows from the fact  $\langle (x - y) - \Pi_K(x - y), \Pi_K(x - y)e_i \rangle = \langle \Pi_{K^*}(y - x)\Pi_K(x - y), e_i \rangle = 0$  by Proposition 2.7. Thus, we proved (ii).  $\square$

We end this section with the following property of the elements in  $K$ .

**Proposition 2.10** *Let  $K$  be a homogeneous cone in  $\mathbb{H}$  and  $x \in K$  with  $x = \sum_{i,j=1}^r x_{ij}$ . If  $x_{kk} = 0$  for some  $k \in \{1, 2, \dots, r\}$ , then*

$$\sum_{j=1}^r x_{kj} + \sum_{i=1}^r x_{ik} = 0.$$

*In particular, if  $x \in K \cap \left(\bigoplus_{i \neq j} \mathcal{A}_{ij}\right)$ , then  $x = 0$ .*

**Proof.** Since  $x \in K$ , there exists  $u \in \mathcal{T}_+$  such that  $x = uu^*$ . Set  $u = \sum_{i,j=1}^r u_{ij}$  with  $u_{ij} = 0$  for every  $i, j \in \{1, 2, \dots, r\}$  such that  $i < j$ . Direct calculation yields

$$x_{kk} = \sum_{j=k}^r u_{kj}u_{kj}^*.$$

Since  $\langle e_k, u_{kj}u_{kj}^* \rangle = \langle u_{kj}, u_{kj} \rangle = \|u_{kj}\|^2$ , by the assumption  $x_{kk} = 0$ , we obtain

$$0 = \langle e_k, x_{kk} \rangle = \sum_{j=k}^r \|u_{kj}\|^2.$$

This along with  $u \in \mathcal{T}_+$  leads to  $u_{kl} = 0$  for every  $l \in \{1, 2, \dots, r\}$ . Therefore, we obtain

$$x_{ik} = \sum_{l=1}^r u_{il}u_{kl}^* = 0, \quad x_{kj} = \sum_{l=1}^r u_{kl}u_{jl}^* = 0, \quad \forall i, j \in \{1, 2, \dots, r\},$$

as desired.  $\square$

### 3 $P$ and $R_0$ properties

We first give the definitions of various  $P(P_0)$  and  $R_0$  properties.

**Definition 3.1** For a continuous function  $F : \mathbb{H} \rightarrow \mathbb{H}$ , we say that it has

(i) the order- $P$  property if for any pair  $x, y \in \mathbb{H}$ ,

$$(x - y) \wedge_K (F(x) - F(y)) \in -(K \cap K^*) \text{ and } (x - y) \vee_K (F(x) - F(y)) \in (K + K^*) \Rightarrow x = y;$$

(ii) the order- $P_0$  property if  $F(x) + \varepsilon B(x)$  has the order  $P$  property for any  $\varepsilon > 0$  where  $B : \mathbb{H} \rightarrow \mathbb{H}$  is a given linear function and satisfies  $\langle x, B(x) \rangle > 0, \langle xB(x), e_i \rangle \geq 0, \forall x \neq 0, \forall i \in \{1, 2, \dots, r\}$ ;

(iii) the  $P$  property if for any pair  $x, y \in \mathbb{H}$ ,

$$\sum_{l \geq j} ([ (x - y)(F(x) - F(y)) ]_{lj} + [ (x - y)(F(x) - F(y)) ]_{lj}^*) - \sum_{i=1}^r [ (x - y)(F(x) - F(y)) ]_{ii} \in -(K + K^*) \Rightarrow x = y;$$

(iv) the  $P_0$  property if  $F(x) + \varepsilon B(x)$  has the  $P$  property for any  $\varepsilon > 0$ ;

(v) the trace- $P$  property if for any pair  $x, y \in \mathbb{H}$  with  $x \neq y$ ,

$$\max_i \langle (x - y)(F(x) - F(y)), e_i \rangle > 0;$$

(vi) the trace- $P_0$  property if  $F(x) + \varepsilon B(x)$  has the  $P$  property for any  $\varepsilon > 0$ ;

(vii) the uniform-trace- $P$  property if there is an  $\alpha > 0$  such that for any pair  $x, y \in \mathbb{H}$ ,

$$\max_i \langle (x - y)(F(x) - F(y)), e_i \rangle \geq \alpha \|x - y\|.$$

In general, we may choose the above  $B$  as the identity transformation.

**Remark 3.1** By Proposition 2.5 (b), the implication condition of the order- $P$  property is equivalent to the following: for any pair  $x, y \in \mathbb{H}$ ,

$$(x - y) \wedge_K (F(x) - F(y)) \in -(K \cap K^*) \text{ and } (F(x) - F(y)) \vee_{K^*} (x - y) \in (K + K^*) \Rightarrow x = y.$$

Note that when  $K$  is self-dual,  $K \cap K^* = K$  and  $K + K^* = K$ . It is easy to see that all the above order- $P$  and  $P$  properties become the order  $P$  and Jordan  $P$  properties given by Tao and

Gowda [26] in the setting of SCCP, respectively. In particular, when  $\mathbb{H} = \mathbb{R}^n$  and  $K = \mathbb{R}_+^n$ , they are all the same as the  $P$  function (see Introduction).

**Remark 3.2** Using the related definitions and Proposition 2.9, we can easily verify the following one-way implications of the properties for nonlinear transformation  $F$ :

$$\begin{aligned} \text{Strong monotonicity} &\Rightarrow \text{uniform-trace-}P &\Rightarrow \text{trace-}P &\Rightarrow \text{trace-}P_0, \\ \text{Strong monotonicity} &\Rightarrow \text{strict monotonicity} &\Rightarrow \text{trace-}P &\Rightarrow \text{order-}P, \\ \text{Strong monotonicity} &\Rightarrow \text{strict monotonicity} &\Rightarrow \text{trace-}P &\Rightarrow P, \\ & & \text{Monotonicity} &\Rightarrow \text{trace-}P_0 &\Rightarrow \text{order-}P_0, \\ & & \text{Monotonicity} &\Rightarrow \text{trace-}P_0 &\Rightarrow P_0. \end{aligned}$$

Here, we say that  $F$  is *monotone* if  $\langle x - y, F(x) - F(y) \rangle \geq 0$ ,  $\forall x, y \in \mathbb{H}$ ;  $F$  is *strictly monotone* if  $\langle x - y, F(x) - F(y) \rangle > 0$ ,  $\forall x \neq y, x, y \in \mathbb{H}$ ; and  $F$  is *strongly monotone with modulus*  $\mu > 0$  if  $\langle x - y, F(x) - F(y) \rangle \geq \mu \|x - y\|^2$ ,  $\forall x, y \in \mathbb{H}$ .

**Remark 3.3** Observe that there are very many possible generalizations of the definition of the order- $P$  property from symmetric cones to homogeneous cones. For instance, for any pair of sets  $\hat{K}$  and  $\check{K}$  such that  $K \cap K^* \subseteq \hat{K}, \check{K} \subseteq K + K^*$ , we say  $F$  has the *order- $P$  property with respect to  $\hat{K}$  and  $\check{K}$*  if

$$(x - y) \wedge_K (F(x) - F(y)) \in -\hat{K} \text{ and } (x - y) \vee_K (F(x) - F(y)) \in \check{K} \Rightarrow x = y;$$

$F$  has the *order- $P_0$  property with respect to  $\hat{K}$  and  $\check{K}$*  if  $F(x) + \varepsilon B(x)$  has the order- $P$  property for any  $\varepsilon > 0$ . In order to get the above implications in Remark 3.2, by Proposition 2.9, we choose  $\hat{K}$  as the dual of  $\check{K}$ . On the other hand, in order to establish the existence result of a solution to HCCP (see Theorem 3.7), we set  $\check{K} := K + K^*$  from the equation (14) in the proof of Lemma 3.6.

Moreover, we may define the *order- $P$  property* of  $F$  by the implication

$$(x - y) \wedge_{K^*} (F(x) - F(y)) \in -(K \cap K^*) \text{ and } (x - y) \vee_{K^*} (F(x) - F(y)) \in (K + K^*) \Rightarrow x = y;$$

and the corresponding order- $P_0$  property. However, in this case, we cannot guarantee the existence result of a solution to HCCP (see Theorem 3.7).

**Remark 3.4** Similarly, there are very many possible generalizations of the definition of the  $P$  property to homogeneous cones. For instance, for every  $\tilde{K}$  such that  $K \cap K^* \subseteq \tilde{K} \subseteq K + K^*$ , we say  $F$  has the  *$P$  property with respect to  $\tilde{K}$*  if

$$\sum_{l \geq j} ([ (x - y)(F(x) - F(y)) ]_{lj} + [ (x - y)(F(x) - F(y)) ]_{lj}^*) - \sum_{i=1}^r [ (x - y)(F(x) - F(y)) ]_{ii} \in -\tilde{K} \Rightarrow x = y;$$

and  $F$  has the  *$P_0$  property with respect to  $\tilde{K}$*  if  $F(x) + \varepsilon B(x)$  has the  $P$  property for any  $\varepsilon > 0$ . Clearly, from the proofs of Lemma 3.6 and Theorem 3.7, we obtain that for any  $\tilde{K}$  if  $F$  has the  $P_0$  property with respect to  $\tilde{K}$  and  $R_0$  properties then the associated HCCPs have solutions. If the answer to our question in Remark 2.2 is “yes”, with the choice of  $\tilde{K} = K + K^*$ , there is some hope for proving  $P \Rightarrow P_0$ .

**Remark 3.5** In the case of SCCP, the order- $P$  property implies the  $P$  (Jordan  $P$ ) property. However, it is not clear whether this is valid for HCCP.

**Definition 3.2** A continuous function  $F : \mathbb{H} \rightarrow \mathbb{H}$  is said to have the  $R_0$  property if the following condition holds: for every sequence  $\{x^{(k)}\} \subset \mathbb{H}$  with

$$\|x^{(k)}\| \rightarrow \infty, \quad \liminf_{k \rightarrow \infty} \frac{x^{(k)}}{\|x^{(k)}\|} \in K, \quad \liminf_{k \rightarrow \infty} \frac{F(x^{(k)})}{\|x^{(k)}\|} \in K^*,$$

we have  $\liminf_{k \rightarrow \infty} \frac{\max_i \langle x^{(k)} F(x^{(k)}), e_i \rangle}{\|x^{(k)}\|^2} > 0$ .

The above definition is motivated by Definition 3.2 of Tao and Gowda [26], which was originally introduced for NCP by Chen and Harker [2]. In the setting of  $\mathbb{R}^n$ , it becomes that a continuous function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  has the  $R_0$  property if any sequence  $\{x^{(k)}\} \subset \mathbb{R}^n$  with

$$\|x^{(k)}\| \rightarrow \infty, \quad \liminf_{k \rightarrow \infty} \frac{\min_i x_i^{(k)}}{\|x^{(k)}\|} \geq 0, \quad \liminf_{k \rightarrow \infty} \frac{\min_i f_i(x^{(k)})}{\|x^{(k)}\|} \geq 0,$$

we have  $\liminf_{k \rightarrow \infty} \frac{\max_i x_i^{(k)} f_i(x^{(k)})}{\|x^{(k)}\|^2} > 0$ . Clearly, when  $f$  is linear the above condition becomes equivalent to the statement that the standard linear complementarity problem  $LCP(f, 0)$  has a unique solution, namely, zero.

Applying Proposition 2.9 and the related definitions we can easily derive two conditions under which the  $R_0$  property holds, which is a generalization of Proposition 3.2 in [26].

**Proposition 3.3** Let  $F : \mathbb{H} \rightarrow \mathbb{H}$  be a continuous function. If  $F$  has either the uniform-trace- $P$  property or satisfies the following implication: for every sequence  $\{x^{(k)}\} \subset \mathbb{H}$  with

$$\|x^{(k)}\| \rightarrow \infty, \quad \liminf_{k \rightarrow \infty} \frac{x^{(k)}}{\|x^{(k)}\|} \in K, \quad \liminf_{k \rightarrow \infty} \frac{F(x^{(k)})}{\|x^{(k)}\|} \in K^*,$$

we have  $\liminf_{k \rightarrow \infty} \frac{\langle x^{(k)} F(x^{(k)}), e_i \rangle}{\|x^{(k)}\|^2} > 0$ , then  $F$  has the  $R_0$  property.

It is well-known that the notion of  $R_0$  property of a function is closely related its coercivity, which plays a central role in describing the boundedness of the solution set to NCP, see, e.g. [7]. In the case of HCCP, we have a similar result.

**Proposition 3.4** Let  $F : \mathbb{H} \rightarrow \mathbb{H}$  be a continuous function. If  $F$  has the  $R_0$  property, then for every  $\delta > 0$ , the set  $\{x \in \mathbb{H} : x \text{ solves } HCCP(F, q), \|q\| \leq \delta\}$  is bounded.

**Proof.** Suppose the set  $\{x \in \mathbb{H} : x \text{ solves } HCCP(F, q), \|q\| \leq \delta\}$  is unbounded. Then, there exist sequences  $\{q^{(k)}\}$  with  $\|q^{(k)}\| \leq \delta$  and  $\{x^{(k)}\}$  with  $\|x^{(k)}\| \rightarrow \infty$  such that

$$x^{(k)} \in K, \quad y^{(k)} = F(x^{(k)}) + q^{(k)} \in K^*, \quad \langle x^{(k)}, y^{(k)} \rangle = 0, \quad \forall k.$$

Since  $\{x^{(k)}\} \subset K$ ,  $\{y^{(k)}\} \subset K^*$  and  $K, K^*$  are closed,  $\liminf_{k \rightarrow \infty} \frac{x^{(k)}}{\|x^{(k)}\|} \in K$  and  $\liminf_{k \rightarrow \infty} \frac{y^{(k)}}{\|y^{(k)}\|} \in K^*$ . Since  $q^{(k)}$  is bounded,  $\liminf_{k \rightarrow \infty} \frac{q^{(k)}}{\|x^{(k)}\|} = 0$ . Thus,

$$\liminf_{k \rightarrow \infty} \frac{F(x^{(k)})}{\|x^{(k)}\|} = \liminf_{k \rightarrow \infty} \frac{F(x^{(k)}) + q^{(k)}}{\|x^{(k)}\|} = \liminf_{k \rightarrow \infty} \frac{y^{(k)}}{\|x^{(k)}\|} \in K^*.$$

This together with the  $R_0$  property of  $F$  gives

$$\liminf_{k \rightarrow \infty} \frac{\max_i \langle x^{(k)} F(x^{(k)}), e_i \rangle}{\|x^{(k)}\|^2} > 0.$$

However, noting that  $\langle x^{(k)}, y^{(k)} \rangle = 0$  and the boundedness  $q^{(k)}$ , by Proposition 2.7, we obtain that for every  $i \in \{1, 2, \dots, r\}$ ,

$$\frac{\langle x^{(k)} F(x^{(k)}), e_i \rangle}{\|x^{(k)}\|^2} = \frac{\langle x^{(k)} y^{(k)}, e_i \rangle}{\|x^{(k)}\|^2} - \frac{\langle x^{(k)} q^{(k)}, e_i \rangle}{\|x^{(k)}\|^2} = -\frac{\langle x^{(k)} q^{(k)}, e_i \rangle}{\|x^{(k)}\|^2} \rightarrow 0 \text{ as } k \rightarrow \infty.$$

This is a contradiction and hence the desired conclusion follows.  $\square$

Before stating our main result in this section, we recall below a useful result from degree theory. The topological degree technique plays an important role in the study of complementarity problems and variational inequality problems, see, e.g, [10, 17, 18, 19, 26, 32, 33]. Let  $\Omega$  be a bounded open set in  $\mathbb{H}$  with its closure  $\text{cl}(\Omega)$  and boundary  $\partial\Omega$ . For a continuous function  $\Phi : \text{cl}(\Omega) \rightarrow \mathbb{H}$  and  $p \notin \Phi(\partial\Omega)$ , we denote  $\text{deg}(\Phi, \Omega, p)$  the (topological) *degree* of  $\Phi$  with respect to  $\Omega$  at  $p$ , see Lloyd [22] for the details.

**Lemma 3.5** (Theorem 2.1.2, [22]) (1) Suppose that  $\Phi, \varphi : \text{cl}(\Omega) \rightarrow \mathbb{H}$  are continuous and  $p \notin \Phi(\partial\Omega)$ . If  $\sup_{x \in \text{cl}(\Omega)} \|\Phi(x) - \varphi(x)\| < \text{dist}(p, \Phi(\partial\Omega))$ , then  $\text{deg}(\varphi, \Omega, p)$  is defined and

$$\text{deg}(\varphi, \Omega, p) = \text{deg}(\Phi, \Omega, p).$$

(2) If  $g_t(x)$  is a homotopy and  $p \notin g_t(\partial\Omega)$  for  $0 \leq t \leq 1$ , then  $\text{deg}(g_t, \Omega, p)$  is independent of  $t \in [0, 1]$ .

The next lemma relies heavily on the invariance of degree under suitable homotopies and generalizes Theorem 3.1 of [26] and its proof.

**Lemma 3.6** Let  $F : \mathbb{H} \rightarrow \mathbb{H}$  be a continuous function, and for every  $\delta > 0$  the set

$$\{x \in \mathbb{H} : x \text{ solves } HCCP(F, q), \|q\| \leq \delta\} \quad (12)$$

is bounded. If  $F$  has either the order- $P_0$  property, or the  $P_0$  property, then for every  $q \in \mathbb{H}$ , the solution set of  $HCCP(F, q)$  is nonempty and bounded.

**Proof.** Choose any  $q \in \mathbb{H}$ . Consider the function

$$\Phi(x) := x \wedge_K (F(x) + q).$$

Define the homotopy

$$G_1(x, t) := x \wedge_K [F(x) + tq + (t-1)F(0)], \quad t \in [0, 1].$$

Clearly,  $G_1(x, 0) = x \wedge_K [F(x) - F(0)]$  and  $G_1(x, 1) = \Phi(x)$  for all  $x$ . By the assumption, the sets  $\{x \in \mathbb{H} : G_1(x, t) = 0\}$  ( $t \in [0, 1]$ ) are uniformly bounded. Thus, we may take a bounded open set  $\Omega \in \mathbb{H}$  such that

$$\bigcup_{t \in [0, 1]} \{x \in \mathbb{H} : G_1(x, t) = 0\} \subseteq \Omega.$$

Then,  $0 \in \Omega$  and  $0 \notin G_1(\partial\Omega, 0)$  since  $G_1(0, 0) = 0$ . Therefore, by Lemma 3.5 (2),

$$\text{deg}(G_1(\cdot, 0), \Omega, 0) = \text{deg}(G_1(\cdot, 1), \Omega, 0) = \text{deg}(\Phi, \Omega, 0).$$

Define  $\varphi_\varepsilon(x) := x \wedge_K [F(x) + \varepsilon B(x) - F(0)]$  for any  $\varepsilon > 0$  where  $B$  is linear and strictly monotone. Note that

$$\begin{aligned} \|\varphi_\varepsilon(x) - G_1(x, 0)\| &= \|x \wedge_K [F(x) + \varepsilon B(x) - F(0)] - x \wedge_K [F(x) - F(0)]\| \\ &= \|\Pi_K[x - (F(x) + \varepsilon B(x) - F(0))] - \Pi_K[x - (F(x) - F(0))]\| \\ &\leq \|[x - (F(x) + \varepsilon B(x) - F(0))] - [x - (F(x) - F(0))]\| \\ &= \|\varepsilon B(x)\|, \end{aligned}$$

where the inequality follows from (4). Since  $\text{dist}(0, G_1(\partial\Omega, 0)) > 0$  by  $0 \notin G_1(\partial\Omega, 0)$ , we pick  $\varepsilon_0 > 0$  such that

$$\sup_{x \in \text{cl}(\Omega)} \|\varphi_\varepsilon(x) - G_1(x, 0)\| < \text{dist}(0, G_1(\partial\Omega, 0)).$$

Then, by Lemma 3.5 (1),  $\deg(G_1(\cdot, 0), \Omega, 0) = \deg(\varphi_\varepsilon, \Omega, 0)$ . So, we obtain

$$\deg(\varphi_\varepsilon, \Omega, 0) = \deg(\Phi, \Omega, 0). \quad (13)$$

For small  $\varepsilon > 0$ , we define the homotopy

$$G_2(x, t) := x \wedge_K [t(F(x) - F(0) + \varepsilon B(x)) + (1 - t)x], \quad t \in [0, 1].$$

Clearly,  $G_2(x, 0) = x \wedge_K x = x$  and  $G_2(x, 1) = \varphi_\varepsilon(x)$  for all  $x$ . We now show that  $0 \notin G_2(\partial\Omega, t)$  for any  $t \in [0, 1]$ . Suppose not, then there exist  $t_0 \in [0, 1]$  and  $x_0 \in \partial\Omega$  such that  $G_2(x_0, t_0) = 0$ . If  $t_0 = 0$ , then  $G_2(x_0, 0) = 0$  means that  $x_0 = 0$ , which contradicts with  $0 \in \Omega$ . We may assume  $t_0 \in (0, 1]$ . By Proposition 2.7,  $G_2(x_0, t_0) = 0$  is equivalent to the following

$$x_0 \in K, \quad F(x_0) - F(0) + \varepsilon B(x_0) + \frac{1 - t_0}{t_0} x_0 \in K^*, \quad \left\langle x_0, F(x_0) - F(0) + \varepsilon B(x_0) + \frac{1 - t_0}{t_0} x_0 \right\rangle = 0.$$

Letting  $\tilde{F}(x) := F(x) + \varepsilon B(x) + (\frac{1}{t_0} - 1)x$ , the above can be written as

$$x_0 \in K, \quad \tilde{F}(x_0) - \tilde{F}(0) \in K^*, \quad \left\langle x_0, \tilde{F}(x_0) - \tilde{F}(0) \right\rangle = 0.$$

Thus, by Propositions 2.5 and 2.7, we obtain

$$\begin{aligned} (x_0 - 0) \wedge_K (\tilde{F}(x_0) - \tilde{F}(0)) &= 0 \in -(K \cap K^*), \\ (x_0 - 0) \vee_K (\tilde{F}(x_0) - \tilde{F}(0)) &= x_0 + (\tilde{F}(x_0) - \tilde{F}(0)) \in K + K^*, \end{aligned} \quad (14)$$

and

$$[(x_0 - 0)(\tilde{F}(x_0) - \tilde{F}(0))]_{lj} = 0, \quad \forall l, j \in \{1, 2, \dots, r\} \text{ such that } l \geq j. \quad (15)$$

If  $F$  has the order- $P_0$  property, then  $\tilde{F}$  has the order- $P$  property. Hence, by (14),  $x_0 = 0$ , a contradiction. If  $F$  has the  $P_0$  property, then  $\tilde{F}$  has the  $P$  property and hence, by (15),  $x_0 = 0$ . This is also a contradiction.

Thus,  $0 \notin G_2(\partial\Omega, t)$ . Again, by Lemma 3.5 (2),

$$\deg(G_2(\cdot, 0), \Omega, 0) = \deg(G_2(\cdot, 1), \Omega, 0) = \deg(\varphi_\varepsilon, \Omega, 0). \quad (16)$$

Notice that  $\deg(G_2(\cdot, 0), \Omega, 0) = 1$ . This together with (13) and (16) yields  $\deg(\Phi, \Omega, 0) = 1$ , which says that  $\Phi(x) = 0$  has a solution. By Proposition 2.7, we proved that  $\text{HCCP}(F, q)$  has a solution. The desired conclusion follows from the assumption (12).  $\square$

We state below our main result in this section.

**Theorem 3.7** *Let  $F : \mathbb{H} \rightarrow \mathbb{H}$  be a continuous function. Suppose that  $F$  has either the order- $P_0$  and  $R_0$  properties, or the  $P_0$  and  $R_0$  properties. Then for every  $q \in \mathbb{H}$ , the solution set of  $\text{HCCP}(F, q)$  is nonempty and bounded.*

**Proof.** It follows immediately from Proposition 3.4 and Lemma 3.6.  $\square$

As a direct consequence of the above theorem, we have the following.

**Corollary 3.8** *Let  $F : \mathbb{H} \rightarrow \mathbb{H}$  be a continuous function. Suppose that  $F$  has either the order- $P_0$  and  $R_0$  properties, or the  $P_0$  and  $R_0$  properties. Then, there exists  $\bar{x} \in \mathbb{H}$  such that*

$$\bar{x} \in \text{int}(K), \quad F(\bar{x}) \in \text{int}(K^*).$$

**Proof.** It follows from Theorem 3.7 that for every  $q \in \mathbb{H}$ , the solution set of  $\text{HCCP}(F, q)$  is nonempty and bounded. Take  $-q_0 \in \text{int}(K^*)$ . Let  $\hat{x}$  be a solution to  $\text{HCCP}(F, q_0)$ . Then, we have  $\hat{x} \in K, \hat{y} := F(\hat{x}) + q_0 \in K^*$ . Therefore,  $F(\hat{x}) = -q_0 + \hat{y} \in \text{int}(K^*)$  and the desired conclusion follows from the continuity of  $F$ .  $\square$

## 4 GUS Property

We say that a continuous transformation  $F : \mathbb{H} \rightarrow \mathbb{H}$  has the *GUS property* if for every  $q \in \mathbb{H}$ ,  $\text{HCCP}(F, q)$  has a unique solution. In this section, we present a necessary condition for the GUS property and show that the trace- $P$  property is sufficient for  $F$  having the GUS property in the setting of HCCP. We also establish an error bound for the HCCP with the uniform-trace- $P$  property.

### 4.1 A necessary condition for the GUS property

Now, we present a necessary condition for the GUS property in the setting of HCCP, which is based on the following lemma.

**Lemma 4.1** *For every  $i \in \{1, 2, \dots, r\}$ , let  $a_{ki} \in \mathcal{A}_{ki}(k < i)$  and  $a_{il} \in \mathcal{A}_{il}(l > i)$  be given and  $\sum_{k=1}^{i-1} a_{ki} + \sum_{l=i+1}^r a_{il} \in \mathbb{H}$ . Then, for every positive scalar  $\lambda$  large enough, we have*

$$e_i + \lambda \sum_{j \neq i} e_j + \sum_{k=1}^{i-1} a_{ki} + \sum_{l=i+1}^r a_{il} \in K \cap K^*.$$

**Proof.** By the assumptions, Lemma 2.8 implies that there exist scalars  $\lambda_j > 0, j \in \{1, 2, \dots, r\} \setminus \{i\}$  sufficiently large such that

$$e_i + \lambda_j e_j + (r-1)[a_{ji} + a_{ij}] \in K \cap K^*, \forall j \neq i.$$

Adding all the above, we have

$$(r-1)e_i + \sum_{j \neq i} \lambda_j e_j + (r-1) \left( \sum_{k=1}^{i-1} a_{ki} + \sum_{l=i+1}^r a_{il} \right) \in K \cap K^*.$$

Thus,

$$e_i + \sum_{j \neq i} \frac{\lambda_j}{r-1} e_j + \sum_{k=1}^{i-1} a_{ki} + \sum_{l=i+1}^r a_{il} \in K \cap K^*.$$

Noting that  $e_j \in K \cap K^*$  and  $K \cap K^*$  is a convex cone, we obtain the desired result by adding corresponding multiples of  $e_j$  ( $j \neq i$ ).  $\square$

Now, we are in a position to prove the necessary condition for the GUS property in the HCCP context, which generalizes Theorem 4.1 of [26] and Theorem 8 of [11].

**Theorem 4.2** *Let  $F$  be a continuous function. If  $F$  has the GUS property, then for every  $j \in \{1, 2, \dots, r\}$ ,*

$$\langle F(e_j) - F(0), e_j \rangle \geq 0.$$

**Proof.** Suppose that there exists an  $e_i \in \mathcal{A}$  such that  $\langle F(e_i) - F(0), e_i \rangle < 0$ . Let

$$F(e_i) - F(0) = \rho_i e_i + \sum_{j \neq i} \rho_j e_j + \sum_{k \neq l} b_{kl}.$$

Thus,  $\rho_i = \frac{\langle F(e_i) - F(0), e_i \rangle}{\|e_i\|^2} < 0$ . Define

$$\tilde{q} := -\rho_i e_i + \lambda \sum_{j \neq i} e_j - \left( \sum_{k=1}^{i-1} b_{ki} + \sum_{l=i+1}^r b_{il} \right).$$

By Lemma 4.1, we may choose sufficiently large scalar  $\lambda > 0$  such that

$$\tilde{q} \in K \cap K^*. \quad (17)$$

Note that  $\tilde{\mathcal{A}} := \bigoplus_{k \neq i, l \neq i} \mathcal{A}_{kl}$  is a subalgebra of  $\mathcal{A}$  and for sufficiently large scalar  $\lambda > 0$ ,

$$\sum_{j \neq i} (\lambda + \rho_j) e_j \in \text{int}(K \cap K^*)$$

and hence  $\sum_{j \neq i} (\lambda + \rho_j) e_j + \sum_{k \neq l, k \neq i, l \neq i} b_{kl} \in K \cap K^*$ . Then

$$F(e_i) - F(0) + \tilde{q} = \sum_{j \neq i} (\lambda + \rho_j) e_j + \sum_{k \neq l, k \neq i, l \neq i} b_{kl} \in K \cap K^*. \quad (18)$$

For  $\text{HCCP}(F, -F(0) + \tilde{q})$  with sufficiently large  $\lambda > 0$ , by (17) and (18), we easily obtain that both  $e_i$  and 0 are its solutions. This is a contradiction.  $\square$

## 4.2 Sufficient conditions for the GUS property

Next, we show that if  $F$  has the trace- $P$  property then the associated  $\text{HCCP}(F, q)$  has the GUS property.

**Theorem 4.3** *Let  $F : \mathbb{H} \rightarrow \mathbb{H}$  be a continuous function. Suppose that  $F$  has the trace- $P$  property. Then for any  $q \in \mathbb{H}$ , the solution set of  $\text{HCCP}(F, q)$  is unique if it has a solution.*

**Proof.** Assume that there exist two solutions  $x, y$  to  $\text{HCCP}(F, q)$ . Then, by Proposition 2.7,  $x, y \in K$ ,  $F(x) + q, F(y) + q \in K^*$ ,  $\langle x(F(x) + q), e_i \rangle = \langle y(F(y) + q), e_i \rangle = 0$ ,  $\forall i \in \{1, 2, \dots, r\}$ .

Thus, we obtain that

$$\begin{aligned} \langle (x - y)(F(x) - F(y)), e_i \rangle &= \langle (x - y)[(F(x) + q) - (F(y) + q)], e_i \rangle \\ &= -\langle x(F(y) + q), e_i \rangle - \langle y(F(x) + q), e_i \rangle \\ &\leq 0, \end{aligned}$$

where the second equality holds by Proposition 2.7 and the inequality holds by Proposition 2.9. Therefore, the trace- $P$  property of  $F$  says  $x = y$ , as desired.  $\square$

As a direct application of the above Theorems 4.3 and 3.7 and the connection between various  $P$  properties, we have the GUS property of the uniform-trace- $P$  transformation.

**Corollary 4.4** *Let  $F : \mathbb{H} \rightarrow \mathbb{H}$  be a continuous function. If  $F$  has the uniform-trace- $P$  property then it has the GUS property.*

### 4.3 Error Bound

We give an error bound for the HCCP with the uniform-trace- $P$  property.

**Theorem 4.5** *Suppose  $F$  has the uniform-trace- $P$  property with modulus  $\alpha > 0$  and is Lipschitz continuous with constant  $\kappa > 0$ . Let  $x^*$  be the unique solution of problem HCCP( $F, q$ ). Then*

$$\frac{1}{2 + \kappa} \|x \wedge_K (F(x) + q)\| \leq \|x - x^*\| \leq \frac{1 + \kappa}{\alpha} \|x \wedge_K (F(x) + q)\|, \quad \forall x \in \mathbb{H}. \quad (19)$$

**Proof.** Notice that  $F(x) + q - x \wedge_K (F(x) + q) = \Pi_{K^*}(F(x) + q - x) \in K^*$ ,  $x - x \wedge_K (F(x) + q) = \Pi_K(x - F(x) - q) \in K$ , and  $x^* \wedge_K (F(x^*) + q) = 0$ . By Propositions 2.7 and 2.9 and the fact that  $x^*$  solves HCCP( $F, q$ ), we obtain

$$\begin{aligned} \langle [F(x) + q - x \wedge_K (F(x) + q)][x^* - x^* \wedge_K (F(x^*) + q)], e_i \rangle &\geq 0, \\ \langle [x - x \wedge_K (F(x) + q)][F(x^*) + q - x^* \wedge_K (F(x^*) + q)], e_i \rangle &\geq 0, \\ \langle [F(x^*) + q - x^* \wedge_K (F(x^*) + q)][x^* - x^* \wedge_K (F(x^*) + q)], e_i \rangle &= 0. \end{aligned}$$

Thus, by direction calculation, we obtain that for every  $i \in \{1, 2, \dots, r\}$ ,

$$\begin{aligned} &\langle [F(x) + q - x \wedge_K (F(x) + q)][x - x \wedge_K (F(x) + q)], e_i \rangle \\ &\geq \langle ([F(x) + q - x \wedge_K (F(x) + q)] - [F(x^*) + q - x^* \wedge_K (F(x^*) + q)]) \\ &\quad ([x - x \wedge_K (F(x) + q)] - [x^* - x^* \wedge_K (F(x^*) + q)]), e_i \rangle \\ &= \langle [F(x) - F(x^*) - x \wedge_K (F(x) + q)][x - x^* - x \wedge_K (F(x) + q)], e_i \rangle \\ &\geq \langle [F(x) - F(x^*)](x - x^*), e_i \rangle - \langle x \wedge_K (F(x) + q)[F(x) - F(x^*) + x - x^*], e_i \rangle \\ &\geq \langle (F(x) - F(x^*))(x - x^*), e_i \rangle - \|x \wedge_K (F(x) + q)\| \cdot (1 + \kappa) \|x - x^*\|, \end{aligned}$$

where the second inequality holds by the fact  $\langle a^2, e_i \rangle \geq 0$  for all  $a \in \mathcal{A}$ ; the last inequality follows from the Lipschitz continuity of  $F$  and  $\|ab\| \leq \|a\|\|b\|$  for any  $a, b \in \mathcal{A}$  by Theorem 2 of [4]. By Propositions 2.5 and 2.7,  $\langle [F(x) + q - x \wedge_K (F(x) + q)][x - x \wedge_K (F(x) + q)], e_i \rangle = 0$ . Thus, we conclude from the above inequality and the uniform-trace- $P$  property of  $F$ ,

$$(1 + \kappa) \|x \wedge_K (F(x) + q)\| \|x - x^*\| \geq \max_i \langle (F(x) - F(x^*))(x - x^*), e_i \rangle \geq \alpha \|x - x^*\|^2.$$

This leads to the right-hand side of inequality (19).

Note that  $\|\Pi_K(y) - \Pi_K(z)\| \leq \|y - z\|$  for any  $y, z \in \mathbb{H}$ . From the Lipschitz continuity of  $F$ , we obtain by direct manipulation that

$$\begin{aligned} \|x \wedge_K (F(x) + q)\| &= \|[x - \Pi_K(x - F(x) - q)] - [x^* - \Pi_K(x^* - F(x^*) - q)]\| \\ &= \|[x - x^*] - [\Pi_K(x - F(x) - q) - \Pi_K(x^* - F(x^*) - q)]\| \\ &\leq \|x - x^*\| + \|\Pi_K(x - F(x) - q) - \Pi_K(x^* - F(x^*) - q)\| \\ &\leq \|x - x^*\| + \|(x - F(x) - q) - (x^* - F(x^*) - q)\| \\ &\leq 2\|x - x^*\| + \|F(x) - F(x^*)\| \\ &\leq (2 + \kappa) \|x - x^*\|. \end{aligned}$$

This means that the left-hand side of inequality (19) holds.  $\square$

## 5 Applications to special HCCP and HCLCP

As applications of the new concepts and new results of the previous sections, we study the existence result for the HCCP with the relaxation transformation and automorphism invariance for HCLCP.

### 5.1 HCCP with the relaxation transformation

As an application of various  $P$  properties, we consider the relaxation transformation in the setting of HCCP. For  $x \in \mathbb{H}$  with  $x := \sum_{i=1}^r x_i e_i + \sum_{i \neq j} x_{ij}$ , define the *relaxation transformation*  $R_f : \mathbb{H} \rightarrow \mathbb{H}$  as

$$R_f(x) := \sum_{i=1}^r \tilde{x}_i e_i + \sum_{i \neq j} x_{ij},$$

where  $f : \mathbb{R}^r \rightarrow \mathbb{R}^r$  is a given continuous function and  $\tilde{x}_i := [f((x_1, x_2, \dots, x_r)^T)]_i$ ,  $i \in \{1, 2, \dots, r\}$ . The relationship between the properties of  $f$  and the properties of  $R_f$  has been very useful in the contexts of SDCP and SCCP, see, e.g., [11, 26]. Here, we study this relationship in the more general setting of HCCP. In this section, let  $B(x) = x$  in the definitions of trace- $P_0$ , order- $P_0$  and  $P_0$  properties.

It is easy to see the following monotonicity properties of  $R_f$  and  $f$ .

**Proposition 5.1** *Let  $f : \mathbb{R}^r \rightarrow \mathbb{R}^r$  be a continuous function and  $R_f$  be defined as above. Then,*

- (a)  $R_f$  is monotone if and only if  $f$  is monotone.
- (b)  $R_f$  is strictly monotone if and only if  $f$  is strictly monotone.
- (c) If  $R_f$  is strongly monotone with modulus  $\mu > 0$  then  $f$  is strongly monotone with modulus  $\mu$ .
- (d) If  $f$  is strongly monotone with modulus  $\mu > 0$  then  $R_f$  is strongly monotone with modulus  $\min\{\mu, 1\}$ .

We now look at the connection between various  $P$  properties of  $R_f$  and  $f$ . Clearly, in this case  $xR_f(x) = R_f(x)x$  for all  $x \in \mathbb{H}$ . So, the  $P$  property of  $R_f$  is equivalent to the following condition: for any pair  $x, y \in \mathbb{H}$ ,

$$(x - y)(F(x) - F(y)) \in -(K + K^*) \Rightarrow x = y.$$

**Proposition 5.2** *The following statements are equivalent:*

- (a)  $f$  is a  $P$  function.
- (b)  $R_f$  has the trace- $P$  property.
- (c)  $R_f$  has the order- $P$  property.
- (d)  $R_f$  has the  $P$  property.

**Proof** To prove (a)  $\Rightarrow$  (b), suppose that  $\langle (x-y)(R_f(x) - R_f(y)), e_i \rangle \leq 0, \forall i \in \{1, 2, \dots, r\}$  holds for  $x = \sum_{i=1}^r x_i e_i + \sum_{i \neq j} x_{ij}$  and  $y = \sum_{i=1}^r y_i e_i + \sum_{i \neq j} y_{ij}$ . To prove (b), we only need to show  $x = y$ . Let  $(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_r)^T := f((x_1, x_2, \dots, x_r)^T)$  and  $(\tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_r)^T := f((y_1, y_2, \dots, y_r)^T)$ . Then  $R_f(x) = \sum_{i=1}^r \tilde{x}_i e_i + \sum_{i \neq j} x_{ij}$  and  $R_f(y) = \sum_{i=1}^r \tilde{y}_i e_i + \sum_{i \neq j} y_{ij}$ . Since  $e_i a = a e_i$  for all  $a \in \mathcal{A}$ , by direct calculation, we have

$$\begin{aligned} & (x-y)(R_f(x) - R_f(y)) \\ &= \sum_{i=1}^r (x_i - y_i)(\tilde{x}_i - \tilde{y}_i)e_i + \sum_{i \neq j} (x_{ij} - y_{ij}) \left[ \sum_{i=1}^r (x_i - y_i)e_i + \sum_{i=1}^r (\tilde{x}_i - \tilde{y}_i)e_i \right] \\ & \quad + \left[ \sum_{i \neq j} (x_{ij} - y_{ij}) \right]^2. \end{aligned} \quad (20)$$

By the definition of  $T$ -algebra, for every  $i \in \{1, 2, \dots, r\}$ ,

$$\langle (x-y)(R_f(x) - R_f(y)), e_i \rangle = (x_i - y_i)(\tilde{x}_i - \tilde{y}_i) + \left\| \sum_{l \neq j} (x_{lj} - y_{lj})e_i \right\|^2.$$

Then, by assumption, we obtain

$$(x_i - y_i)(\tilde{x}_i - \tilde{y}_i) + \sum_{l, j \neq i} (\|x_{lj} - y_{lj}\|^2 + \|x_{li} - y_{li}\|^2) \leq 0, \forall i \in \{1, 2, \dots, r\}. \quad (21)$$

This means  $(x_i - y_i)(\tilde{x}_i - \tilde{y}_i) \leq 0$ , and hence  $x_i = y_i$  and  $\tilde{x}_i = \tilde{y}_i$  for every  $i \in \{1, 2, \dots, r\}$  by the  $P$  property of  $f$ . Thus, by (21),  $x_{ij} = y_{ij}$  for all  $i, j \in \{1, 2, \dots, r\}$ . Hence  $x = y$  and (b) holds.

Both implications (b)  $\Rightarrow$  (c) and (b)  $\Rightarrow$  (d) are obvious. It remains to show (c)  $\Rightarrow$  (a) and (d)  $\Rightarrow$  (a).

To prove (c)  $\Rightarrow$  (a), suppose

$$\begin{pmatrix} x_1 - y_1 \\ \vdots \\ x_r - y_r \end{pmatrix} \circ \left[ f \begin{pmatrix} x_1 \\ \vdots \\ x_r \end{pmatrix} - f \begin{pmatrix} y_1 \\ \vdots \\ y_r \end{pmatrix} \right] \leq 0. \quad (22)$$

Define  $x := \sum_{i=1}^r x_i e_i + \sum_{i \neq j} x_{ij}$ ,  $y := \sum_{i=1}^r y_i e_i + \sum_{i \neq j} y_{ij}$ . Then,

$$x - y = \sum_{i=1}^r (x_i - y_i)e_i, \quad R_f(x) - R_f(y) = \sum_{i=1}^r (\tilde{x}_i - \tilde{y}_i)e_i.$$

Let  $z_i := x_i - y_i$  and  $\tilde{z}_i := \tilde{x}_i - \tilde{y}_i$ . By (22),  $z_i \tilde{z}_i \leq 0$  for every  $i \in \{1, 2, \dots, r\}$ , or equivalently,

$$\max\{z_i, \tilde{z}_i\} \geq 0, \quad \min\{z_i, \tilde{z}_i\} \leq 0. \quad (23)$$

Direct calculation yields

$$\begin{aligned} (x-y) \wedge_K (R_f(x) - R_f(y)) &= \sum_{i=1}^r (z_i - (z_i - \tilde{z}_i)_+) e_i = \sum_{i=1}^r (\min\{z_i, \tilde{z}_i\}) e_i, \\ (x-y) \vee_K (R_f(x) - R_f(y)) &= \sum_{i=1}^r (\tilde{z}_i + (z_i - \tilde{z}_i)_+) e_i = \sum_{i=1}^r (\max\{z_i, \tilde{z}_i\}) e_i. \end{aligned}$$

Thus, by (23), we obtain

$$(x - y) \wedge_K (R_f(x) - R_f(y)) \in -(K \cap K^*), \quad (x - y) \vee_K (R_f(x) - R_f(y)) \in K + K^*. \quad (24)$$

Therefore,  $x = y$  by the order- $P$  property of  $R_f$  and  $x_i = y_i, \forall i \in \{1, 2, \dots, r\}$ . By (22),  $f$  is a  $P$  function and (a) holds.

Likewise, to prove (d)  $\Rightarrow$  (a), suppose that (22) holds and  $x, y$  are given as in the above case. By direct calculation, we obtain  $(x - y)(R_f(x) - R_f(y)) = \sum_{i=1}^r (x_i - y_i)(\tilde{x}_i - \tilde{y}_i)e_i$ . By (22),  $(x - y)(R_f(x) - R_f(y)) \in -(K \cap K^*) \subseteq -(K + K^*)$ . This leads to  $x = y$  by the  $P$  property of  $R_f$ . Thus,  $x_i = y_i$  for every  $i \in \{1, 2, \dots, r\}$  and (a) is established.  $\square$

In the setting of SCCP, the  $P_0$  property of  $f : \mathbb{R}^r \rightarrow \mathbb{R}^r$  implies the  $P_0$  property of  $R_f$ , see [26] for details. The next result says that this is still true in the more general setting of HCCP.

**Proposition 5.3** *The following statements are equivalent:*

- (a)  $f$  is a  $P_0$  function.
- (b)  $R_f$  has the order- $P_0$  property.
- (c)  $R_f$  has the  $P_0$  property.
- (d)  $R_f$  has the trace- $P_0$  property.

**Proof.** Suppose (a) holds. To prove (b) and (c), it suffices to establish (d) by the arguments after Definition 3.1. That is,  $R_f(x) + \varepsilon x$  has the trace- $P$  property for all  $\varepsilon > 0$ . Assume that  $\langle (x - y)[(R_f(x) + \varepsilon x) - (R_f(y) + \varepsilon y)], e_i \rangle \leq 0, \forall i \in \{1, 2, \dots, r\}$  holds for  $x = \sum_{i=1}^r x_i e_i + \sum_{i \neq j} x_{ij}$  and  $y = \sum_{i=1}^r y_i e_i + \sum_{i \neq j} y_{ij}$ . We need to show  $x = y$ . In the same way as in the proof of Proposition 5.2, let  $(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_r)^T := f((x_1, x_2, \dots, x_r)^T)$  and  $(\tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_r)^T := f((y_1, y_2, \dots, y_r)^T)$ . Since  $R_f(x) + \varepsilon x = \sum_{i=1}^r (\tilde{x}_i + \varepsilon x_i)e_i + (1 + \varepsilon) \sum_{i \neq j} x_{ij}$  and  $R_f(y) + \varepsilon y = \sum_{i=1}^r (\tilde{y}_i + \varepsilon y_i)e_i + (1 + \varepsilon) \sum_{i \neq j} y_{ij}$ , by direct calculation, we have

$$\begin{aligned} & (x - y)[(R_f(x) + \varepsilon x) - (R_f(y) + \varepsilon y)] \\ &= \sum_{i=1}^r (x_i - y_i)[(\tilde{x}_i - \tilde{y}_i) + \varepsilon(x_i - y_i)]e_i + \sum_{i \neq j} (x_{ij} - y_{ij}) \left[ (1 + \varepsilon) \sum_{i=1}^r (x_i - y_i)e_i + \sum_{i=1}^r (\tilde{x}_i - \tilde{y}_i)e_i \right] \\ & \quad + (1 + \varepsilon) \left[ \sum_{i \neq j} (x_{ij} - y_{ij}) \right]^2. \end{aligned}$$

Then, by assumption, for every  $i \in \{1, 2, \dots, r\}$ ,

$$\begin{aligned} & \langle (x - y)[(R_f(x) + \varepsilon x) - (R_f(y) + \varepsilon y)], e_i \rangle \\ &= (x_i - y_i)[(\tilde{x}_i - \tilde{y}_i) + \varepsilon(x_i - y_i)] + (1 + \varepsilon) \sum_{l, j \neq i} (\|x_{lj} - y_{lj}\|^2 + \|x_{li} - y_{li}\|^2) \leq 0. \quad (25) \end{aligned}$$

This means  $(x_i - y_i)[(\tilde{x}_i - \tilde{y}_i) + \varepsilon(x_i - y_i)] \leq 0$ , and hence  $x_i = y_i$  and  $\tilde{x}_i = \tilde{y}_i$  for every  $i \in \{1, 2, \dots, r\}$  by the  $P_0$  property of  $f$ . Thus, by (25),  $x_{ij} = y_{ij}$  for all  $i, j \in \{1, 2, \dots, r\}$ . Hence  $x = y$  and the desired claim holds.

Next, we prove (b)  $\Rightarrow$  (a). As in the proof of Proposition 5.2, suppose

$$\begin{pmatrix} x_1 - y_1 \\ \vdots \\ x_r - y_r \end{pmatrix} \circ \left[ f \begin{pmatrix} x_1 \\ \vdots \\ x_r \end{pmatrix} - f \begin{pmatrix} y_1 \\ \vdots \\ y_r \end{pmatrix} + \varepsilon \begin{pmatrix} x_1 - y_1 \\ \vdots \\ x_r - y_r \end{pmatrix} \right] \leq 0. \quad (26)$$

Define  $x := \sum_{i=1}^r x_i e_i + \sum_{i \neq j} x_{ij}$ ,  $y := \sum_{i=1}^r y_i e_i + \sum_{i \neq j} x_{ij}$ . Then,

$$x - y = \sum_{i=1}^r (x_i - y_i) e_i, \quad (R_f(x) + \varepsilon x) - (R_f(y) + \varepsilon y) = \sum_{i=1}^r [(\tilde{x}_i - \tilde{y}_i) + \varepsilon(x_i - y_i)] e_i.$$

Let  $z_i := x_i - y_i$  and  $\tilde{z}_i := \tilde{x}_i - \tilde{y}_i$ . By (26),  $z_i(\tilde{z}_i + \varepsilon z_i) \leq 0$  for every  $i \in \{1, 2, \dots, r\}$ , or equivalently,

$$\max\{z_i, \tilde{z}_i + \varepsilon z_i\} \geq 0, \quad \min\{z_i, \tilde{z}_i + \varepsilon z_i\} \leq 0. \quad (27)$$

Direct calculation yields

$$\begin{aligned} & (x - y) \wedge_K [(R_f(x) + \varepsilon x) - (R_f(y) + \varepsilon y)] \\ &= \sum_{i=1}^r [z_i - (z_i - (\tilde{z}_i + \varepsilon z_i))_+] e_i = \sum_{i=1}^r (\min\{z_i, \tilde{z}_i + \varepsilon z_i\}) e_i, \end{aligned}$$

and

$$\begin{aligned} & (x - y) \vee_K [(R_f(x) + \varepsilon x) - (R_f(y) + \varepsilon y)] \\ &= \sum_{i=1}^r [\tilde{z}_i + \varepsilon z_i + (z_i - (\tilde{z}_i + \varepsilon z_i))_+] e_i = \sum_{i=1}^r (\max\{z_i, \tilde{z}_i + \varepsilon z_i\}) e_i. \end{aligned}$$

Thus, by (27), we obtain

$$\begin{aligned} (x - y) \wedge_K [(R_f(x) + \varepsilon x) - (R_f(y) + \varepsilon y)] &\in -(K \cap K^*), \\ (x - y) \vee_K [(R_f(x) + \varepsilon x) - (R_f(y) + \varepsilon y)] &\in K + K^*. \end{aligned}$$

Therefore,  $x = y$  by the order- $P_0$  property of  $R_f$  and  $x_i = y_i$ ,  $\forall i \in \{1, 2, \dots, r\}$ . By (26),  $f$  is a  $P_0$  function and (a) holds.

Similarly, to prove (c)  $\Rightarrow$  (a), suppose that (26) holds and  $x, y$  are given as in the above case. By direct calculation, we obtain

$$(x - y)[(R_f(x) + \varepsilon x) - (R_f(y) + \varepsilon y)] = \sum_{i=1}^r (x_i - y_i)[(\tilde{x}_i - \tilde{y}_i) + \varepsilon(x_i - y_i)] e_i.$$

By (26), we obtain  $(x - y)[(R_f(x) + \varepsilon x) - (R_f(y) + \varepsilon y)] \in -(K \cap K^*) \subseteq -(K + K^*)$ . This leads to  $x = y$  and  $x_i = y_i$  for every  $i \in \{1, 2, \dots, r\}$  by the  $P_0$  property of  $R_f$ . So, we proved (a).  $\square$

Moreover, for the  $R_0$  property of  $f$  and the solution set of  $\text{HCCP}(R_f, q)$ , we have the following proposition.

**Proposition 5.4** *Suppose that  $f : \mathbb{R}^r \rightarrow \mathbb{R}^r$  has the  $R_0$  property. Then, for every  $\delta > 0$  the set*

$$\{x \in \mathbb{H} : x \text{ solves } \text{HCCP}(R_f, q), \|q\| \leq \delta\}$$

*is bounded.*

**Proof.** Suppose the set  $\{x \in \mathbb{H} : x \text{ solves } \text{HCCP}(R_f, q), \|q\| \leq \delta\}$  is unbounded. Then there exist sequences  $\{q^{(k)}\}$  with  $\|q^{(k)}\| \leq \delta$  and  $\{x^{(k)}\}$  with  $\|x^{(k)}\| \rightarrow \infty$  such that

$$x^{(k)} \in K, \quad y^{(k)} = R_f(x^{(k)}) + q^{(k)} \in K^*, \quad \langle x^{(k)}, y^{(k)} \rangle = 0, \quad \forall k. \quad (28)$$

Let  $x^{(k)} = \sum_{i=1}^r x_i^{(k)} e_i + \sum_{i \neq j} x_{ij}^{(k)}$ . By  $\{x^{(k)}\} \subset K$ ,  $x_i^{(k)} \geq 0$  for every  $i \in \{1, 2, \dots, r\}$ . Let  $q^{(k)} = \sum_{i=1}^r q_i^{(k)} e_i + \sum_{i \neq j} q_{ij}^{(k)}$ . Similarly, by  $R_f(x^{(k)}) = \sum_{i=1}^r \tilde{x}_i^{(k)} e_i + \sum_{i \neq j} x_{ij}^{(k)}$  and  $\{R_f(x^{(k)}) + q^{(k)}\} \subset K^*$ , we have  $\tilde{x}_i^{(k)} + q_i^{(k)} \geq 0$  for every  $i \in \{1, 2, \dots, r\}$ . Thus,

$$x_i^{(k)} (\tilde{x}_i^{(k)} + q_i^{(k)}) \geq 0, \quad \forall i \in \{1, 2, \dots, r\}. \quad (29)$$

Let  $z^{(k)} := (x_1^{(k)}, x_2^{(k)}, \dots, x_r^{(k)})^T$  and  $\tilde{z}^{(k)} := f(z^{(k)}) = (\tilde{x}_1^{(k)}, \tilde{x}_2^{(k)}, \dots, \tilde{x}_r^{(k)})^T$ . From (28) and (29), the boundedness of  $q^{(k)}$  and the fact that

$$\langle x^{(k)}, y^{(k)} \rangle = \sum_{i=1}^r x_i^{(k)} (\tilde{x}_i^{(k)} + q_i^{(k)}) + \sum_{i \neq j} \left( \|x_{ij}^{(k)}\|^2 + \langle x_{ij}^{(k)}, q_{ij}^{(k)} \rangle \right), \quad (30)$$

we obtain that  $\{x_{ij}^{(k)}\}$  is bounded and  $\|z^{(k)}\| \rightarrow \infty$  as  $\|x^{(k)}\| \rightarrow \infty$ . Note that  $\liminf_{k \rightarrow \infty} \frac{x_i^{(k)}}{\|z^{(k)}\|} \geq 0$  and  $\liminf_{k \rightarrow \infty} \frac{\tilde{z}_i^{(k)}}{\|z^{(k)}\|} = \liminf_{k \rightarrow \infty} \frac{\tilde{z}_i^{(k)} + q_i^{(k)}}{\|z^{(k)}\|} \geq 0$ . Then, by the  $R_0$  property of  $f$ , we have

$$\liminf_{k \rightarrow \infty} \frac{\max_i x_i^{(k)} f_i(z^{(k)})}{\|z^{(k)}\|^2} > 0.$$

However, by (28) and (30) and Proposition 2.7,

$$0 = \liminf_{k \rightarrow \infty} \frac{\max_i \langle x_i^{(k)}, y_i^{(k)} \rangle}{\|z^{(k)}\|^2} = \liminf_{k \rightarrow \infty} \frac{\max_i x_i^{(k)} \tilde{x}_i^{(k)}}{\|z^{(k)}\|^2} = \liminf_{k \rightarrow \infty} \frac{\max_i z_i^{(k)} f_i(z^{(k)})}{\|z^{(k)}\|^2}.$$

This is a contradiction and hence the desired conclusion holds.  $\square$

We end this section by presenting the following existence result for  $\text{HCCP}(R_f, q)$ , which generalizes Proposition 5.2 of [26].

**Theorem 5.5** *Suppose that  $f : \mathbb{R}^r \rightarrow \mathbb{R}^r$  has the  $P_0$  and the  $R_0$  properties. Then, for any  $q \in \mathbb{H}$ , the solution set of  $\text{HCCP}(R_f, q)$  is nonempty and bounded.*

**Proof.** It is immediate from Theorem 3.7 and Proposition 5.3.  $\square$

## 5.2 Automorphism invariance

This section deals with the *automorphism invariant properties* for HCLCP. Automorphism invariant properties have been considered in the special cases of semidefinite linear complementarity problem (SDLCP) and symmetric cone linear complementarity problem (SCLCP), see, e.g., [12, 13].

Recall that an invertible linear transformation  $A : \mathbb{H} \rightarrow \mathbb{H}$  is said to be an *automorphism of the cone  $K$*  (cone automorphism) if  $A(K) = K$ . Let  $\text{Aut}(K)$  and  $\text{Aut}(K^*)$  denote the sets of all automorphisms of  $K$  and  $K^*$ , respectively. Clearly, for every  $A \in \text{Aut}(K)$ ,  $A^{-1} \in \text{Aut}(K)$  and  $A^T \in \text{Aut}(K^*)$ .

Given HCLCP( $L, q$ ), we say that  $L$  has the *automorphism invariant property*  $\chi$  if for every  $A \in \text{Aut}(K)$ ,  $\hat{L} := A^T L A$  has the property  $\chi$ . Let  $\text{SOL}(L, q)$  denote the set of all solutions to HCLCP( $L, q$ ).

**Definition 5.6** For a linear transformation  $L : \mathbb{H} \rightarrow \mathbb{H}$ , we say  $L$  has

- (i) the  $R_0$  property if  $\text{SOL}(L, q) = \{0\}$ .
- (ii) the  $Q$  property if  $\text{SOL}(L, q) \neq \emptyset$  for all  $q \in \mathbb{H}$ .
- (iii) the GUS property if  $\text{SOL}(L, q)$  is a singleton for all  $q \in \mathbb{H}$ .
- (iv) the Lipschitzian GUS property if  $L$  has the GUS property and the solution map  $q \mapsto \text{SOL}(L, q)$  is Lipschitzian.

For the  $T$ -algebra  $\mathcal{A}$  with rank  $r$ , define an index set  $J \subseteq \{1, 2, \dots, n\}$ . Let

$$\mathcal{A}^J := \bigoplus_{i,j \in J} \mathcal{A}_{ij}, \quad \mathbb{H}^J := \mathcal{A}^J \cap \mathbb{H}.$$

Then,  $\mathcal{A}^J$  is a subalgebra of  $\mathcal{A}$  with rank  $|J|$  with the unit element  $\sum_{i \in J} e_i$ , and  $\mathbb{H}^J$  is the subspace of ‘‘Hermitian’’ elements in  $\mathcal{A}^J$ .

Let  $P^J$  denote the orthogonal projection from  $\mathcal{A}$  onto  $\mathcal{A}^J$ . For a linear transformation  $L : \mathbb{H} \rightarrow \mathbb{H}$ , a *principal subtransformation*  $L^J$  of  $L$  is defined by

$$L^J := P^J L : \mathbb{H}^J \rightarrow \mathbb{H}^J.$$

**Definition 5.7** For a linear transformation  $L : \mathbb{H} \rightarrow \mathbb{H}$ , we say  $L$  has the *ultra- $P$  property* (respectively, *ultra-trace- $P$  property*, *ultra-order- $P$  property* and *ultra-GUS property*) if for every  $A \in \text{Aut}(K)$ , every principal subtransformation of  $\hat{L} = A^T L A$  has the  $P$  property (respectively, *ultra-trace- $P$  property*, *ultra-order- $P$  property* and *ultra-GUS property*).

As in Section 3, it is easy to verify the similar one-way implications of the above properties for linear transformation from the related definitions:

$$\begin{aligned} \text{Strong monotonicity} &\Rightarrow \text{strict monotonicity} \Rightarrow \text{ultra-trace-}P \Rightarrow \text{ultra-order-}P, \\ \text{Strong monotonicity} &\Rightarrow \text{strict monotonicity} \Rightarrow \text{ultra-trace-}P \Rightarrow \text{ultra-}P, \\ \text{Strong monotonicity} &\Rightarrow \text{strict monotonicity} \Rightarrow \text{ultra-GUS} \Rightarrow \text{GUS}. \end{aligned}$$

The following result is a generalization of Theorem 5.1 of [13] under the cone automorphisms.

**Theorem 5.8** *The  $R_0, Q, GUS$  and Lipschitzian GUS properties are automorphism invariant.*

**Proof.** For  $HCLCP(L, q)$  and  $\hat{L} := ALA$  with  $A \in \text{Aut}(K)$ , let  $x \in \text{SOL}(L, q)$ . Then,  $x \in K$ ,  $y := L(x) + q \in K^*$  and  $\langle x, y \rangle = 0$ . Set  $\hat{x} := A^{-1}(x)$ . We have  $x = A(\hat{x})$  and  $A^T(y) = A^T L A(\hat{x}) + A^T q = \hat{L} + A^T q$ . Since  $A^{-1}(x) \in K$ ,  $A^T(y) \in K^*$  and  $\langle A^{-1}(x), A^T(y) \rangle = \langle x, y \rangle = 0$ , we have  $\hat{x} \in \text{SOL}(\hat{L}, A^T q)$ .

On the other hand, let  $\bar{x} \in \text{SOL}(\bar{L}, A^T q)$ . Then,  $\bar{x} \in K$ ,  $\bar{y} := \hat{L}(\bar{x}) + A^T q \in K^*$  and  $\langle \bar{x}, \bar{y} \rangle = 0$ . Since  $A(\bar{x}) \in K$ ,  $(A^T)^{-1}(\bar{y}) = L(A(\bar{x})) + q \in K^*$  and  $\langle A(\bar{x}), (A^T)^{-1}(\bar{y}) \rangle = \langle \bar{x}, \bar{y} \rangle = 0$ , we have  $A(\bar{x}) \in \text{SOL}(L, q)$ . Thus, we proved

$$\text{SOL}(L, q) = A(\text{SOL}(\hat{L}, A^T q)), \quad \text{or} \quad A^{-1}(\text{SOL}(L, q)) = \text{SOL}(\hat{L}, A^T q).$$

Therefore, the conclusion follows immediately.  $\square$

Note that the proof of the above theorem is very elementary and it does not use the homogeneity of the underlying cone  $K$ . However, since the automorphism groups of homogeneous cones are very rich, they provide the most interesting and powerful applications of this theorem.

We also easily obtain the following automorphism invariance fact.

**Proposition 5.9** *The ultra- $P, ultra-trace-}P, ultra-order-}P$  and *ultra-GUS properties are automorphism invariant.**

**Proof.** It follows immediately from the related definitions and Theorem 5.8.  $\square$

## 6 Final Remarks

In this paper, by employing the  $T$ -algebraic characterization of homogeneous cones we prove that if a continuous function has either the order- $P_0$  and  $R_0$ , or the  $P_0$  and  $R_0$  properties then all the associated HCCPs have solutions. In particular, if a continuous function has the trace- $P$  property then the associated HCCP has a unique solution (if any); if it has the uniform-trace- $P$  property then the associated HCCP has the GUS property. A necessary condition is presented for a nonlinear transformation having the GUS property in HCCP. Moreover, we establish a global error bound for HCCP with the uniform-trace- $P$  property. Finally, as applications, we consider the HCCP with the relaxation transformation on a  $T$ -algebra and automorphism invariant properties for HCLCP.

Many of our results apply to the more general setting of arbitrary convex cones (the order- $P$  property, order- $P_0$  property, trace- $P$  property, and  $R_0$  property). Further generalizations of similar results and theory to the arbitrary convex cone setting is a good direction for future research. The design of algorithms for HCCP (and beyond) and a study of their mathematical and computational properties provide other interesting future research avenues.

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