Strengthening lattice-free cuts using non-negativity

Ricardo Fukasawa Oktay Günlük

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Abstract

In recent years there has been growing interest in generating valid inequalities for mixed-integer programs using sets with 2 or more constraints. In particular, Andersen et al. (2007) and Borozan and Cornuéjols (2007) study sets defined by equations that contain exactly one integer variable per row. The integer variables are not restricted in sign. Cutting planes based on this approach have already been computationally studied by Espinoza [15] for general mixed-integer problems and there is ongoing computational research in this area.

In this paper, we extend the model studied in the earlier papers and require the integer variables to be non-negative. We extend the results in Andersen et al. (2007) and Borozan and Cornuéjols (2007) to our case and show that cuts generated by their approach can be strengthened by using the non-negativity of the integer variables. In particular, it is possible to obtain cuts which have negative coefficients for some variables.

Keywords: Mixed integer programming, valid inequalities, lattice-free polyhedra.

1 Introduction

Given a mixed-integer program (MIP) and a basic feasible solution to its linear programming (LP) relaxation, one can define a relaxation of the feasible solution set of the form

$$X = \left\{ (x,s) \in \mathbb{Z}^m \times \mathbb{R}^n_+ \ : \ x_i - \sum_{j=1}^{n} a_{ij} s_j \ = \ f_i \ \text{ for } i \in \{1,\ldots,m\} \right\}$$

which is obtained by starting with the associated simplex tableau and (i) deleting rows associated with basic continuous variables, (ii) relaxing integrality of the non-basic variables and (iii) relaxing the non-negativity of basic variables. Notice that variables x can be projected out by requiring s to satisfy $f_i + \sum_{j=1}^{n} a_{ij} s_j \in \mathbb{Z}$ for all i. This set can also be viewed as a continuous relaxation of the corner polyhedra of Gomory [16].

In a recent paper Andersen et al. [2] study the set X when $m = 2$ and show that all valid inequalities for X can be represented by maximal lattice-free convex sets in $\mathbb{R}^2$. Later Borozan and Cornuéjols [7] extended this and showed that minimal valid inequalities for the semi-infinite relaxation of X are in one-to-one correspondence with maximal lattice-free convex sets in $\mathbb{R}^m$ that contain f (provided that f is not on the boundary). In particular, they used the fact that all maximal lattice-free convex sets in $\mathbb{R}^m$ are polyhedra with at most 2^m facets [20, 4]. In addition, Cornuéjols and Margot [8] extended the results in [2] and studied conditions under which valid inequalities for the set X (when $m = 2$) become facet defining. More recently, Andersen et al. [3] extended their earlier work by considering upper bounds on some of the continuous variables. There has been some initial computational work by Espinoza [15] and there is ongoing computational work by other groups [1, 10] that use the results in [2, 7] to produce cutting planes for MIPs.

In this paper, we study the set

$$X^+ := \left\{(x,s) \in \mathbb{Z}^m_+ \times \mathbb{R}^n_+ \;:\; x_i - \sum_{j=1}^{n} a_{ij}s_j \;=\; f_i \;\text{ for } i \in \{1,\ldots,m\}\right\}$$

which contains non-negative points in X. As $X^+ \subseteq X$, it gives a tighter relaxation of MIPs for which integer basic variables are required to be non-negative.

Our main result in this paper is to show inequalities derived in [7] (using maximal lattice-free convex sets) can be strengthened using the fact that x variables are required to be non-negative in X^+. This strengthening, for example, leads to minimal valid inequalities of the form $\alpha x \geq 1$ where α can have negative components. The following example emphasizes the difference between valid inequalities for the sets X and X^+.

Example 1.1 *Let $r_1, r_1, r_2, r_3, r_4, r_5, f \in \mathbb{R}^2$ be defined as follows:*

$$r_1 = \begin{pmatrix} -\frac{1}{4} \\ \frac{3}{4} \end{pmatrix} \quad r_2 = \begin{pmatrix} -\frac{1}{4} \\ -\frac{5}{4} \end{pmatrix} \quad r_3 = \begin{pmatrix} \frac{7}{4} \\ -\frac{5}{4} \end{pmatrix} \quad r_4 = \begin{pmatrix} \frac{5}{4} \\ -\frac{5}{4} \end{pmatrix} \quad r_5 = \begin{pmatrix} \frac{3}{4} \\ -\frac{5}{4} \end{pmatrix} \quad \text{and} \quad f = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \end{pmatrix}$$

and consider the following set,

$$X = \left\{(x,s) \in \mathbb{Z}^2_+ \times \mathbb{R}^5_+ \;:\; \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \sum_{j=1}^{5} r_j s_j = f\right\}$$

defined by 2 rows. Using the results in [8, 2], it is possible to show that the following inequality:

$$s_1 + s_2 + s_3 + s_4 + s_5 \geq 1$$

is valid and facet-defining for X. However, using the non-negativity of the x variables in $X^+ = X \cap \mathbb{R}^7_+$, it is possible to show that the following stronger inequality:

$$s_1 + s_2 + s_3 - s_5 \geq 1$$

is valid (and facet defining) for X^+. We will come back to this example in Section 2.

The rest of the paper is organized as follows: In Section 2, we define the semi-infinite extension of X^+ where we essentially study the set X^+ when it has infinitely many variables, one for each rational coefficient vector. For this extension, we characterize the basic properties of minimal valid functions [1], relate them to convex sets that do not contain non-negative integer points in their interior and show that certain polyhedral sets lead to minimal valid functions. In Section 3, we focus on the semi-infinite extension of X^+ when it is defined by two rows and give a complete characterization of minimal valid functions and how they are related to convex sets that do not contain non-negative integer points in their interior. In Section 4, we show how to use nonnegativity to strengthen valid inequalities for X based on maximal lattice-free convex sets to obtain valid inequalities for X^+.

2 The semi-infinite extension of X^+

In this section, we study the semi-infinite extension of X^+ and show basic properties of minimal valid functions for it. The semi-infinite extension of X^+ is the set

$$R_f^+ \;=\; \Big\{(x,s) \in \mathbb{Z}_+^m \times \mathbb{J} \;:\; x - \sum_{r\in\mathbb{Q}^m} r s_r \;=\; f\Big\}$$

where $\mathbb{J} = \{s \in \mathbb{R}^{\mathbb{Q}^m} \;:\; s \text{ has finite support}\}$ and $f \geq 0$. A point s is said to have finite support if $s_r > 0$ for a finite number of $r \in \mathbb{Q}^m$. Note that R_f^+ can be obtained by restricting the semi-infinite extension of X to non-negative values of x. More precisely, the semi-infinite extension of X is

$$R_f \;=\; \Big\{(x,s) \in \mathbb{Z}^m \times \mathbb{J} \;:\; x - \sum_{r\in\mathbb{Q}^m} r s_r \;=\; f\;\Big\}.$$

The set R_f has been studied by Borozán and Cornuejols [7] who show that there is a bijection relating minimal valid functions for R_f to maximal lattice-free convex sets in $\mathbb{R}^m$ that contain f. We define valid functions and lattice-free sets more precisely in Sections 2.1 and 2.2, respectively.

Our main observations in this section is that most of the fundamental results known to hold for R_f can be extended to R_f^+. In particular, we establish a relationship between minimal valid functions for R_f^+ and maximal convex sets without non-negative integer points in their interior, which we call *maximal non-negative lattice-point free convex sets*. We are, however, only able to show that such relationship is bijective when $m = 2$. When $m \geq 3$, we show that given a polyhedral maximal NLPF convex set, one can construct a minimal valid function but we are not able to show that any minimal valid function can be constructed using a polyhedral maximal NLPF convex set.

2.1 Valid functions for R_f^+

We say that a function $\psi : \mathbb{Q}^m \to \mathbb{R}$ is a *valid function* for R_f^+ if

$$\sum_{r\in\mathbb{Q}^m} \psi(r) s_r \geq 1$$

for all $(x,s) \in R_f^+$. Any valid inequality for R_f that is violated by the point $(x,s) = (f,0)$ can be written in this form, see [7] and [2]. Similarly, valid inequalities for R_f^+ that are violated by

[1] A minimal valid function is a function that gives a valid cut that is not dominated by any other cut

$(f,0)$ can also be written in this form. Note that the x variables do not appear in the inequality as they can be projected out using the equations defining R_f^+. After doing so, it is easy to see that the right hand side of the inequality has to be strictly positive if it is violated by $(f,0)$. We also note that we only consider finite functions ψ since, in the context of generating cutting planes for mixed-integer programming, functions that can assume the value $\pm\infty$ are not useful in practice.

We say that ψ is a *minimal* valid function if it is a valid function and there is no other valid function ψ' such that (i) $\psi(r) \geq \psi'(r)$ for all $r \in \mathbb{Q}^m$, and (ii) $\psi(r) > \psi'(r)$ for some $r \in \mathbb{Q}^m$. In the context of cutting planes, a minimal valid function is analogous to a non-dominated inequality, therefore it is natural to only focus on minimal valid functions.

For the sake of completeness, we define the following: A function $f : \mathbb{Q}^m \to \mathbb{R}$ is called

(i) *subadditive,* if $f(x') + f(x'') \geq f(x' + x'')$ for all $x', x'' \in \mathbb{Q}^m$.

(ii) *positively homogeneous,* if $f(\alpha x') = \alpha f(x')$ for all $x' \in \mathbb{Q}^m$ and $\alpha \in \mathbb{Q}_+$.

(iii) *convex,* if $\alpha f(x') + (1-\alpha) f(x'') \geq f(\alpha x' + (1-\alpha)x'')$ for all $x', x'' \in \mathbb{Q}^m$ and $\alpha \in [0,1] \cap \mathbb{Q}$.

We next show that minimal valid functions satisfy all of the above properties.

Lemma 2.1 *If ψ is a minimal valid function for R_f^+ then ψ is (i) subadditive, (ii) positively homogeneous, and (iii) convex.*

Proof. The proof essentially summarizes and adopts proofs of Lemmas 2.2, 2.3, 2.4 and 2.5 in [7]. (i) Assume that ψ is not subadditive, then $\psi(r') + \psi(r'') < \psi(r' + r'')$ for some $r', r'' \in \mathbb{Q}^m$. Define $\phi : \mathbb{Q}^m \to \mathbb{R}$ to be identical to ψ with the exception that at point $r' + r''$ it assumes the value $\phi(r' + r'') = \psi(r') + \psi(r'')$. We will show that ϕ is valid, and therefore ψ can not be minimal, a contradiction. If ϕ is not valid there exists a point $(x', s') \in R_f^+$ such that $\sum_{r \in \mathbb{Q}^m} \phi(r) s'_r < 1$. But in this case ψ can not be valid either as $\sum_{r \in \mathbb{Q}^m} \phi(r) s'_r = \sum_{r \in \mathbb{Q}^m} \psi(r) s''_r < 1$ where $(x', s'') \in R_f^+$ and s'' is obtained from s' by reducing its $(r' + r'')$th component to zero and increasing the r'th and r''th components by $s'_{(r'+r'')}$. Therefore, ϕ is indeed valid, and ψ is not minimal.
(ii) As ψ is subadditive, we have that $\psi(r) + \psi(0) \geq \psi(r) \Rightarrow \psi(0) \geq 0$. Let $(\bar{x}, \bar{s}) \in R_f^+$. Since s has finite support and ψ is finite, we know that $\sum \psi(r) s_r < +\infty$. Note that the point $(\bar{x}, \tilde{s})$ defined by $\tilde{s}_r := \bar{s}_r$ for $r \neq 0$ and $\tilde{s}_0 = 0$ also belongs to R_f^+. Hence, $0 + \sum_{r \neq 0} \psi(r) s_r \geq 1$. Therefore ψ is still a valid function if we change $\psi(0) = 0$, so minimality of ψ implies $\psi(0) = 0$.

Therefore, if $\alpha = 0$ then $\psi(\alpha r) = \alpha \psi(r)$ for all $r \in \mathbb{Q}^m$. Assume that $\psi(\alpha r') \neq \alpha \psi(r')$ for some $\alpha > 0$ and $r', \alpha r' \in \mathbb{Q}^m$. Let $\beta = \min\{\psi(\alpha r')/\alpha, \psi(r')\}$ and define $\phi : \mathbb{Q}^m \to \mathbb{R}$ be same as ψ except let $\phi(\alpha r') = \alpha\beta$ and $\phi(r') = \beta$. As in the first part of the proof, it is straight forward to reach a contradiction by observing that ϕ is a valid function for R_f^+ provided that ψ is valid.
(iii) Notice that ψ is positively homogeneous and therefore for all $\alpha \in [0,1]$ and $r', r'' \in \mathbb{Q}^m$

$$\alpha\psi(r') + (1-\alpha)\psi(r'') = \psi(\alpha r') + \psi((1-\alpha) r'') \geq \psi(\alpha r' + (1-\alpha) r'')$$

where the last inequality follows from subadditivity. ∎

In [7], Borozan and Cornuéjols show that all valid functions for R_f are not only subadditive, positively homogeneous, and convex but also non-negative as well. We emphasize that not all valid functions for R_f^+ are non-negative. In Section 2.3, we describe a family of valid functions that assume negative values for some $r \in \mathbb{Q}^m$.

The above lemma states that all valid functions for R_f^+ are (i) subadditive and (ii) positively homogeneous (convexity is a consequence of those two properties). We next address the reverse question, namely, when is a subadditive and positively homogeneous function valid for R_f^+.

Lemma 2.2 *If ψ is positively homogeneous and subadditive, then it is valid for R_f^+ if and only if $\psi(x - f) \geq 1$ for all $x \in \mathbb{Z}_+^m$.*

Proof. The only if part is straight forward: if $\psi(\bar{x} - f) < 1$ for some $\bar{x} \in \mathbb{Z}_+^m$, define $\bar{s}$ to have all zero components except $\bar{s}_{(\bar{x}-f)} = 1$. We therefore have $(\bar{x}, \bar{s}) \in R_f^+$ and yet $\sum_{r \in \mathbb{Q}^m} \psi(r)\bar{s}_r < 1$, a contradiction.

For the if part, note that for all $(\bar{x}, \bar{s}) \in R_f^+$ we have $\bar{x} \in \mathbb{Z}_+^m$ and $\sum_{r \in \mathbb{Q}^m} r\bar{s}_r = \bar{x} - f$. First using homogeneity and then using subadditivity, we have $\sum_{r \in \mathbb{Q}^m} \psi(r)\bar{s}_r = \sum_{r \in \mathbb{Q}^m} \psi(\bar{r}s_r) \geq \psi(\sum_{r \in \mathbb{Q}^m} r\bar{s}_r)$. This implies that $\sum_{r \in \mathbb{Q}^m} \psi(r)\bar{s}_r \geq 1$ and therefore ψ is a valid function for R_f^+. ∎

2.2 NLPF sets and minimal valid functions for R_f^+

We next study some convex sets which are closely related with minimal valid functions for R_f^+. We call a set $S \subset \mathbb{R}^m$ *non-negative lattice-point free* (NLPF) if $int(S) \cap \mathbb{Z}_+^m = \emptyset$, where $int(S)$ denotes the interior of the set S.

For a given function $\psi : \mathbb{Q}^m \to \mathbb{R}$ we define a closed set in $\mathbb{R}^m$ associated with the function as follows:

$$S(\psi, f) := cl\Big(\big\{x \in \mathbb{Q}^m : \psi(x - f) \leq 1\big\}\Big)$$

where $cl(\cdot)$ is the closure operator. Note that if ψ is convex, the corresponding set $S(\psi, f)$ is also convex. Using this definition, Lemma 2.2 can be re-stated in terms of NLPF convex sets as follows:

Remark 2.3 *If ψ is positively homogeneous and subadditive, then it is valid for R_f^+ if and only if $S(\psi, f)$ is a NLPF set.*

Moreover, remember that the proof of Lemma 2.1 shows that if the function ψ is positively homogeneous and subadditive, then it is a convex function. As all minimal valid functions for R_f^+ are positively homogeneous and subadditive, we also observe that $S(\psi, f)$ is convex for all minimal valid functions ψ, and so we have a first relationship to NLPF convex sets.

We next present a more detailed relationship between minimal valid functions ψ for R_f^+ and the NLPF convex set $S(\psi, f)$ that will help develop our results later on in the paper. In particular, we show that for $r \in \mathbb{Q}^m$, the value of $\psi(r)$ depends on where r lies with respect to the recession cone of $S(\psi, f)$. For a set B, let $RC(B)$ denote the recession cone of B and $RC^o(B)$ denote the boundary of the recession cone of B. More precisely, $RC^o(B) = RC(B) \setminus int(RC(B))$.

Lemma 2.4 *Let $f \in \mathbb{Q}^m$ and $\psi : \mathbb{Q}^m \to \mathbb{R}$ be a minimal valid function for R_f^+. Then $f \in int(S(\psi, f))$. Moreover, for every $r \in \mathbb{Q}^m$, the function ψ satisfies the following:*

(i) if $r \in RC(S(\psi, f))$ then $\psi(r) \leq 0$,

(ii) if $r \in RC^o(S(\psi, f))$ then $\psi(r) = 0$,

(iii) if $r \notin RC(S(\psi, f))$ then $0 < \psi(r) = 1/\max\{\lambda \in \mathbb{R}_+ \,:\, f + \lambda r \in S\}$, and,

(iv) if $r \geq 0$ then $\psi(r) \geq 0$.

Proof. To simplify notation, let $S = S(\psi, f)$, $RC = RC(S(\psi, f))$, and $RC^o = RC^o(S(\psi, f))$. We start with showing that $\psi(r) < \infty$ for all $r \in \mathbb{Q}^m$ implies that $f \in int(S)$. Let e_d be the unit vector with a 1 in the d-th component and 0 everywhere else. If $\psi(e_d) = 0$ then $\psi(f + e_d - f) = \psi(e_d) = 0$ so $f + e_d \in S$. If $\psi(e_d) \neq 0$, since $|\psi(e_d)| < \infty$, we have that $1 \geq \frac{1}{|\psi(e_d)|}\psi(e_d) = \psi\left(\frac{1}{|\psi(e_d)|}e_d\right) = \psi\left(f + \frac{1}{|\psi(e_d)|}e_d - f\right)$ and hence $f + \frac{1}{|\psi(e_d)|}e_d \in S$. Since the same argument is valid for all e_d and $-e_d$ for all $d = 1, \ldots, m$, we have that there exists $\epsilon > 0$ such that $f \pm \epsilon e_d \in S$ for all $d = 1, \ldots, m$ and hence $f \in int(S)$. We next prove (i), (ii), (iii) and (iv).

(i) Consider $r \in RC$. As $f \in S$, we have $f + \lambda r \in S$ for all $\lambda \in \mathbb{Q}_+$ implying $\psi(f + \lambda r - f) \leq 1$. Hence $\psi(\lambda r) = \lambda \psi(r) \leq 1$. Since λ can be arbitrarily large, we have $\psi(r) \not> 0$, or equivalently, $\psi(r) \leq 0$.

(ii) We first show that $\psi(r) > 0$ when $r \notin RC$. For the sake of contradiction assume that $\psi(r) \leq 0$. Then for any $x \in S \cap \mathbb{Q}^m$ and any $\lambda \in \mathbb{Q}_+$ we have that $\psi(x + \lambda r - f) \leq \psi(x - f) + \lambda\psi(r) \leq \psi(x - f) \leq 1$, therefore $x + \lambda r \in S$. Since S is convex, $x + \lambda r \in S$ for all $\lambda \in \mathbb{R}_+$ and hence $r \in RC$.

Now consider $r \in RC^o$ and note as $r \in RC$ we have $\psi(r) \leq 0$. Suppose, for the sake of contradiction, that $\psi(r) = -\beta$ for some $\beta > 0$. Since $r \in RC^o$, there exists a nonzero vector $v \notin RC$ such that $r + \delta v \notin RC$ for all $\delta > 0$. Now choose a δ' such that $0 < \delta' < \beta/\psi(v)$ and remember that $v \notin RC$ implies $0 < \psi(v) < +\infty$. As $r + \delta' v \notin RC$ we have $f + \lambda(r + \delta' v) \notin S$ for some sufficiently large $\lambda > 0$. In other words, $\psi(\lambda(r + \delta' v)) > 1$. As ψ is subadditive and positively homogeneous, we also have

$$\lambda\psi(r) + \lambda\delta'\psi(v) \geq \psi(\lambda(r + \delta' v)) > 1 \Rightarrow \psi(r) \geq 1/\lambda - \delta'\psi(v) > 1/\lambda - \beta \geq -\beta,$$

which is a contradiction and therefore $\psi(r) = 0$.

(iii) Finally, we consider $r \notin RC$. Notice that we have already shown in part (ii) that $\psi(r) > 0$ and therefore,

$$1 = \frac{1}{\psi(r)}\psi(r) = \psi(\frac{1}{\psi(r)}r) = \psi(f + \frac{1}{\psi(r)}r - f)$$

implying $f + r/\psi(r) \in S$ and hence

$$1/\psi(r) \leq \bar{\lambda} = \max\{\lambda \in R_+ \,:\, f + \lambda r \in S\}.$$

Now if $\bar{\lambda} > 1/\psi(r)$, we have that $\psi(f + \bar{\lambda} r - f) = \psi(\bar{\lambda} r) = \bar{\lambda}\psi(r) > 1$, a contradiction. Therefore, $\bar{\lambda} = 1/\psi(r)$.

(iv) Let $(\bar{x}, \bar{s}) \in R_f^+$. Let $\tilde{r} \in \mathbb{Q}_+^m$ and let $D \in \mathbb{Z}_+$ be such that $\tilde{r}D \in \mathbb{Z}_+^m$. Let M be an arbitrary positive integer number and $(\tilde{x}, \tilde{s}) \in R_f^+$ be defined as $\tilde{x} := \bar{x} + MD\tilde{r}$, $\tilde{s}_{\tilde{r}} := \bar{s}_{\tilde{r}} + MD\tilde{r}$, $\tilde{s}_r := \bar{s}_r$ for all $r \neq \tilde{r}$. But then $1 \leq \sum_{r\in\mathbb{Q}^m} \psi(r)\tilde{s}_r = \sum_{r\in\mathbb{Q}^m} \psi(r)\bar{s}_r + MD\psi(\tilde{r})$. Hence $\psi(\tilde{r}) \geq \frac{1-\sum_{r\in\mathbb{Q}^m} \psi(r)\bar{s}_r}{MD}$ for all $M \in \mathbb{Z}_+$ and hence $\psi(\tilde{r}) \geq 0$. ∎

Notice that the first part of the proof of Lemma 2.4 can be easily extended to show that, even if we allow ψ to take on the value ∞, we have $\psi < \infty$ if and only if $f \in int(S(\psi, f))$. As we assume ψ to be finite, we only consider maximal NLPF convex sets that contain f in their interior. We remark that in the context of X and R_f, the analogous assumption is not too restrictive – Zambelli [25] showed that all cutting-planes for X can be generated using maximal lattice-free convex sets that contain f in the interior.

2.3 A minimal valid function for R_f^+

We have so far established that, given a minimal valid function for R_f^+, one can obtain an associated NLPF convex set $S(\psi, f)$ containing f in its interior. We now focus on studying the reverse question, that is, given a NLPF convex set with f in its interior, can we obtain an associated valid function and if so, is it minimal?

We start by defining a mapping from polyhedral NLPF convex sets to valid functions for R_f^+. Throughout this section we assume that B is a polyhedral set that satisfies the following two properties: (i) it contains f in its interior, (and therefore it is also full-dimensional) and (ii) it does not contain any non-negative integer points in its interior. Therefore, B can be represented as

$$B = \{x \in \mathbb{R}^m : a_i^T x \leq b_i, \forall i = 1, \ldots, k\},$$

where (i) $a_i^T f < b_i$ for all $i \in I := \{1, \ldots, k\}$, and (ii) $int(B) \cap \mathbb{Z}_+ = \emptyset$. In addition, we assume that all of the inequalities used in the description of B are facet defining.

Let $\psi_B : \mathbb{Q}^m \to \mathbb{R}$ be defined as follows:

$$\psi_B(r) := \max_{i\in I} \left\{r^T \hat{a}_i\right\} \quad (1)$$

where $\hat{a}_i = a_i/(b_i - a_i^T f)$.

We now show that the function ψ_B defined in (1) satisfies the desired property, that is, ψ_B is valid for R_f^+ if B is NLPF.

Lemma 2.5 *If B is NLPF, then the function ψ_B is valid for R_f^+.*

Proof. Clearly ψ_B is positively homogenous. We next show that it is also subadditive: Let $r^1, r^2 \in \mathbb{Q}^m$ and let $\psi_B(r^1 + r^2) = \hat{a}_l^T(r^1 + r^2)$ for some $l \in I$. Then

$$\psi_B(r^1) + \psi_B(r^2) = \max_{i\in I}\{\hat{a}_i^T r^1\} + \max_{i\in I}\{\hat{a}_i^T r^2\} \geq \hat{a}_l^T r^1 + \hat{a}_l^T r^2 = \hat{a}_l^T(r^1 + r^2) = \psi_B(r^1 + r^2).$$

Therefore, ψ_B is subadditive and by Lemma 2.2 and Remark 2.3, it is valid if and only if $S(\psi_B, f)$ is NLPF. Let $r \in S(\psi_B, f)$ and note that for all $i \in I$ we have

$$1 \geq \psi(r - f) \geq \hat{a}_i^T(r - f) = (a_i^T r - a_i^T f)/(b_i - a_i^T f) \;\Rightarrow\; b_i - a_i^T f \geq a_i^T r - a_i^T f \;\Rightarrow\; b_i \geq a_i^T r$$

and therefore $r \in B$. As r was chosen arbitrarily, we have $S(\psi_B, f) \subseteq B$ and therefore $S(\psi_B, f)$ is NLPF and the proof is complete. ∎

Note that it is possible to extend the last argument in the proof to show the following fact that will be useful in later proofs.

Remark 2.6 $S(\psi_B, f) = B$.

Proof. Let $r \in B \cap \mathbb{Q}^m$ and therefore $a_i^T r \leq b_i$ for all $i \in \{1, \dots, k\}$. Let $\psi(r - f) = \hat{a}_t^T (r - f)$ for some $t \in \{1, \dots, k\}$. Then,

$$\psi(r - f) = (a_t^T r - a_t^T f)/(b_t - a_t^T f) \leq (b_t - a_t^T f)/(b_t - a_t^T f) = 1$$

and therefore $r \in S(\psi_B, f)$. Since $B \cap \mathbb{Q}^m \subseteq S(\psi_B, f)$ then $B = cl(B \cap \mathbb{Q}^m) \subseteq S(\psi_B, f) \Rightarrow S(\psi_B, f) = B$. ∎

The following simple observation will be used next in the proof of Lemma 2.8.

Remark 2.7 *A vector $r \in \mathbb{Q}^m$, is contained in $RC(B)$ if and only if $a_i^T r \leq 0$ for all $i \in I$, and $r \in int(RC(B))$ if and only if $a_i^T r < 0$ for all $i \in I$.*

Note that, it also follows from the above remark that $\psi_B(r) < 0$ if and only if $r \in int(RC(B))$ and, in light of Lemma 2.4, this means that $r \in int(RC(B)) \Rightarrow r \ngeq 0$. So we were able to derive a geometric property of polyhedral NLPF sets by using the function ψ_B.

We next show that the function ψ_B is actually a minimal valid function if B is maximal.

Lemma 2.8 *If B is maximal NLPF, then ψ_B is a minimal valid function for R_f^+.*

Proof. Suppose not and let ψ be a minimal valid function for R_f^+ such that $\psi \leq \psi_B$ and $\psi(\bar{r}) < \psi_B(\bar{r})$ for some $\bar{r} \in \mathbb{Q}^m$. We next consider two cases.

<u>*Case 1:*</u> $\bar{r} \notin RC(B)$.

For simplicity, let $S = S(\psi, f)$. By Lemma 2.4, we have $\psi(\bar{r}), \psi_B(\bar{r}) > 0$ and by positive homogeneity of ψ_B and ψ, we have that there exist $\mu > \lambda > 0$ such that $\psi_B(\lambda\bar{r}) = \psi(\mu\bar{r}) = 1$. Let $\bar{x} = f + \lambda\bar{r}$ and let $\bar{\bar{x}} = f + \frac{\mu+\lambda}{2}\bar{r}$. Then $\psi_B(\bar{\bar{x}} - f) > 1$, which implies that $\bar{\bar{x}} \notin B$. But $\psi(\bar{\bar{x}} - f) < 1$, which implies $\bar{\bar{x}}$ is in the interior of $cl(S)$. It follows that B is strictly contained in $cl(S)$. By Remark 2.3, $cl(S)$ is a NLPF set. This contradicts the assumption that B is a maximal NLPF set. Therefore $\psi(r) = \psi_B(r)$ for all $\bar{r} \notin RC(B)$.

<u>*Case 2:*</u> $\bar{r} \in RC(B)$.

First note that for all $i \in I$ there exists a vector $v^i \in \mathbb{Q}^m \setminus RC(B)$ such that $\psi_B(v^i) = \hat{a}_i^T v^i \geq \hat{a}_t^T v^i$ for all $t \in I$. To show that v^i exists, we use the fact that $a_i^T x \leq b_i$ is facet defining for B and therefore there exists a point x^i such that $a_i^T x^i = b_i$ and $a_t^T x^i \leq b_t$ for all $t \neq i$. Then $v^i = x^i - f$ satisfies the desired properties as

$$\hat{a}_i^T v^i = \frac{a_i^T x^i - a_i^T f}{b_i - a_i^T f} = 1 \ \text{ and } \ \hat{a}_t^T v^i = \frac{a_t^T x^i - a_t^T f}{b_t - a_t^T f} \leq \frac{b_t - a_t^T f}{b_t - a_t^T f} = 1$$

for all $t \in I$. The fact that $v^i \notin RC(B)$ follows from the fact that $a_i^T v^i = \hat{a}_i^T v^i (b_i - a_i^T f) > 0$.

If $\bar{r} \in bd(RC(B))$, we have that $a_t^T \bar{r} \leq 0$ for all $t \in I$, with $a_i^T \bar{r} = 0$ for some $i \in I$. In this case, $\psi_B(\bar{r}) = \hat{a}_i^T \bar{r} = 0$. Note that $a_i^T(v^i + \bar{r}) = a_i^T v^i > 0$ and hence $(v^i + \bar{r}) \notin RC(B)$. Moreover, note that

$$\hat{a}_i^T(v^i + \bar{r}) = \frac{a_i^T v^i}{b_i - a_i^T f} = 1 \text{ and } \hat{a}_t^T(v^i + \bar{r}) = \frac{a_t^T v^i + a_t^T \bar{r}}{b_i - a_i^T f} \leq \frac{a_t^T v^i}{b_i - a_i^T f} \leq 1$$

for all $t \in I$ and therefore $\psi_B(v^i + \bar{r}) = \hat{a}_i^T(v^i + \bar{r})$. Since ψ is minimal, it is subadditive and hence $\psi(v^i) + \psi(\bar{r}) \geq \psi(v^i + \bar{r})$. But then, $\psi(\bar{r}) \geq \psi(v^i + \bar{r}) - \psi(v^i) = \psi_B(v^i + \bar{r}) - \psi_B(v^i) = 0 = \psi_B(\bar{r}) > \psi(\bar{r})$, a contradiction. Hence $\psi(\bar{r}) = \psi_B(\bar{r})$ for all $\bar{r} \notin int(RC(B))$.

If $\bar{r} \in int(RC(B))$, let i be such that $\psi_B(\bar{r}) = \hat{a}_i^T \bar{r}$. Since $\bar{r} \in int(RC(B))$, we have $a_i^T \bar{r} < 0$. By the choice of v^i, we have that $a_i^T v^i = \hat{a}_i^T v^i (b_i - a_i^T f) > 0$. Let $\alpha = |a_i^T \bar{r}| / a_i^T v^i$ and note that $a_i^T(\bar{r} + \alpha v^i) = 0$ implying that $(\bar{r} + \alpha v^i) \notin int(RC(B))$ and hence $\psi(\bar{r} + \alpha v^i) = \psi_B(\bar{r} + \alpha v^i) \geq 0$. As ψ is valid and therefore subadditive, we have

$$\psi(\bar{r}) + \psi(\alpha v^i) = \psi(\bar{r}) + \psi_B(\alpha v^i) = \psi(\bar{r}) + \alpha \hat{a}_i^T v^i \geq \psi(\bar{r} + \alpha v^i).$$

As $\psi(\bar{r} + \alpha v^i) = \psi_B(\bar{r} + \alpha v^i) \geq 0$, we have

$$\psi(\bar{r}) + \alpha \hat{a}_i^T v^i \geq 0 \implies \psi(\bar{r}) \geq -\alpha \hat{a}_i^T v^i = \hat{a}_i^T \bar{r} = \psi_B(\bar{r}) > \psi(\bar{r}),$$

again a contradiction. ■

We end this section revisiting the example presented in Section 1 to illustrate how the function ψ_B defined in (1) can lead to valid (and sometimes facet defining) inequalities for X^+ that dominate the ones obtained by the results in [7, 8].

Example 1.1 (continued) *Remember the set*

$$X = \left\{(x,s) \in \mathbb{Z}_+^2 \times \mathbb{R}_+^5 \; : \; \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \sum_{i=1}^{5} r_i s_i = f \right\} \tag{2}$$

where f and r_i are defined in Section 1. As shown in Figure 1(a), the triangle T defined by the corner points p_1, p_2, p_3 is a maximal lattice point free set in $\mathbb{R}^2$. Notice that $p_i = f + r_i$ for $i = 1, \dots, 5$ and consequently it follows from the results in [8] that the inequality $s_1 + s_2 + s_3 + s_4 + s_5 \geq 1$ is valid and facet-defining for X.

In comparison, notice that the translated cone C (shown in Figure 1(b)) defined by the rays $\overrightarrow{p_1p_2}$ and $\overrightarrow{p_1p_3}$ is a maximal NLPF set. This set can be written as

$$C = \{x \in \mathbb{R}^2 : -x_1 \leq 0, x_1 + x_2 \leq 1\}$$

and notice that $p_4 \in RC^o(C)$ and $p_5 \in RC(C) \setminus RC^o(C)$. The set C leads to the minimal valid function

$$\psi_C(r) \;=\; \max\left\{\frac{r^T \cdot [-1,0]^T}{0 - [-1,0] \cdot f}\,,\; \frac{r^T \cdot [1,1]^T}{1 - [1,1] \cdot f}\right\} \;=\; \max\left\{r^T \cdot \begin{bmatrix} -4 \\ 0 \end{bmatrix},\; r^T \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix}\right\}$$

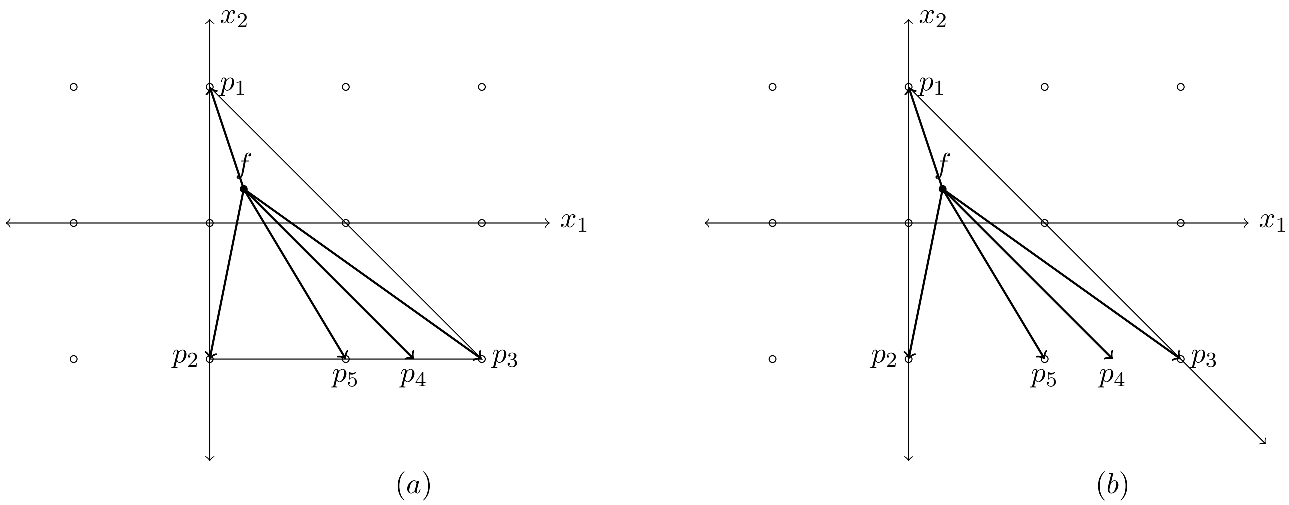

Figure 1: (a) A maximal lattice point free set T and
. (b) a maximal NLPF set C in $\mathbb{R}^2$, both containing $f > 0$.

which gives the following stronger valid inequality for $X^+ = X \cap \mathbb{R}^+_7$,

$$s_1 + s_2 + s_3 - s_5 \geq 1.$$

Furthermore, this inequality is facet defining as the dimension of X^+ is 5, and the following 5 affinely independent points are in X^+ and satisfy the inequality as equality: $p_1 = [0, 1; 1, 0, 0, 0, 0]$, $p_2 = [0, 0; 1/2, 1/2, 0, 0, 0]$, $p_3 = [1, 0; 1/2, 0, 1/2, 0, 0]$, $p_4 = [1, 0; 1, 0, 0, 4/5, 0]$, $p_5 = [1, 0; 3, 0, 0, 0, 2]$.

Notice that $r_5 \in int(RC(C))$ and this is the reason why s_5 can get a negative coefficient in the cut (see Lemma 2.4). In fact, it also follows from Lemma 2.4 that to get negative coefficients, we must have $r \not\geq 0$.

2.3.1 Properties of ψ_B

In this Section, we focus our attention on some properties and geometric interpretations of the function ψ_B. Moreover some of the Lemmas from this section will be helpful in future sections.

We start with observing that the definition of ψ_B given by (1) coincides with the function used in [7] to map lattice-point free convex sets to minimal valid functions for X. The following remark follows from Lemma 2.4, Remark 2.6, and the fact that ψ_B is positively homogeneous and subadditive.

Remark 2.9 *If B is a polyhedral NLPF, then for $r \notin int(RC(B))$*

$$\psi_B(r) := \begin{cases} 0 & \text{if } r \in RC^o(B) \\ 1/\max\{\lambda \in \mathbb{R}_+ \ : \ f + \lambda r \in B\} & \text{if } r \notin RC(B). \end{cases}$$

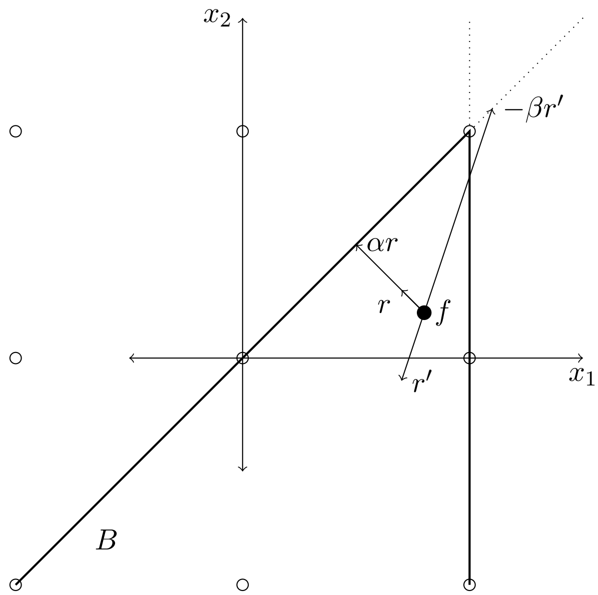

Figure 2: A polyhedral NLPF set B and the function values $\psi_B(r) = \frac{1}{\alpha}$ and $\psi_B(r') = -\frac{1}{\beta}$

Notice that if B is a polyhedral lattice point free set, then $int(RC(B)) = \emptyset$, since such sets cannot have full-dimensional recession cones. Therefore the function ψ_B is identical to the function defined in [7] when B is lattice point free.

Remark 2.9 gives a geometric interpretation of the value of $\psi_B(r)$ when $r \notin int(RC(B))$. When $r \in int(RC(B))$, remember that we have $\psi_B(r) < 0$. In this case, we can give the following geometric description of ψ_B stated in Lemma 2.10. Figure 2 shows these geometric interpretations.

Lemma 2.10 *Let $r \in int(RC(B))$. Then $\psi_B(r) = -1/\lambda'$ where $\lambda' > 0$ is the largest scalar for which there exists at least one index $i \in I$ such that $a_i^T(f - \lambda' r) \leq b_i$.*

Proof. Let $\hat{\lambda} = -1/\psi_B(r)$ and let $l \in \arg\max_{i \in I} \left\{a_i^T r/(b_i - a_i^T f)\right\}$. Therefore

$$a_l^T(f - \hat{\lambda} r) = a_l^T f + \frac{1}{\psi_B(r)} a_l^T r = a_l^T f + \frac{b_l - a_l^T f}{a_l^T r} a_l^T r = b_l$$

and therefore we have $a_i^T(f - \hat{\lambda} r) \leq b_i$ for at least one $i \in I$. Now let $\lambda > \hat{\lambda}$ and as $r \in int(RC(B))$ we have $a_i^T r < 0$ for all $i \in I$ and

$$a_i^T(f - \lambda r) = a_i^T f - \lambda a_i^T r > a_i^T f + \frac{1}{\psi_B(r)} a_i^T r \geq a_i^T f + \frac{b_i - a_i^T f}{a_i^T r} a_i^T r = b_i.$$

Therefore, if $\lambda > \hat{\lambda}$, the condition $a_i^T(f - \lambda r) \leq b_i$ is not satisfied by any $i \in I$, implying that $\hat{\lambda}$ indeed is the largest scalar for which $a_i^T(f - \hat{\lambda} r) \leq b_i$ for at least one $i \in I$. ∎

We end this section by noting that for $r \in RC(B)$, the value of $\psi_B(r)$ is determined by the inequalities that define facets of $RC(B)$. Notice that this is the reason why we need a non-redundant inequality description of B. This property will be used later in Section 4 to strengthen inequalities for R_f^+ .

Lemma 2.11 *Let $r \in RC(B)$. Then $\psi_B(r) = a_l^T r/(b_l - a_l^T f)$ for some $l \in I$ such that $a_l^T x \leq 0$ is facet-defining for $RC(B)$.*

Proof. Since $r \in RC(B)$, we have by Lemma 2.4 that $\psi(r) \leq 0$. If $r \notin int(RC(B))$, then $a_l^T r = 0$ for some facet $a_l^T x \leq 0$ of $RC(B)$ and $\psi_B(r) = \max_{i \in I} \left\{a_i^T r/(b_i - a_i^T f)\right\} = 0$ and the result follows. Thus, we may assume $r \in int(RC(B))$. Let $I^c \subseteq I$ be such that $a_i^T x \leq 0$ defines a facet of $RC(B)$ if and only if $i \in I^c$. If $\psi_B(r) > a_i^T r/(b_i - a_i^T f)$ for all $i \in I^c$, then let $j \notin I^c$ be such that $\psi_B(r) = a_j^T r/(b_j - a_j^T f)$. Since $r \in int(RC(B))$ we have that $a_i^T r < 0$ for all $i \in I^c$ and since $a_j^T r \leq 0$ is not a facet of $RC(B)$, there exists $\lambda \in \mathbb{R}_+^{|I^c|}$ such that $\lambda \neq 0$ and $a_j = \sum_{i \in I^c} \lambda_i a_i$. Thus $a_j r < 0$. Moreover since $(b_i - a_i^T f) > 0$ we have that $\psi_B(r) < 0$ and hence $b_j = a_j^T (f + \frac{1}{\psi_B(r)} r)$ and $b_i < a_i^T (f + \frac{1}{\psi_B(r)} r)$ for all $i \in I^c$. However, for all $j \notin I^c$, we have that there exist $\mu_i \geq 0, \forall i \in I^c$ such that $a_j = \sum_{i \in I^c} \mu_i a_i$. Therefore $b_j = a_j^T (f + \frac{1}{\psi_B(r)} r) = \sum_{i \in I^c} \mu_i a_i^T (f + \frac{1}{\psi_B(r)} r) \geq \sum_{i \in I^c} \mu_i b_i$. But this contradicts the fact that $a_j^T x \leq b_j$ defines a facet of B. Therefore, there exists $l \in I^c$ such that $a_l^T (f - \lambda r) = b_l$. ∎

3 Special case: $m = 2$

In the previous section we have established that minimal valid functions for R_f^+ give rise to NLPF convex sets and maximal polyhedral NLPF sets give minimal valid functions for R_f^+. In particular, this last fact shows that there exists a mapping from maximal polyhedral NLPF sets to minimal valid functions and such mapping is injective (follows from Lemma 2.9). Moreover, Remark 2.6 shows that $S(\psi_B, f)$ gives the inverse mapping. To complete the picture, we would like to see if this mapping is a bijection, that is, if all minimal valid functions ψ for R_f^+ can be defined by a polyhedral maximal NLPF set of the form $B = S(\psi, f)$.

We are able to answer this question for $m = 2$, which is the case we consider in this section. Let $B \subseteq \mathbb{R}^2$ be a full-dimensional closed convex set (not necessarily polyhedral) that is NLPF and has f in its interior. Note that even if B is not polyhedral, any cone in $\mathbb{R}^2$ is polyhedral. Furthermore, if the recession cone of B is full-dimensional, that is, if $int(RC(B)) \neq \emptyset$, then $RC(B) = \{x \in \mathbb{R}^2 : c_i^T x \leq 0, i = 1, 2\}$, where $c_i^T x \leq 0$ defines a facet of $RC(B)$ for $i \in J = \{1, 2\}$. In this case, let $d_i = \sup\{c_i^T x : x \in B\}$ for $i \in J$ and note that $B \subseteq C = \{x \in \mathbb{R}^2 : c_i^T x \leq d_i, i \in J\}$. We now let $\hat{c}_i = c_i/(d_i - c_i^T f)$ and note that, as f is in the interior of B, we have $c_i^T f < d_i$. We now extend the definition of the function ψ_B in two dimensions as follows:

$$\psi_B(r) := \begin{cases} \max\limits_{i \in J} \left\{r^T \hat{c}_i\right\} & \text{if } r \in int(RC(B)) \\ 1/\max\{\lambda \in \mathbb{R}_+ \ : \ f + \lambda r \in B\} & \text{if } r \notin RC(B) \\ 0 & \text{otherwise.} \end{cases}$$

It is not hard to show that if B is polyhedral, the above definition coincides with the one in Section 2.3. To see that $\psi_B(r)$ is subadditive and positively homogeneous, just notice that $B =$

$\{x : c_i^T x \leq d_i, i \in I\}$, where I is a (possibly infinite) index set. Then

$$\psi_B(r) = \sup_{i \in I \cup J} \{\hat{c}_i^T r\}$$

and the result follows. We next show that essentially all minimal valid functions have the form above. Then in the next section, we will actually show that all maximal NLPF convex sets in $\mathbb{R}^2$ are in fact polyhedral and conclude afterwards with the desired result that all minimal valid functions can be defined as the function ψ_B for a polyhedral maximal NLPF set.

Lemma 3.1 *Let $\psi : \mathbb{Q}^2 \to \mathbb{R}$ be a minimal valid function such that the NLPF set $B = S(\psi, f)$ contains f in its interior. Then $\psi = \psi_B$.*

Proof. By Lemma 2.4 we have $\psi(r) = \psi_B(r)$ for all $r \notin int(RC(B))$. We therefore consider $r \in int(RC(B))$. Let $\epsilon \in (0, 1/2)$. We first construct vectors $v^i \notin RC(B)$, for $i \in J$, that satisfy the following properties: (i) $\hat{c}_i^T v^i \geq \hat{c}_t^T v^i$ for all $t \in J$ and (ii) $\psi_B(v^i) \leq \hat{c}_i^T v^i + \epsilon$. Remember that $d_i = \sup\{c_i^T x : x \in B\}$ and as $\epsilon(d_i - c_i^T f) > 0$ there exists $x^i \in B$ such that $c_i^T x^i \geq d_i - \epsilon(d_i - c_i^T f)$. As $x^i \in B$, we also have $c_t^T x^i \leq d_t$ for all $t \in J$. Furthermore, as $\{x \in \mathbb{R}^2 : c_i^T x \leq 0, i \in J\}$ is a minimal polyhedral representation of $RC(B)$, it is possible to pick points $r^i \in RC(B)$ for all $i \in I$ such that (i) $c_i^T r^i = 0$ and (ii) $c_t^T r^i < 0$ for all $t \neq i$. Now let $\lambda > -\epsilon(d_t - c_t^T f)/c_t^T r^i$ for all $t \neq i$, and note that $v^i = x^i + \lambda r^i - f$ satisfies the first desired property as

$$\hat{c}_i^T v^i = \frac{c_i^T x^i - c_i^T f}{d_i - c_i^T f} \geq \frac{d_i - c_i^T f}{d_i - c_i^T f} - \epsilon = 1 - \epsilon = \frac{d_t - c_t^T f}{d_t - c_t^T f} - \epsilon \geq \frac{c_t^T x^i + \lambda c_t^T r^i - c_t^T f}{d_t - c_t^T f} = \hat{c}_t^T v^i.$$

Also note that as $c_i^T v^i = \hat{c}_i^T v^i (d_i - c_i^T f) > 0$, it follows that $v^i \notin RC(B)$. Moreover, as $v^i + f = x^i + \lambda r^i$ where $x^i \in B$ and $r^i \in RC(B)$, we have $f + v^i \in B$, implying $\max\{\lambda \in \mathbb{R}_+ : f + \lambda v^i \in B\} \geq 1$ and therefore $\psi_B(v^i) \leq 1 \leq \hat{c}_i^T v^i + \epsilon$.

Let i be such that $\psi_B(r) = \hat{c}_i^T r$. Since $r \in int(RC(B))$, we have $c_i^T r < 0$ and remember that $c_i^T v^i = \hat{c}_i^T v^i (d_i - c_i^T f) > 0$. Let $\alpha = |c_i^T r| / c_i^T v^i$ and note that $c_i^T (r + \alpha v^i) = 0$ implying that $(r + \alpha v^i) \notin int(RC(B))$ and hence $\psi(r + \alpha v^i) = \psi_B(r + \alpha v^i) \geq 0$. Also remember that $v^i \notin RC(B)$ and therefore $\psi(v^i) = \psi_B(v^i)$. As ψ is a minimal valid function, it is subadditive, and therefore have

$$\psi(r) + \alpha \hat{c}_i^T v^i \geq \psi(r) + \alpha \psi_B(v^i) - \alpha\epsilon = \psi(r) + \psi(\alpha v^i) - \alpha\epsilon \geq \psi(r + \alpha v^i) - \alpha\epsilon \geq -\alpha\epsilon$$

implying

$$\psi(r) \geq -\alpha \hat{c}_i^T v^i - \alpha\epsilon = \hat{c}_i^T r - \alpha\epsilon = \psi_B(r) - \alpha\epsilon.$$

Notice that since $\hat{c}_i^T v^i \geq 1 - \epsilon > 1/2$, we have that $\alpha = |c_i^T r| / c_i^T v^i = |\hat{c}_i^T r| / \hat{c}_i^T v^i \leq 2|\hat{c}_i^T r|$. Hence $\psi(r) \geq \psi_B(r) - 2|\hat{c}_i^T r|\epsilon$. Since this is valid for any $\epsilon > 0$, it follows that $\psi(r) \geq \psi_B(r)$. ∎

3.1 Maximal NLPF sets in $\mathbb{R}^2$

We next study maximal NLPF convex sets in $\mathbb{R}^2$ and show that they are polyhedra with a small number of facets. More precisely, the main result of this section is the following theorem, which is similar to theorems by Bell [6], Doignon [14], Lovász [20], and Scarf [22] for maximal lattice-free convex sets. We will use this result in Section 3.2 to characterize $S(\psi, f)$ for minimal valid functions ψ.

Theorem 3.2 *A maximal NLPF set in $\mathbb{R}^2$ is either a full-dimensional polyhedron with at most 4 facets or an irrational hyperplane.*

The rest of this section is devoted to the proof of Theorem 3.2. We first study full-dimensional maximal NLPF convex sets and show that we can restrict ourselves to the case where such sets contain a strictly positive point in the interior.

Lemma 3.3 *Let $K \subseteq \mathbb{R}^m$ be a full-dimensional maximal NLPF convex set. If there does not exist a point $f > 0$ in $int(K)$, then K is a half-space.*

Proof. Notice first that if K contains a point $f' > 0$, then K contains a point $f > 0$ in its interior. Indeed pick $y \in int(K)$ and since $f_\lambda = \lambda y + (1-\lambda)f' \in int(K)$ for all $\lambda \in (0,1)$, we can pick λ arbitrarily close to 1 such that $f_\lambda > 0$.

Therefore, there exists a supporting hyperplane $ax = b$ for K such that $ax \leq b$ for all $x \in K$ and $ax \geq b$ for all $x \in cl(\{x \in \mathbb{R}^m : x > 0\}) = \{x \in \mathbb{R}^m : x \geq 0\}$. But then $\{x \in \mathbb{R}^m : ax \leq b\} \supseteq K$ and does not contain any nonnegative integer points in its interior, hence by maximality of K, $K = \{x \in \mathbb{R}^m : ax \leq b\}$. ∎

We now show that, independent of the dimension m, maximal NLPF convex sets are polyhedral under certain conditions on their recession cones.

Lemma 3.4 *Let $K \subseteq \mathbb{R}^m$ be a maximal NLPF convex set. If $RC(K) \cap \mathbb{R}^m_+ = \{0\}$, then K is polyhedral.*

Proof. First note that $K \neq \emptyset$ as it is maximal. Moreover, $K \cap \mathbb{R}^m_+$ cannot be empty, otherwise the convex hull of K and the origin contains K and is a NLPF convex set, a contradiction. Since $K \cap \mathbb{R}^m_+ \neq \emptyset$, we have that the condition $RC(K) \cap \mathbb{R}^m_+ = \{0\}$ is equivalent to $K \cap \mathbb{R}^m_+$ is bounded.

As $K \cap \mathbb{R}^m_+$ is bounded, there exists numbers $u_i \in R_+$ for all $i \in I = \{1, \ldots, n\}$ such that $x_i \leq u_i$ for all $x \in K \cap \mathbb{R}^m_+$. For $i \in I$, define the sets $C_i = \{x \in \mathbb{R}^m \ : \ x \geq 0, \ x_i \geq u_i + 1\}$. Note that if K is a NLPF set, so is its closure and therefore by maximality, K has to be closed. Therefore K and all C_i are non-empty, convex and closed sets. Furthermore, for all $i \in I$ the sets K and C_i are pairwise disjoint and have no common directions of recession.

Therefore, for each $i \in I$ there exists a hyperplane $(\alpha^i)^T x = \beta^i$ that strongly separates K and C_i (see, for example, [21] Separation Theorems). In other words, there exists $\alpha^i \in \mathbb{R}^m$ and $\beta^i \in \mathbb{R}$ such that for all $x' \in K$ and $x'' \in C_i$ we have $(\alpha^i)^T x' < \beta^i$ and $(\alpha^i)^T x'' > \beta^i$. Notice that for all $i, j \in I$ the unit direction e_j is a direction of recession for C_i and therefore $\alpha^i \geq 0$ for all $i \in I$.

As $K \cap \mathbb{R}^m_+$ is not empty, there exists some $\bar{x} \in K \cap \mathbb{R}^m_+$. Combining this with $\alpha^i \geq 0$ and $(\alpha^i)^T \bar{x} < \beta^i$, we therefore have $\beta^i > 0$ for all $i \in I$. Finally, let $\tilde{x}^i$ be a vector of all zeroes except the i'th component which is equal to $u_i + 1$. Note that $\tilde{x}^i \in C_i$ and as $(\alpha^i)^T \tilde{x}^i > \beta^i > 0$, we have $\alpha^i_i > 0$.

Now, let $\bar{\alpha} = \sum_{i \in I} \alpha^i$ and $\bar{\beta} = \sum_{i \in I} \beta^i$ and note that $\bar{\alpha}^T x < \bar{\beta}$ for all $x \in K$. Therefore, $K \cap \mathbb{R}^m_+ \subseteq X = \{x \in \mathbb{R}^m \ : \ x \geq 0, \ \bar{\alpha}^T x \leq \bar{\beta}\}$. Let $X^L = X \cap Z^m_+$ be the collection of lattice points in X and note that X^L contains a finite number of points as $\bar{\alpha} > 0$ and $\bar{\beta} > 0$. As K does not contain non-negative lattice points in its interior, for each $p \in X^L$, there exists a closed half-space defined by $(\alpha^p)^T x \leq \beta^p$ that contains K and has p on its boundary. Therefore the following polyhedral set

$$P = \{x \in \mathbb{R}^m \ : \ \bar{\alpha}^T x \leq \bar{\beta}, \ (\alpha^p)^T x \leq \beta^p \text{ for all } p \in X^L\}$$

contains K and does not contain any non-negative lattice points. As K is assumed to be maximal, $K = P$ and the proof is complete. ∎

Notice that if B is a full-dimensional maximal NLPF set with $dim(RC(B)) = 0$, then $RC(B) \cap \mathbb{R}^m_+ = \{0\}$ and hence Lemma 3.4 implies that B is polyhedral. Lemmas 3.5 and 3.6 complete the proof that B is polyhedral by considering other possible dimensions of $RC(B)$.

Lemma 3.5 *Let $S \subseteq \mathbb{R}^2$ be a NLPF set such that there is a point $f > 0$ in its interior. If $dim(RC(S)) = 2$ then $RC(S) \cap \mathbb{R}^2_+ = \{0\}$.*

Proof. Suppose there is a vector $v \in RC(S)$ such that $v \geq 0$ and $v \neq 0$. Since $v \neq 0$, we may assume, by symmetry, that $v_1 = 1$. Since $RC(S)$ is full-dimensional, there exists a nonzero vector u such that $u_1 = 0$ and such that $v + \epsilon u \in RC(S)$ for some ϵ small enough. Now, for any $\alpha > 0$ we have that $w = f + \alpha v \in S$ and $z = f + \alpha(v + \epsilon)u \in S$. But then choose $\alpha > 1/|\epsilon u_2|$ such that $f_1 + \alpha \in \mathbb{Z}_+$. Then $|w_2 - z_2| = |\alpha \epsilon u_2| > 1$. Since $w_1 = z_1 = f_1 + \alpha \in \mathbb{Z}_+$, then we have a nonnegative integer point in the interior of the line segment between w and z and hence a nonnegative integer point in the interior of S, which is a contradiction. ∎

Lemma 3.6 *Let $S \subseteq \mathbb{R}^2$ be a maximal NLPF set that contains a point $f > 0$ in its interior. If $dim(RC(S)) = 1$, then S is a polyhedron.*

Proof. If for all $r \in RC(S)$ we have that $r \not\geq 0$, then by Lemma 3.4 the result follows. Thus, we may assume that there exists $r \in RC(S)$ such that $r \geq 0$. In addition, we can assume that there exists a point $\bar{y} \in \mathbb{Z}^2$ in the interior of S such that $\bar{y} \not\geq 0$, since otherwise, S is maximal lattice-free and hence, by [20], it is polyhedral. We will next show that if all these assumptions are made, then S has a nonnegative lattice point in its interior, which is a contradiction.

<u>*Case 1:*</u> r has one zero component.

Without loss of generality, assume that $r_1 = 0$. In addition, after scaling, we can assume that $r_2 = 1$. In this case, if $\bar{y}_1 \geq 0$, then $\bar{y} + |\bar{y}_2| r$ is a nonnegative integer point in the interior of S, which is a contradiction. Therefore, we may assume that $\bar{y}_1 < 0$. But since $f > 0$ is a point in the interior of S, then there exists a point w in the interior of S such that $w_1 = 0$. But then there exists $\lambda > 0$ such that $w + \lambda r$ is a nonnegative integer point in the interior of S.

Case 2: $r > 0$.

If r is rational, then we may assume that r is integer and thus, there exists $\lambda \in \mathbb{Z}_+$ such that $\bar{y} + \lambda r$ is a nonnegative integer point in the interior of S. Thus, we may assume that r is not rational. Without loss of generality, let $r_1 = 1$.

Now consider the line $-r_2 x_1 + x_2 = b$ generated by $\bar{y} + \lambda r$ for $\lambda \in \mathbb{R}$. Note that r_2 and $b = -r_2 \bar{y}_1 + \bar{y}_2$ are irrational numbers. From the approximation of r_2 by continued fractions (see for instance [23]), it follows that there exists a sequence (p_n, q_n) such that $p_n \in \mathbb{Z}_+$ and $q_n \in \mathbb{Z}_+$ and $\lim_{n\to\infty} p_n = \lim_{n\to\infty} q_n = \infty$ and such that $0 \leq \frac{p_n}{q_n} - r_2 \leq \frac{1}{q_n^2}$. Since $\bar{y}$ is in the interior of S, there exists $\epsilon > 0$ such that if $||x - \bar{y}||_2 \leq \epsilon$, then $x \in int(S)$.

But then, pick n large enough such that $1/q_n < \epsilon$ and $p_n > |\bar{y}_2|$, $q_n > |\bar{y}_1|$. Notice that the point $w = (\bar{y}_1 + q_n, \bar{y}_2 + p_n)$ is a nonnegative integer point. Moreover, $w = x + q_n r$ where $x = (\bar{y}_1, \bar{y}_2 + p_n - r_2 q_n)$ and since $0 \leq \frac{p_n}{q_n} - r_2 \leq \frac{1}{q_n^2} \Rightarrow 0 \leq p_n - r_2 q_n \leq \frac{1}{q_n} < \epsilon$, we have that $||x - \bar{y}||_2 \leq \epsilon$ so $x \in int(S)$. This in turn implies that $w \in int(S)$, which contradicts the fact that S does not have nonnegative integer points in its interior. ∎

Lemmas 3.4, 3.5 and 3.6 show that in $\mathbb{R}^2$ any maximal NLPF set that contains $f > 0$ in its interior is polyhedral. The following corollary follows immediately from the proofs of Lemmas 3.5 and 3.6 and will be used to bound the number of facets that such a maximal NLPF set has.

Corollary 3.7 *If $S \subseteq \mathbb{R}^2$ is a full-dimensional maximal NLPF set then S is either a maximal lattice-free convex set or $RC(S) \cap \mathbb{R}^2_+ = \{0\}$.*

We now use Corollary 3.7 to show that, when $m = 2$, maximal NLPF sets have a nonnegative integer point in the relative interior of each of their facets, which will imply that there are at most 4 facets. Notice that the fact that a polyhedral maximal NLPF set has at most 2^m facets for general m can be proven by adapting the proof of a theorem in Schrijver [24] (credited to Bell [6], Doignon [14] and Scarf [22]) directly, without using this fact. However, this fact is helpful in identifying when a NLPF set is not maximal and will be used in Section 4 where we are concerned with strengthening inequalities that are defined by non-maximal NLPF sets.

Lemma 3.8 *Let $B = \{x \in \mathbb{R}^2 : a_i^T x \leq b_i, \forall i = 1, \ldots, k\}$ be a full-dimensional maximal NLPF set. Then there exists a nonnegative integer point in the relative interior of each one of its facets.*

Proof. If B is a maximal lattice-free convex set, then the result was proven by Bell [6], Doignon [14] and Scarf [22], so we may assume that B is not a maximal lattice-free convex set and hence, by Corollary 3.7, $RC(B) \cap \mathbb{R}^2_+ = \{0\}$. Without loss of generality we assume the inequality description of B is minimal and each inequality describes a facet. We also assume that $b_i \in \mathbb{Q}$ for all $i \in I = \{1, \ldots, k\}$. Consider the face $F_j = B \cap \{x \in \mathbb{R}^2 : a_j^T x = b_j\}$ defined by the jth inequality and assume that F_j does not contain a nonnegative integer point in its relative interior. Let $F_j^+ = F_j \cap \mathbb{R}^2_+$. Notice that $RC(F_j^+) \subseteq RC(B) \cap \mathbb{R}^2_+$ and hence $RC(F_j^+) = \{0\}$ so F_j^+ is bounded. We next consider 2 cases.

<u>*Case 1:*</u> $a_j \in \mathbb{Q}^2$. In this case, let τ be such that $\tau a_j \in \mathbb{Z}^2$ and consider replacing $a_j^T x \leq b_j$ in the description of B with $\tau a_j^T x \leq \tau b_j + 1/2$. Clearly, the new set contains B strictly and is NLPF, a contradiction.

<u>*Case 2:*</u> $a_j \notin \mathbb{Q}^2$. Consider the set

$$\Delta \;=\; \{x \in \mathbb{R}^m_+ \;:\; a_i^T x \leq b_i \forall i \in I \setminus \{j\},\; a_j^T x > b_j,\; a_j^T x \leq b_j + 1\}$$

and let $T = \Delta \cap \mathbb{Z}^m$. Notice that T is the set of all nonnegative integer points that would be included in B if b_j is increased by 1.

Now $RC(\Delta) = RC(F_j^+)$ so Δ is bounded and hence T is finite. If $T = \emptyset$ then replacing b_j in the description of B with $b_j + 1$ gives a strictly larger NLPF convex set, a contradiction. If $T \neq \emptyset$ then let $\hat{b}_j = \min_{x \in T}\{a_j^T x\} > b_j$ and note that in this case, we can replace b_j in the description of B with $\hat{b}_j$ to obtain a strictly larger NLPF convex set, again a contradiction. ∎

From Lemma 3.8 it is straightforward to obtain a bound on the number of facets of maximal NLPF sets by the following simple argument due to Bell [6] (also see Borozan and Cornuéjols [7]).

Lemma 3.9 *Let $B \in \mathbb{R}^2$ be a full-dimensional maximal NLPF convex set. Then it is a polyhedron with at most 4 facets.*

Proof. Each facet F of B has a point x^F in its relative interior. If there are more than 4 facets, two nonnegative integral points x^F and $x^{F'}$ must be identical modulo 2. Then their middle point $\frac{1}{2}(x^F + x^{F'})$ is integral, nonnegative and interior, which is a contradiction. ∎

The following lemma is true for any arbitrary number of rows, and is just stated here for completeness of the characterization of maximal NLPF sets in $\mathbb{R}^2$.

Lemma 3.10 *Let S be a maximal NLPF set that is not full-dimensional. Then S is an irrational hyperplane.*

Proof. If S is not full dimensional then all $x \in S$ satisfy $a^T x = b$ for some $b \in \mathbb{R}$ and $a \in \mathbb{R}^m$. Therefore $S \subseteq \{x \in \mathbb{R}^m \;:\; a^T x = b\}$ and as S is maximal NLPF, $S = \{x \in \mathbb{R}^m \;:\; a^T x = b\}$. If b is not integral, it is possible to rewrite the equation defining S as $(1/b)a^T x = 1$ and therefore, without loss of generality, we assume that $b \in \mathbb{Z}$. Now, if a is rational there exists a large enough $\tau \in \mathbb{Z}$ such that $\tau a \in \mathbb{Z}^m$. In this case, $S \subset \{x \in \mathbb{R}^m \;:\; \tau a^T x \geq \tau b,\, \tau a^T x \leq \tau b + 1\}$ which contradicts the maximality of S. Therefore, $a \notin \mathbb{Q}^m$, and S indeed is an irrational hyperplane. ∎

3.2 Minimal valid functions for $m = 2$

We are finally ready to characterize minimal valid functions for R_f^+ by relating them to maximal NLPF convex sets, as stated in the following theorem.

Theorem 3.11 *Let $\psi : \mathbb{Q}^2 \to \mathbb{R}$ be a minimal valid function for R_f^+. If $S(\psi, f)$ contains f in its interior, then $S(\psi, f)$ is a maximal NLPF convex set.*

Proof. Let $B = S(\psi, f)$ and note that as ψ is a valid function, B is NLPF. Also remember that, by Lemma 3.1, $\psi = \psi_B$. For the sake of contradiction, assume that B is not maximal, and let B' be a maximal NLPF set strictly containing B. If $int(RC(B)) = \emptyset$, then $\psi_{B'}$ dominates ψ_B, a contradiction. Therefore, we assume that $RC(B)$ is full-dimensional.

Let $RC = \{x \in \mathbb{R}^2 : c_i^T x \leq 0, i \in J\}$ be a minimal description of the recession cone of B and let $d_i = \sup\{c_i^T x : x \in B\}$ for $i \in J$. Notice that $B \subseteq C = \{x \in \mathbb{R}^2 : c_i^T x \leq d_i, i \in J\}$. Now let $B'' = B' \cap C$ and note $RC(B'') = RC$ and therefore $\psi_B(r) = \psi_{B''}(r)$ for all $r \in RC$. Furthermore, as $B'' \supset B$, by Lemma 2.4, $\psi_{B''}(r) \leq \psi_B(r)$ for all $r \notin RC$. As ψ is minimal, $\psi = \psi_{B''}$ and hence $B = B''$. Therefore, B is polyhedral as $B = B' \cap C$ where both B' and C are polyhedral.

Let $B = \{x : a_i^T x \leq b_i, i \in I\}$. By Lemma 3.8, a polyhedral set is maximal NLPF, if and only if, there exists a nonnegative integer point in the relative interior of each one of its facets. As B is not maximal, for some $t \in I$, the facet $\mathcal{F}_t$ defined by $a_t^T x \leq b_t$ does not contain any nonnegative integer points in its relative interior. If $\mathcal{F}_t$ is bounded, that is if $a_t^T x \leq 0$ does not define a facet of RC, then for some $\epsilon > 0$, the set $\bar{B} = \{x : a_i^T x \leq b_i, i \in I \setminus \{t\}\ ;\ a_t^T x \leq b_t + \epsilon\}$ is NLPF. Therefore, $\psi_{\bar{B}}$ is a valid function and as $\bar{B} \supset B$, $\psi_{\bar{B}}$ dominates ψ_B, a contradiction. Hence, we assume that $\mathcal{F}_t$ is unbounded and $a_t^T x \leq 0$ defines a facet of RC.

We now argue that RC has at least two facets. If not then $RC = \{x \in \mathbb{R}^2 : a_t^T x \leq 0\}$. As $f = (f_1, f_2)^T \notin \mathbb{Z}^2$, not all of the following four points are the same

$$p_1 = \begin{pmatrix} \lfloor f_1 \rfloor \\ \lfloor f_2 \rfloor \end{pmatrix}, p_2 = \begin{pmatrix} \lfloor f_1 \rfloor \\ \lceil f_2 \rceil \end{pmatrix}, p_3 = \begin{pmatrix} \lceil f_1 \rceil \\ \lfloor f_2 \rfloor \end{pmatrix}, p_4 = \begin{pmatrix} \lceil f_1 \rceil \\ \lceil f_2 \rceil \end{pmatrix}$$

and since $f > 0$, they are non-negative and integral. Furthermore, $f \in conv(p_1, p_2, p_3, p_4)$ and hence, there exist two distinct points $p', p'' \in \{p_1, p_2, p_3, p_4\}$ such that $a_t^T p' \leq a_t^T f \leq a_t^T p''$. This implies that $a_t^T (p' - f) \leq 0$ and hence $r' = p' - f \in RC(B)$. As $f \in int(B)$, we have that $p' = f + r' \in int(B)$, contradicting the fact that B is NLPF.

Therefore, RC indeed has at least two facets. Let $a_k^T x \leq 0$ define a different facet of RC than $a_t^T x \leq 0$. This also implies that $a_k^T x \leq b_k$ defines an unbounded facet of B. Now, in the linear description of B, replace $a_t^T x \leq b_t$ by $(a_t + \epsilon a_k)^T x \leq b_t + \epsilon b_k$ for some small $\epsilon > 0$ and call the resulting set B^ϵ. Clearly, $B^\epsilon \supset B$. In addition, if ϵ is small enough the new inequality is facet defining for B^ϵ and also it induces a facet of $RC(B^\epsilon)$.

We next show that, if $\epsilon > 0$ is sufficiently small, then B^ϵ would be NLPF. To see this, note that, by Lemma 3.5, B does not have nonnegative rays in its recession cone RC, and therefore, there exists $\epsilon' > 0$ such that for every $\epsilon < \epsilon'$ we have that B^ϵ also has no nonnegative rays in its recession cone. Therefore, if $\epsilon > 0$ is small enough, $B^\epsilon \cap R_+^2$ is bounded and therefore $B^\epsilon \cap \mathbb{Z}_+^2$ is finite. Let $U = (B^\epsilon \setminus B) \cap \mathbb{Z}_+^2$ and note that for all points $x \in U$ we have (i) $a_t^T x > b_t$ and (ii) $a_k^T x < b_k$. Let

$$\beta = \min_{x \in U} \{a_t^T x - b_t\} \quad \text{ and } \alpha = \max_{x \in U} \{b_k - a_k^T x\}$$

and reduce ϵ, if necessary, so that $\epsilon < \beta/\alpha$. If B^ϵ is not NLPF, there is a nonnegative integer point $y \in int(B^\epsilon)$. As $\mathcal{F}_t$, the face of B defined by $a_t^T x \leq b_t$, has no integer points by assumption,

$a_t^T y > b_t$ and therefore $y \in U$. But then,

$$(a_t + \epsilon a_k)^T y < b_t + \epsilon b_k \Rightarrow a_t^T y - b_t < \epsilon(b_k - a_k^T y) < \epsilon\alpha < \beta \leq a_t^T y - b_t.$$

This is a contradiction and therefore $int(B^\epsilon) \cap Z_+^2 = \emptyset$ and B^ϵ is NLPF.

As the final step, we will next show that ψ_{B^ϵ} dominates ψ_B which will imply that ψ_B can not be minimal, a contradiction. First note that as B^ϵ is larger than B, we have $\psi_{B^\epsilon}(r) \leq \psi_B(r)$ for all $r \notin RC(B^\epsilon)$. Moreover, $RC(B^\epsilon) \supsetneq RC$ and therefore $\psi_{B^\epsilon}(r) < \psi_B(r)$ for all $r \in RC(B^\epsilon) \setminus int(RC)$. Finally, for $r \in int(RC)$, first note that

$$\psi_B(r) = \max\left\{\gamma, \frac{r^T a_t}{b_t - f^T a_t}\right\} \text{ and } \psi_{B^\epsilon}(r) = \max\left\{\gamma, \frac{r^T(a_t + \epsilon a_k)}{(b_t + \epsilon b_k) - f^T(a_t + \epsilon a_k)}\right\}$$

where $\gamma = \max_{i \in J \setminus \{t\}} \left\{r^T \hat{c}_i\right\}$. First note that as $f \in int(B)$,

$$(b_t + \epsilon b_k) - f^T(a_t + \epsilon a_k) = (b_t - f^T a_t) + \epsilon(b_k - f^T a_k) < b_t - f^T a_t.$$

In addition for $r \in int(RC)$ we have $r^T a^k < 0$ and therefore $r^T(a_t + \epsilon a_k) < r^T a_t$ implying that $\psi_B(r) \geq \psi_{B^\epsilon}(r)$. Therefore, ψ_{B^ϵ} indeed dominates ψ_B which contradicts the starting assumption that ψ_B is minimal. ■

3.3 Geometry of NLPF sets in $\mathbb{R}^2$

So far in this section we established a strong relationship between minimal functions and maximal NLPF sets for $m = 2$. In particular, Theorem 3.11 shows that any minimal function is generated by a maximal NLPF set, which by Theorem 3.2 is polyhedral and therefore, by Lemma 3.8, has at most 4 facets. In other words, if $\psi : \mathbb{Q}^2 \to \mathbb{R}$ is a minimal valid function for R_f^+, then $\psi = \psi_B$ where B is full-dimensional and has a minimal description

$$B = \{x \in \mathbb{R}^2 : a_i^T x \leq b_i, \forall i = 1, \ldots, k\}$$

with $k \leq 4$. We next show that if $k = 4$ then B is a maximal lattice point free set.

Lemma 3.12 *Let B be a maximal NLPF set in $\mathbb{R}^2$ that contains a point $f > 0$ in its interior. If B has 4 facets, then it contains no lattice points in its interior and therefore it is a maximal lattice point free set. Furthermore, B is bounded.*

Proof. Assume that B contains lattice points in its interior and let $\bar{x}$ be one such point with the property that $d(x) = \max\{-x_1, -x_2\}$ is smallest. As B is NLPF, $x \notin \mathbb{R}_+^2$ and $d(x) > 0$. As every facet of B has to have a non-negative lattice point in its relative interior by Lemma 3.8, B has to contain, on its boundary, 4 non-negative lattice points with all 4 possible odd/even parity. Let $y \in B$ be a non-negative lattice point that has the same odd/even parity as x and notice that $z = x/2 + y/2$ is integral and $z \in int(B)$ and therefore z is not a non-negative lattice point. But then, as $y \geq 0$ we have $d(z) \leq d(x/2) < d(x)$, a contradiction. Therefore, B is a maximal lattice

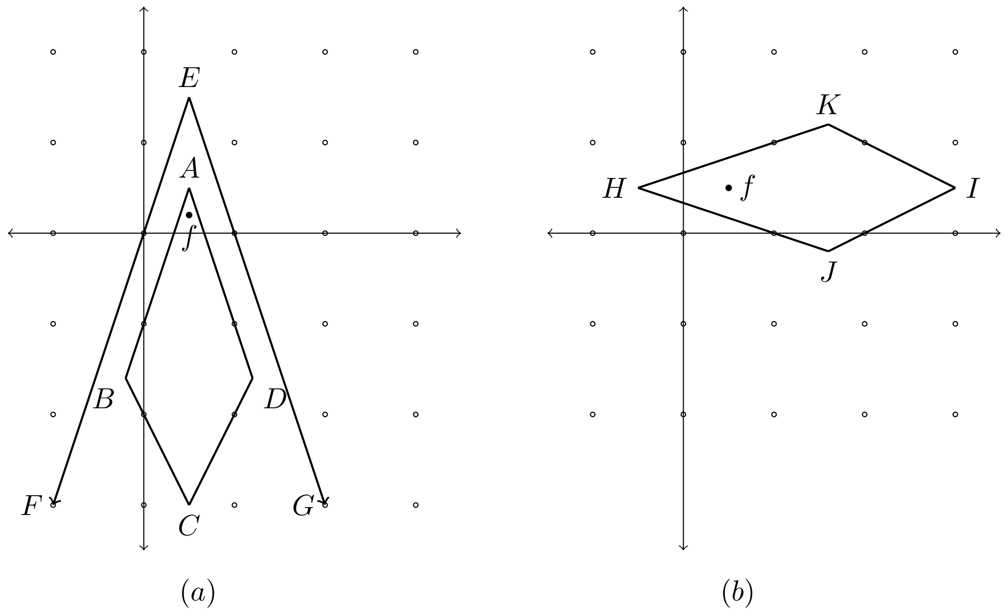

Figure 3: (a) A maximal lattice free quadrilateral contained in a positive lattice free cone, and,
. (b) a maximal lattice free quadrilateral which is also maximal NLPF.

point free set with 4 facets and as such it has to be a quadrilateral (see [2]) and therefore, it is bounded. ■

Remember that R_f is the relaxation of R_f^+ where integer variables are not required to be non-negative, see [7]. Also remember that minimal valid inequalities for R_f are defined by maximal lattice point free sets. More precisely, if B is a maximal lattice point free set, then ψ_B is a minimal valid function for R_f, and if ψ is a minimal valid function for R_f, then $S(\psi, f)$ is a maximal lattice point free set.

From a practical point of view, Lemma 3.3 implies that any minimal valid inequality $\psi : \mathbb{Q}^2 \rightarrow \mathbb{R}$ for R_f^+ is also valid and minimal for R_f provided that $S(\psi, f)$ is a quadrilateral. The converse, however, is not true as minimal valid inequalities for R_f that are associated with quadrilaterals might need to be strengthened to obtain minimal valid inequalities for R_f^+. To see this point notice that the maximal lattice point free quadrilateral $Q = conv\{A, B, C, D\}$ shown in Figure 3(a) is strictly contained in the maximal NLPF cone K defined as the convex hull of the rays $\overrightarrow{EF}$ and $\overrightarrow{EG}$. The maximal lattice point free quadrilateral $Q' = conv\{H, I, J, K\}$ shown in Figure 3(b), on the other hand, is both maximal lattice point free and maximal NLPF and therefore the function $\psi_{Q'}$ is a minimal valid function for both sets R_f^+ and R_f.

The cone K shown in Figure 3(a) also shows another difference between maximal NLPF and maximal lattice point free sets. In the case of maximal lattice point free sets, if a set is full dimensional and unbounded, then it is a split which does not have a full-dimensional recession cone. On the contrary, as shown in Figure 3(a), maximal NLPF sets can have full-dimensional recession cones.

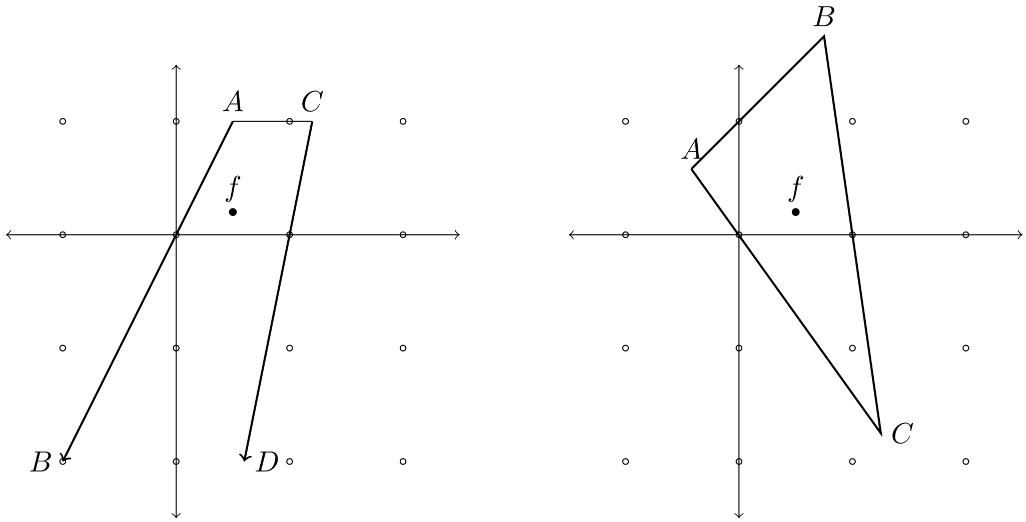

Figure 4: Bounded and unbounded maximal NLPF sets with 3 facets

When a maximal NLPF set has 3 facets it can be a bounded set (triangle) or an unbounded set. In both cases, the set might contain lattice points and therefore might lead to minimal valid inequalities that are not valid for R_f. Figure 4 shows these cases. Finally, if a maximal NLPF set has 2 facets, it can be a split, in which case the set is also maximal lattice point free, or, it might be a translated cone as shown in Figure 3(a).

4 Strengthening valid inequalities for R_f^+ .

In the previous section we showed that when $m = 2$, it is sufficient to consider polyhedral NLPF sets to obtain all minimal valid functions for X^+ . This result motivates the following question addressed in this section: given a polyhedral NLPF set $B \subset \mathbb{R}^m$ which is not maximal, how can one obtain a NLPF set $B' \supsetneq B$ such that $\psi_{B'} \leq \psi_B$? One possibility is to start with a maximal lattice-free convex set as in Example 1.1 and try to obtain a NLPF set that strictly contains it. It is important to note, however, that $B' \supsetneq B$ does not imply $\psi_{B'} \leq \psi_B$. In other words, larger sets do not necessarily lead to better valid inequalities. This is an important difference between the relaxation R_f^+ studied in this paper and relaxation R_f studied in [7]. The following example illustrates this fact.

Example 4.1 *Let $f = (0.8, 0.2)$ and consider the following two NLPF sets $B = \{x \in \mathbb{R}^2 : -x_1 + x_2 \leq -1/2\ ;\ x_1 \leq 1\}$ and $B' = \{x \in \mathbb{R}^2 : -x_1 + x_2 \leq 0\ ;\ x_1 \leq 1\}$. Figure 5(a) illustrates this example. Notice that $B' \supsetneq B$ and both sets contain f in their interior. For $\bar{r} = (-0.3, -0.9)$ we have that*

$$\begin{aligned}
\psi_B(\bar{r}) &= \max\{\frac{(-1,1)^T(-0.3,-0.9)}{-0.5-(-1,1)^T(0.8,0.2)}, \frac{(1,0)^T(-0.3,-0.9)}{1-(1,0)^T(0.8,0.2)}\} = \max\{-6,-1.5\} = -1.5, \ \textit{and} \\
\psi_{B'}(\bar{r}) &= \max\{\frac{(-1,1)^T(-0.3,-0.9)}{0-(-1,1)^T(0.8,0.2)}, \frac{(1,0)^T(-0.3,-0.9)}{1-(1,0)^T(0.8,0.2)}\} = \max\{-1,-1.5\} = -1
\end{aligned}$$

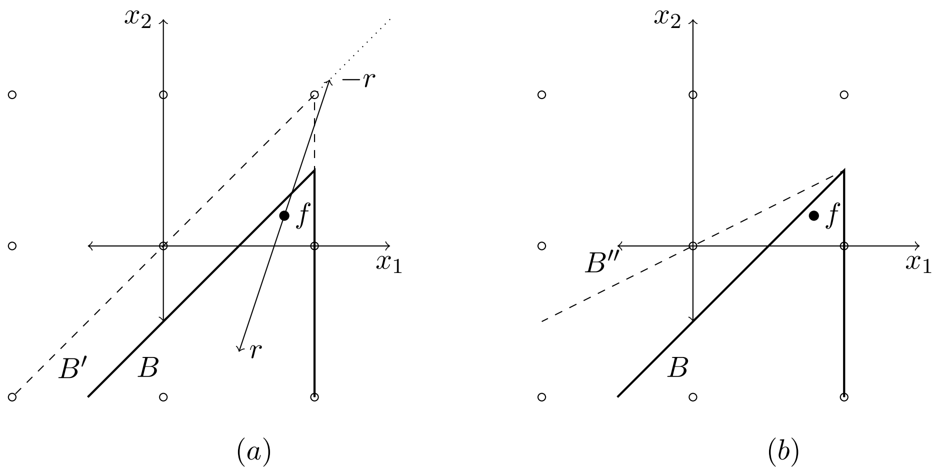

Figure 5: Two NLPF sets B' and B'' that contain B. $\psi_{B'} \not\leq \psi_B$ whereas $\psi_{B''} \leq \psi_B$.

and therefore $\psi_B(\bar{r}) < \psi_{B'}(\bar{r})$ even though B' contains B.

Geometrically, what is happening is that the coefficient of $r \in RC(B)$ is defined by first taking the vector $-r$ and finding the largest scalar $\alpha(B, r)$ such that $f + \alpha(B, r) \cdot (-r)$ satisfies one of the constraints defining B at equality (and possibly violating other constraints). Then $\psi_B(r) = -\frac{1}{\alpha(B,r)}$ (see Lemma 2.10). From Figure 5(a), we can see that $\alpha(B, r) < \alpha(B', r)$ and hence $\psi_B(r) < \psi_{B'}(r)$.

As all minimal functions are associated with maximal sets in $\mathbb{R}^2$, there exists a different maximal NLPF set $B'' \supsetneq B$ that gives a stronger valid inequality.

The set $B'' = \{x \in \mathbb{R}^2 : -1/2x_1 + x_2 \leq 0 \; ; \; x_1 \leq 1\} \supset B$ shown in Figure 5(b), on the other hand, gives a valid inequality that dominates ψ_B. The fact that $\psi_{B''} \leq \psi_B$ follows from Lemma 4.2.

We next give sufficient conditions under which $B' \supseteq B$ implies that $\psi_{B'}$ dominates ψ_B (i.e. $\psi_{B'} \leq \psi_B$ and $\psi(\bar{r}) < \psi_B(\bar{r})$ for some $\bar{r} \in \mathbb{Q}^m$). We assume that all polyhedral descriptions given are minimal, in other words, all inequalities given define facets of the corresponding polyhedra.

Lemma 4.2 *Let $B = \{x \in \mathbb{R}^m : a_i^T x \leq b_i, i \in I\}$ be a NLPF set such that $0 < f \in int(B)$ and let $B' \supsetneq B$. If $int(RC(B)) \neq \emptyset$ then let $RC(B) = \{x \in \mathbb{R}^m : a_i^T x \leq 0, i \in I^c\}$ where $I^c \subseteq I$. If one of the following conditions hold, then $\psi_{B'}$ dominates ψ_B.*
(i) $int(RC(B)) = \emptyset$.
(ii) B' is obtained from B by dropping a constraint, i.e., $B' = \{x \in \mathbb{R}^m : a_i^T x \leq b_i, i \in I \setminus \{k\}\}$.
(iii) B' is obtained from B by relaxing a constraint that does not give a facet of the recession cone, i.e., $B' := \{x \in \mathbb{R}^m : a_i^T x \leq b_i, i = I \setminus \{k\} \; ; \; a_k^T x \leq b_k + \epsilon\}$ where $k \in I \setminus I^c$ and $\epsilon > 0$.
(iv) B' is obtained from B by rotating a facet-defining inequality of B using another one, i.e., $B' := \{x \in \mathbb{R}^m : (a_l + \epsilon a_k)^T x \leq b_l + \epsilon b_k \; ; \; a_i^T x \leq b_i, i \in I \setminus \{l\}\}$ where $l, k \in I$ and $\epsilon > 0$.

Proof. Let $\bar{x} \in B' \setminus B$ and define $\bar{r} = \bar{x} - f$ so that $f + \bar{r} \in B' \setminus B$. Note that $\bar{r} \notin RC(B)$ as $f + \bar{r} \notin B$. By Lemma 2.4, for this choice of $\bar{r}$ we have $\psi_{B'}(\bar{r}) < \psi_B(\bar{r})$. We next consider each case separately and show that $\psi_{B'} \leq \psi_B$ also holds. Recall that we defined $\hat{a}_i = a_i/(b_i - a_i^T f)$.

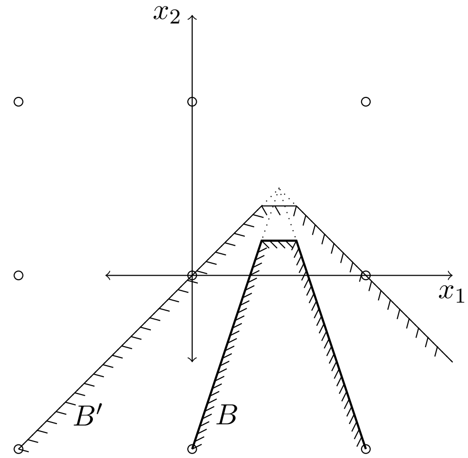

Figure 6: Example of a NLPF set B' that contains another NLPF set B such that $\psi_{B'} \leq \psi_B$.

(i) Follows directly from Lemma 2.4 as $\psi_B \geq 0$ when $int(RC(B)) = \emptyset$.
(ii) Follows from the definition of ψ as $\psi_B(r) = \max_{i \in I} \left\{ r^T \hat{a}_i \right\}$ and $\psi_{B'}(r) = \max_{i \in I \setminus \{k\}} \left\{ r^T \hat{a}_i \right\}$.
(iii) As $B' \supseteq B$, Lemma 2.4 implies that $\psi_{B'}(r) \leq \psi_B(r)$ for all $r \notin int(RC(B))$. In addition, for $r \in int(RC(B))$ by Lemma 2.11, $\psi_B(r) = \hat{a}_j^T r$ for some $j \in I^c$ and as $RC(B) = RC(B')$, we have $\psi_{B'}(r) = \psi_B(r)$.
(iv) Let $c = (a_l + \epsilon a_k)$ and $d = b_l + \epsilon b_k$. We will show that $c^T r/(d - c^T f) \leq \max\{\hat{a}_l^T r, \hat{a}_k^T r\}$. This fact, together with the fact that $\psi_{B'}(r) = \max\{c^T r/(d - c^T f), \max_{I \setminus \{l\}} \hat{a}_i^T r\}$ implies that $\psi_{B'}(r) \leq \psi_B(r)$. Suppose, for the sake of contradiction that $\hat{c}^T r = c^T r/(d - c^T f) > \max\{\hat{a}_l^T r, \hat{a}_k^T r\}$. Then $\hat{c}^T r > \hat{a}_l^T r$ implies

$$\frac{a_l^T r + \epsilon a_k^T r}{(b_l - a_l^T f) + \epsilon (b_k - a_k^T f)} > \frac{a_l^T r}{b_l - a_l^T f}.$$

As all the denominators are positive, we have

$$a_l^T r(b_l - a_l^T f) + \epsilon a_k^T r(b_l - a_l^T f) > a_l^T r(b_l - a_l^T f) + \epsilon a_l^T r(b_k - a_k^T f)$$

and hence $a_k^T r(b_l - a_l^T f) > a_l^T r(b_k - a_k^T f)$. Similarly, $\hat{c}^T r > \hat{a}_k^T r$ implies $a_k^T r(b_l - a_l^T f) < a_l^T r(b_k - a_k^T f)$, which is a contradiction and thus the result follows. ∎

Figure 1 shows an example of B and B' satisfying conditions (i) and (ii) of Lemma 4.2, while Figure 6 shows an example of B and B' that satisfy conditions (iii) and (iv). In particular note that if one obtains a NLPF convex set B' that contains a lattice-free split B, this will satisfy condition (i) and hence the intersection cut obtained from B' will strictly dominate the one obtained from B. Figure 7 shows an example of this case. Also note that the set B' in Figure 5(a) does not satisfy condition (iii) as the relaxed constraint of B is associated with a facet of $RC(B)$. The set B'' in Figure 5(b), however, satisfies condition (iv), as it is obtained by rotating one facet defining inequality of B is using another facet-defining inequality.

Notice that Lemma 4.2 states conditions under which $B' \supseteq B$ gives a function $\psi_{B'} \leq \psi_B$. However, such a function $\psi_{B'}$ is not useful for generating valid inequalities for R_f^+ unless B' is

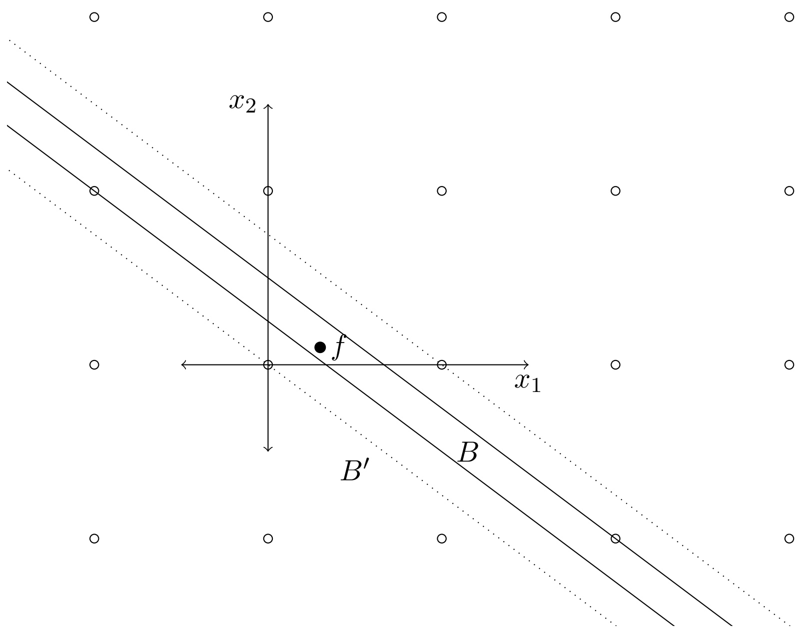

Figure 7: Example of a lattice-free split B and a NLPF split B' that strictly contains it

NLPF. Therefore, one needs to be able to check if B' is NLPF in order to apply Lemma 4.2 to strengthen a valid inequality for R_f^+. In general, checking this condition can be as difficult as solving an IP in m dimensions. However, there are some sufficient conditions that can be checked that guarantee that B' is NLPF.

We next identify simple conditions under which dropping a constraint from B leads to a NLPF set. We are not able to establish easily checkable conditions for the remaining operations described in Lemma 4.2. Formally, let $B = \{x \in \mathbb{R}^m : a_i^T x \leq b_i, \forall i = 1, \ldots, k\}$ be a polyhedral NLPF set that contains $f > 0$ in its interior (we also assume that all inequalities describing B define facets). Let

$$B^j = \{x \in \mathbb{R}^m : a_i^T x \leq b_i, \forall i \in \{1, \ldots, k\} \setminus \{j\}\}$$

denote the polyhedron obtained by dropping the jth inequality and let

$$F^j = \{x \in B : a_j^T x = b_j\}$$

denote the facet defined by the jth inequality of B. It is easy to see that B^j can not be NLPF if F^j contains a nonnegative integer point in its relative interior. The following observation establishes the reverse condition.

Lemma 4.3 *Assume that F^k does not contain a nonnegative integer point in its relative interior. In addition, if $\mathbb{Z}_+^m \cap int(B^k) \subseteq \{x : a_k^T x \leq b_k\}$, then $\mathbb{Z}_+^m \cap int(B^k) = \emptyset$, that is, B^k is NLPF.*

Proof. If $\mathbb{Z}^m_+ \cap int(B^k) \neq \emptyset$, let $y \in \mathbb{Z}^m_+ \cap int(B^k)$ and note that by assumption $a_k^T y \leq b_k$. If $a_k^T y < b_k$, then $y \in int(B)$, which contradicts the fact that B is NLPF. Hence $a_k^T y = b_k$ and y has to be in the relative interior of F^k, again a contradiction. ∎

Note that $\mathbb{Z}^m_+ \cap int(B^k) \subseteq \mathbb{R}^m_+ \cap B^k$. Based on this observation and Lemma 4.3, we next present two conditions that can be checked easily to verify that B^k is NLPF.

Corollary 4.4 *Assume that F^k does not contain a nonnegative integer point in its relative interior. Then B^k is a NLPF provided that $\mathbb{R}^m_+ \cap B^k \subseteq \{x : a_k^T x \leq b_k\}$.*

Also note that if $a_k \leq 0$ and $b_k \geq 0$, then $\mathbb{R}^m_+ \subseteq \{x : a_k^T x \leq b_k\}$ and the above condition holds trivially. Another condition that can be checked is the following.

Lemma 4.5 *If $F^k \cap \mathbb{R}^m_+ = \emptyset$ then B^k is NLPF.*

Proof. Suppose not. Then there exists $y \in \mathbb{Z}^m_+$ such that $a_k^T y > b_k$ and $a_j^T y < b_j$ for all $j = 1, \ldots, k-1$. In addition, as $f > 0$ is in the interior of B, we have that $a_j^T f < b_j$ for all $j = 1, \ldots, k$. But then for all $\lambda \in [0,1]$ we have that $x^\lambda = \lambda f + (1-\lambda) y$ satisfies $a_j^T x^\lambda < b_j$ for all $j = 1, \ldots, k-1$ and $x^\lambda \geq 0$. Moreover there exists λ such that $a_k^T x^\lambda = b_k$, but this contradicts the assumption that $F^k \cap \mathbb{R}^m_+ = \emptyset$ ∎

Remember Example 1.1 and note that the inequality that was dropped to obtain the maximal NLPF set satisfies the conditions of both Corollary 4.4 and Lemma 4.5. Also note that in order to apply Lemma 4.3 or Corollary 4.4, one needs to check if $int(F^k)$ contains nonnegative integer points, which requires solving an integer program in $\mathbb{R}^m$. The condition $F^k \cap \mathbb{R}^m_+ = \emptyset$ in Lemma 4.5, however, can be checked by solving a linear program.

5 Conclusion

In this paper, we defined a new relaxation for mixed-integer sets and studied valid inequalities associated with it. Our relaxation can be seen as a tightening of the relaxation defined by Borozan and Cornuéjols [7] and Andersen et al. [2]. The difference between the two relaxations is the presence of non-negativity constraints in our set. In this respect, the difference between the two relaxations is similar to the difference between the master equality polyhedron [9] that we studied recently and the cyclic group polyhedron of Gomory. In both cases, exploiting non-negativity leads to stronger inequalities.

Even though some of our results generalize easily for $m > 2$ constraints, there are others that we were not able to extend. For instance, for $m > 2$, are maximal NLPF sets polyhedral? If so, do they always have a nonnegative integer point in the relative interior of each of their facets? Moreover, can there be minimal functions that arise from non-maximal positive lattice-free convex sets? Even though we only derived a one-to-one correspondence between maximal NLPF sets and minimal functions for $m = 2$, we believe such correspondence also exists for $m > 2$.

Finally, notice that the nonnegativity on the integer variables is an arbitrary choice of constraints. In principle, one could impose any additional set of constraints to the integer variables and use this additional information to strengthen the inequalities obtained. A case of particular interest is when the basic variables are all between given bounds $[0, u]$ (for example binary variables) and hence we only need to focus on convex sets that don't have integer points in $[0, u]$ in their interior. We believe, for instance that Theorems 3.2 and 3.11 can be generalized for such cases.

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