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A VaR Black-Litterman Model for the Construction of Absolute Return Fund-of-Funds

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Abstract This research was motivated by our work with the private investment group of an international bank. The objective is to construct fund-of-funds (FoF) that follow an absolute return strategy and meet the requirements imposed by the Value-at-Risk (VaR) market risk measure. We propose the VaR-Black Litterman model which accounts for the VaR and trading (diversification, buy-in threshold, liquidity, currency) requirements. The model takes the form of a probabilistic integer, non-convex optimization problem. We develop a solution method to handle the computational tractability issues of this problem. We first derive a deterministic reformulation of the probabilistic problem, which, depending on the information on the probability distribution of the FoF return, is the equivalent or a close approximation of the original problem. We then show that the continuous relaxation of the reformulated problem is a nonlinear and convex optimization problem for a wide range of probability distributions. Finally, we use a specialized nonlinear branch-and-bound algorithm which implements the new portfolio return branching rule to construct the optimal FoF. The practical relevance of the model and solution method is shown by their use by the private investment group of a financial institution for the construction of four FoFs that are now traded worldwide. The computational study attests that the proposed algorithmic technique is very efficient, outperforming, in terms of both speed and robustness, three state-of-the-art alternative solution methods and solvers.

Keywords Portfolio Optimization · Probabilistic Programming · VaR · Funds-of-Funds · Black-Litterman · Absolute Return · Trading Constraints

1 Introduction

This study consists in the construction of long-only absolute return fund-of-funds (FoF) for the Private Banking Division of a major financial institution. The project is part of the institution's recent initiative to enhance its portfolio construction framework and to equip managers with sophisticated optimization models and tools to deal with the unintended movements of financial markets. More specifically, the fund construction model presented in the paper is used by the Private Banking division for its new "Absolute Return" investment program which aims at extending the availability of absolute return financial products to individual investors.

The portfolio selection discipline goes back to the Markowitz mean-variance portfolio optimization model [31] which is based on the trade-off between risk and return and in which the diversification principle plays a dominant role. The mean-variance portfolio selection model is a quadratic optimization problem that defines the proportion of capital to be invested in each considered asset. Since Markowitz's work, many other portfolio optimization models have been proposed. The main motivations have been threefold and relate to

– the mitigation of the impact of the estimation risk [5]: many empirical studies (see, e.g., [11, 33]) have shown that the optimal portfolio is extremely sensitive to the estimation of the inputs, and, in particular, that errors in the estimation of the expected returns have a much larger impact than those in the estimation of the variances and covariances. Asset managers would thus rather trade-off some return for a more secure portfolio that performs well under a wider

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set of realizations of the random variables. The need for constructing portfolios that are much less impacted by inaccuracies in the estimation of the first two moments of the asset return is clear and has fueled the development of robust optimization [13,15,19,44], stochastic programming [8,43], Bayesian [6,23,25,40] and robust statistics-based [45] portfolio optimization models which refine the estimation of the parameters and/or effectively consider them as random variables;
- the accounting of other risk measures which can be symmetric or asymmetric and that can target maximum return or index tracking objectives [37]. The choice of a risk measure for the construction of a portfolio has been analyzed under different angles in the literature, including the coherence of risk measures [1,28], convexity properties [14,17] and computational tractability [30];
- the tackling of practical restrictions. These restrictions can come from trading requirements [8,36], such as the holding of positions in a minimum (and/or maximum) number of industrial sectors or asset categories (cardinality constraints), the purchase of shares by large lots (roundlot constraints), the requirement to avoid very small positions (buy-in threshold constraints), the payment of transaction costs, the impact of a trade on the value of a stock, the turnover of the portfolio, and from international regulations such as the Basel Accord requirements [2].

These developments have greatly amended the nature of portfolio optimization and have made the formulation of the portfolio selection model and the finding of its optimal solution much more challenging. In many cases, asset managers have to choose between solving the actual problem inaccurately and solving an approximation of the problem accurately. We present below the specifics (pursued strategy, risk measure, financial industry requirements) of the FoF optimization model proposed in this paper.

Rather than investing in individual securities, a *fund-of-fund (FoF)* is a fund that invests in other, diversified and complementary sub-funds. FoF are very attractive to investors since they all constitute unique asset allocation products, allowing investors to diversify their capital amongst different managers' styles, while keeping an eye on risk exposure. FoF are positioned to benefit from both downward and upward market trends and are expected to provide stabler returns due to their diversification width.

Long-only absolute return fund-of-funds take only long positions, seeking for undervalued securities and limiting volatility and downside risk by holding cash, fixed income or other basic asset classes. They can use options, futures and other derivatives to hedge risk and gain exposure for underlying physical investments but not for speculative purposes, and pursue strategies that are intended to result in positive returns under all market conditions, in stark contrast to other funds which pursue relative return strategies. A fund following an absolute return strategy is one striving for absolute return targets: the success of the fund is examined by checking whether the fund is worth more today than at any point in the past, and not whether its return exceeds that of the benchmarked index.

The risk measure elicited to implement the absolute return strategy is the Value-at-Risk (VaR) criterion [39]. The risk exposure of the FoF is mitigated by limiting from above the variance of the FoF, and the capital preservation is accounted for through a Value-at-Risk constraint, which is also one of the conditions imposed by Basel II [2] that promulgated the VaR criterion [26] as the standard risk measure used by financial organizations.

The following features play a crucial role for the construction of the constrained FoF optimization model. First, in order to obtain a robust parameter estimation of the assets' potential, the estimation of their expected value is based on both quantitative market information and the opinions of experts using the Black-Litterman method [6]. The diversification objective is supported by constraints that require to hold positions in certain asset classes and that impose lower- and upper-bounds on the amount of the positions per asset category, per geographical region, and per currency. The proportion of detained assets with low liquidity is also constrained. Buy-in threshold constraints [36] effectively limit the monitoring costs by imposing a lower bound on each individual detained position.

In this study, we develop a new asset allocation model which integrates the above-mentioned diverse and complicated constraints, and we provide a solution method for the underlying optimization model. The resulting portfolio optimization model is a probabilistic integer programming problem and is thereafter referred to as the *VaR Black-Litterman* FoF model.

The paper is organized as follows. Section 2 provides the step-by-step formulation of the model and the description of the input parameters (especially those used for the implementation of the Black-Litterman model) and their evaluation. In Section 3, we discuss the amenability of the VaR Black-Litterman model to a deterministic equivalent (or approximation) formulation that we solve using a mixed-integer nonlinear solution technique. The computational tractability and efficiency of the solution method is analyzed in Section 4. Section 5 provides concluding remarks.

2 Formulation of the VaR Black-Litterman FoF Model

2.1 Asset universe

The bank's financial market specialists have selected the assets that can possibly be included in the FoF. The assets belong to seven asset classes which are themselves decomposed into subclasses (Table 1). Assets are traded in three currencies ($US, Euro, Japanese Yen) and have different liquidity levels.

Classes	Subclasses
Short-term deposits	
Bonds	Government Bonds Inflation-Linked Bonds Investment-Grade Corporate Bonds High Yield Corporate Bonds Structured Credits Convertible Bonds Emerging Market Bonds
Equities	Europe North America Asia
Commodities	Energy Metals Agricultural Live Stock
Real Estate	Europe North America Asia
Currencies	Euro US$ Japanese Yen
Specialized Funds	Equity Hedge Directional Trading Event Driven Relative Value

Table 1 Asset/Fund Classes and Subclasses

2.2 Model description

The FoF can include any of the n selected assets which have random returns. A first estimate of the expected return of an asset is obtained through historical time-series. The objective function of the FoF consists of maximizing the quarterly expected return of the FoF

$$\max\ (\mu)^T x \tag{1}$$

where x and μ are both n-dimensional vectors and the symbol ^{T} refers to the transposed operation. We thereafter refer to μ_i as the average return of asset i and to x_i as the fraction of the available capital invested in asset i. Note that we formulate the FoF optimization model here in terms of the expected return estimated through time-series. In Section 2.3 where the Black-Litterman approach is discussed, we shall describe how a refined estimate of the expected return can be obtained and shall provide the associated reformulation the FoF optimization problem.

The optimization problem is a constrained one subject to the satisfaction of a set of linear, one linear integer, one probabilistic, and one quadratic constraints. The first subset of *linear* constraints are the no-short selling constraints (2) which enforce the long-only feature of the FoF:

$$x \geq 0\ . \tag{2}$$

Note that, although the constructed fund is a long fund, it could detain positions in other funds which use short-selling.

The second linear constraint is the budget constraint (3) according to which the entirety of the capital is invested:

$$\sum_{i=1}^{n} x_i = 1 \tag{3}$$

Each asset belongs to a certain asset class and subclass (Table 1) and is traded in a certain currency. *Class* (4), *subclass* (5) and *currency* (6) *diversification constraints* impose a lower ($\underline{b_k}, \underline{b_{k,s}}, \underline{c_k}$) and an upper ($\overline{b_k}, \overline{b_{k,s}}, \overline{c_k}$) bounds on the quantity of the capital invested per class, subclass and currency. Denoting by C_k the asset class k, by $C_{k,s}$ the subclass s within class k, and by R_l the currency l, the class, subclass and currency diversification constraints are formulated as the linear inequalities (4), (5) and (6):

$$\underline{b_k} \leq \sum_{j \in C_k} x_j \leq \overline{b_k},\ \forall k \tag{4}$$

$$\underline{b_{k,s}} \leq \sum_{j \in C_{k,s}} x_j \leq \overline{b_{k,s}},\ \forall s \in k, \forall k \quad . \tag{5}$$

$$\underline{c_l} \leq \sum_{j \in R_l} x_j \leq \overline{c_l},\ \forall l \tag{6}$$

Liquidity constraints ensure that a minimal proportion of the capital is invested in assets with (strong) weekly or monthly liquidity. Representing by L_k the liquidity k (k=daily, weekly or monthly), the liquidity constraints read:

$$\underline{l_k} \leq \sum_{j \in L_k} x_j,\ \forall k \ . \tag{7}$$

Linear integer constraints take the form of linear inequalities involving integer decision variables and result from trading requirements. In the problem motivating this study, *buy-in threshold constraints* (see, e.g., [24, 36]) are introduced in order to prevent the holding of very small active positions. It is known that some portfolio models occasionally return an optimal portfolio containing very small investments in a (large) number of securities. Such a portfolio has limited impact on the total performance of the portfolio and involves substantial costs for maintaining it (brokerage fees, bid-ask spreads, monitoring costs, etc.). Moreover, small positions have very poor liquidity. This motivates the incorporation of buy-in threshold constraints which impede the holding of an active position representing strictly less than a prescribed proportion x_{min} of the available capital. We introduce a binary variable $\delta_j \in \{0, 1\}, j = 1, \ldots, n$ for each asset. Constraints

$$x_j \leq \delta_j \ ,\ \ j = 1, \ldots, n \tag{8}$$

$$\delta_j \in \{0, 1\} \ ,\ \ j = 1, \ldots, n \tag{9}$$

force δ_j to be equal to 1 if the investor detains shares of asset j (i.e., $x_j > 0$). The addition of

$$x_{min}\ \delta_j \leq x_j \ ,\ j = 1, \ldots n \tag{10}$$

does not permit the holding of small positions.

The *quadratic* constraint ensures that the volatility of the portfolio does not exceed a prescribed maximal value s. Denoting by σ_i the standard deviation of the return of asset i and by ρ_{ij} the correlation between funds i and j, the variance of the portfolio is $Var[x] = x^T \Sigma x$ with Σ being the variance-covariance matrix, and the volatility constraint is

$$x^T \Sigma x \leq s \ . \tag{11}$$

Each element $\Sigma_{ij} = \rho_{ij} \sigma_i \sigma_j$ of Σ represents the covariance between i and j.

The capital preservation objective is ensured through a *Value-at-Risk* (VaR) constraint that limits the magnitude of the loss to happen with a specified probability level $1 - p$ (e.g., 95%) over a certain period of time (e.g., one year) and that takes the form of a *probabilistic* constraint [27]. Denoting by Y the random variable representing the loss on the fund value and by K the initial value of the fund, the value-at-risk p-VaR of the fund loss at the p% probability level is:

$$VaR_p(Y) = \min\{\gamma : P(Y \geq \gamma) \leq 1 - p\} \ , \tag{12}$$

and the corresponding VaR constraint is written:

$$P(Y \leq VaR_p(Y)) \geq p$$

where $\mathcal{P}$ refers to a probability measure.

In our case, the VaR constraint is defined as the requirement for the probability of the FOF's loses exceeding 5% (or respectively 10%) of the capital invested in the FoF to be at most equal to 5% and therefore reads:

$$P(Y \leq \beta \cdot K) \geq 0.95 \tag{13}$$

where $\beta = 0.05$ or 0.1.

Following the Basel Accord, the VaR criterion has become the standard risk measure to define the market risk exposure of a financial position. The introduction of the VaR constraint specified as indicated above is in accordance with the BASEL II credit risk requirements imposed by the Basel Bank of International Settlements [2]. Second, we note that the diversification constraints can compensate for a structural limit of the VaR criterion. It is well known that VaR does not have the sub-additivity property, which is one of the four properties to be satisfied to qualify as a coherent risk measure [1]. Therefore, VaR does not respect the axiom according to which diversification reduces risk. A consequence of this is that portfolios or funds constructed by using the VaR market risk criterion are sometimes concentrated in a few positions. The diversification constraints can remedy to this issue.

2.3 Revision of expected return estimate

As aforementioned, investment professionals criticize optimal mean-variance portfolios for they are sometimes counter-intuitive: a small change in the problem inputs (predominantly, the estimate of the expected returns and their variances and covariances) significantly alter the composition of the optimal portfolio. In order to alleviate the impact of errors due to this so-called estimation risk, we use in this paper more robust estimates of the expected returns by combining quantitative return data and the opinions of experts. This is achieved by using the Black-Litterman framework [6] which integrates the investor's economic reasoning and overcomes the problems of unintuitive, highly-concentrated portfolios and input-sensitivity [18, 42]. Instead of solely relying on historical numbers to predict future returns, the Black-Litterman approach uses a Bayesian approach to combine the subjective views of an expert regarding the relative or absolute future performances of specific assets or asset classes with the market equilibrium vector of expected returns (the prior distribution) to form a new, mixed estimate of expected returns. The outcome is a revised vector of expected returns which is obtained by tilting the prior, market equilibrium-based estimate of the returns in the direction of assets favored in the views expressed by the investor. The extent of deviation from equilibrium depends on the degree of confidence the investor has in each view. While in a standard portfolio optimization problem, the necessary inputs are the expected returns μ of the n different assets and the covariances among them, the Black-Litterman approach requires more inputs as described below.

The Black-Litterman first requires the computation of a vector of neutral positions w_i which represent the standard investment behavior of a standard investor. Usually, in the Black-Litterman context, each position is set equal to the relative market capitalization weight x^{cap} of the corresponding asset. In market equilibrium, this weighting scheme implies a corresponding vector of expected returns, the so-called vector π of *market equilibrium expected returns* which is defined as the one that would clear the market were all investors having identical views [6] and whose value, as explained below, is obtained through a reverse optimization procedure.

In the portfolio selection literature, the optimal investment policy x^* of a risk-neutral investor with quadratic utility function, or assuming Gaussian returns π, is defined as the optimal solution of the unconstrained maximization problem

$$\max_x \; x^T \pi - \frac{x^T \Sigma x}{2} \; .$$

This establishes the relationship between the optimal positions x^* and the implied vector of expected returns π

$$x^* = \Sigma^{-1} \pi \; .$$

The expression Σ^{-1} refers to the inverse of the matrix Σ.

In the Black-Litterman context, we set $x^* = x^{cap}$ and we proceed backward to find the implied vector of market equilibrium expected returns:

$$\pi = \Sigma x^{cap} \; .$$

The approach is linked to the capital asset pricing model (CAPM) according to which prices adjust until, in market equilibrium, the expected returns are such that the demand for these assets exactly matches the available supply.

The Black-Litterman approach does not consider that the vector of market equilibrium return is known but instead assumes that it is a random variable following a multivariate normal distribution with mean π and variance $\tau\Sigma$. The Black-Litterman approach posits that the covariance matrix of expected returns is proportional to the one of historical returns, rescaled only by a shrinkage factor τ which is strictly positive and lower than 1 since the uncertainty of the mean is lower than that of the returns themselves. Different approaches have been proposed to set the value τ. Following [7], we interpret $\tau\Sigma$ here as the standard error of estimate of the vector of implied equilibrium returns, and, therefore, set the value of τ equal to 1 divided by the number of observations (i.e., realizations of past returns).

If the investor does not hold a view about expected returns, he or she should simply hold the market portfolio. However, given the economic situation, an investor or expert generally has expectations about short-term returns that differ from those implied by the current market clearing conditions. The Black-Litterman model gives the possibility to integrate experts' opinions in the derivation of the optimal investment strategy. One of the more challenging aspects of the Black-Litterman approach is to transform the stated *views* and the *degree of confidence* the expert has in them into the inputs used in the Black-Litterman formula.

Let k be the number of views expressed as linear combinations of the expected returns of the assets. The matrix notation for the views is given by

$$P\tilde{\mu} = q - \epsilon \tag{14}$$

where P is a $[k \times n]$ matrix, $\tilde{\mu}$ is the n-dimensional revised vector of expected returns, and q and e are k-dimensional vectors. The first view is represented as a linear combination of expected returns in the first row of P. The value of this first view is given by the first element of q, plus an error term ϵ_1 which represents the degree of uncertainty about the first view. The error term vector ϵ is a normally distributed random vector with mean 0 and variance Ω. The standard Black-Litterman model considers that all views are independently drawn from the future return distribution; therefore, Ω is a diagonal matrix: $\omega_{ij} = 0, \forall i \neq j$.

Note that views can be expressed as *absolute* return expectations for individual assets or as *relative* return expectations comparing the returns of several assets or aggregates of assets. To illustrate this, we consider:

– the relative view stating that, at a 90% confidence level, the Dow Jones' return will exceed that of the S&P's 500 index by 1% to 3%, and
– the absolute view stating that the return of the NASDAQ Composite Index will be, at a confidence level of 95%, between 12% and 15%.

Denoting by π_1 (resp., π_2, π_3) the expected return of the Dow Jones (resp., S&P's 500, NASDAQ), and assuming that the return spread is normally distributed on the interval for both views, the standard deviation $\sqrt{\omega_{11}}$ of the error term ϵ_1 pertaining to the first view is equal to: $\sqrt{\omega_{11}} = 0.61\% = \frac{0.02}{1.64*2}$. This is obtained by interpreting the relative view statement as a 90% confidence interval with a width of 2% and centered on 2%, and by considering symmetric confidence bands around normally distributed returns (Figure 1).

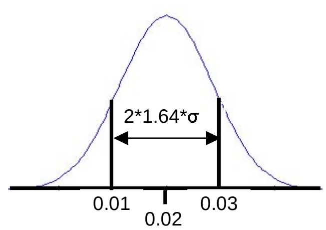

Fig. 1 Variance of Views

We proceed similarly to obtain the value of $\sqrt{\omega_{22}} = 0.765\% = \frac{0.03}{1.96*2}$. Evidently, the higher (lower) the degree of confidence in a view i, the smaller (larger) the value of ω_{11} and the more (less) the expected returns of the asset returns will be revised in the direction of the view.

In matrix notations, we have:

$$P = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad q = \begin{bmatrix} 0.02 \\ 0.135 \end{bmatrix}, \quad \Omega = \begin{bmatrix} (0.0061)^2 & 0 \\ 0 & (0.00765)^2 \end{bmatrix}.$$

The revised vector of expected returns is obtained by taking into account two probabilistic sources of information, i.e., the vector of market equilibrium expected returns and the views of experts, which are both expressed as multivariate normal probability distributions. The mechanism quantifies the impact of a second source of information, i.e., the posterior (experts' views) on one's a priori belief (market equilibrium returns). The output is the updated vector of expected returns which is itself a multivariate normal random variable. Figure 2 summarizes the different steps of the Black-Litterman approach.

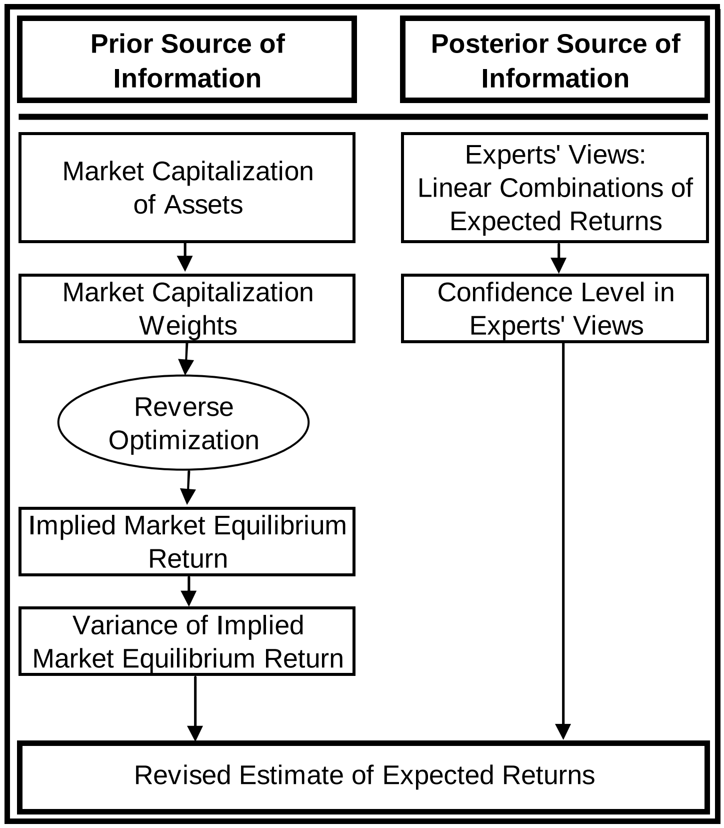

Fig. 2 Black-Litterman Flow Chart

We distinguish the two following cases for the computation of the revised vector of expected returns.

1. **Proposition 1** *If the experts have 100% confidence level in their views and not all of them are absolute ones, then*

$$\tilde{\mu} = E[\mu|views] = \pi + \tau\Sigma P^T[P\tau\Sigma P^T]^{-1}[q - P\pi]\ . \tag{15}$$

Proof : The optimal value of the revised vector $\tilde{\mu}$ of expected returns is the one that minimizes its variance with respect to the market equilibrium returns subject to the set of constraints: $P\tilde{\mu} = q$. The omission of ϵ in the above constraint results from the certainty in the views, which implies that the diagonal elements of the variance-covariance matrix Ω of the error term ϵ are all equal to 0.
This can be formulated as the following nonlinear optimization programming problem:

$$\begin{aligned} &\min\ [\tilde{\mu} - \pi]^T \cdot (\tau\Sigma)^{-1} \cdot [\tilde{\mu} - \pi] \\ &\text{subject to } P\tilde{\mu} = q \\ &\tilde{\mu} \in \mathcal{R}^n \end{aligned} \tag{16}$$

whose Lagrangian is

$$L = [\tilde{\mu} - \pi]^T \cdot (\tau\Sigma)^{-1} \cdot [\tilde{\mu} - \pi] - \lambda(P\tilde{\mu} - q)$$

where λ is the Lagrangian multiplier.
From the first-order conditions of optimality, we have:

$$\frac{\partial L}{\partial \tilde{\mu}} = 0 = 2(\tau\Sigma)^{-1}(\tilde{\mu} - \pi) - \lambda P \tag{17}$$

$$\frac{\partial L}{\partial \lambda} = 0 = P\tilde{\mu} - q \tag{18}$$

From (17), we obtain

$$\tilde{\mu} = \pi + \frac{(\tau\Sigma)P\lambda}{2}$$

and, substituting this value for $\tilde{\mu}$ in (18), we have

$$\lambda = 2(q - P\pi)(P^T(\tau\Sigma)P)^{-1} .$$

Substituting the above value of λ in (17) returns (15) which was set out to prove. □

2. **Proposition 2** *If the experts do not have 100% confidence level in their views, then*

$$\tilde{\mu} = E[\pi|views] = \pi + \tau\Sigma P^T[P\tau\Sigma P^T + \Omega]^{-1}[q - P\pi] . \tag{19}$$

Proof : From the a priori, market equilibrium-based estimate (defined as a stochastic variable following a normal distribution with mean π and variance $\tau\Sigma$) of the vector of expected returns, we have

$$\pi = \tilde{\mu} + v \tag{20}$$

with $v \rightsquigarrow N(0, \tau\Sigma)$, and, from the experts' estimate of the vector of expected returns, we have (14) where $\epsilon \rightsquigarrow N(0, \Omega)$.
Using the notations

$$Y = \begin{bmatrix} \pi \\ q \end{bmatrix} , \quad X = \begin{bmatrix} I \\ P^T \end{bmatrix} , \quad W = \begin{bmatrix} \tau\Sigma & 0 \\ 0 & \Omega \end{bmatrix}$$

with I referring to the identity matrix of appropriate dimension, we can rewrite

$$\pi = \tilde{\mu} + v$$
$$q = P\tilde{\mu} + \epsilon$$

as

$$Y \ = \ X\tilde{\mu} \ + \ d$$

where $d \rightsquigarrow N(0, W)$.
The generalized least square estimator of $\tilde{\mu}$ is given by:

$$\tilde{\mu} \ = \ (X^T W^{-1} X)^{-1} \ X^T W^{-1} Y$$

which can be successively transformed as:

$$\begin{aligned} \tilde{\mu} &= \left[\begin{bmatrix} I \ P^T \end{bmatrix} \begin{bmatrix} \tau\Sigma & 0 \\ 0 & \Omega \end{bmatrix}^{-1} \begin{bmatrix} I \\ P \end{bmatrix} \right]^{-1} \times \left[\begin{bmatrix} I \ P^T \end{bmatrix} \begin{bmatrix} \tau\Sigma & 0 \\ 0 & \Omega \end{bmatrix}^{-1} \begin{bmatrix} \pi \\ q \end{bmatrix} \right] \\ &= \left[\begin{bmatrix} (\tau\Sigma)^{-1} \ \ P^T\Omega^{-1} \end{bmatrix} \begin{bmatrix} I \\ P \end{bmatrix} \right]^{-1} \times \left[\begin{bmatrix} (\tau\Sigma)^{-1} \ \ P^T\Omega^{-1} \end{bmatrix} \begin{bmatrix} \pi \\ q \end{bmatrix} \right] \\ &= \left[(\tau\Sigma)^{-1} \ + \ P^T\Omega^{-1}P \right]^{-1} \times \left[(\tau\Sigma)^{-1}\pi + \ P^T\Omega^{-1}q \right] . \end{aligned} \tag{21}$$

Using the Woodbury matrix identity [46,20] stating that

$$[A + UCV]^{-1} = A^{-1} - A^{-1}U \left[C^{-1} + VA^{-1}U \right]^{-1} VA^{-1}$$

where A, C, U and V are matrices of appropriate dimensionality, we have

$$\left[(\tau\Sigma)^{-1} \ + \ P^T\Omega^{-1}P \right]^{-1} \ = \ \tau\Sigma - \tau\Sigma P^T \left[\Omega + P\tau\Sigma P^T \right]^{-1} P\tau\Sigma ,$$

and, thus,

$$\begin{aligned} & \left[(\tau\Sigma)^{-1} \ + \ P^T\Omega^{-1}P \right]^{-1} \ \times \ \left[(\tau\Sigma)^{-1}\pi + \ P^T\Omega^{-1}q \right] \\ = \ & \pi + \tau P^T\Sigma\Omega^{-1}q - \tau\Sigma P^T \left[\Omega + P\tau\Sigma P^T \right]^{-1} P\pi - \tau\Sigma P^T \left[\Omega + P\tau\Sigma P^T \right]^{-1} P\tau\Sigma P^T\Omega^{-1}q \\ = \ & \pi - \tau\Sigma P^T \left[\Omega + P\tau\Sigma P^T \right]^{-1} P\pi + \tau\Sigma P^T \left[\Omega^{-1} - \left[\Omega + P\tau\Sigma P^T \right]^{-1} P\tau\Sigma P^T\Omega^{-1} \right] q . \end{aligned} \tag{22}$$

Using the equality below

$$\Omega^{-1} - \left[\Omega + P\tau\Sigma P^T\right]^{-1} P\tau\Sigma P^T \Omega^{-1} \;=\; \frac{\Omega^{-1}\left[\Omega + P\tau\Sigma P^T\right] - P\tau\Sigma P^T\Omega^{-1}}{\left[\Omega + P\tau\Sigma P^T\right]} \;=\; \left[\Omega + P\tau\Sigma P^T\right]^{-1}$$

to replace the right-hand side term in (22), we obtain (19). □

It can be seen that the error term vector ϵ does not directly enter the Black-Litterman formula. However, the variance of each error term does. We use the revised vector of expected returns in the optimization problem, and the objective function (1) of the VaR Black-Litterman model thus becomes $\max\ \tilde{\mu}^T x$. Evidently, if the investor has no opinion about the future return behavior of assets, then $\tilde{\mu} = \pi$.

2.4 Model formulation and features

Below, we provide the complete formulation of the VaR Black-Litterman FoF optimization model:

$$\begin{aligned}
&\max\ \tilde{\mu}^T x \\
\text{subject to}\ &\sum_{i=1}^{n} x_i = 1 \\
&\underline{b_k} \leq \sum_{j \in C_k} x_j \leq \overline{b_k}, && \forall k \\
&\underline{b_{k,s}} \leq \sum_{j \in C_{k,s}} x_j \leq \overline{b_{k,s}}, && \forall s \in k, \forall k \\
&\underline{c_l} \leq \sum_{j \in R_l} x_j \leq \overline{c_l}, && \forall l \\
&\underline{l_k} \leq \sum_{j \in L_k} x_j, && \forall k \\
&x_j \leq \delta_j, && j = 1, \ldots, n \\
&x_{min}\, \delta_j \leq x_j, && j = 1, \ldots n \\
&x^T \Sigma x \leq s \\
&P(Y \leq \beta \cdot K) \geq 0.95 \\
&\delta \in \{0,1\}^n \\
&Y \in \mathcal{R} \\
&x \in \mathcal{R}_+^n
\end{aligned} \quad . \qquad (23)$$

The probabilistic integer problem above is non-convex. The derivation of a solution method enabling to solve it to optimality poses a very challenging computational problem, especially if the asset universe comprises a moderate to large number of assets.

3 Solution Method

The solution method involves two steps. The first one consists in the reformulation of the VaR constraint and in the derivation of a deterministic equivalent or approximation for the stochastic VaR Black-Litterman optimization problem (23). We study the computational tractability and the convexity properties of the reformulated deterministic problem. The second step is devoted to the development of a specialized branch-and-bound algorithm, called *portfolio return* branch-and-bound algorithm, allowing for the numerical solution of the reformulated deterministic problem.

3.1 Derivation of Deterministic Equivalent or Approximation

The VaR constraint (13) that forces the loss to not exceed β (i.e., $\beta = 5\%$ or 10%) of the FoF's initial value with a probability at least equal to $p = 95\%$ can be restated as the insurance that the FoF will have a return larger than or equal to -$\beta\%$ with a probability larger than or equal to 95%. Therefore, (13) is equivalent to:

$$P(\xi^T x \geq -\beta) \geq p \tag{24}$$

where β is a fixed parameter taking value 0.1 or 0.05, ξ is the n-dimensional vector of stochastic returns and $\xi^T x$ is the random return generated by the FoF.

Let $F_{(x)}$ be the cumulative probability distribution of an n-variate standard normal variable $\frac{\xi^T x - \mu^T x}{x^T \Sigma x}$ representing the normalized return of the FoF. Note that $F_{(x)}$ is the probability distribution of a random variable (i.e., the normalized portfolio return) with mean 0 and standard deviation 1 regardless of the holdings of the portfolio, but, as indicated by the subscript $_{(x)}$ in the notation $F_{(x)}$, the form of the probability distribution depends on the holdings and the probability distribution of their returns.

The probability $P(\xi^T x \geq -\beta)$ in the VaR constraint above (24) can be successively rewritten as:

$$\begin{aligned} &\mathcal{P}(\xi^T x \geq -\beta) \\ =&\mathcal{P}\left(\frac{\xi^T x - \mu^T x}{x^T \Sigma x} \geq \frac{-\beta - \mu^T x}{\sqrt{x^T \Sigma x}}\right) \\ =&1 - F_{(x)}\left(\frac{-\beta - \mu^T x}{\sqrt{x^T \Sigma x}}\right) \end{aligned} \tag{25}$$

and the VaR constraint (24) reads:

$$\begin{aligned} &1 - F_{(x)}\left(\frac{-\beta - \mu^T x}{\sqrt{x^T \Sigma x}}\right) \geq 0.95 \\ \Leftrightarrow\ & F_{(x)}\left(\frac{-\beta - \mu^T x}{\sqrt{x^T \Sigma x}}\right) \leq 0.05 \\ \Leftrightarrow\ & \frac{-\beta - \mu^T x}{\sqrt{x^T \Sigma x}} \leq F_{(x)}^{-1}(0.05) \\ \Leftrightarrow\ & \mu^T x + F_{(x)}^{-1}(0.05)\sqrt{x^T \Sigma x} \geq -\beta \end{aligned} \tag{26}$$

where $F_{(x)}^{-1}(0.05)$ is the 5^{th}-quantile of the distribution F. Clearly, (26) is the deterministic equivalent of the VaR constraint and is fully defined upon the determination of a single parameter, i.e. the quantile of the probability distribution. We provide below the deterministic equivalent formulation of the problem (23):

$$\begin{aligned} \max\ & \tilde{\mu}^T x \\ \text{subject to}\ & Ax \leq b \\ & x_j \leq \delta_j && j = 1, \ldots, n \\ & x_{min}\, \delta_j \leq x_j && j = 1, \ldots, n \\ & x^T \Sigma x \leq s \\ & \mu^T x + F_{(x)}^{-1}(0.05)\sqrt{x^T \Sigma x} \geq -\beta \\ & \delta \in \{0,1\}^n \\ & x \in \mathcal{R}_+^n \end{aligned} \tag{27}$$

where $Ax \leq b$ is a compact representation of the set of linear constraints (3)-(7).

It is straightforward to see that the value of quantile determines the convexity nature of the optimization problem, which has a tremendous impact on the computational tractability of the problem at hand. In fact, the left-hand side of the constraint (26) is a concave function if $F_{(x)}^{-1}(0.05)$ is negative. If this condition is verified, the feasibility set of the continuous relaxation of (27) is convex. Note that this is the case for a wide range of probability distributions: Bonami and Lejeune [8] have shown that the quantile of $F^{-1}_{(x)}(1-\beta)$ is negative for any symmetric and right-skewed probability distribution for any value of $p > 0.5$. Typically, the enforced probability level is high, ranging between $[0.7, 0.99]$.

Note also that, under the assumption that each asset has a normal probability distribution for its return, the probability distribution of the portfolio return is also Gaussian, and the exact numerical value of the quantile can be easily obtained. However, for most markets, the normal assumption is too strong in view of the prevalence of fat-tailed return, skewness and larger than normal dependence among extreme return events [34].

If the exact form of the probability distribution of the portfolio return is unknown, one cannot derive a deterministic equivalent for the probabilistic constraint (24), but can resort to well-known probability inequalities to derive surrogate deterministic constraints which closely approximate the original stochastic one (24).

Proposition 3 *The VaR constraint*

$$P(\xi^T x \geq -\beta) \geq p$$

is implied by the inequality

$$\mu^T x - \sqrt{\frac{p}{1-p}}\sqrt{x^T \Sigma x} \geq -\beta \ . \tag{28}$$

Proof : Consider the random vector Y such that $Y^T x$ has the same mean ($\mu^T x$) and variance ($x^T \Sigma x$) as $\xi^T x$. This is obtained by setting $Y^T x = (2\,\mu^T - \xi^T)x$. It follows that:

$$\mathcal{P}\left(Y^T x - \mu^T x > \mu^T x + \beta\right) = \mathcal{P}\left(\xi^T x - \mu^T x < -\beta - \mu^T x\right) = 1 - \mathcal{P}\left(\xi^T x - \mu^T x \geq -\beta - \mu^T x\right) \ .$$

Cantelli's inequality [10] (also known as the one-sided Chebychev's inequality) states that:

$$P(Y^T x - \mu^T x \geq t) \leq \frac{x^T \Sigma x}{x^T \Sigma x + t^2} \ . \tag{29}$$

Setting $t = \beta + \mu^T x$, we have

$$P(Y^T x - \mu^T x \geq \mu^T x + \beta) \leq \frac{x^T \Sigma x}{x^T \Sigma x + (\beta + \mu^T x)^2} \ , \tag{30}$$

and

$$1 - \mathcal{P}\left(\xi^T x - \mu^T x \geq -\beta - \mu^T x\right) \leq \frac{x^T \Sigma x}{x^T \Sigma x + (\beta + \mu^T x)^2} \ . \tag{31}$$

Therefore, the deterministic inequality

$$1 - \frac{x^T \Sigma x}{x^T \Sigma x + (\beta + \mu^T x)^2} \geq p.$$

which can be rewritten as

$$\mu^T x - \sqrt{\frac{p}{1-p}}\ \sqrt{x^T \Sigma x} \geq -\beta$$

implies the VaR constraint $P(\xi^T x \geq -\beta) \geq p$, which was set out to prove. □

The deterministic inequality is an approximation of and enforces stricter requirements than the VaR constraint. Tighter approximations can be derived if additional knowledge about the features of the probability distribution of the portfolio return is available.

Proposition 4 *If the probability distribution of the portfolio return can be assumed to be symmetric, then the VaR constraint*

$$P(\xi^T x \geq -\beta) \geq p$$

is implied by the inequality

$$\mu^T x - \sqrt{\frac{1}{2(1-p)}}\sqrt{x^T \Sigma x} \geq -\beta \ . \tag{32}$$

Proof : This result is obtained by using the symmetric version of the Chebychev's inequality which states that:

$$P(Y^T x - \mu^T x \geq t) \leq \frac{x^T \Sigma x}{2t^2} \; . \tag{33}$$

The proof is derived the same way as in Proposition 1.

Proposition 5 *If the probability distribution of the portfolio return can be assumed to be symmetric and unimodal, then the VaR constraint*

$$P(\xi^T x \geq -\beta) \geq p$$

is implied by the inequality

$$\mu^T x - \sqrt{\frac{2}{9(1-p)}}\sqrt{x^T \Sigma x} \geq -\beta \; . \tag{34}$$

Proof : This result is obtained by using the Camp-Meidell's inequality [9,32] which states that:

$$P(Y^T x - \mu^T x \geq t) \leq \frac{2x^T \Sigma x}{9t^2} \; . \tag{35}$$

The proof is derived the same way as in Proposition 1.

The three deterministic approximations of the Var constraint are second-order cone constraints and the associated feasibility set is therefore convex. The results derived above in this section show that the continuous relaxation of the absolute return VaR Black-Litterman optimization problem is convex for a wide range of probability distributions, and that for any portfolio, and thus for any portfolio return distribution with finite first and second moments, a convex approximation can be obtained.

3.2 Portfolio return branching strategy

The deterministic equivalent of the absolute return VaR Black-Litterman optimization problem is solved using a specialized non-linear branch-and-bound algorithm which is based on a new branching strategy called *portfolio return*. We use the open-source mixed-integer non-linear programming (MINLP) solver Bonmin [4] in which the new branching rule is implemented. At each node of the branch-and-bound tree, the interior-point solver Ipopt [3] is used to solve the nonlinear continuous relaxations of the integer problems. We refer the reader to [41] for a detailed description of non-linear branch-and-bound algorithms and to [8] for an application of this technique to the solution of a particular type of stochastic portfolio optimization problems accounting for trading restrictions.

Using the interior-point solver Ipopt, the algorithm first solves the continuous relaxation of the VaR Black-Litterman optimization problem in which all integrality constraints are removed. Let (x^*, δ^*) be the optimal solution of the continuous relaxation. If all δ^* are integer valued, (x^*, δ^*) is the optimal solution and the problem is solved. Otherwise, i.e., if at least one of the integer variables (δ_i) has a fractional value ($\delta_i^* \notin \mathcal{Z}$) in the optimal solution, one ($\delta_{\hat{i}}$) of them is selected for *branching*, and two nodes are created where the upper and lower bounds on $\delta_{\hat{i}}$ are set to $\lfloor \delta_{\hat{i}}^* \rfloor$ and $\lceil \delta_{\hat{i}}^* \rceil$, respectively, and the two corresponding sub-problems are put in a list of open nodes. We obtain the following disjunctive problem with one second-order cone and one quadratic constraints

$$\begin{aligned}
\max\ & \tilde{\mu}^T x \\
\text{subject to}\ & Ax \leq b \\
& x_j \leq \delta_j && j = 1, \ldots, n \\
& x_{min}\, \delta_j \leq x_j && j = 1, \ldots, n \\
& x^T \Sigma x \leq s \\
& \mu^T x + F_{(x)}^{-1}(0.05)\sqrt{x^T \Sigma x} \geq -\beta \\
& (\delta_{\hat{i}} = 0) \vee (\delta_{\hat{i}} = 1) \\
& x \geq 0
\end{aligned} \tag{36}$$

to which correspond the following two nodes

$$
\begin{array}{lrl}
\max & \tilde{\mu}^T x & \\
\text{subject to} & Ax & \leq b \\
& x_j & \leq \delta_j,\ j = 1, \ldots, n \\
& x_{min}\,\delta_j & \leq x_j,\ j = 1, \ldots, n \\
& x^T \Sigma x & \leq s \\
& \mu^T x + F^{-1}_{(x)}(0.05)\sqrt{x^T \Sigma x} & \geq -\beta \\
& 0 & \geq x_{\hat{i}} \\
& \delta & \geq 0 \\
& \delta & \leq 1 \\
& x & \geq 0
\end{array}
\quad (37) \quad \text{and} \quad
\begin{array}{lrl}
\max & \tilde{\mu}^T x & \\
\text{subject to} & Ax & \leq b \\
& x_j & \leq \delta_j,\ j = 1, \ldots, n \\
& x_{min}\,\delta_j & \leq x_j,\ j = 1, \ldots, n \\
& x^T \Sigma x & \leq s \\
& \mu^T x + F^{-1}_{(x)}(0.05)\sqrt{x^T \Sigma x} & \geq -\beta \\
& x_{\hat{i}} & \leq x_{min} \\
& \delta & \geq 0 \\
& \delta & \leq 1 \\
& x & \geq 0
\end{array}
\quad (38)
$$

The above formulations of the two sub-problems result from the mapping between integer variables $\delta[i]$ and assets i (and positions x_i), and the definition of the buy-in threshold constraint (10) which makes the enforcement of $0 \geq x_{\hat{i}}$ and $x_i \leq x_{min}$ equivalent to imposing $0 \geq \delta_i$ and $\delta_i \leq 1$ (in (36)).

The choice of the variable to branch on strongly impacts the computational efficiency of the branch-and-bound algorithm. The default rule is to branch with respect to the variable which has the largest fractional part, but this rule often turns out to be very moderately efficient. In this paper, the branch on variable is determined by the portfolio return branching strategy, which is *dynamic*, (since it iteratively, i.e., at each node, revises the branching priorities), and *integrated* (since their update accounts for the structure of the portfolio at the current node in the tree). The definition of the branching priorities is carried out by associating a number π_i to each integer decision, and the solver, at each node, branches with respect to the integer variable having the largest priority. Within the portfolio return branching strategy, we assess how the restoration of the integrality condition alters the value of the objective function (i.e., the portfolio return). The variable whose integer feasibility restoration has the largest impact on the portfolio return receives the highest priority, and is the one with respect to which we branch. More precisely, denoting by z^*, z_1^* and z_2^* the optimal value of the continuous relaxation of the VaR Black-Litterman problem, (37) and (38) respectively, we calculate π_i for the integer variables that have a fractional value in the optimal solution of the continuous relaxation of the VaR Black-Litterman problem as follows:

$$\pi_i = \left|z^* - z_1^*\right| + \left|z^* - z_2^*\right| \ , \tag{39}$$

and the branch on variable $\delta_{\hat{i}}$ is the one having the largest π_i.

At each node, a sub-problem is chosen according to the process described above from the list of open ones, and the continuous relaxation of the current node is solved, thus providing a lower bound. The enumeration at the current node is stopped if any of the three following conditions happen: *(i)* the continuous relaxation is infeasible; *(ii)* the optimal solution of the continuous relaxation is not better than the value of the best integer feasible solution found so far; *(iii)* the optimal solution of the continuous relaxation is integer feasible. By iterating the process a search tree is created and the algorithm continues until the list of open sub-problems is empty.

4 Computational Efficiency

The computational framework is the following. The optimization models are coded using the AMPL programming language [16]. The portfolio return solution method is implemented using the COmputational INfrastructure for Operations Research (COINOR) open source optimization environment: the source code of the COINOR solvers is available to users under the Common Public License, which allows users to read, modify, or improve and redistribute the software. We implement the branching rule within the MINLP solver Bonmin [4] and we solve the continuous relaxations of the successive integer programming problems with the Ipopt [3] solver. From a computational point of view, the fact that Bonmin and Ipopt are both open-source publicly available solvers is very interesting: it allowed the implementation of the new portfolio return branching strategy. Data related to the returns and the market capitalization of the assets were collected from the Bloomberg platform.

To evaluate the computational contribution of our solution method, we compare its results with those obtained with:

- the standard branch-and-bound algorithm of the `Bonmin` solver,
- the `MINLPBB` solver [35] (which is another MINLP solver),
- the `CPLEX 11` [22] solver.

The computational tests were performed on an IBM IntellistationZ Pro with an Intel Xeon 3.2GHz CPU, 2 gigabytes of RAM and running Linux Fedora Core 3. We have tested our algorithm on two sets of 12 problem instances. In the first set, β is set equal to 10% while, in the other, it is equal to 5%. Within a set, the problem instances differ in terms of the expressed views, the value of the authorized variance s, the bounds on the diversification and the liquidity constraints, and the type of approximation used for the VaR constraint.

To compare the performance and reliability of the proposed solution with the four algorithms we use as benchmarks, we draw the performance profile [38] of each of them. The curve associated with a solution method, say *PR*, indicates the proportion of problem instances solved to optimality using *S1* within a factor m of the time required by the fastest solution method. Denoting by $t^*(P)$ the time needed by the fastest algorithm to solve problem P, P is solved within a factor m by all solution approaches requiring less than $m\ t^*(P)$. It follows that, at $m = 1$, the plotted point for *S1* returns the fraction of problem instances on which *PR* is the fastest. The plotted point for *BC1* at $m = 20$ represents the proportion of problems that could be solved using up to 20 times more time than the fastest of the three solution methods. In Figures 3, 4 and 5, we use the acronyms `PR`, `BN`, `BB` and `CP` to respectively refer to the portfolio return branching strategy, `Bonmin`'s standard branch-and-bound algorithm, the `MINLPBB` solver and the CPLEX solver.

Figures 3 and 4 clearly show that the portfolio return solution technique strictly dominates the three other solution methods. The portfolio return solution technique solves to optimality all problem instances and is the fastest on all instances. For the problems in which $\beta = 10\%$ (Figure 3), the portfolio return solution technique is:

1. 5 times faster than `BN` (resp., `BB` and `CP`) in 91.67% (resp., 91.67%, 100%) of the instances;
2. 10 times faster than `BN` (resp., `BB` and `CP`) in 33.33% (resp., 41.67%, 33.33%) of the instances;
3. 20 times faster than `BN` (resp., `BB` and `CP`) in 8.33% (resp., 16.67%, 8.33%) of the instances;

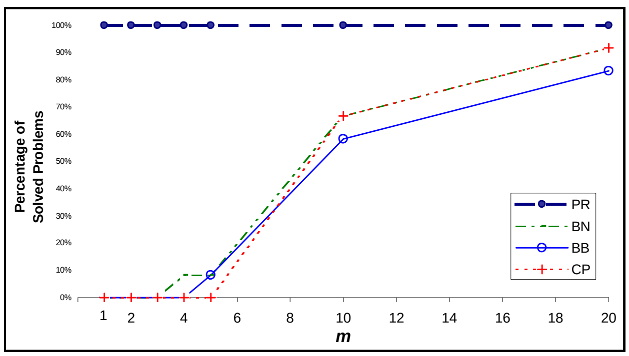

Fig. 3 Performance Profile I

The results are almost identical for the problems in which the VaR is equal to 95% (Figure 4). For those instances, the portfolio return branching rule is

1. 5 times faster than `BN` (resp., `BB` and `CP`) in 91.67% (resp., 91.67%, 83.67%) of the instances;
2. 10 times faster than `BN` (resp., `BB` and `CP`) in 33.33% (resp., 33.33%, 41.67%) of the instances;
3. 20 times faster than `BN` (resp., `BB` and `CP`) in 8.33% (resp., 8.33%, 16.67%) of the instances;

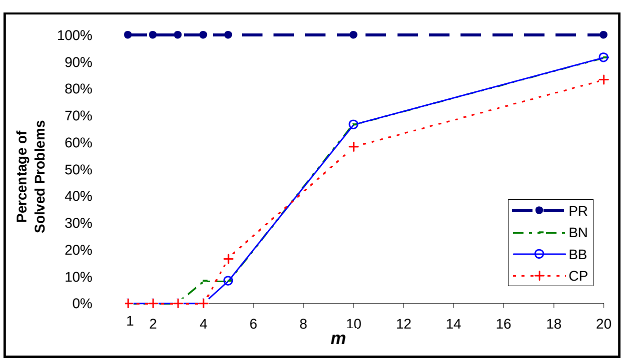

Fig. 4 Performance Profile II

Figure 5 shows that the gain in computational time of the portfolio return branching rule must be associated with the fact that it requires the solution of a much smaller number of subproblems (nodes), as compared to the `BN` (resp,, `BB` and CP) solution methods. Clearly, the search initiated by the portfolio return method allows a much faster convergence towards the optimal solution, and, on average, permits to divide the number of nodes to be processed by more than:

1. 10 (resp., 7) for $p = 90\%$ (resp., $p = 95\%$), as compared to the average number of nodes required by BN;
2. 27 (resp., 15) for $p = 90\%$ (resp., $p = 95\%$), as compared to the average number of nodes required by BB;
3. 6 (resp., 4) for $p = 90\%$ (resp., $p = 95\%$), as compared to the average number of nodes required by CP.

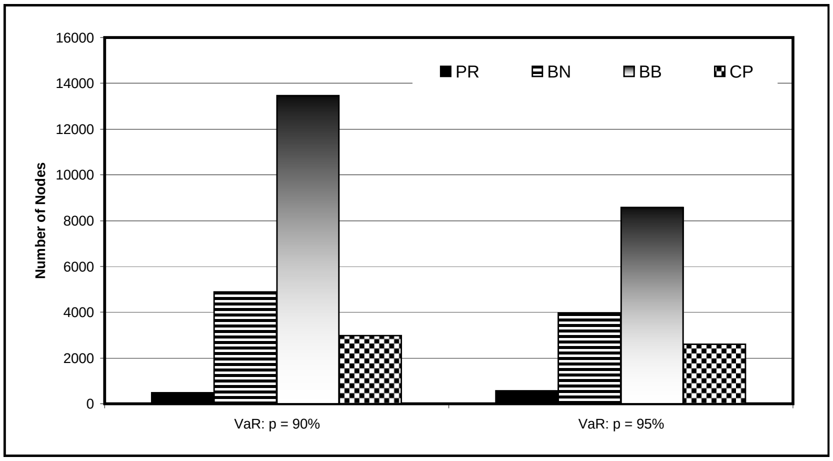

Fig. 5 Average Number of Subproblems per Solution Method

5 Conclusion

This research was initiated by our collaboration with a financial institution whose objective was the construction of fund-of-funds responding to specific industry requirements. The first contribution of this paper is the derivation of the new VaR Black-Litterman FoF model for the construction of fund-of-funds targeting an absolute return strategy. In order to circumvent or at least alleviate the problems associated with the estimation risk, the asset returns are approximated through the combination of the market equilibrium based returns and the opinions of experts by using the Black-Litterman approach. Moreover, the resulting vector of estimated returns is implicitly assumed to be stochastic by the VaR constraint, which prescribes the construction of a FoF having an expected return not falling below -5 or -10%, with a probability at least equal to 95%. The model also accounts for the handling of specific trading constraints and takes the form of a very complex stochastic integer programming problem.

The second contribution is the derivation of deterministic equivalent or approximations for the VaR Black-Litterman model. We further show that, for a wide range of probability distributions, those deterministic reformulations are convex, which is critical for the computational tractability, the numerical solution of the problem, and for the use of the proposed model to asset universe comprising a large number of possible investment vehicles. The approximations of the deterministic equivalents are obtained through the use of the Cantelli, the one-sided symmetric Chebychev, and the Camp-Meidell probability inequalities, and their tightness depends on the assumed properties of the probability distribution of the fund-of-funds return.

Our third contribution is the development of a solution methodology that proves to be:

– *robust*: it allows the finding of the optimal portfolio for all the considered instances. To appraise the significance of this result, we refer the reader to [29]. In this very recent paper, an heuristic method is proposed to solve numerically portfolio optimization problems containing non-linear convex constraints and integer decision variables. The method proposed in [29] allows the finding of near-optimal solutions for optimization models in which the number of assets does not exceed 100;
– *fast and computationally tractable*: it is much faster than the three state-of-the-art tested algorithmic methods;
– *adaptability*: it is based on open-source optimization solver and can therefore be easily adapted to and/or supplemented by recent algorithmic developments;
– *relevant for and responding to the standards of the industry*: the proposed solution method was implemented and used to construct four absolute return fund-of-funds that re traded on the major stock markets.

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