On Cone of Nonsymmetric Positive Semidefinite Matrices

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Abstract

In this paper, we analyze and characterize the cone of nonsymmetric positive semidefinite matrices (NS-psd). Firstly, we study basic properties of the geometry of the NS-psd cone and show that it is a hyperbolic but not homogeneous cone. Secondly, we prove that the NS-psd cone is a maximal convex subcone of $P_0$-matrix cone which is not convex. But the interior of the NS-psd cone is not a maximal convex subcone of $P$-matrix cone. As the byproducts, some new sufficient and necessary conditions for a nonsymmetric matrix to be positive semidefinite are given. Finally, we present some properties of metric projection onto the NS-psd cone.

Keywords: Nonsymmetric positive semidefinite matrix; hyperbolic cone; facial structure; maximal convex subcone; $P_0$-matrix; projection.

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1 Introduction

We consider the space of $n \times n$ real matrices, denoted by $\mathcal{M}^n$, with the trace inner product

$$\langle X, Y \rangle := tr(X^T Y)$$

for $X, Y \in \mathcal{M}^n$ and the induced Frobenius matrix norm $\|X\| = \sqrt{tr(X^T X)}$.

A nonsymmetric matrix $X \in \mathcal{M}^n$ is called positive semidefinite (NS-psd for short) if $u^T X u \geq 0$ for all $u \in \mathbb{R}^n$, and called positive definite if $u^T X u > 0$ for all $0 \neq u \in \mathbb{R}^n$. We use $\mathcal{M}^n_+$ to denote the set of all nonsymmetric positive semidefinite matrices in $\mathcal{M}^n$, and $\mathcal{M}^n_{++}$ to denote the set of all nonsymmetric positive definite matrices in $\mathcal{M}^n$. Then $\mathcal{M}^n_+$ is a closed convex cone and $\mathcal{M}^n_{++}$ is the interior of $\mathcal{M}^n_+$.

Let $\mathcal{S}^n$ be the subspace of $n \times n$ symmetric matrices in $\mathcal{M}^n$. Correspondingly, let $\mathcal{S}^n_+$ denote the cone of positive semidefinite matrices in $\mathcal{S}^n$, and $\mathcal{S}^n_{++}$ denote the cone of positive definite matrices in $\mathcal{S}^n$. $\mathcal{S}^n_+$ is a closed convex cone and its interior is $\mathcal{S}^n_{++}$.

It’s well known that $\mathcal{S}^n_+$, as an very important non-polyhedral convex cone, has nice geometric properties and arises in many areas, including engineering, statistics, and system and

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control theory, etc. Linear optimization problem over $S^n_+$, known as semidefinite programming (SDP), plays a fundamental role in mathematical programming, see, e.g., [21, 2, 7]. However, compared with the $S^n_+$, the NS-psd cone $M^n_+$ hasn’t been well studied on convex analysis. Actually, $S^n_+$ and $M^n_+$ are very different in many aspects. Let us observe the following four examples:

- $S^n_+$ is a self-dual homogenous cone in $S^n$, but $M^n_+$ is a hyperbolic cone and not a homogenous cone in $M^n$ (see Theorem 3.2).
- A matrix $X \in S^n_+$ is invertible if and only if it belongs to the interior of $S^n_+$, whereas an invertible matrix $X \in M^n_+$ doesn’t imply that $X$ is in the interior of $M^n_+$. In fact, if we take $X = \begin{pmatrix} 0 & 1 \\ -1 & 0 & 1 \end{pmatrix} \in M^2_+$, then for any $\epsilon > 0$, we have
  \[
  \begin{pmatrix} 0 & 1 \\ -1 & 0 & 1 \end{pmatrix} + \epsilon \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \epsilon & 1 \\ -1 & 1 \end{pmatrix} \in M^2_+,
  \]
  meanwhile
  \[
  \begin{pmatrix} 0 & 1 \\ -1 & 0 & 1 \end{pmatrix} - \epsilon \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -\epsilon & 1 \\ -1 & -\epsilon \end{pmatrix} \notin M^2_+.
  \]
  This means that $\begin{pmatrix} 0 & 1 \\ -1 & 0 & 1 \end{pmatrix}$ belongs to the boundary of $M^2_+$ but it is invertible.
- It’s well known that a symmetric matrix belongs to $S^n_+$ (resp. $S^n_{++}$) if and only if it is a $P_0$ (resp. $P$)-matrix, but the equivalence fails when symmetry assumption is dropped (see Example 3.3.2 in [5]).
- $S^n_+$ and $M^n_+$ are subsets of $P_0$-matrix cone ($P_0$ for short). For $S^n_+$, $bd(S^n_+) \subseteq bd(P_0)$, which implies $bd(S^n_+) \cap int(P_0) = \emptyset$. While, for $M^n_+$, $bd(M^n_+) \cap int(P_0) \neq \emptyset$ (see Proposition 4.1 (ii)).

In this paper, we will take a close look at the NS-psd cone in the view of convex analysis. First, we study the facial structure of $M^n_+$ in Section 3, which is a representative property for closed convex cones. We’ll show that $M^n_+$ is a hyperbolic cone but not a homogeneous cone in $M^n$, while $S^n_+$ is a self-dual homogenous cone in $S^n$. In Section 4, we study the relationship between the NS-psd cone and the $P_0$-matrix cone, where the latter one is a very important class of matrices in linear complementarity theory and it contains $M^n_+$ as a proper subclass. By proving some fundamental and interesting results about matrix determinant and the boundary properties of $P_0$ and $M^n_+$, we obtain that $M^n_+$ is a maximal convex subcone of $P_0$, however, $M^n_{++}$ is not a maximal convex subcone of $P$. Some necessary and sufficient conditions for a matrix to be NS-psd are also presented. Finally, we study the metric projection onto $M^n_+$ in Section 5, including the strong semismoothness, explicit formulas of directional derivative and Clarke’s generalized Jacobian of the projection, which extend a series of results in [12, 17, 18].

2
2 Preliminaries

In this section, we review some concepts and properties about convex cones in a finite-dimensional real vector space \( \mathcal{E} \) equipped with an inner product \( \langle \cdot , \cdot \rangle \) and the induced norm \( \| \cdot \| \).

**Convex cone** A convex cone \( \mathcal{K} \subseteq \mathcal{E} \) is a nonempty set that is closed under nonnegative linear combination of all its members, i.e., \( \lambda \mathcal{K} \subseteq \mathcal{K}, \mathcal{K} + \mathcal{K} \subseteq \mathcal{K}, \forall \lambda \geq 0 \). The biggest subspace contained in \( \mathcal{K} \) is called the *linearity space* of \( \mathcal{K} \), denoted by \( L(\mathcal{K}) \). If \( L(\mathcal{K}) = \{0\} \), we call the convex cone \( \mathcal{K} \) is *pointed*. In other words, a convex cone \( \mathcal{K} \) is pointed if it has no lines. If a closed convex pointed cone has nonempty interior, we call it a *proper cone*.

Given two nonempty convex cones \( \mathcal{K}_1 \) and \( \mathcal{K}_2 \) in \( \mathcal{E} \), let \( \mathcal{K}_1 \oplus \mathcal{K}_2 \) denote the direct sum of \( \mathcal{K}_1 \) and \( \mathcal{K}_2 \), i.e., each vector \( x \in \mathcal{K}_1 + \mathcal{K}_2 \) can be expressed uniquely in the form \( x = y + z \) where \( y \in \mathcal{K}_1, z \in \mathcal{K}_2 \). One special case of \( \mathcal{K}_1 \oplus \mathcal{K}_2 \) is when \( \langle \mathcal{K}_1, \mathcal{K}_2 \rangle = 0 \).

Given a set \( \mathcal{C} \subseteq \mathcal{E} \), the dual cone of \( \mathcal{C} \) is defined as \( \mathcal{C}^* = \{ x : \langle x, y \rangle \geq 0, \forall y \in \mathcal{C} \}. \) The convex hull of \( \mathcal{C} \) is denoted by \( \text{conv}(\mathcal{C}) \). \( \text{cone}(\mathcal{C}) \) denotes the convex cone generated by \( \mathcal{C} \). We let \( \text{int}(\mathcal{C}), \text{cl}(\mathcal{C}), \text{bd}(\mathcal{C}) \) and \( \text{ri}(\mathcal{C}) \) denote the interior, closure, boundary, relative interior and relative boundary of \( \mathcal{C} \), respectively. And we use \( \mathcal{C}_1 \subseteq \mathcal{C}_2 \) denote a proper subset, i.e., \( \mathcal{C}_1 \subset \mathcal{C}_2 \).

**Typical closed convex cones** A closed pointed convex cone \( \mathcal{K} \subseteq \mathcal{E} \) with nonempty interior is *homogeneous* if for any \( x, y \in \text{int}(\mathcal{K}) \) there exists an invertible linear mapping \( g \) such that \( g(\mathcal{K}) = \mathcal{K} \) and \( g(x) = y \), i.e., the group of automorphisms of \( \mathcal{K} \) acts transitively on the interior of \( \mathcal{K} \). If \( \mathcal{K} \) is homogenous and \( \mathcal{K}^* = \mathcal{K} \) (self dual), we call \( \mathcal{K} \) a *symmetric cone*. It’s well known that \( S^+_n \) is a symmetric cone in \( S^n \). And there are several other common symmetric cones, such as the nonnegative orthant \( (R^n_+ \), the Lorentz cone \( \text{(i.e., second order cone)}, \) and so on. For more details about homogeneous and symmetric cones, see [6, 8, 20, 19], etc.

Besides the homogenous cone, there exists a more general closed convex cone, called hyperbolic cone. It is defined as follows. Given a homogeneous polynomial \( p \) of degree \( n \) on \( \mathcal{E} \), \( p \) is called to be hyperbolic with respect to the direction \( d \in \mathcal{E} \), if \( p(d) \neq 0 \) and the polynomial \( t \mapsto p(td + x) \) has only real roots for every \( x \in \mathcal{E} \). The associated hyperbolic cone of the hyperbolic polynomial \( p \) with direction \( d \) is defined as the set of all such \( x \) that the univariate polynomial \( \lambda \mapsto p(\lambda d - x) \) has only nonnegative roots, where \( \lambda \mapsto p(\lambda d - x) \) is called the characteristic polynomial of \( x \). Hyperbolic cones contain homogeneous cones as a subclass. For more details see [1, 14, 9] and references therein.

**Faces of a closed convex cone** Let \( \mathcal{K} \subseteq \mathcal{E} \) be a closed convex cone. A convex subset \( \mathcal{F} \subseteq \mathcal{K} \) is a *face* of \( \mathcal{K} \), denoted by \( \mathcal{F} \preceq \mathcal{K} \), if

\[
x \in \mathcal{F}, \ 0 \preceq_{\mathcal{K}} y \preceq_{\mathcal{K}} x \ \Rightarrow \ \text{cone} \{ \{ y \} \} \subseteq \mathcal{F},
\]

where \( \preceq_{\mathcal{K}} \) denotes the partial order with respect to \( \mathcal{K} \), that is, \( x_1 \preceq_{\mathcal{K}} x_2 \) means that \( x_2 - x_1 \in \mathcal{K} \). Equivalently, \( \mathcal{F} \preceq \mathcal{K} \) if \( x + y \in \mathcal{F}, x \in \mathcal{K} \), and \( y \in \mathcal{K} \) implies that \( x \in \mathcal{F} \) and \( y \in \mathcal{F} \). If \( \mathcal{F} \preceq \mathcal{K} \) but \( \mathcal{F} \neq \mathcal{K} \), we write \( \mathcal{F} \prec \mathcal{K} \). If \( \emptyset \neq \mathcal{F} \preceq \mathcal{K} \), then \( \mathcal{F} \) is a *proper face* of \( \mathcal{K} \). Every proper face of \( \mathcal{K} \) belongs to \( \text{rb}(\mathcal{K}) \). The complementary or conjugate face of \( \mathcal{F} \preceq \mathcal{K} \), denoted by \( \mathcal{F}^c \), is defined as \( \mathcal{F}^c = \mathcal{K}^* \cap \mathcal{F}^\perp \). For \( \mathcal{C} \subseteq \mathcal{K} \), we let \( \mathcal{F}(\mathcal{C}, \mathcal{K}) \) denote the smallest face that contains \( \mathcal{C} \), i.e., \( \mathcal{F}(\mathcal{C}, \mathcal{K}) \) is the intersection of all faces containing \( \mathcal{C} \). Followings are two important properties.
about facial structure of the closed convex cone $K$ (see [15] or [16]):

$$\mathcal{F} = \mathcal{F}(C,K) \text{ if and only if } ri(C) \subseteq ri(\mathcal{F});$$ (2.1)

$$U := \{ri(\mathcal{F}), \mathcal{F} \prec K\} \text{ is a partition of } rb(K).$$ (2.2)

Here, (2.2) means that the elements of $U$ are pairwise disjoint and cover $bd(K)$. Due to (2.1), it holds for any $\bar{x} \in ri(C)$ that

$$F(C,K) = \{y \in K : \alpha \bar{x} - y \in K, \exists \alpha > 0\}.\quad (2.3)$$

The ray generated by $0 \neq x \in K$ is called an extreme ray if $cone\{x\} \subseteq K$. Every extreme ray is a one-dimensional face. And a zero-dimensional face is called an extreme point, or a vertex. We use $Exe(K)$ denote the set of extreme rays of $K$. For $S^n_+$,

$$Exe(S^n_+) = \{uu^T : 0 \neq u \in \mathbb{R}^n\}.\quad (2.4)$$

If the closed convex cone $K$ is not pointed, then it has no extreme ray and no extreme point. Conversely (see Section 2.8, [7]),

*Every proper cone is equivalent to the convex hull of its extreme points and extreme rays.*

(2.5)

A face $F \subseteq K$ is an exposed face if it is the intersection of $K$ with a hyperplane. If every face of $K$ is exposed, we call $K$ the facially exposed. Further, $K$ is called a nice cone if $F^* = K^* + F^\perp$ for all $F \subseteq K$. All nice cones are facially exposed (see [11]). And all proper faces of hyperbolic cones are exposed (Theorem 23, [14]), so do homogeneous cones.

**Basic notations**

$X \succeq 0$ : $X$ is a nonsymmetric positive semidefinite matrix.

$K_1 \setminus K_2$ : difference of two sets $K_1$ and $K_2$, i.e., $\{x \in K_1 : x \notin K_2\}$.

$AS^n$ : the subspace of antisymmetric matrices.

$N(A)$ : the null space of a linear operator or a matrix $A$.

$K^\perp$ : the orthogonal complement of $K$ in $M^n$.

$K^{\perp\perp}$ : the orthogonal complement of $K$ in $S^n$.

$span(K)$ : linear space spanned by set $K$.

$E^{ij}$ : the matrix in $M^n$ with $(i,j)$th element being 1, all else being zeros.

$X_{\alpha\beta} : (X_{ij})_{i \in \alpha, j \in \beta}$, where $\alpha, \beta$ are subsets of $\{1, \cdots, n\}$.

$I$ : identity matrix of size depending on the context.

$diag(X)$ : a vector generated by the diagonal elements of $X \in M^n$.

$X^*$ : the classic adjoint matrix of $X \in M^n$, i.e., the transpose of the matrix formed by taking the cofactor of each element of $X$.

3 The geometry of NS-psd cone

In this section, we study some basic properties of $M^n_+$, mainly on its facial structure of it.
Since any real square matrix $A \in \mathcal{M}^n$ has a representation in terms of its symmetric and antisymmetric parts by
\[ A = \frac{A + A^T}{2} + \frac{A - A^T}{2}, \] (3.1)
the antisymmetric part vanishes under quadratic form, i.e., $u^T A - A^T u = 0 \ \forall u \in \mathbb{R}^n$, and the symmetric part has a role determining positive semidefiniteness, we easily obtain the following basic facts and proposition.

**Fact 1** $X \succeq 0 \iff X^T \succeq 0 \iff X + X^T \succeq 0$.

**Fact 2** $\mathcal{M}^n = S^n \oplus A S^n$.

**Fact 3** $\{ X \in \mathcal{M}^n : X + X^T \in C \} = C \oplus A S^n$ for any subset $C \subseteq S^n$.

**Proposition 3.1** In space $\mathcal{M}^n$, the following statements are true:

(i) $\mathcal{M}^n_+ = S^n_+ \oplus A S^n$.

(ii) $L(\mathcal{M}^n_+) = A S^n$.

(iii) $(\mathcal{M}^n_+)^* = S^n_+$.

**Proof.** (i) Direct result of Fact 3 by taking $C = S^n_+$.

(ii) Since $S^n_+$ is a pointed cone, i.e., $S^n_+$ contains no line. The biggest subspace in $\mathcal{M}^n_+$ is just $A S^n$ by (i).

(iii) It’s known that $S^n_+$ is self dual in $S^n$. Then using the property $(K_1 + K_2)^* = K_1^* \cap K_2^*$ for any cones $K_1$ and $K_2$ ([16]), and by item (i), we have
\[ (\mathcal{M}^n_+)^* = (S^n_+ \oplus A S^n)^* = (S^n_+)^* \cap (A S^n)^* = (S^n_+)^* \cap (A S^n)^\perp = (S^n_+)^* \cap S^n = S^n_+. \]
This completes the proof. \qed

Clearly, $\mathcal{M}^n_+$ is not a symmetric cone since $\mathcal{M}^n_+ \neq (\mathcal{M}^n_+)^*$. Also, due to the above statement (ii), $\mathcal{M}^n_+$ is not a pointed cone. This implies that $\mathcal{M}^n_+$ is not a homogeneous cone. Actually, it is a hyperbolic cone.

**Theorem 3.2** $\mathcal{M}^n_+$ is a hyperbolic cone and not a homogeneous cone.

**Proof.** Let $P(X) = \det(\frac{X + X^T}{2})$, $X \in \mathcal{M}^n$. Then $P(X)$ is a homogeneous polynomial of degree $n$ on $\mathcal{M}^n$. Since a real symmetric matrix has only real eigenvalues,
\[ P(X + t I) = \det(t I + \frac{X + X^T}{2}) = 0 \]
has only real roots for all $X \in \mathcal{M}^n$. Thus, $P(X)$ is a hyperbolic polynomial with respect to the identity matrix $I$. Let $\Lambda_1(X), \Lambda_2(X), \cdots, \Lambda_n(X)$ denote $n$ roots of $P(\lambda I - X) = \det(\lambda I - \frac{X + X^T}{2}) = 0$. Due to the fact that
\[ X \in \mathcal{M}^n_+ \iff \frac{X + X^T}{2} \in S^n_+, \]
we conclude that $\mathcal{M}^n_+ = \{ X \in \mathcal{M}^n : \Lambda_i(X) \geq 0, i = 1, 2, \cdots, n \}$, i.e., $\mathcal{M}^n_+$ is a hyperbolic cone of $P$ with direction $I$. \qed
From (2.5), Proposition 3.1 and Theorem 3.2, we know that all proper faces of $M^n_+$ are exposed and $M^n_+$ has no extreme ray and no extreme point.

To establish the facial structure of $M^n_+$, we first present the following two lemmas.

**Lemma 3.3** Given closed convex cones $K_1 \subset E$ and $K_2 \subset E$, $(K_1, K_2) = 0$. If $x \in K_1, y \in K_2, C_1 \subseteq K_1, C_2 \subseteq K_2$, and $x + y \in C_1 \oplus C_2$, then there holds $x \in C_1, y \in C_2$.

**Proof.** By the given, there exist $x_1 \in C_1, y_1 \in C_2$ such that $x + y = x_1 + y_2$. From $(\text{span}(K_1), y) = (\text{span}(K_1), y_2) = 0$, we obtain
\[
\langle \text{span}(K_1), x \rangle = \langle \text{span}(K_1), x + y \rangle = \langle \text{span}(K_1), x_1 + y_2 \rangle = \langle \text{span}(K_1), x_1 \rangle,
\]
i.e., $(\text{span}(K_1), x - x_1) = 0$. As $x - x_1 \in \text{span}(K_1)$, we have
\[
\langle x - x_1, x - x_1 \rangle = 0,
\]
which implies that $x = x_1 \in C_1$. Similarly, we have $y \in C_2$. This finishes the proof. \qed

**Lemma 3.4** If $K = K_1 \oplus K_2$, where $K_1, K_2$ are two closed convex cones in $E$, and $K \ni x = x_1 + x_2, x_1 \in K_1, x_2 \in K_2$, then $F(x, K) = F(x_1, K_1) \oplus F(x_2, K_2)$.

**Proof.** We first show that $(F(x_1, K_1) \oplus F(x_2, K_2)) \subseteq K$. Let $F := F(x_1, K_1) \oplus F(x_2, K_2)$ and $y, z \in K$ satisfying $y + z \in F$. Then there exist $y_1, z_1 \in K_1, y_2, z_2 \in K_2$ such that $y = y_1 + y_2, z = z_1 + z_2$. So, $(y_1 + z_1) + (y_2 + z_2) = y + z \in F = F(x_1, K_1) \oplus F(x_2, K_2)$. Noting that $y_1 + z_1 \in K_1, y_2 + z_2 \in K_2$, by Lemma 3.3 we have $y_1 + z_1 \in F(x_1, K_1), y_2 + z_2 \in F(x_2, K_2)$. Using the definition of face, we immediately imply that
\[
y_1, z_1 \in F(x_1, K_1), y_2, z_2 \in F(x_2, K_2).
\]
Thus,
\[
y = y_1 + y_2 \in F, \quad z = z_1 + z_2 \in F,
\]
which means that $F = F(x_1, K_1) \oplus F(x_2, K_2)$ is a face of $K$.

Now we prove the desired result.

1. “\(\subseteq\)” : Noting that $x = x_1 + x_2 \in (F(x_1, K_1) \oplus F(x_2, K_2))$, we have $F(x, K) \subseteq F(x_1, K_1) \oplus F(x_2, K_2)$ due to the definition of minimal face.

2. “\(\supseteq\)” : By (2.3), we have
\[
F(x_1, K_1) = \{ y \in K_1 : \alpha x_1 - y \in K_1, \exists \alpha > 0 \},
\]
\[
F(x_2, K_2) = \{ y \in K_2 : \alpha x_2 - y \in K_2, \exists \alpha > 0 \}.
\]
Let $y_1 \in F(x_1, K_1), y_2 \in F(x_2, K_2)$. Then there exist $\alpha_1 > 0, \alpha_2 > 0$ such that $\alpha_1 x_1 - y_1 \in K_1, \alpha_2 x_2 - y_2 \in K_2$. Let $\alpha := \max\{\alpha_1, \alpha_2\}$. Thus
\[
\alpha (x_1 + x_2) - (y_1 + y_2) = (\alpha_1 x_1 - y_1) + (\alpha - \alpha_1) x_1 + (\alpha_2 x_2 - y_2) + (\alpha - \alpha_2) x_2 \in K_1 \oplus K_2,
\]
i.e.,
\[
\alpha x - (y_1 + y_2) \in K,
\]
which yields $y_1 + y_2 \in F(x, K)$. So $F(x_1, K_1) \oplus F(x_2, K_2) \subseteq F(x, K)$. The proof is complete. \qed
Utilizing Lemma 3.4, we give out the following results.

**Theorem 3.5** In space $\mathcal{M}^n$, the following statements are true:

1. $(\mathcal{F} \oplus AS^n) \leq \mathcal{M}^n_+, \forall \mathcal{F} \leq S^n_+; \text{ adversely, } \forall \mathcal{F} \leq \mathcal{M}^n_+, \exists \mathcal{F}_1 \leq S^n_+ \text{ s.t. } \mathcal{F} = \mathcal{F}_1 \oplus AS^n$.

2. $(\mathcal{F}, \mathcal{M}^n_+) = \{Y \geq 0 : N(Y + Y^T) \supseteq N(X + X^T), X \in \mathcal{M}^n_+ \}$.

3. $\mathcal{F}^* = (\mathcal{M}^n_+)^* + \mathcal{F}^\perp$, for any $\mathcal{F} \leq \mathcal{M}^n_+$.

4. $bd(\mathcal{M}^n_+) = bd(S^n_+) \oplus AS^n$.

**Proof.** We present two existing results in $\mathcal{M}^n$ (see [7] or [21]):

\begin{equation}
\mathcal{F}(X, S^n_+) = \{Y \in S^n_+ : N(Y) \supseteq N(X)\}, \tag{3.2}
\end{equation}

\begin{equation}
\mathcal{F}^* = S^n_+ + \mathcal{F}^\perp, \forall \mathcal{F} \subseteq S^n_+. \tag{3.3}
\end{equation}

Then we prove (i)-(iv):

(i) Let $\mathcal{F} \leq S^n_+$ and take $X \in \mathcal{M}^n$ such that $\frac{X + XT}{2} \in ri(\mathcal{F})$. Due to (2.1), we have $\mathcal{F} = \mathcal{F}(\frac{X + XT}{2}, S^n_+)$. Note that for any subspace $\mathcal{L} \subset \mathcal{M}^n$, $\mathcal{F}(X, \mathcal{L}) = \mathcal{L}, \forall X \in \mathcal{L}$. From Lemma 3.4, we obtain

\begin{align*}
\mathcal{F}(X, \mathcal{M}^n_+) &= \mathcal{F}(\frac{X + XT}{2} + \frac{X - XT}{2}, \mathcal{M}^n_+) \\
&= \mathcal{F}(\frac{X + XT}{2}, S^n_+) \oplus \mathcal{F}(\frac{X - XT}{2}, AS^n) \\
&= \mathcal{F} \oplus AS^n.
\end{align*}

That is, $\mathcal{F} \oplus AS^n \leq \mathcal{M}^n_+$.

Adversely, for each $\mathcal{F} \subseteq \mathcal{M}^n_+$, taking $Y \in ri(\mathcal{F})$, we have $\mathcal{F} = \mathcal{F}(Y, \mathcal{M}^n_+) = \mathcal{F}(\frac{Y + Y^T}{2}, S^n_+) \oplus AS^n = \mathcal{F}_1 \oplus AS^n$ where $\mathcal{F}_1 = \mathcal{F}(\frac{Y + Y^T}{2}, S^n_+)$.  

(ii) By Lemma 3.4, Fact 3 and (3.2), we have

\begin{align*}
\mathcal{F}(X, \mathcal{M}^n_+) &= \mathcal{F}(\frac{X + XT}{2}, S^n_+) \oplus \mathcal{F}(\frac{X - XT}{2}, AS^n) \\
&= \mathcal{F}(X + XT, S^n_+) \oplus AS^n \\
&= \{Y \in \mathcal{M}^n : Y^T \in \mathcal{F}(X + XT, S^n_+)\} \\
&= \{Y \geq 0 : N(Y + Y^T) \supseteq N(X + X^T)\}.
\end{align*}

(iii) Let $\mathcal{F} \leq \mathcal{M}^n_+$. Due to above result (i), there exists $\mathcal{F}_1 \leq S^n_+$ such that $\mathcal{F} = \mathcal{F}_1 \oplus AS^n$. By (3.3), we have $\mathcal{F}^* = (\mathcal{F}_1 \oplus AS^n)^* = \mathcal{F}_1^* \cap S^n_+ = S^n_+ + \mathcal{F}_1^\perp$. Moreover, $(\mathcal{M}^n_+)^* + \mathcal{F}^\perp = S^n_+ + (\mathcal{F}_1 \oplus AS^n)^\perp = S^n_+ + (\mathcal{F}_1^\perp \cap S^n_+) = S^n_+ + \mathcal{F}_1^\perp$. Hence, $\mathcal{F}^* = (\mathcal{M}^n_+)^* + \mathcal{F}^\perp$.

(iv) By (2.2), the boundary $bd(\mathcal{M}^n_+)$ consists of all the relative interior of proper faces in $\mathcal{M}^n_+$. Using the result (i), we immediately get the result (iv). □

Theorem 3.5 (iii) tells us that $\mathcal{M}^n_+$ is a nice cone in $\mathcal{M}^n$. And the fact $bd(\mathcal{M}^n_+) = bd(S^n_+) \oplus AS^n$ from Theorem 3.5 (iv) implies that $int(\mathcal{M}^n_+) = int(S^n_+) \oplus AS^n$, which further means that for any given matrix $X \in \mathcal{M}^n_+$, $X$ belongs to $\mathcal{M}^n_+$ if and only if $(X + X^T)$ is invertible.

From Proposition 3.1, Theorems 3.2 and 3.5, we can see the difference between the psd cone and the NS-psd cone in geometry.
4 Relation with $P_0$-matrix cone

A matrix $X \in \mathcal{M}^n$ is said to be a $P_0$ (resp. $P$)-matrix if all its principal minors are nonnegative (resp. positive). Let $\mathcal{P}_0$ and $\mathcal{P}$ denote the sets of $P_0$-matrices and $P$-matrices respectively [10], i.e.,

$$
\mathcal{P}_0 := \{ X \in \mathcal{M}^n : \det(X_{\alpha\alpha}) \geq 0, \ \forall \ \alpha \subseteq \{1, \ldots, n\} \},
\mathcal{P} := \{ X \in \mathcal{M}^n : \det(X_{\alpha\alpha}) > 0, \ \forall \ \alpha \subseteq \{1, \ldots, n\} \}.
$$

Then, they have following properties ([10], or see Section 3, [5]):

- $X \in \mathcal{P}_0 \iff \forall \ \alpha \subseteq \{1, \ldots, n\}$, all real eigenvalues of $X_{\alpha\alpha}$ are nonnegative.
- $X \in \mathcal{P} \iff \forall \ \alpha \subseteq \{1, \ldots, n\}$, all real eigenvalues of $X_{\alpha\alpha}$ are positive.
- $X \in \mathcal{P}_0 \iff \ \forall \ \varepsilon > 0, \ X + \varepsilon I \in \mathcal{P}$.

Further, there exist

(i) $\mathcal{P}_0 \cap S^n = S^n_+,$  $\mathcal{P} \cap S^n = S^n_{++}$,
(ii) $\mathcal{P}_0 \supset M^n_+, \ \mathcal{P} \supset M^n_{++}$,
(iii) $\mathcal{P} = \text{int}(\mathcal{P}_0)$.

Obviously, $\mathcal{P}_0$ is a cone in $\mathcal{M}^n$, but it’s not convex. For example, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ are $P_0$ matrices, but their sum $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ doesn’t belong to $\mathcal{P}_0$.

Followings are some basic facts about geometry of $\mathcal{P}_0$.

**Proposition 4.1** The following statements are right:

(i) $\text{conv}(\mathcal{P}_0) = \text{cone}(\mathcal{P}_0) = \{ X \in \mathcal{M}^n : \text{diag}(X) \geq 0 \}$.
(ii) $\text{bd}(M^n_+) \not\subseteq \mathcal{P}$, $\text{bd}(M^n_+) \cap \mathcal{P} \neq \emptyset$.

**Proof.** (i) Given matrices $E^{ij}, 1 \leq i, j \leq n$ and $-E^{kl}, 1 \leq k \neq l \leq n$. It’s trivial that they are all $P_0$-matrices. Thus

$$
\text{conv}(\mathcal{P}_0) = \text{cone}(\mathcal{P}_0) \supseteq \text{cone}(\{ E^{ij}, 1 \leq i, j \leq n \} \cup \{ -E^{kl}, 1 \leq k \neq l \leq n \})
= \{ \sum_{1 \leq i,j \leq n} \lambda_{ij}E^{ij} + \sum_{1 \leq k \neq l \leq n} \mu_{kl}(-E^{kl}) : \lambda_{ij}, \mu_{kl} \geq 0, \forall i, j, k, l \}
= \{ \sum_{1 \leq i \leq n} \lambda_{ii}E^{ii} + \sum_{1 \leq i \neq j \leq n} \eta_{ij}E^{ij} : \eta_{ij} \in \mathbb{R}, \lambda_{ii} \geq 0, \forall i, j \}
= \{ X \in \mathcal{M}^n : \text{diag}(X) \geq 0 \}.
$$

Since every $P_0$-matrix must be with nonnegative diagonal elements by its definition, i.e., $\mathcal{P}_0 \subseteq \{ X \in \mathcal{M}^n : \text{diag}(X) \geq 0 \}$, there holds

$$
\text{conv}(\mathcal{P}_0) \subseteq \text{conv}(\{ X \in \mathcal{M}^n : \text{diag}(X) \geq 0 \}) = \{ X \in \mathcal{M}^n : \text{diag}(X) \geq 0 \}.
$$
Thus \( \text{conv}(P_0) = \{ X \in M^n : \text{diag}(X) \geq 0 \} \). The proof of statement (i) is complete.

(ii) By Theorem 3.5 (iv), \( bd(M^n_+) = bd(S^n_+) \oplus A S^n_+ \supset bd(S^n_+) \). Note that \( bd(S^n_+) = \{ X \in S^n_+ : \det X = 0 \} \notin P \).

We know \( bd(M^n_+) \notin P \). To prove \( bd(M^n_+) \cap P \neq \emptyset \), we just need to find an element \( X \) in \( P \) and \( bd(M^n_+) \). For example, take an upper triangular matrix

\[
X = \begin{pmatrix}
1 & 2 & \cdots & 2 \\
& \ddots & \ddots & \vdots \\
& & \ddots & 2 \\
& & & 1
\end{pmatrix}
\]

\( \in P \). Then \( X + X^T \) belongs to \( bd(S^n_+) \), which means \( X \in bd(M^n_+) \). This completes the proof. \( \Box \)

Since \( P_0 \) is not a convex cone, we are interested in the maximal convex subcone contained in \( P_0 \) whose definition is introduced as below.

**Definition 4.2** Given a cone \( C \subseteq E \). A subset \( D \subseteq C \) is said to be a maximal convex subcone of \( C \) if it is a convex cone and there are no other convex cones in \( C \) containing \( D \). In other words, there isn’t such \( x \in C \setminus D \) that \( \text{cone}(x \cup D) \subseteq C \).

Because the convexity for a cone is equivalent to the closedness under nonnegative linear combination of any two elements in it, by the above definition, a convex cone \( D \) is a maximal convex subcone of \( C \) if and only if

\[
\forall x \in (C \setminus D), \exists y \in D \text{ such that } x + y \notin C. \tag{4.1}
\]

In other words, a convex cone \( D \) is a maximal convex subcone of \( C \) if and only if

\[
x + y \in C, \forall y \in D \Rightarrow x \in D. \tag{4.2}
\]

The implication (4.2) tells us that \( D \) can’t be expanded to a larger convex cone than itself in \( C \).

Obviously, if cone \( C \) is not empty, the maximal convex subcone of \( C \) must exist. And, for a convex cone, its maximal convex subcone is just itself. For a general nonconvex cone, its maximal convex subcones are not always unique.

Now, we investigate the relationship between the NS-psd cone and \( P_0 \)-matrix cone in low-dimensional space \( M^2 \).

**Proposition 4.3** Let \( X \in M^2 \). The following statements are true:

(i) \( \det(X + dd^T) = \det X + d^T X^* d, \forall d \in \mathbb{R}^2 \).
(ii) \( X \succeq 0 \iff X^* \succeq 0 \iff \det(X + dd^T) \geq 0, \forall d \in \mathbb{R}^2 \).
(iii) \( M^2_+ \) is a maximal convex subcone of \( P_0 \).
Proof. (i) For any $d \in \mathbb{R}^2$, expanding $\det(X + dd^T)$, we have

$$
\det(X + dd^T) = \det \begin{pmatrix} X_{11} + d_1 d_1 & X_{12} + d_1 d_2 \\ X_{21} + d_2 d_1 & X_{22} + d_2 d_2 \end{pmatrix}
$$

$$
= (X_{11}X_{22} - X_{12}X_{21}) + X_{11}d_2^2 - (X_{12} + X_{21})d_1d_2 + X_{22}d_1^2
$$

$$
= \det X + d^T \begin{pmatrix} X_{22} & -X_{12} \\ -X_{21} & X_{11} \end{pmatrix} d
$$

$$
= \det X + d^T X^* d.
$$

(ii) The first “$\iff$” is due to the following fact:

$$
X^* = \begin{pmatrix} X_{22} & -X_{12} \\ -X_{21} & X_{11} \end{pmatrix} \succeq 0
$$

$$
\iff \begin{pmatrix} X_{22} & -\frac{X_{12} + X_{21}}{2} \\ -\frac{X_{12} + X_{21}}{2} & X_{11} \end{pmatrix} \succeq 0
$$

$$
\iff \begin{pmatrix} X_{11} & X_{12} + X_{21} \\ X_{12} + X_{21} & X_{22} \end{pmatrix} \succeq 0
$$

$$
\iff \frac{X + X^T}{2} \succeq 0 \iff X \succeq 0.
$$

For the second “$\iff$”, the necessity is due to

$$
X^* \succeq 0 \Rightarrow X \succeq 0 \Rightarrow X + dd^T \succeq 0, \forall d \in \mathbb{R}^2 \Rightarrow \det(x + dd^T) \geq 0, \forall d \in \mathbb{R}^2.
$$

For the sufficiency, by (i), we have

$$
\det X + d^T X^* d \geq 0, \forall d \in \mathbb{R}^2.
$$

which implies

$$
d^T X^* d \geq 0, \forall d \in \mathbb{R}^2,
$$

otherwise, if $d^T X^* d < 0$ for some $\hat{d} \in \mathbb{R}^2$, then

$$
(\lambda \hat{d})^T X^* (\lambda \hat{d}) = \lambda^2 \hat{d}^T X^* \hat{d} \to -\infty, \text{ when } \lambda \to \infty.
$$

Thus $X^* \succeq 0$. This completes the proof of statement (ii).

(iii) In $\mathcal{M}^2$, suppose that $\mathcal{M}^2_+$ is not a maximal convex subcone of $\mathcal{P}_0$. Then by (4.1),

$$
\exists X \in (\mathcal{P}_0 \setminus \mathcal{M}^2_+), \text{ s.t. } X + Y \in \mathcal{P}_0, \forall Y \in \mathcal{M}^2_+.
$$

Hence, for all $d \in \mathbb{R}^2$, $X + dd^T \in \mathcal{P}_0$, i.e., $\det(X + dd^T) \geq 0$. By (ii), we get $X \succeq 0$, which contradicts the known fact $X \in (\mathcal{P}_0 \setminus \mathcal{M}^2_+)$. So we conclude that $\mathcal{M}^2_+$ is a maximal convex subcone of $\mathcal{P}_0$ in $\mathcal{M}^2$. $\square$
We’ll try to generalize the above results to high-dimensional space \( M^n (n > 2) \) in the rest of this section.

**Proposition 4.4** Let \( X \in M^n \) with \( n > 2 \). The following statements hold:

(i) \( \det(X + dd^T) = \det X + d^T X d, \ \forall d \in \mathbb{R}^n. \)

(ii) If \( \det X > 0 \), then

\[ X \geq 0 \iff X^* \geq 0 \iff \det(X + dd^T) \geq 0, \forall d \in \mathbb{R}^n. \]

**Proof.** (i) Clearly, \((X + dd^T)_{ij} = X_{ij} + d_i d_j, \ \forall i, j = 1, \ldots, n. \) Then

\[
\det(X + dd^T) = \sum_{j_1,j_2,\ldots,j_n} (-1)^{\tau(j_1,j_2,\ldots,j_n)} (X_{1j_1} + d_1 d_{j_1})(X_{2j_2} + d_2 d_{j_2}) \cdots (X_{nj_n} + d_n d_{j_n})
\]

\[
= \sum_{j_1,j_2,\ldots,j_n} \left[ (-1)^{\tau(j_1,j_2,\ldots,j_n)} \sum_{k=0,\ldots,n} \sum_{\tau_1,\tau_2,\ldots,\tau_k} (X_{\iota_1,k_1} X_{\iota_2,k_2} \cdots X_{\iota_k,k_k} d_{k+1} d_{j_{k+1}} \cdots d_n d_{j_n}) \right]
\]

\[
= \sum_{j_1,j_2,\ldots,j_n} \left[ (-1)^{\tau(j_1,j_2,\ldots,j_n)} \prod_{i=1}^n X_{ij_i} + \sum_{i=1}^n d_{d_{j_i}} \sum_{k=1}^{n-2} \sum_{\{\iota_1,\ldots,\iota_k\} \subseteq \{1,\ldots,n\}} \left( \prod_{t \in \mathcal{N} \setminus \{1,\ldots,k\}} d_{t_{j_t}} \right) \left( \prod_{t \in \{1,\ldots,k\}} X_{t_{j_t}} \right) \right]
\]

\[
= \sum_{j_1,j_2,\ldots,j_n} \left[ (-1)^{\tau(j_1,j_2,\ldots,j_n)} \prod_{i=1}^n X_{ij_i} \right] + \sum_{j_1,j_2,\ldots,j_n} \left[ (-1)^{\tau(j_1,j_2,\ldots,j_n)} \prod_{i=1}^n d_{d_{j_i}} \right]
\]

\[
+ \sum_{j_1,j_2,\ldots,j_n} \left[ (-1)^{\tau(j_1,j_2,\ldots,j_n)} \sum_{k=1}^{n-2} \sum_{\{\iota_1,\ldots,\iota_k\} \subseteq \{1,\ldots,n\}} \left( \prod_{t \in \mathcal{N} \setminus \{1,\ldots,k\}} d_{t_{j_t}} \right) \left( \prod_{t \in \{1,\ldots,k\}} X_{t_{j_t}} \right) \right]
\]

\[
+ \sum_{j_1,j_2,\ldots,j_n} \left[ (-1)^{\tau(j_1,j_2,\ldots,j_n)} \sum_{i=1}^n d_{d_{j_i}} \prod_{t \neq i} X_{t_{j_t}} \right]
\]

\[
= \det X + \det dd^T + \sum_{k=1}^{n-2} \sum_{\{\iota_1,\ldots,\iota_k\} \subseteq \{1,\ldots,n\}} \sum_{j_1,j_2,\ldots,j_n} \left[ (-1)^{\tau(j_1,j_2,\ldots,j_n)} \left( \prod_{t \in \mathcal{N} \setminus \{1,\ldots,k\}} d_{t_{j_t}} \right) \left( \prod_{t \in \{1,\ldots,k\}} X_{t_{j_t}} \right) \right]
\]

\[
+ \sum_{i=1}^n \sum_{j_1,j_2,\ldots,j_n} \left[ (-1)^{\tau(j_1,j_2,\ldots,j_n)} d_{d_{j_i}} \prod_{t \neq i} X_{t_{j_t}} \right]
\]

\[
= \det X + \det dd^T + \sum_{k=1}^{n-2} \sum_{\{\iota_1,\ldots,\iota_k\} \subseteq \{1,\ldots,n\}} A_{\{1,\ldots,k\}} + \sum_{i=1}^n B_i, \quad \text{(4.3)}
\]
where, \( j_1 j_2 \cdots j_n \) is a permutation of \( 12 \cdots n \), \( \tau(j_1 j_2 \cdots j_n) \) denotes the inverse ordinal number of this permutation, \( \mathcal{N} \) denotes the set \( \{1, 2, \cdots, n\} \), and
\[
A_{\{l_1, \cdots, l_k\}} := \sum_{j_1 j_2 \cdots j_n} \left( (-1)^{\tau(j_1 j_2 \cdots j_n)} \prod_{t \in \mathcal{N} \setminus \{l_1, \cdots, l_k\}} d_{t j_i} \prod_{t \in \{l_1, \cdots, l_k\}} X_{t j_i} \right),
\]
\[
B_i := \sum_{j_1 j_2 \cdots j_n} \left( (-1)^{\tau(j_1 j_2 \cdots j_n)} d_{i j_i} \prod_{t \neq i} X_{t j_i} \right).
\]

Note that, \( A_{\{l_1, \cdots, l_k\}} = \det Y \), where \( Y = (Y_{ij})_{n \times n} \), and \( Y_{ij} = \left\{ \begin{array}{cl} X_{t j_i}, & t \in \{l_1, \cdots, l_k\} \\ d_{i j_i}, & t \in \mathcal{N} \setminus \{l_1, \cdots, l_k\} \end{array} \right\} \).

When \( 1 \leq k \leq n - 2 \), \( Y \) has at least two rows whose components are proportional. Thus
\[
A_{\{l_1, \cdots, l_k\}} = \det Y = 0, \quad \forall 1 \leq k \leq n - 2.
\]

Meanwhile,
\[
B_i = \sum_{j_1 j_2 \cdots j_n} (-1)^{\tau(j_1 j_2 \cdots j_n)} X_{1 j_1} \cdots X_{l_{j_i} - 1 j_i} d_{i j_i} d_{j_i j_i} X_{i + 1 j_i + 1} \cdots X_{n j_n}
\]
\[
= \det \begin{pmatrix} X_{11} & \cdots & X_{1n} \\ \vdots & \ddots & \vdots \\ X_{n1} & \cdots & X_{nn} \end{pmatrix}
\]
\[
= d_i d_1 (X^*)_{1i} + \cdots + d_i d_n (X^*)_{ni} \quad \text{(expanding by the } i\text{-th row),}
\]
from which we conclude that
\[
\sum_{i=1}^{n} B_i = \sum_{1 \leq i, j \leq n} (X^*)_{ij} d_i d_j = d^T X^* d.
\]

Above all, due to (4.3), we obtain \( \det(X + d d^T) = \det X + d^T X^* d \).

(ii) The first \( "\leftrightarrow" \): If \( X \geq 0 \), then \( X + d d^T \geq 0 \) for all \( d \in \mathbb{R}^n \). Thus \( \det(X + d d^T) \geq 0 \), for all \( d \in \mathbb{R}^n \). By statement (i), we have
\[
\det X + d^T X^* d \geq 0, \forall d \in \mathbb{R}^n \Rightarrow d^T X^* d \geq 0, \forall d \in \mathbb{R}^n \Rightarrow X^* \succeq 0.
\]

If \( X^* \succeq 0 \), applying the above implication again, we have \( (X^*)^* \succeq 0 \). Since \( \det X > 0 \) and
\[
(X^*)^* = \det X^* \cdot (X^*)^{-1} = \det[\det X \cdot X^{-1}] \cdot (\det X \cdot X^{-1})^{-1} = (\det X)^{n-2} X,
\]
we have \( X \succeq 0 \).

The second \( "\leftrightarrow" \): By (i) and the first \( "\leftrightarrow" \), we easily obtain the desired result. \( \square \)
Proposition 4.4 can be generalized to any principal submatrix of $X$ by replacing $X$ with $X_{aa}$ ($\alpha \subset \{1, \ldots, n\}$). And Proposition 4.4 (i) is the generalization of Proposition 4.3 (i) to case $n > 2$. However, Proposition 4.3 (ii) is no longer correct for $n > 2$, because $\text{rank}(X^*) = 0$ whenever $\text{rank}(X) \leq n - 2$, which means “$X^* \succeq 0 \Rightarrow X \succeq 0$” for $n > 2$. Coming with it, is it still true that $M_n^\alpha$ is a maximal convex subcone of $P_0$ for general $n$? The answer is affirmative.

In order to prove it, we present several basic facts on theory of maximal convex subcones.

Let $Mcs(l, C)$ denote the collection of maximal convex subcones of $C$ that contain $l$, where $l \subset C$, $C$ is a cone in $E$. And we call $D$ a maximal convex cone generated by $l$ in $C$ if $D \in Mcs(l, C)$. Since $\text{cone}(l)$ is the smallest convex cone containing $l$, every maximal convex cone generated by $l$ in $C$ (if it exists) contains $\text{cone}(l)$ as a subset, the maximal convex cones generated by $l$ in $C$ equal the maximal convex cones generated by $\text{cone}(l)$ in $C$, i.e.,

$$Mcs(l, C) = Mcs(\text{cone}(l), C).$$

(4.4)

Apparently, $Mcs(l, C) = \{\emptyset\}$ if $\text{cone}(l) \not\subset C$. Two evident facts can be directly derived from Definition 4.2:

(i) If $l_1 \subseteq l_2$, then $Mcs(l_1, C) \supseteq Mcs(l_2, C)$. \hspace{1cm} (4.5)

(ii) If $D \in Mcs(l, C)$, $D \subseteq K \subseteq C$, and $K$ is a convex cone, then $K = D$. \hspace{1cm} (4.6)

**Lemma 4.5** Suppose that $Mcs(l, C) = \{D\}$. Then the following statements hold:

(i) $Mcs(D_1, C) = \{D\}$, for any subset $D_1 \subset C$ such that $l \subseteq D_1 \subseteq D$.

(ii) If $l$ is a convex cone, then $x + y \in C$, $\forall y \in l \Rightarrow x \in D$.

**Proof.** (i) By the fact $Mcs(l, C) = \{D\}$ and (4.5), we have

$$\{D\} = Mcs(D, C) \subseteq Mcs(D_1, C) \subseteq Mcs(l, C) = \{D\}.$$

So we get the proof of statement (i).

(ii) By contradiction, suppose that $x \notin D$ and $x + y \in C$ for any $y \in l$. Since $l$ is a convex cone and $C$ is a cone, it follows that, for any nonnegative integer $k$,

$$\{\mu x + \sum_{i=1}^{k} \lambda_i y_i : y_i \in l, \mu \geq 0, \lambda_i \geq 0, i = 1, \ldots, k\} \subseteq C.$$

In other words,

$$\text{cone}(\{x\} \cup l) \subseteq C.$$

Since $\text{cone}(\{x\} \cup l)$ is a convex cone and $\text{cone}(\{x\} \cup l) \supset l$, there exists $D_1 \in Mcs(l, C)$ such that $D_1 \supseteq \text{cone}(\{x\} \cup l)$. Noting that $x \notin D$, we know $D_1 \neq D$, which contradicts $Mcs(l, C) = \{D\}$. So we get the proof of statement (ii).
Lemma 4.6 Suppose that $\mathcal{Mcs}(l, C) \neq \{\emptyset\}$, where $ri(C) = ri(cl(C))$ and $l = ri(l) \subseteq ri(C)$. Let $K$ be a subset of $E$. If $D_1 \subseteq K$ holds for any $D_1 \in \mathcal{Mcs}(l, C)$, then $D_2 \subseteq cl(K)$ holds for any $D_2 \in \mathcal{Mcs}(l, cl(C))$.

Proof. By contradiction, we assume that there exists $\hat{D}_2 \in \mathcal{Mcs}(l, cl(C))$ such that $\hat{D}_2 \not\subseteq cl(K)$. We’ll show that $ri(\hat{D}_2) \not\subseteq K$.

If $ri(\hat{D}_2) \subseteq K$, by convexity of $\hat{D}_2$, we have

$$\hat{D}_2 \subseteq cl(\hat{D}_2) = cl(ri(\hat{D}_2)) \subseteq cl(K),$$

which contradicts $\hat{D}_2 \not\subseteq cl(K)$. So $ri(\hat{D}_2) \not\subseteq K$.

Note that $ri(\hat{D}_2) \subseteq ri(cl(C)) = ri(C) \subseteq C$ and $ri(\hat{D}_2)$ is a convex cone which contains $ri(l) = l$. So there exists $\hat{D}_1 \in \mathcal{Mcs}(l, C)$ such that $ri(\hat{D}_2) \subseteq \hat{D}_1$. Since $ri(\hat{D}_2) \not\subseteq K$, we have $\hat{D}_1 \not\subseteq K$. This contradicts the precondition that $\hat{D}_1 \subseteq K$ since $\hat{D}_1 \in \mathcal{Mcs}(l, C)$. The proof is complete.

Utilizing the Lemma 4.6 and Proposition 4.4, we are ready to prove the main result of this section.

Theorem 4.7 $M^n_+$ is the unique maximal convex subcone generated by $S^n_{++}$ in $P_0$.

Proof. First we show that

$$\mathcal{D} \subseteq M^n_+ \text{ for any } \mathcal{D} \in \mathcal{Mcs}(S^n_{++}, P). \quad \text{(4.8)}$$

For any $\mathcal{D} \in \mathcal{Msc}(S^n_{++}, P)$, take $0 \neq d \in \mathbb{R}^n$. There exists $n - 1$ vectors $v_1, \cdots, v_{n-1} \in \mathbb{R}^n$ such that

$$\begin{bmatrix} v_1, \cdots, v_{n-1}, \frac{d}{\|d\|_2} \end{bmatrix}$$

forms an orthogonal matrix. In this case, for any $\lambda > 0, \lambda_i > 0, i = 1, \cdots, n - 1, \sum_{i=1}^{n-1} \lambda_i v_i v_i^T + \lambda d d^T$ is a symmetric matrix whose eigenvalues are all positive, i.e.,

$$\{ \sum_{i=1}^{n-1} \lambda_i v_i v_i^T + \lambda d d^T : \lambda_i > 0, \lambda > 0, i = 1, \cdots, n - 1 \} \subset S^n_{++}.$$

Taking any $X \in \mathcal{D}$, by the convexity of $\mathcal{D}$ and $S^n_{++} \subseteq \mathcal{D}$, we have

$$\{ X + \sum_{i=1}^{n-1} \lambda_i v_i v_i^T + \lambda d d^T : \lambda_i > 0, \lambda > 0, i = 1, \cdots, n - 1 \} \subset \mathcal{D},$$

which implies that

$$\det(X + \sum_{i=1}^{n-1} \lambda_i v_i v_i^T + \lambda d d^T) > 0, \forall \lambda_i > 0, \lambda > 0, i = 1, \cdots, n - 1.$$
Taking limit when $\lambda_i \to 0$, $i = 1, \ldots, n - 1$, and fixing $\lambda = 1$, we have
\[
\det(X + dd^T) \geq 0.
\]
By the arbitrariness of $d \in \mathbb{R}^n$, we obtain
\[
\det(X + dd^T) \geq 0, \text{ for all } d \in \mathbb{R}^n.
\]
For $n = 2$, this implies $X \succeq 0$ by Proposition 4.3. For $n > 2$, noting that $X \in \mathcal{P}$ means $\det X > 0$, by Proposition 4.4 (ii), we also get $X \succeq 0$. Hence, $D \subseteq \mathcal{M}^n_+.$

Seeing that $ri(\mathcal{P}) = ri(cl(\mathcal{P}))$ and $S^n_{++} = ri(S^n_+) \subset ri(\mathcal{P}_0) = \mathcal{P}$, then applying Lemma 4.6 to (4.8), we have
\[
D \subseteq cl(\mathcal{M}^n_+) \text{ for any } D \in \mathcal{M}cs(S^n_{++}, cl(\mathcal{P})),
\]
\[
\text{i.e.,} \quad D \subseteq \mathcal{M}^n_+ \text{ for any } D \in \mathcal{M}cs(S^n_{++}, \mathcal{P}_0). \tag{4.9}
\]
Since $\mathcal{M}^n_+$ is a convex cone and $\mathcal{M}^n_+ \subseteq \mathcal{P}_0$, applying the fact (4.6) to (4.9), it holds that
\[
D = \mathcal{M}^n_+ \text{ for any } D \in \mathcal{M}cs(S^n_{++}, \mathcal{P}_0).
\]
That is to say
\[
\mathcal{M}cs(S^n_{++}, \mathcal{P}_0) = \{\mathcal{M}^n_+\}. \tag{4.10}
\]
This completes the proof. $\square$

Consequently, we obtain the following corollary.

**Corollary 4.8** Let $X \in \mathcal{M}^n$. The following statements are true:

(i) $X \in \mathcal{M}^n_+$ if and only if $X + Y \in \mathcal{P}_0$ for all $Y \in S^n_{++}$, i.e.,
\[
X \succeq 0 \Leftrightarrow \det(X + Y)_{\alpha\alpha} \geq 0, \forall Y \in S^n_{++}, \forall \alpha \subseteq \{1, \cdots, n\}.
\]

(ii) $\mathcal{M}cs(Exe(S^n_+), \mathcal{P}_0) = \{\mathcal{M}^n_+\}$.

**Proof.** (i) The necessity of statement (i) is trivial. The sufficiency is the straight result of (4.10) and Lemma 4.5 (ii).

(ii) Combining with (4.10) and Lemma 4.5 (i), we have
\[
\mathcal{M}cs(S^n_+, \mathcal{P}_0) = \{\mathcal{M}^n_+\}.
\]
And due to the fact that $cone(Exe(S^n_+)) = cone(\{dd^T : d \in \mathbb{R}^n\}) = S^n_+$ and by (4.4), we have
\[
\mathcal{M}cs(S^n_+, \mathcal{P}_0) = \mathcal{M}cs(Exe(S^n_+), \mathcal{P}_0).
\]
Hence
\[
\mathcal{M}cs(Exe(S^n_+), \mathcal{P}_0) = \{\mathcal{M}^n_+\}.
\]
The proof is complete. $\square$
However, \( M_{++}^n \) is not a maximal convex subcone of \( \mathcal{P} \).

**Theorem 4.9** \( M_{++}^n \) is not a maximal convex subcone of \( \mathcal{P} \). Therefore, if \( X \in \mathcal{P} \), then
\[
\det(X + Y)_{\alpha\alpha} > 0, \forall \ Y \in S^n_{++}, \forall \alpha \subseteq \{1, \cdots, n\} \neq X > 0.
\]

**Proof.** By Proposition 4.1 (ii), \( bd(M_{++}^n) \cap \mathcal{P} \neq \emptyset \). Let \( X \in bd(M_{++}^n) \cap \mathcal{P} \). Then \( cone(\{X\}) \subset bd(M_{++}^n) \cap \mathcal{P} \). Moreover,
\[
M_{++}^n \subset (M_{++}^n \cup cone(\{X\})) \subset \mathcal{P},
\]
where \((M_{++}^n \cup cone(\{X\}))\) is a convex cone since for any \( Y \in M_{++}^n, Z \in bd(M_{++}^n), Y + Z \in M_{++}^n \). Thus, \( M_{++}^n \) is not a maximal convex subcone of \( \mathcal{P} \).

We end this section by stating some other maximal convex subcones in \( \mathcal{P}_0 \).

**Theorem 4.10** Let \( \mathcal{M}_u = \{X \in M^n : \text{\text{diag}}(X) \geq 0, X_{ij} = 0, i > j\} \), \( \mathcal{M}_l = \{X \in M^n : \text{\text{diag}}(X) \geq 0, X_{ij} = 0, i < j\} \). Then both \( \mathcal{M}_u \) and \( \mathcal{M}_l \) are maximal convex subcones of \( \mathcal{P}_0 \).

**Proof.** It’s clear that \( \mathcal{M}_u, \mathcal{M}_l \) are two convex cones in \( M^n \). And for any \( X \in \mathcal{M}_u \) or \( \mathcal{M}_l \), any \( \alpha \subseteq \{1, \cdots, n\} \), the eigenvalues of \( X_{\alpha\alpha} \) are exactly the diagonal elements of \( X_{\alpha\alpha} \). So all eigenvalues of \( X_{\alpha\alpha} \) with \( \alpha \subseteq \{1, \cdots, n\} \) are nonnegative, which means \( X \in \mathcal{P}_0 \). This further implies that \( \mathcal{M}_u, \mathcal{M}_l \) are two convex subcones of \( \mathcal{P}_0 \).

Next, we just prove the maximal convexity of \( \mathcal{M}_u \) in \( \mathcal{P}_0 \). The proof for \( \mathcal{M}_l \) is in the similar way. As we know, the maximal convexity of \( \mathcal{M}_u \) in \( \mathcal{P}_0 \) is equivalent to
\[
\forall \ X \in \mathcal{P}_0 \setminus \mathcal{M}_u, \exists Y \in \mathcal{M}_u, \text{s.t.} \ (X + Y) \notin \mathcal{P}_0.
\]

Take any \( X \in \mathcal{P}_0 \setminus \mathcal{M}_u \), which implies that \( X_{kl} \neq 0 \) for some \( k > l \). Choose \( Y \in \mathcal{M}_u \) satisfying \( X_{kl}Y_{lk} > 0 \) and \( (X_{ll} + Y_{ll})(X_{kk} + Y_{kk}) < X_{kk}X_{lk} + X_{kl}Y_{lk} \). Such \( Y \) always exists because of the arbitrariness of \( Y_{lk} \) (one just needs to make \( |Y_{lk}| \) big enough such that the right hand side of the above inequality is bigger enough than the left). Let \( \alpha = \{l, k\} \), it follows
\[
\det(X + Y)_{\alpha\alpha} = \det\left(\begin{array}{ccc}
X_{ll} + Y_{ll} & X_{lk} + Y_{lk} \\
X_{kl} & X_{kk} + Y_{kk}
\end{array}\right)
= (X_{ll} + Y_{ll})(X_{kk} + Y_{kk}) - (X_{kl}X_{lk} + X_{kl}Y_{lk}) < 0,
\]
which implies \( X + Y \notin \mathcal{P}_0 \). So \( \mathcal{M}_u \) is a maximal convex subcone of \( \mathcal{P}_0 \).

Above all, the sets \( \mathcal{M}_u, \mathcal{M}_l \) and \( M_{++}^n \) are members of \( \mathcal{Mcs}(I_+, \mathcal{P}_0) \), where
\[
I_+ := \{X \in M^n : \text{\text{diag}}(X) \geq 0, X_{ij} = 0 \ \forall i \neq j\}
\]
is the intersection of \( \mathcal{M}_u, \mathcal{M}_l \) and \( M_{++}^n \). Therefore,
\[
\mathcal{Mcs}(I_+, \mathcal{P}_0) \supset \{\mathcal{M}_u, \mathcal{M}_l, M_{++}^n\} \supset \{M_{++}^n\} = \mathcal{Mcs}(S^n_{++}, \mathcal{P}_0).
\]
This inclusion is consistent with the fact (4.5), which says that the smaller \( l \) is, the larger \( \mathcal{Mcs}(l, \mathcal{C}) \) is.
5 Projection onto NS-psd cone

Let $\Pi_C : E \rightarrow E$ denote the metric projection of $x$ onto $C$, where $C \subseteq E$ is a nonempty closed convex set. Then, for any $x \in E$,

$$\Pi_C(x) = \arg\min \left\{ \frac{1}{2} \|x - y\|^2 : y \in C \right\}.$$  

Equivalently,

$$\langle \Pi_C(x) - y, \Pi_C(x) - x \rangle \leq 0, \forall y \in C. \quad (5.1)$$

It’s well known that $\Pi_C(\cdot)$ is unique and contractive, i.e., $\|\Pi_C(x) - \Pi_C(y)\| \leq \|x - y\|$, $\forall x, y \in E$. Let $\text{dist}(x, C) := \min \left\{ \|x - y\| : y \in C \right\}$. Then $\text{dist}(x, C) = \|x - \Pi_C(x)\|$.

For the projection onto $M^n_+$, Qi and Sun have already given its expression as follows (see Section 4.3, [13])

$$\Pi_{M^n_+}(X) = \Pi_{S^n_+}(\frac{X + X^T}{2}) + \frac{X - X^T}{2}. \quad (5.2)$$

From the positive homogeneity of $\Pi_K(\cdot)$ for any closed convex cone $K$, we immediately get

$$\text{dist}(X, M^n_+) = \frac{1}{2} \text{dist}(X + X^T, S^n_+). \quad (5.3)$$

We now discuss the tangent cone and second order tangent set of $M^n_+$. For the closed convex set $C \subseteq E$, the tangent cone of $C$ at $x \in C$ is defined as (see Section 2.2.4, [4])

$$T_C(x) := \left\{ y \in E : \text{dist}(x + ty, C) = o(t), t \geq 0 \right\}.$$

And the inner and outer second order tangent sets of $C$ at $x \in C$ in direction $h \in E$ are respectively defined by (see Section 3.2.1, [4])

$$T^{i,2}_C(x, h) := \left\{ y \in E : \text{dist}(x + th + \frac{1}{2}t^2y, C) = o(t^2), t \geq 0 \right\}$$

and

$$T^{o,2}_C(x, h) := \left\{ y \in E : \exists t_k \downarrow 0, \text{dist}(x + t_k h + \frac{1}{2}t_k^2y, C) = o(t_k^2), t_k \geq 0 \right\}.$$ 

Obviously $T^{i,2}_C(x, h) \subseteq T^{o,2}_C(x, h)$.

From Example 3.40 of [4], we know that in space $S^n$, the inner and outer second order tangent sets for $S^n_+$ are the same at any point and in any direction, and their explicit formulas are presented therein. For convenience, let $T^{i,2}_{S^n_+}(\cdot, \cdot)$ and $T^{o,2}_{S^n_+}(\cdot, \cdot)$ denote the tangent cone, inner and outer second order tangent sets of $S^n_+$ restricted in space $S^n$, respectively. Then $T^{i,2}_{S^n_+}(X, H) = T^{o,2}_{S^n_+}(X, H)$ for any $X \in S^n$ and any $H \in S^n$. For the tangent cone and second order tangent sets of $M^n_+$ in space $M^n$, they have the following forms.

**Theorem 5.1** For any $X \in M^n$, the following statements hold:

(i) $T_{M^n_+}(X) = T^{i,2}_{S^n_+}(X + X^T) \oplus AS^n$.

(ii) $T^{o,2}_{M^n_+}(X, H) = T^{i,2}_{M^n_+}(X, H) = T^{o,2}_{S^n_+}(X + X^T, H + H^T) \oplus AS^n$. 

17
The following statements hold:

\[ \partial \]

Furthermore, we have the following conclusions.

\[ \text{semismooth on } S \]

\[ \text{directionally differentiable everywhere in } (ii) \]

\[ \Pi \]

\[ J \]

\[ \text{with} \]

\[ \text{from (5.3) and Fact 3, we have} \]

\[ \text{Proof.} \]

\[ F\text{-differentiable almost everywhere. Let} \]

\[ x \]

\[ f \]

\[ \text{real vector spaces equipped with an inner product } \langle \cdot , \cdot \rangle \text{ and the induced norm } \| \cdot \|. \]

\[ \text{We say that} \]

\[ f : X \to Y \]

\[ \text{is directionally differentiable at } \]

\[ x, \] \text{and} \[ f(x + h) = f(x) + f'(x; h) + o(\|h\|), \]

\[ h \in X. \]

\[ \text{In addition, if } f' \text{ is linear and continuous, then } f \text{ is said to be } F(\text{Fréchet})\text{-differentiable at } x. \]

\[ \text{Suppose that} \]

\[ f : X \to Y \]

\[ \text{is a locally Lipschitz function. Thus } f \]

\[ \text{is } F\text{-differentiable almost everywhere in } X \]

\[ \text{from the well-known Rademacher’s theorem that every locally Lipschitz continuous function is} \]

\[ F\text{-differentiable almost everywhere.} \]

\[ \text{Let } D_f \text{ denote the set of points where } f \text{ is } F\text{-differentiable in } X. \]

\[ \text{Then the Clarke’s generalized Jacobian of } f \text{ at } x \text{ is defined as} \]

\[ \partial f(x) := \text{conv}\{\partial_B f(x)\} \]

\[ \text{with} \]

\[ \partial_B f(x) := \{ \lim_{k \to \infty} J f(y^k) : y^k \in D_f, y^k \to x \}, \]

\[ \text{where } J f(y^k) \text{ denotes the F-derivative of } f \text{ at } y^k. \]

\[ \text{Bonnans et al. [3] showed that } \Pi_{S^n} (\cdot) \text{ is directionally differentiable everywhere in } S^n. \]

\[ \text{Sun and Sun [17] proved that } \Pi_{S^n} (\cdot) \text{ is strongly semismooth on } S^n. \]

\[ \text{Qi and Sun [13] showed the strong semismoothness of } \Pi_{M^n} (\cdot) \text{ over } M^n. \]

\[ \text{Furthermore, we have the following conclusions.} \]

\[ \textbf{Theorem 5.2} \]

\[ \text{The following statements hold:} \]

\[ (i) \] \text{\( \Pi_{M^n} (\cdot) \) is directionally differentiable everywhere in } M^n, \text{ and the directional derivative} \]

\[ \Pi'_{M^n} (X; H) = \Pi'_{S^n} \left( \frac{X+X^T}{2}; \frac{H+H^T}{2} \right) + \frac{H-H^T}{2}, \]

\[ H \in M^n. \]

\[ (ii) \] \text{For any } V_1 \in \partial_B \Pi_{M^n} (X) \text{ (resp. } \partial \Pi_{M^n} (X)), \text{ there exists } V_2 \in \partial_B \Pi_{S^n} \left( \frac{X+X^T}{2} \right) \text{ (resp.} \]

\[ \partial \Pi_{S^n} \left( \frac{X+X^T}{2} \right)), \text{ such that} \]

\[ V_1(H) = V_2 \left( \frac{H+H^T}{2} \right) + \frac{H-H^T}{2}, \]

\[ H \in M^n. \]
Proof. (i) Taking $X \in M^n$, by the definition of directional derivative and from (5.2), we have

$$
\Pi'_{M^n}(X; H) = \lim_{t \downarrow 0} \frac{\Pi_{M^n}(X+tH) - \Pi_{M^n}(X)}{t} = \lim_{t \downarrow 0} \frac{\Pi_{S^n}(x^T + \frac{X+H}{2}t) - \Pi_{S^n}(\frac{X+X^T}{2} + \frac{H+H^T}{2})}{t}
$$

The result (i) is proved.

(ii) Let $D_M$ and $D_S$, respectively, denote the sets of points in $M^n$ and $S^n$ where $\Pi_{M^n}(\cdot)$ and $\Pi_{S^n}(\cdot)$ are almost everywhere F-differentiable. Since $\Pi_{M^n}(\cdot)$ and $\Pi_{S^n}(\cdot)$ are almost everywhere F-differentiable.

Proof. Taking any $V_1 \in \partial B \Pi_{M^n}(X)$, there exists a sequence $\{X^k\}$ in $D_M$ converging to $X$ such that $V_1 = \lim_{k \to \infty} J \Pi_{M^n}(X^k)$. Then by (5.5), it follows that for any $H \in M^n$,

$$
V_1(H) = \lim_{k \to \infty} J \Pi_{M^n}(X^k)(H) = \lim_{k \to \infty} \Pi'_{M^n}(X^k; H) = \lim_{k \to \infty} \Pi'_{S^n}(\frac{X^k+X^T}{2}; \frac{H+H^T}{2}) + \frac{H-H^T}{2} = [\lim_{k \to \infty} J \Pi_{S^n}(\frac{X^k+X^T}{2})(\frac{H+H^T}{2}) + \frac{H-H^T}{2}].
$$

Letting $V_2 = \lim_{k \to \infty} J \Pi_{S^n}(\frac{X^k+X^T}{2})$, we obtain the desired result. Taking the convex hull of the above limit points will yield the corresponding result for $\partial \Pi_{M^n}(X)$. Here, we omit the proof.

Utilizing Theorem 5.2 and combining with the results in [17, 18, 12], we can obtain the explicit formulas for the directional derivative and Clarke's generalized Jacobian of $\Pi_{M^n}(\cdot)$.

Let $X \in M^n$ with the spectral decomposition $\frac{X+X^T}{2} = \Lambda P P^T$, where $\Lambda$ is the diagonal matrix of eigenvalues of $\frac{X+X^T}{2}$ and $P$ is a corresponding orthogonal matrix of orthonormal eigenvectors. Denote three index sets of positive, zero and negative eigenvalues of $\frac{X+X^T}{2}$, respectively, by

$$
\alpha := \{i : \lambda_i(\frac{X+X^T}{2}) > 0\}, \quad \beta := \{i : \lambda_i(\frac{X+X^T}{2}) = 0\}, \quad \gamma := \{i : \lambda_i(\frac{X+X^T}{2}) < 0\}.
$$

Rearrange $\Lambda$ as

$$
\begin{bmatrix}
\Lambda_\alpha & 0 & 0 \\
0 & \Lambda_\beta & 0 \\
0 & 0 & \Lambda_\gamma
\end{bmatrix}
$$

and $P$ as $[P_\alpha, P_\beta, P_\gamma]$, where $P_\alpha \in \mathbb{R}^{n \times |\alpha|}$, $P_\beta \in \mathbb{R}^{n \times |\beta|}$, $P_\gamma \in \mathbb{R}^{n \times |\gamma|}$. And define the matrix $U \in S^n$ with entries

$$
U_{ij} = \max\{\lambda_i(\frac{X+X^T}{2}), 0\} + \max\{\lambda_j(\frac{X+X^T}{2}), 0\} \quad \text{if } |\lambda_i(\frac{X+X^T}{2})| + |\lambda_j(\frac{X+X^T}{2})| > 0.
$$

where $0/0$ is also defined to be 1. Using Sun and Sun's results (Corollary 10 and Lemma 11,
where $\tilde{H} := P^T(H + H^T/2)P$, and $\circ$ denotes the Hadamard product. Clearly, from the above expression (5.6) and Theorem 5.2 (i), we claim that

$$\Pi_{M^n_+}^\prime(X; H) = P \begin{bmatrix} \tilde{H}_{\alpha\alpha} & \tilde{H}_{\alpha\beta} & U_{\alpha\gamma} \odot \tilde{H}_{\alpha\gamma} \\ \tilde{H}_{\alpha\beta}^T & \Pi_{S^+_{n\mid\beta}}(H_{\beta\beta}) & 0 \\ \tilde{H}_{\alpha\gamma}^T \odot U^T_{\alpha\gamma} & 0 & 0 \end{bmatrix} + \hat{H}_{\beta\beta} \odot \tilde{H}_{\alpha\gamma}^T \odot U^T_{\alpha\gamma} \cdot 0 \cdot 0 \cdot \left( P^T + \frac{H - H^T}{2} \right).$$

Moreover we obtain:

- $\Pi_{M^n_+}^\prime(\cdot)$ is F-differentiable at $X \in M^n$ if and only if $X + X^T$ is nonsingular.
- The directional derivative $\Pi_{M^n_+}^\prime(X; \cdot)$ is F-differentiable at $H \in M^n$ if and only if $\hat{H}_{\beta\beta}$ is nonsingular.
- For any $V \in \partial B\Pi_{M^n_+}^\prime(X)$ (resp. $\partial B\Pi_{M^n_+}^\prime(\cdot)$), there exists a $W \in \partial B\Pi_{S^+_{n\mid\beta}}(0)$ (resp. $\partial B\Pi_{S^+_{n\mid\beta}}(0)$) such that

$$V(H) = P \begin{bmatrix} \hat{H}_{\alpha\alpha} & \hat{H}_{\alpha\beta} & W(H_{\beta\beta}) & U_{\alpha\gamma} \odot \hat{H}_{\alpha\gamma} \\ \hat{H}_{\alpha\beta}^T & \Pi_{S^+_{n\mid\beta}}(H_{\beta\beta}) & 0 & 0 \\ \hat{H}_{\alpha\gamma}^T \odot U^T_{\alpha\gamma} & 0 & 0 & 0 \end{bmatrix} + \hat{H}_{\beta\beta} \odot \hat{H}_{\alpha\gamma}^T \odot U^T_{\alpha\gamma} \cdot \left( P^T + \frac{H - H^T}{2} \right).$$

Conversely, for any $W \in \partial B\Pi_{S^+_{n\mid\beta}}(0)$ (resp. $\partial B\Pi_{S^+_{n\mid\beta}}(0)$), there exists a $V \in \partial B\Pi_{M^n_+}^\prime(X)$ (resp. $\partial B\Pi_{M^n_+}^\prime(\cdot)$) such that the above equation holds.

### 6 Conclusions

In this paper, we mainly have given a convex analysis on the NS-psd cone from three aspects that are presented in Sections 3, 4 and 5. Especially, we have given some conditions on determining the positive semidefiniteness of a nonsymmetric matrix in Propositions 4.3, 4.4 and Corollary 4.8. The research results are useful for us to study nonlinear optimization problems over the NS-psd cone, and may help us to open a typical instance for hyperbolic cone programming, where the hyperbolic cone is a research-worthy and more general kind of convex cones so far as we’ve known.

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