The Maximum Flow Problem with Disjunctive Constraints

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Abstract

We study the maximum flow problem subject to binary disjunctive constraints in a directed graph: A *negative disjunctive constraint* states that a certain pair of arcs in a digraph cannot be simultaneously used for sending flow in a feasible solution. In contrast to this, *positive disjunctive constraints* force that for certain pairs of arcs at least one arc has to carry flow in a feasible solution. It is convenient to represent the negative disjunctive constraints in terms of a so-called *conflict graph* whose vertices correspond to the arcs of the underlying graph, and whose edges encode the constraints. Analogously we represent the positive disjunctive constraints by a so-called *forcing graph*.

For conflict graphs we prove that the maximum flow problem is strongly $\mathcal{NP}$-hard, even if every connected component of the conflict graph is a path of length two. In contrast to this we show that for forcing graphs the problem can be solved efficiently if fractional flow values are allowed. If on the other hand the flow values are required to be integral we provide the sharp line between polynomially solvable and strongly $\mathcal{NP}$-hard instances.

Keywords: maximum flow problem; conflict graph; binary constraints

1 Introduction

We study two interesting variants of the maximum flow problem (MF) on directed graphs. These variants consist of *binary disjunctive constraints* on certain pairs of arcs.

- A negative disjunctive constraint expresses an incompatibility or a conflict between the two arcs in a pair. From each conflicting pair, at most one arc can carry flow in a feasible solution.

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- A positive disjunctive constraint enforces that in a feasible solution flow has to be sent over at least one arc from the underlying pair.

Throughout this paper we will represent these binary disjunctive constraints by means of an undirected constraint graph: Every vertex of the constraint graph corresponds to an arc of the original digraph, and every edge corresponds to a binary constraint. In the case of negative disjunctive constraints this constraint graph will be called *conflict graph*, and in the case of positive disjunctive constraints this graph will be called *forcing graph*.

For a formal definition of the *maximum flow problem with conflict graph* ($MFCG$) and the *maximum flow problem with forcing graph* ($MFFG$) we introduce the *maximum flow problem* in the standard way (cf. [1]). Let $G = (N, A)$ be a directed connected graph with n nodes and m arcs. Every arc $a = (i, j)$ has associated a positive integer capacity u_{ij} (resp. u_a). W.l.o.g we further assume that the indegree and outdegree of each node $v \in N \setminus \{s, t\}$ is greater than 0. Furthermore, one node s is designated as source and one node t as sink. The goal is to maximize the total flow sent from s to t not exceeding the capacities on any arc and keeping flow balance in every node. We will call the resulting optimal flow value f_{MF}.

$$(MF) \qquad \max f$$

$$\text{s.t.} \quad \begin{array}{rcrcrl} \sum_{j:(i,j)\in A} x_{ij} & - & \sum_{j:(j,i)\in A} x_{ji} & = & 0 & \forall\, i \in N \setminus \{s,t\} \\ \sum_{j:(s,j)\in A} x_{sj} & - & \sum_{j:(j,s)\in A} x_{js} & = & f & \\ \sum_{j:(t,j)\in A} x_{tj} & - & \sum_{j:(j,t)\in A} x_{jt} & = & -f & \\ & & x_{ij} & \geq & 0 & \end{array}$$

Adding to this formulation (MF) the negative disjunctive structure defined by a conflict graph $H = (A, E)$ with vertices corresponding to the arcs of G gives $MFCG$:

$$(MFCG)$$

$$(a, \bar{a}) \in E \implies (x_a = 0 \vee x_{\bar{a}} = 0) \qquad (1)$$

$MMFG$ adds to the formulation (MF) the following constraints induced by a forcing graph $H = (A, E)$:

$$(MFFG)$$

$$(a, \bar{a}) \in E \implies (x_a + x_{\bar{a}} > 0) \qquad (2)$$

For the case of integer flow values these constraints become $x_a + x_{\bar{a}} \geq 1$, which corresponds to the similar disjunctive constraints introduced for other combinatorial optimization problems in the literature.

We will also use a generalized version of MF, namely *maximum flow problem with lower bounds* ($MFLB$), where for certain arcs (i, j) the feasible flow has to achieve a given amount l_{ij}:

$$0 \leq l_{ij} \leq x_{ij} \leq u_{ij}$$

In order to identify feasible solutions of $MFLB$, it is useful to transform this problem into a *circulation problem* (CP) by introducing an arc of infinite capacity from t to s (cf. [1, Sec. 6.7]) and looking for a feasible flow f fulfilling the following conditions:

$$\sum_{j:(i,j)\in A} x_{ij} \quad - \quad \sum_{j:(j,i)\in A} x_{ji} \quad = \quad 0 \qquad \forall\, i \in N$$
$$0 \le l_{ij} \le x_{ij} \le u_{ij}$$

Note that $MFLB$ can be solved in the following way: First CP is transformed to a *feasible flow* problem. For this problem any max-flow algorithm can find a feasible solution or determine that no such solution exists. If a feasible solution for CP has been found, one can apply any maximum flow algorithm (that is based on residual capacities) with a slight modification on $MFLB$. The alteration consists of modifying the residual capacities with the help of the feasible solution found for CP ([1], chapter 6.7).

As usual we define an s-t cut $(S, \bar{S})$ as a partition of the node set N into two sets S and $\bar{S} = N \setminus S$ with $s \in S$ and $t \in \bar{S}$. The capacity $c(S, \bar{S})$ of a cut is defined as

$$c(S, \bar{S}) = \sum_{(i,j)\in(S,\bar{S})} u_{ij}.$$

For a problem with lower bounds we define the capacity $u(S, \bar{S})$ of an s-t cut as

$$u(S, \bar{S}) = \sum_{(i,j)\in(S,\bar{S})} u_{ij} - \sum_{(i,j)\in(\bar{S},S)} l_{ij}. \tag{3}$$

For use as a conflict resp. forcing graph we introduce the following terminology:

Definition 1 *A 2-ladder is an undirected graph whose components are paths of length one, i.e. isolated edges connecting pairs of vertices.*

In this paper we will characterize the complexity of $MFCG$ and $MFFG$. For $MFCG$ we show that the problem is already strongly $\mathcal{NP}$-hard if the digraph G consists only of disjoint paths from s to t and the conflict graph consists of a 2-ladder. Both structures seem to be most elementary non-trivial graphs for $MFCG$.

If we allow arbitrary nonnegative flow values to fulfill (2) in MMFG, it can shown that there are two possible solution scenarios: Either we have an optimal solution value equal to f_{MF}, i.e. we can send the full maximum flow f_{MF} such that the conditions of the forcing graph are fulfilled. Or these conditions require a rerouting of the flow such that the flow is diminished by some arbitrarily small $\varepsilon > 0$ yielding an optimal solution value of $f_{MF} - \varepsilon$. We will show that it is polynomially decidable which of the two cases occurs.

If we restrict flow values of MMFG to be integer we can show that a class of elementary instances of MMFG consisting of disjoint paths from s to t is (trivially) solvable in polynomial time but adding just one particular arc makes this class strongly $\mathcal{NP}$-hard.

1.1 Related Literature

Conflict graphs were considered for many other combinatorial optimization problems: Recently results were derived for the classical 0-1 knapsack problem with conflict graphs. While this problem is strongly $\mathcal{NP}$-hard for arbitrary conflict graphs, it was shown in [10] that pseudo-polynomial time algorithms (and also fully polynomial time approximation schemes) exist if the given conflict graph is a tree, a graph of bounded treewidth or a chordal graph.

Bin packing problems with special classes of conflict graphs were considered from an approximation point of view by [9] and [8]. Complexity results for different classes of conflict graphs for a scheduling problem under makespan minimization are given in [2]. In [5] complexity results as well as pseudo-polynomial time algorithms were derived for the transportation problem with conflict graphs. The minimum spanning tree problem with conflict constraints was studied in [12] and [3]. The latter paper also discusses the maximum matching problem and the shortest path problem and considers both positive and negative disjunctive constraints.

2 $MFCG$ is Strongly $\mathcal{NP}$-hard

In this section we derive a strongly $\mathcal{NP}$-hardness result even for extremely simply structured instances of $MFCG$. In particular, the results holds even for networks G consisting only of disjoints paths from s to t. Obviously, the maximum flow value f_{MF} can be computed trivially on such instances by taking the minimum of the capacities on each path and summing up over all paths. It seems hard to imagine an even simpler non-trivial flow network. For the conflict graph it suffices to use a 2-ladder which is the simplest meaningful disjunctive constraint structure.

Our proof uses a reduction of the strongly $\mathcal{NP}$-complete *independent set problem* (cf. [4]). Given an undirected graph Γ with vertex set V the independent set problem asks for a subset of vertices $V' \subseteq V$ of cardinality at least K such that no two vertices in V' are adjacent. Such a subset is called an independent set (IS). Let $N(j) \subseteq V$ denote the neighborhood of vertex j in G, i.e. all the vertices in $V \setminus \{j\}$ that are adjacent to j.

2.1 The Digraph G_{MFCG}

We construct the digraph G_{MFCG} for our flow network in the following way (see Figure 2.1): Introduce two special vertices s and t and for each vertex $j \in V$ a directed path P_j of length $|N(j)|$ from s to t. Denote the $|N(j)|$ arcs of this path as e_{ji} in arbitrary order where $i \in N(j)$. Obviously, each edge (i, j) in Γ implies in G_{MFCG} the occurrence of an arc e_{ji} in P_j and of e_{ij} in P_i. Now define a 2-ladder conflict graph G_{DIS} whose connected components, i.e. isolated edges, have as vertices exactly these arcs e_{ij} and e_{ji} of G_{MFCG} induced by an edge $(i, j) \in \Gamma$. All arc capacities are set to one. Let I be the instance of $MFCG$ defined by G_{MFCG} and G_{DIS}.

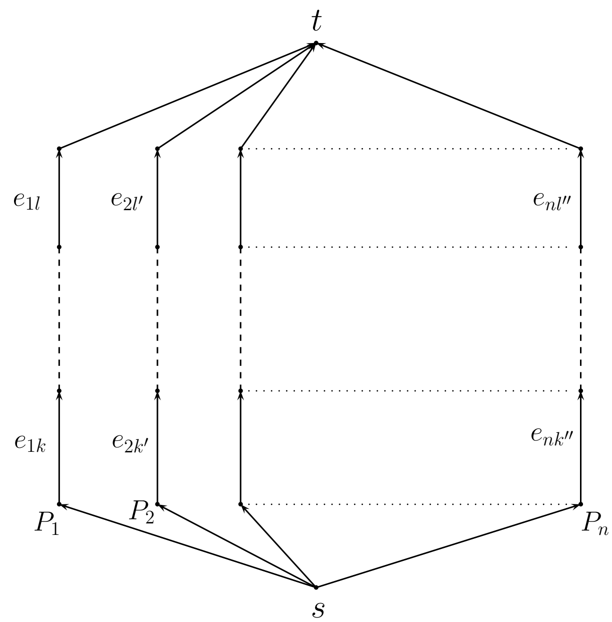

Figure 1: The digraph G_{MFCG} induced by a graph Γ.

Theorem 1 *$MFCG$ is strongly $\mathcal{NP}$-hard, even if the conflict graph is a 2-ladder and the network consists only of disjoint paths.*

Proof. We show that the following equivalence holds:

$$\exists \text{ a flow } f \text{ in } I \text{ with value } \geq K \Longleftrightarrow \exists \text{ IS } V' \text{ in } \Gamma \text{ with cardinality } \geq K.$$

"$\Longleftarrow$": For every vertex $j \in V'$ send one unit of flow over the path P_j, for all other vertices no flow is sent over P_j. Clearly, this gives a flow with value $|V'| \geq K$. If $j \in V'$ by definition of (IS) no vertex $i \in N(j)$ can be in V'. Therefore, if there is an arc e_{ji} in P_j, no flow will be sent over arc e_{ij} of P_i.

Hence, the conditions of the conflict graph G_{DIS} are satisfied and the described flow is feasible for I.

"$\Longrightarrow$": If f includes some flow over path P_j add vertex j to the independent set of Γ. Since all arc capacities are one at least K paths must contribute to flow f and hence the constructed vertex set has cardinality $\geq K$. Since f fulfills the conditions of the conflict graph G_{DIS} a flow over path P_j forbids a flow over any path P_i for $i \in N(j)$. Hence, the resulting vertex set is independent. □

Note that this reduction preserves the solution values and can easily be extended to the optimization version of the two problems.

Corollary 1 *MFCG is at least as hard to approximate as Maximum Independent Set.* □

3 *MFFG* with Integer Flow Values

In this section we assume that all flow values are integral and hence the positive disjunctive constraint (2) becomes $x_a + x_{\bar{a}} \geq 1$. The case of arbitrary flow values will be treated in Section 4.

For integer flow values we can give a sharp line between polynomially solvable and strongly $\mathcal{NP}$-hard instances of *MMFG*. Let I be an instance defined by a digraph $G = (N, A)$ consisting of disjoint paths between a source node s and a sink node t and let $H = (A, E)$ be an arbitrary forcing graph for G. Trivially sending as much flow as possible over each path solves the general maximum flow problem and also fulfills all positive disjunctive constraints (at least one unit of flow is routed over every arc) thus giving an optimal solution for I.

Adding to such an instance just one new arc which destroys the disjoint paths structure makes the problem already strongly $\mathcal{NP}$-hard, even if the forcing graph is again a 2-ladder. Thus considering Theorem 2, *MFFG* is "slightly easier" than *MFCG*.

In our construction we reduce the *vertex cover problem* on an undirected graph Γ with vertex set V to an instance I of *MMFG* that is defined as follows: The digraph G_{MFFG} is constructed in the same way as G_{MFCG} in Section 2 (recall Figure 2.1) with one additional node v that is joined only to s by the new arc (v, s). v becomes the new source node of the network. The arc (v, s) gets a capacity value k. The forcing graph G_{DIS} is identical to the conflict graph of Section 2.

Vertex cover asks for subset of vertices $V' \subseteq V$ such that for each edge (i, j) of Γ at least one of the two vertices i and j is in V'. Similar to the reduction of the independent set problem in Section 2 the vertex cover is now reproduced by the forcing graph G_{DIS} as in the proof of Theorem 2. Since vertex cover is a minimization problem we set the capacity k of the new arc (v, s) in a first step equal to one and test if the instance I is feasible. If not, we augment the capacity k by one and iteratively solve I until a feasible solution is found for the first time. The corresponding capacity k' is reported as the value of the minimum vertex cover in Γ.

Theorem 2 *MFFG is strongly $\mathcal{NP}$-hard, even if the conflict graph is a 2-ladder and at least as hard to approximate as the minimum vertex cover problem.* □

4 *MFFG* with Arbitrary Flow Values

As pointed out in Section 1 there are two possible solution scenarios for *MMFG* with arbitrary nonnegative flow values. Either the maximum flow with value f_{MF} on the digraph $G = (N, A)$ can be rerouted such that the conditions imposed by the forcing graph $H = (A, E)$ are fulfilled or the required diversion of tiny amounts of flows to previously "empty" arcs causes a marginal decrease of the overall flow value yielding an optimal solution value of $f_{MF} - \varepsilon$ for some arbitrarily small $\varepsilon > 0$. We will show that it is polynomially decidable which of the two cases occurs.

4.1 A Feasible Solution for *MFFG*

We will first construct a feasible solution for *MMFG* with value $f_{MF} - \varepsilon$. In the following subsection we will show how to decide whether also a solution with value f_{MF} exists. It will be convenient to recall some basic results for the *maximum flow problem with lower bounds* (*MFLB*) which can be found in [1, Sec. 6.7] with detailed proofs. First note that *MFLB* admits a feasible solution if and only if the corresponding circulation problem *CP* admits a feasible solution.

Theorem 3 *(Circulation Feasibility Conditions). [1, Th. 6.11] A circulation problem with nonnegative lower bounds on arc flows is feasible if and only if for every set S of nodes*

$$\sum_{(i,j)\in(\bar{S},S)} l_{ij} \leq \sum_{(i,j)\in(S,\bar{S})} u_{ij}.$$

Now we recall an analogon of the classical max-flow min-cut theorem for the case with bounded arc flows and cut capacities defined according to (3).

Theorem 4 *(Generalized Max-Flow Min-Cut Theorem). [1, Th. 6.10] For capacities $u(S, \bar{S})$ of an s-t cut $(S, \bar{S})$ in a network with both lower and upper bounds on arc flows the maximum value of flow from node s to node t equals* *the minimum capacity among all s-t cuts (given that a feasible flow exists in the network).*

Our $MFFG$ problem is transformed into an $MFLB$ problem by introducing a lower bound of ε' for every arc that appears in a positive disjunctive relation as a vertex of H. If we choose $\varepsilon' < \frac{1}{|A|}$ and apply Theorem 3 we immediately get that a feasible solution for the transformed problem exists, since all capacities have positive integer values. By applying Theorem 4 to the transformed problem we get the following.

Theorem 5 *There exists a feasible solution for $MFFG$ with solution value $f_{MF} - \varepsilon$ for arbitrarily small $\varepsilon > 0$.* □

4.2 Deciding the Optimal Value of $MFFG$

In [11] it is described how to detect all minimum *s-t* cuts in a digraph. Since there may be exponentially many of them, they cannot be all listed explicitly in polynomial time, but it can be decided efficiently if a given arc belongs to some minimum cut. In [6] the approach introduced in [11] is elaborated in more detail. For our purpose it is important that a directed acyclic graph $DAG_{s,t}$ (a so-called Picard-Queyranne Directed Acyclic Graph) is generated from a maximum flow in the original digraph G and that $DAG_{s,t}$ contains all minimum *s-t* cuts in its structure. Every node of $DAG_{s,t}$ corresponds to a subset of nodes of G. Moreover, the nodes of $DAG_{s,t}$ induce a partition of the nodes of G.

By the construction of $DAG_{s,t}$ the node containing t has in-degree 0, the node containing s has out-degree 0. A *closure* of $DAG_{s,t}$ is a subset $\mathcal{C}$ of the nodes of $DAG_{s,t}$ with the following property: For every node $A \in \mathcal{C}$ if there is an arc from node A to some node B in $DAG_{s,t}$ then also $B \in \mathcal{C}$. It was shown that all the nodes in G induced by a closure of $DAG_{s,t}$ containing s and not t correspond to the set S of a minimal *s-t* cut $(S, \bar{S})$ in G.

Therefore we solve in a first step the relaxed version MF of $MFFG$ in order to find the optimal flow value f_{MF}. Then we consider the corresponding Picard-Queyranne Graph $DAG_{s,t}$. By the fact that all capacities in G are integer, any cut that is not minimal must have a value $\geq (f_{MF} + 1)$. Now consider all violated positive disjunctive constraints, i.e. all edges $E' \subseteq E$ in the forcing graph H where neither of the two arcs of A incident to such an edge in E' carries any flow in f_{MF}. We will distinguish the following two cases:

Case 1. $\forall\ e \in E'$ joining two arcs of G: For at least one of these arcs, say $a = (i, j)$, both vertices i and j are in the same subset of nodes of G corresponding to a node of $DAG_{s,t}$.

Then introducing a lower bound of ε for the capacity of all such arcs (i, j) gives an instance of $MFLB$ which fulfills the positive disjunctive constraints and keeps the same solution value f_{MF}. This follows from the property of $DAG_{s,t}$: Since i and j belong to the same node of $DAG_{s,t}$ the capacities u_{ij} did not contribute to any minimal cut. Therefore, the arcs (i, j) only contributed to cuts with value $\geq (f_{MF} + 1)$. Moreover, only on these arcs lower bounds $l_{ij} = \varepsilon$ were introduced. Now considering the cut capacities $u(S, \bar{S})$ of the new problem $MFLB$, every cut containing these modified arcs still has a value $\geq (f_{MF} + 1) - \varepsilon|A|$. By choosing ε sufficiently small, a feasible solution with value f_{MF} is derived.

Case 2. $\exists\ e \in E'$ joining two arcs $a = (i, j)$ and $a' = (i', j')$ of G such that i and j as well as i' and j' are in different subsets of nodes of G induced by the nodes of $DAG_{s,t}$.

Then by the properties of $DAG_{s,t}$ there exists minimum cuts $(S, \bar{S})$, resp. $(S', \bar{S}')$, for each of the two arcs a, resp. a', with $j \in S$ and $i \in \bar{S}$, resp. $j' \in S'$ and $i' \in \bar{S}'$, since both (i, j) and (i', j') carry no flow in f_{MF}.

But then the problem can be similarly transformed to an $MFLB$ instance where the new lower bounds $l_{ij} = \varepsilon$, resp. $l_{i'j'} = \varepsilon$, do contribute to the minimum cut $u(S, \bar{S})$, resp. $u(S', \bar{S}')$, and decrease its value thus reducing the optimal flow value. It also follows by the structure of these minimum cuts that no feasible solution to $MFFG$ with value f_{MF} can exist.

Theorem 6 *For $MFFG$ with arbitrary flow values it can be decided in polynomial time if the optimal flow value corresponds to the value f_{MF} of the relaxed problem MF or if it equals $f_{MF} - \varepsilon$ for some arbitrarily small $\varepsilon > 0$.* □

Remark 1 The assumption of integer capacities is not restrictive in the following sense: If we allow rational capacities we could transform every instance by scaling into one with integer capacities. However if we allow real capacity values, by the above arguments deciding whether the optimal flow value equals f_{MF} or $f_{MF} - \varepsilon$ is still possible. However in this case there is no obvious way to bound ε (like $\varepsilon' < \frac{1}{|A|}$ in the case of integer capacities). By knowledge of the second smallest cut value in G the value of ε could be chosen appropriately, however we are not aware of a polynomial time algorithm that computes the second smallest cut value. Note that this problem is different from the computation of the second best cut (counting also different solutions with the same value) which is solvable in polynomial time as shown in [7].

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