Optimality Conditions and Duality for Nonsmooth Multiobjective Optimization Problems with Cone Constraints and Applications *

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May 31, 2010

Abstract: In this work, a nonsmooth multiobjective optimization problem involving generalized invexity with cone constraints and Applications (for short, (MOP)) is considered. The Kuhn-Tucker necessary and sufficient conditions for (MOP) are established by using a generalized alternative theorem of Craven and Yang. The relationship between weakly efficient solutions of (MOP) and vector valued saddle points of its Lagrange function is developed. Furthermore, We investigate the Mond-Weir type duality results for (MOP) and the relationships between weakly efficient solutions of (MOP) and solutions of Hartman-Stampacchia weak vector quasi-variational inequalities (for short, (HSVQI)) and Hartman-Stampacchia nonlinear weak vector quasi-variational inequalities (for short, (HSNVQI)). As an application, we also prove the existence of solutions for (HSVQI) and (HSNVQI) by the Kuhn-Tucker sufficient conditions. These results extend and improve corresponding results of others.

Key Words and Phrases: Nonsmooth multiobjective optimization problem, Hartman-Stampacchia weak vector quasi-variational inequalities, Hartman-Stampacchia nonlinear weak vector quasi-variational inequalities, Weak (strong, converse) duality, Generalized cone-invex function, Optimality conditions.

2000 Mathematics Subject Classification: 90C29, 90C46.

1 Introduction

The weak minimum (weakly efficient, weak Pareto) solution is an important con-

*This work was supported by the NSFC (70771080),

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cept in mathematical models, economics, decision theory, optimal control and game theory (see, for example, [10]). In most works, an assumption of convexity was made for the objective functions. Very recently, some generalized convexity has received more attention (see, for example, [2, 9, 10, 14, 15, 16, 18, 19, 20]). A significant generalization of convex functions is invex function introduced first by Hanson [14], which has greatly been applied in nonlinear optimization and other branches of pure and applied sciences.

As known to all, some alternative theorems play crucial roles in deriving necessary and sufficient conditions for (MOP). In [11], Craven and Yang presented generalized cone-invex functions and established a generalized alternative theorem involving nonsmooth functions. In 2004, Yang, Yang and Teo [22] presented two theorems of alternative with generalized subconvexlikeness and applied them to establish duality theorems and a saddle-point type optimality condition.

In [20], by introducing the notion of Q-nonsmooth pseudoinvexity, Suneja, Khurana and Vani established some necessary and sufficient optimality conditions for (MOP) involving Clarke's generalized gradient. However, there is a flaw in the proof of Theorem 3.1 via a generalized alternative theorem of Craven and Yang.

Inspired and motivated by above works, the purpose of this paper is to investigate a nonsmooth multiobjective problems involving generalized cone-invex functions with cone constraints. The Kuhn-Tucker (saddle type) necessary and sufficient conditions for (MOP) are established by using a generalized alternative theorem of Craven and Yang [11] under the assumptions of generalized cone invexity in the sense of Clarke generalized directional derivative, which is different from that defined in [20]. The Mond-Weir type duality results for (MOP) are presented. We also develop the relationships between weakly efficient solutions of (MOP) and solutions of Hartman-Stampacchia weak vector quasi-variational inequalities (for short, (HSVQI)) and Hartman-Stampacchia nonlinear weak vector quasi-variational inequalities (for short, (HSNVQI)), which is distinct from [12, 13, 16, 17, 19]. Furthermore, we study the existence of solutions for (HSVQI) and (HSNVQI) by the Kuhn-Tucker sufficient conditions. These results extend and improve corresponding results of [1, 3, 7, 13, 16, 17, 19] to nonsmooth case.

The remaining of this paper is organized as follows. In Section 2, we introduce preliminary knowledge. In Section 3, we investigate the Kuhn-Tucker necessary and sufficient optimality conditions for (MOP) under some suitable conditions. The relationship between the vector valued saddle points and weakly efficient solutions of (MOP) is developed in Section 4. The Mond-Weir type duality results for (MOP) is studies in Section 5. As an application, the existence of solutions for Hartman-Stampacchia weak vector quasi-variational inequalities and Hartman-Stampacchia nonlinear weak vector quasi-variational inequalities is given in Section 6. Finally, we conclude the paper in Section 7.

2 Preliminaries

Let R^n be the n-dimensional Euclidean space, and $R^n_+ = \{x = (x_1, \cdots, x_n)^T : x_l \geq 0, l = 1, \cdots, n\}$, where the superscript T denotes the transpose. A nonempty subset G of topological vector space X is said to be a cone if $\lambda G \subset G$ for all $\lambda > 0$. G is called a convex cone if G is a cone and $G + G \subset G$, G is called a pointed cone if G is a cone and $G \bigcap (-G) = \{0\}$.

Throughout this paper, without other specifications, let $Q \subset R^k$ and $S \subset R^m$ be closed convex cones with nonempty interior. Let $\eta : R^n \times R^n \to R^n$ with $\eta(x, x_0) \neq 0$ for some $x \neq x_0$. Let $f = (f_1, \cdots, f_k)^T : R^n \to R^k, g = (g_1, \cdots, g_m)^T : R^n \to R^m$ and let $e = (1, \cdots, 1)^T \in R^k$, where f_i and g_j are locally Lipschitz, $i \in \{1, \cdots, k\}$ and $j \in \{1, \cdots, m\}$. The dual cone of $K \subset R^n$ is denoted by

$$K^* = \{u \in R^n : x^T u \geq 0, \forall x \in K\}.$$

The multiobjective optimization problem (for short, (MOP)) is defined as follows:

$$\min f(x)$$
$$\text{subject to} \qquad -g(x) \in S.$$

Denote by F the feasible set of (MOP) . It is easy to check that

$$F = \{x \in R^n : \mu^T g(x) \leq 0, \forall \mu \in S^*\}.$$

We first recall some definitions and lemmas which are needed in the main results of this work.

Definition 2.1. A point $x_0 \in F$ is said to be a weakly efficient (weak minimum) solution of (MOP) if,

$$f(x) - f(x_0) \notin -\text{int}Q, \forall x \in F.$$

Denote by F^w the weakly efficient solutions set of (MOP) .

Definition 2.2.[9] A real-valued function $\ell : R^n \to R$ is said to be locally Lipschitz if for each $z \in R^n$, there exist a positive constant κ and a neighborhood N of z such that, for each $x, y \in N$,

$$|\ell(x) - \ell(y)| \leq \kappa \|x - y\|,$$

where $\|\cdot\|$ denotes any norm in R^n. The Clarke [9] generalized subgradient of ℓ at z is denoted by

$$\partial \ell(z) = \{\xi \in R^n : \ell^\circ(z; d) \geq \xi^T d, \forall d \in R^n\},$$

where $\ell^\circ(z; d) = \limsup_{y \to z, t \to 0} \frac{\ell(y+td) - \ell(y)}{t}$.

Clearly, $\ell^\circ(z; d) = \max\{\xi^T d : \xi \in \partial \ell(z)\}$.

Definition 2.3. Let $p : R^n \to R^k$. The generalized subgradient of p at $z \in R^n$ is the set

$$\partial p(z) = \{(\zeta_1, \cdots, \zeta_k)^T : \zeta_i \in \partial p_i(z), i = 1, \cdots, k\},$$

where $\partial p_i(z)$ is the Clarke's generalized subgradient of $p_i (i \in \{1, \cdots, k\})$ at $z \in R^n$.

Definition 2.4. [11] f is called generalized Q-invex at $z \in R^n$ if, there exists $\eta : R^n \times R^n \to R^n$ such that for every $x \in R^n$ and $\zeta \in \partial f(z)$,

$$f(x) - f(z) - \zeta\eta(x, z) \in Q.$$

f is called generalized Q-pseudoinvex at $z \in R^n$ if, there exists $\eta : R^n \times R^n \to R^n$ such that for every $x \in R^n$ and $\zeta \in \partial f(z)$,

$$f(x) - f(z) \in -\text{int}Q \Rightarrow \zeta\eta(x, z) \in -\text{int}Q.$$

f is called generalized Q-(pseudo)invex with respect to η if, it is generalized Q-(pseudo)invex at each point $z \in R^n$ with respect to η.

Remark 2.1. f is generalized Q-invex and g is generalized S-invex with respect to the same $\eta : R^n \times R^n \to R^n$ if and only if (f, g) is generalized $Q \times S$-invex with respect to $\eta : R^n \times R^n \to R^n$.

Remark 2.2. In [20], Suneja, Khurana and Vani defined the following pseudoinvexity: f is called Q-nonsmooth pseudoinvex at $z \in R^n$ if, there exists $\eta : R^n \times R^n \to R^n$ such that for every $x \in R^n$,

$$f(x) - f(z) \in -\text{int}Q \Rightarrow f^\circ(z; \eta(x, z)) \in -\text{int}Q,$$

where $f^\circ(z; \eta(x, z)) = (f_1^\circ(z; \eta(x, z)), \cdots, f_k^\circ(z; \eta(x, z)))^T$.

It is easy to see that if $Q = R_+^k$, then Q-nonsmooth pseudoinvexity implies generalized Q-pseudoinvexity; if $Q = -R_+^k$, then generalized Q-pseudoinvexity implies Q-nonsmooth pseudoinvexity.

Definition 2.5. (MOP) is said to satisfy the extended constraint qualification if, there exists $x \in R^n$ with $\mu^T g(x) \geq 0$, for all $\mu \in S^*$ such that $\xi_1, \cdots, \xi_m$ are linearly independent, for any $\xi_i \in \partial g_i(x)$, $i \in \{1, \cdots, m\}$.

Definition 2.6. (f, g) is called KT-(Q, S)-pseudoinvex at y if, there exists $\eta : R^n \times R^n \to R^n$ such that for every $x \in R^n, x \neq y$,

$$f(x) - f(y) \in -\text{int}Q \Rightarrow \zeta\eta(x, y) \in -\text{int}Q, \xi\eta(x, y) \in -S, \forall\zeta \in \partial f(y), \xi \in \partial g(y).$$

(f, g) is called KT-(Q, S)-pseudoinvex with respect to η if, it is KT-(Q, S)-pseudoinvex at each point $z \in R^n$ with respect to η.

Clearly, if f and g are generalized Q-pseudoinvex and generalized S-pseudoinvex with respect to η at y, then (f, g) is called KT-(Q, S)-pseudoinvex with respect to η at y.

Remark 2.3. If f and g are differentiable on F, then the generalized Q-invexity reduces to the Q-invexity [7, 17]; Moreover, if $Q = R^k_+$ and $S = R^m_+$, then the KT-(Q, S)-pseudoinvexity reduces to the KT-pseudoinvexity [4].

Lemma 2.1. [9] If $p_i : R^n \to R (i \in \{1, \cdots, k\})$ are locally Lipschitz, then the following statements are true:

(i) $\partial(\Sigma_{i=1}^k p_i)(z) \subset \Sigma_{i=1}^k \partial p_i(z), \forall z \in R^n$;

(ii) $\partial(\Sigma_{i=1}^k t_i p_i)(z) \subset \Sigma_{i=1}^k t_i \partial p_i(z), \forall z \in R^n$, where $t = (t_1, \cdots, t_k)^T \in R^k$.

It is easy to prove the following results:

Proposition 2.1. Let $\tau \in Q^*$ and $p = (p_1, \cdots, p_k)^T : R^n \to R^k$. If $p_i : R^n \to R$ is locally Lipschitz for each $i \in \{1, \cdots, k\}$, then $\tau^T p$ is locally Lipschitz on R^n.

Lemma 2.2. [10] Let $\Delta \subset R^k$ be a convex cone with $\text{int}\Delta \neq \emptyset$ and Δ^* the dual cone of Δ. Then the following statements are true:

(i) If $u \in \text{int}\Delta$, then $x^T u > 0$ for all $x \in \Delta^* \setminus \{0\}$;

(ii) If $x \in \text{int}\Delta^*$, then $x^T u > 0$ for all $u \in \Delta \setminus \{0\}$.

Lemma 2.3. [11] Let f be generalized Q-invex at $z \in R^n$ with respect to $\eta : R^n \times R^n \to R^n$, where $Q \subset R^k$ is a closed convex cone with nonempty interior. Then only one of the following two systems holds:

(i) There exists $x \in R^n$ such that $f(x) - f(z) \in -\text{int}Q$;

(ii) There exists $\lambda \in Q^* \setminus \{0\}$, such that $(\lambda^T \vartheta)(R^n) \subset R_+$, where $\vartheta(\cdot) = f(\cdot) - f(z)$.

Lemma 2.4. [7, 10] Let K be a convex cone of topological vector space X with $\text{int}K \neq \emptyset$. Then, for any $x, y \in X$, the following statements are true:

(i) $y - x \in K$ and $y \in -K$ imply $x \in -K$;

(ii)$y - x \in K$ and $y \in -\text{int } K$ imply $x \in -\text{int } K$;

(iii)$y - x \in K$ and $x \notin -\text{int } K$ imply $y \notin -\text{int } K$.

In the rest of this paper, we always denote by the following

Assumption A : Let f be generalized Q-invex and g generalized S-invex with respect to the same $\eta : R^n \times R^n \to R^n$.

3 Optimality Conditions for (MOP)

In this section, we shall investigate the Kuhn-Tucker necessary and sufficient optimality conditions for (MOP) under some suitable conditions.

Theorem 3.1.(Kuhn-Tucker necessary optimality condition) Let **Assumption A** hold at $\bar{x} \in F$. Suppose that the extended constraint qualification is satisfied at $\bar{x} \in F^w$. Then there exist $\lambda \in Q^* \setminus \{0\}$ and $\mu \in S^*$ such that

$$0 \in \partial f(\bar{x})^T \lambda + \partial g(\bar{x})^T \mu; \quad \mu^T g(\bar{x}) = 0,$$

where $\partial f(\bar{x})^T = \{\zeta^T : \zeta \in \partial f(\bar{x})\}$.

Proof. Let $\bar{x} \in F^w$. Then

$$f(x) - f(\bar{x}) \notin -\text{int}Q, \forall x \in F.$$

In view of $-g(\bar{x}) \in S$, if $g(\bar{x}) - g(x) \in S$, for any given $x \in R^n$ then $-g(x) \in S$. Consequently, there is no $x \in R^n$ such that

$$(f(x) - f(\bar{x}), g(x) - g(\bar{x})) \in -\text{int}(Q \times S).$$

Since f is generalized Q-invex and g is generalized S-invex with respect to the same η at $\bar{x} \in F$, from Lemma 2.3, there exist $\lambda \in Q^*$ and $\mu \in S^*$ with $(\lambda, \mu) \neq 0$ such that

$$\lambda^T (f(x) - f(\bar{x})) + \mu^T (g(x) - g(\bar{x})) \geq 0, \forall x \in R^n,$$

and so

$$\lambda^T f(x) + \mu^T g(x) \geq \lambda^T f(\bar{x}) + \mu^T g(\bar{x}), \forall x \in R^n. \tag{3.1}$$

We declare that $\lambda \neq 0$. In fact, if $\lambda = 0$, then

$$\mu \neq 0, \mu^T g(x) \geq \mu^T g(\bar{x}), \forall x \in R^n,$$

and thus

$$0 \in \partial(\mu^T g)(\bar{x}).$$

From Lemma 2.1, we have

$$\begin{aligned} 0 &\in \partial(\mu^T g)(\bar{x}) \\ &= \partial(\sum_{j=1}^{m} \mu_j g_j)(\bar{x}) \\ &\subset \sum_{j=1}^{m} \mu_j \partial g_j(\bar{x}) \\ &= \partial g(\bar{x})^T \mu, \end{aligned}$$

which contradicts with the fact that the extended constraint qualification holds at $\bar{x}$. Thus (3.1) allows to

$$0 \in \partial(\lambda^T f + \mu^T g)(\bar{x})$$

and consequently from Lemma 2.1 it follows that

$$\begin{aligned} 0 &\in \partial(\lambda^T f + \mu^T g)(\bar{x}) \\ &= \partial(\sum_{i=1}^{k} \lambda_i f_i + \sum_{j=1}^{m} \mu_j g_j)(\bar{x}) \\ &= \partial(\sum_{i=1}^{k} \lambda_i f_i)(\bar{x}) + \partial(\sum_{j=1}^{m} \mu_j g_j)(\bar{x}) \\ &\subset \sum_{i=1}^{k} \lambda_i \partial f_i(\bar{x}) + \sum_{j=1}^{m} \mu_j \partial g_j(\bar{x}) \\ &= \partial f(\bar{x})^T \lambda + \partial g(\bar{x})^T \mu. \end{aligned}$$

By the Definition 2.5, we obtain

$$\mu^T g(\bar{x}) \geq 0.$$

Notice that $-g(\bar{x}) \in S$ implies that $\mu^T g(\bar{x}) \leq 0$, we have $\mu^T g(\bar{x}) = 0$. This completes the proof.

Remark 3.1. Compared with Theorem 3.2 in [20], the Kuhn-Tucker necessary optimality condition for (MOP) is derived by using the extended constraint qualification, but not the generalized Slater constraint qualification.

If $\bar{x} \in F^w$ is an interior point of F, without any constraint qualifications, the following Kuhn-Tucker necessary optimality conditions hold:

Theorem 3.2.(Kuhn-Tucker necessary optimality condition) Let f be generalized Q-invex with respect to $\eta : R^n \times R^n \to R^n$ at $\bar{x} \in F^w$ with $\bar{x} \in \text{int}F$. Then there exist $\lambda \in Q^* \setminus \{0\}$ and $\mu \in S^*$ such that

$$0 \in \partial f(\bar{x})^T \lambda + \partial g(\bar{x})^T \mu; \quad \mu^T g(\bar{x}) = 0.$$

Proof. Let $\bar{x} \in F^w$. Then

$$f(x) - f(\bar{x}) \notin -\text{int}Q, \forall x \in F.$$

In view of $\bar{x} \in \text{int}F$, it suffices to prove that

$$F = \{x : \forall \tilde{x} \in R^n, \exists t_{\tilde{x}} \in [0, 1), x = \tilde{x} + t_{\tilde{x}}(\bar{x} - \tilde{x})\}.$$

Therefore, for each $x \in R^n$,

$$f(x + t_x(\bar{x} - x)) - f(\bar{x}) \notin -\text{int}Q.$$

Since $f_i (i = 1, 2, ..., k)$ is locally Lipschitz,

$$f(x) - f(\bar{x}) = \lim_{t_x \to 0} f(x + t_x(\bar{x} - x)) - f(\bar{x}) \in R^k \setminus (-\text{int}Q),$$

that is,

$$f(x) - f(\bar{x}) \notin -\text{int}Q, \forall x \in R^n.$$

Thus, there is no $x \in R^n$ such that

$$(f(x) - f(\bar{x})) \in -\text{int}Q.$$

Since f is generalized Q-invex with respect to η at $\bar{x}$, from Lemma 2.3, there exist $\lambda \in Q^* \setminus \{0\}$ such that

$$\lambda^T f(x) \geq \lambda^T f(\bar{x}), \forall x \in R^n.$$

Thus $0 \in \partial(\lambda^T f)(\bar{x})$. Moreover, we get

$$0 \in \partial f(\bar{x})^T \lambda.$$

It is easy to verify that there exists $\mu \in S^*$ such that $0 \in \partial g(\bar{x})^T \mu$ and $\mu^T g(\bar{x}) = 0$. Therefore there exist $\lambda \in Q^* \setminus \{0\}$ and $\mu \in S^*$ such that

$$0 \in \partial f(\bar{x})^T \lambda + \partial g(\bar{x})^T \mu.$$

This completes the proof.

Theorem 3.3.(Kuhn-Tucker sufficient optimality condition) Let f be generalized Q-pseudoinvex and g be generalized S-invex with respect to the same $\eta : R^n \times R^n \to R^n$ at $\bar{x} \in F$. Assume that there exist $\lambda \in Q^* \setminus \{0\}$ and $\mu \in S^*$ such that

$$0 \in \partial f(\bar{x})^T \lambda + \partial g(\bar{x})^T \mu; \quad \mu^T g(\bar{x}) = 0. \tag{3.2}$$

Then $\bar{x} \in F^w$.

Proof. Let $\bar{x} \in F$. Suppose to the contrary that $\bar{x} \notin F^w$, then there exists $\hat{x} \in F$ such that

$$f(\hat{x}) - f(\bar{x}) \in -\text{int}Q.$$

By the generalized Q-pseudoinvexity of f with respect to η at $\bar{x} \in F$, we get

$$\zeta\eta(\hat{x}, \bar{x}) \in -\text{int}Q, \forall \zeta \in \partial f(\bar{x}).$$

Since $\lambda \in Q^* \setminus \{0\}$, from Lemma 2.2 one has

$$\lambda^T \zeta\eta(\hat{x}, \bar{x}) < 0, \forall \zeta \in \partial f(\bar{x}). \tag{3.3}$$

It follows from (3.2) that there exists $\bar{\xi} \in \partial g(\bar{x})$ such that

$$\mu^T \bar{\xi} \eta(\hat{x}, \bar{x}) > 0. \tag{3.4}$$

Notice that g is generalized S-invex with respect to η at $\bar{x} \in F$. It follows that

$$g(\hat{x}) - g(\bar{x}) - \xi \eta(\hat{x}, \bar{x}) \in S, \forall \xi \in \partial g(\bar{x})$$

and therefore

$$\mu^T g(\hat{x}) - \mu^T g(\bar{x}) - \mu^T \bar{\xi} \eta(\hat{x}, \bar{x}) \geq 0,$$

or equivalently,

$$\mu^T \bar{\xi} \eta(\hat{x}, \bar{x}) \leq \mu^T g(\hat{x}) \leq 0.$$

which contradicts with (3.4). Thus $\bar{x} \in F^w$. This completes the proof.

Remark 3.2. The form of the Kuhn-Tucker necessary and sufficient optimality conditions for (MOP) presented in this section is more reasonable than that given in [20].

Remark 3.3. The assumption that generalized Q-pseudoinvexity of f with respect to the above η at $\bar{x}$ in Theorem 3.2 can be replaced by generalized Q-invexity of f with respect to the η at $\bar{x}$.

Theorem 3.4.(Kuhn-Tucker sufficient optimality condition) Let (f, g) be KT-(Q, S)-pseudoinvex with respect to the same $\eta : R^n \times R^n \to R^n$ at $\bar{x} \in F$. Assume that there exist $\lambda \in Q^* \setminus \{0\}$ and $\mu \in S^*$ such that

$$0 \in \partial f(\bar{x})^T \lambda + \partial g(\bar{x})^T \mu; \quad \mu^T g(\bar{x}) = 0. \tag{3.5}$$

Then $\bar{x} \in F^w$.

Proof. Let $\bar{x} \in F$. Suppose to the contrary that $\bar{x} \notin F^w$, then there exists $\hat{x} \in F$ such that

$$f(\hat{x}) - f(\bar{x}) \in -\text{int} Q.$$

By the KT-(Q,S)-pseudoinvexity of (f,g) with respect to η at $\bar{x}\in F$, we get

$$\zeta\eta(\hat{x},\bar{x})\in-\mathrm{int}Q,\xi\eta(\hat{x},\bar{x})\in-S,\forall\zeta\in\partial f(\bar{x}),\xi\in\partial g(\bar{x}).$$

Moreover, one has

$$(\lambda^T\zeta+\mu^T\xi)\eta(\hat{x},\bar{x})<0,\forall\zeta\in\partial f(\bar{x}),\xi\in\partial g(\bar{x}).$$

which contradicts with (3.5). Thus $\bar{x}\in F^w$. This completes the proof.

We now give an example to illustrate the obtained results.

Example 3.1. Let $R^n=R^k=R^m=R^2$ and $Q=S=[0,+\infty)\times(-\infty,0]$. Let $f:R^n\to R^k$ and $g:R^n\to R^m$. We consider the following problem (MOP):

$$\begin{aligned}&\min f(x):=(2x_1,-3x_2^2)^T\\ \text{subject to}\quad &g(x):=(x_1^2+4x_1-5,-x_2)^T\in-S.\end{aligned}$$

Simple computation allows that the feasible solutions set $F=\{(x_1,x_2)^T\in R^2:-5\le x_1\le 1,x_2\le 0\}$, and the weakly efficient solutions set $F^w=\{(x_1,x_2)^T:x_1=-5$, or $x_2=0\}$. One can easy verify that f and g are generalized $Q-$invex and generalized $S-$invex with respect to $\eta(x,\bar{x}):=(2^{x_2\bar{x}_2}(x_1-\bar{x}_1),x_2-\pi^{x_1x_2-\bar{x}_2})^T$ at $\bar{x}$, respectively, and the extended constraint qualifications hold at $\bar{x}$, where $x=(x_1,x_2)^T$ and $\bar{x}=(-5,0)^T$. Clearly, there exist $\bar{\lambda}=(3,0)^T\in Q^*$ and $\bar{\mu}=(1,0)^T\in S^*$ such that

$$0\in\partial f(\bar{x})^T\bar{\lambda}+\partial g(\bar{x})^T\bar{\mu};\quad \bar{\mu}^Tg(\bar{x})=0.$$

4 Relationship between Vector valued Saddle Points and Weakly Efficient Solutions for (MOP)

In this section, let $Q=R^k_+$. We shall develop the relationship between vector valued saddle points and weakly efficient solutions for (MOP) under some suitable assumptions. We now propose the Lagrange function for (MOP) :

$$L(x,\mu):=(L_1(x,\mu),L_2(x,\mu),...,L_k(x,\mu))=f(x)+\mu^Tg(x)e,\forall x\in F,\mu\in S^*.$$

where $L_i(x,\mu) = f_i(x) + \mu^T g(x), i \in \{1,2,...,k\}$ and $e = (1,1,...,1)^T \in R^k$.

Definition 4.1. A point $(\bar{x},\bar{\mu}) \in F \times S^*$ is called a vector valued saddle point of the Lagrange function $L(x,\mu)$ if

(i) $L(\bar{x},\mu) - L(\bar{x},\bar{\mu}) \notin \text{int}Q, \forall \mu \in S^*$;

(ii)$L(x,\bar{\mu}) - L(\bar{x},\bar{\mu}) \notin -\text{int}Q, \forall x \in F$.

Theorem 4.1. Let $(\bar{x},\bar{\mu}) \in F \times S^*$ be a vector valued saddle point of $L(x,\mu)$. Then $\bar{x} \in F^w$.

Proof. Let $(\bar{x},\bar{\mu}) \in F \times S^*$ be a vector valued saddle point of $L(x,\mu)$. Therefore, from Definition 4.1, there exists $i \in \{1,2,...,k\}$ such that

$$L_i(\bar{x},\mu) - L_i(\bar{x},\bar{\mu}) \leq 0, \forall \mu \in S^*.$$

Moreover, one has

$$\mu^T g(\bar{x}) - \bar{\mu}^T g(\bar{x}) \leq 0, \forall \mu \in S^*,$$

which implies $\bar{\mu}^T g(\bar{x}) = 0$.

Suppose to the contrary that $\bar{x} \notin F^w$. Then there exists $x_0 \in F$ such that

$$f(x_0) - f(\bar{x}) \in -\text{int}Q. \tag{4.1}$$

Since $\bar{\mu}^T g(x_0) \leq 0$, from (4.1), we have

$$f(x_0) + \bar{\mu}^T g(x_0) - f(\bar{x}) - \bar{\mu}^T g(\bar{x}) \in -\text{int}Q,$$

which contradicts Definition 4.1. This completes the proof.

Theorem 4.2. Let **Assumption A** hold at $\bar{x} \in F^w$. Suppose that the extended constraint qualification is satisfied at $\bar{x}$. Then there exists $\bar{\mu} \in S^*$ such that $(\bar{x},\bar{\mu})$ is a vector valued saddle point for $L(x,\mu)$.

Proof. Let $\bar{x} \in F^w$. By Theorem 3.1, there exist $\bar{\lambda} \in Q^* \setminus \{0\}$ and $\bar{\mu} \in S^*$ such that

$$0 \in \partial f(\bar{x})^T \bar{\lambda} + \partial g(\bar{x})^T \bar{\mu}; \quad \bar{\mu}^T g(\bar{x}) = 0. \tag{4.2}$$

Since $\mu^T g(\bar{x}) \leq 0$ for all $\mu \in S^*$, then

$$\begin{aligned} L(\bar{x},\mu) - L(\bar{x},\bar{\mu}) &= \mu^T g(\bar{x})e - \bar{\mu}^T g(\bar{x})e \\ &= \mu^T g(\bar{x})e \notin \text{int}Q. \end{aligned}$$

Suppose to the contrary that $(\bar{x},\bar{\mu})$ is not vector valued saddle point of $L(x,\mu)$. Then, there exists $x^0 \in F$ such that

$$L(x^0,\bar{\mu}) - L(\bar{x},\bar{\mu}) = f(x^0) + \bar{\mu}^T g(x^0)e - f(\bar{x}) - \bar{\mu}^T g(\bar{x})e \in -\text{int}Q.$$

Without loss of generality, let $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2, ..., \bar{\lambda}_k)^T$ with $\sum_{i=1}^k \bar{\lambda}_i = 1$. Then

$$\bar{\lambda}^T f(x^0) - \bar{\lambda}^T f(\bar{x}) + \bar{\mu}^T g(x) - \bar{\mu}^T g(\bar{x}) < 0. \tag{4.3}$$

Since **Assumption A** hold at $\bar{x}$, one has

$$f(x^0) - f(\bar{x}) - \zeta\eta(x^0,\bar{x}) \in Q, \forall \zeta \in \partial f(\bar{x}),$$

and so,

$$g(x^0) - g(\bar{x}) - \xi\eta(x^0,\bar{x}) \in S, \forall \xi \in \partial g(\bar{x}).$$

Moreover, we have

$$\bar{\lambda}^T f(x^0) - \bar{\lambda}^T f(\bar{x}) - \bar{\lambda}^T \zeta\eta(x^0,\bar{x}) \geq 0, \forall \zeta \in \partial f(\bar{x}),$$

and so,

$$\bar{\mu}^T g(x^0) - \bar{\mu}^T g(\bar{x}) - \bar{\mu}^T \xi\eta(x^0,\bar{x}) \geq 0, \forall \xi \in \partial g(\bar{x}).$$

Therefore,

$$\bar{\lambda}^T f(x^0) - \bar{\lambda}^T f(\bar{x}) + \bar{\mu}^T g(x^0) - \bar{\mu}^T g(\bar{x}) \geq (\bar{\lambda}^T \zeta + \bar{\mu}^T \xi)\eta(x^0,\bar{x}), \forall \zeta \in \partial f(\bar{x}), \xi \in \partial g(\bar{x}).$$

It follows from (4.2) that

$$\bar{\lambda}^T f(x^0) - \bar{\lambda}^T f(\bar{x}) + \bar{\mu}^T g(x^0) - \bar{\mu}^T g(\bar{x}) \geq 0,$$

which contradicts (4.3). This completes the proof.

Theorem 4.3. Let **Assumption A** hold at $\bar{x} \in F$. Suppose that the extended constraint qualification is satisfied at $\bar{x}$ and there exist $\bar{\lambda} \in Q^* \setminus \{0\}$ and $\bar{\mu} \in S^*$ such that

$$0 \in \partial f(\bar{x})^T \bar{\lambda} + \partial g(\bar{x})^T \bar{\mu}; \quad \bar{\mu}^T g(\bar{x}) = 0.$$

Then $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is a vector valued saddle point of $L(x, \mu)$.

Proof. It directly from Theorems 3.3 and 4.2. This completes the proof.

Example 4.1. Let $Q = S = R^2_+$. Let $f : R^2 \to R^2$ and $g : R^2 \to R^2$. We consider the following problem (MOP):

$$\min f(x) := (2x_1^4, 3x_2^6)^T$$
$$\text{subject to} \quad g(x) := (x_1^2, x_2^2)^T \in -S.$$

It is easy to see that the feasible solutions set $F = \{0\}$, and the weakly efficient solutions set $F^w = \{0\}$. It is easy to verify that f and g are generalized $Q-$invex and generalized $S-$invex with respect to $\eta(x, \bar{x}) := ((x_1 - \bar{x}_1)^3, (x_2 - \bar{x}_2)^2)^T$ at $\bar{x}$, respectively, where $x = (x_1, x_2)^T$ and $\bar{x} = (\bar{x}_1, \bar{x}_2)^T = 0$. We now construct the Lagrange function

$$L(x, \mu) := f(x) + \mu^T g(x)e = (2x_1^4 + \mu_1 x_1^2 + \mu_2 x_2^2, 3x_2^6 + \mu_1 x_1^2 + \mu_2 x_2^2), \forall x \in F, \mu =\in S^*,$$

where $\mu = (\mu_1, \mu_2)^T$. Simple computation allows that $\bar{\mu} = (1, 3)^T \in S^*$ such that $(\bar{x}, \bar{\mu})$ is vector valued saddle point of $L(x, \mu)$.

5 Mond-Weir Type Duality for (MOP)

In this section, we shall discuss a Mond-Weir type dual problem for (MOP) with generalized cone-invex functions (for short, (MOPD)) given by

$$\max f(y)$$

$$\text{Subject to} \quad 0 \in \partial f(y)^T \lambda + \partial g(y)^T \mu, \tag{5.1}$$

$$\mu^T g(y) \geq 0, \lambda \in Q^* \setminus \{0\}, \mu \in S^*. \tag{5.2}$$

Denote by F^D the feasible set of (MOPD).

Definition 5.1. A point $(x_0, \lambda_0, \mu_0) \in F^D$ is called a weakly efficient solution of (MOPD) if,

$$f(y) - f(x_0) \notin \text{int}Q, \forall (y, \lambda, \mu) \in F^D.$$

Denote by F_w^D the weakly efficient solutions set of (MOPD).

Theorem 5.1. (Weak duality) Let $x \in F$ and $(y, \lambda, \mu) \in F^D$. Assume that at least one of the following statements hold:

(i) **Assumption A** hold at y;

(ii) (f, g) is KT- (Q, S)-pseudoinvex with respect to $\eta : R^n \times R^n \to R^n$ at y.

Then

$$f(x) - f(y) \notin -\text{int}Q.$$

Proof. Let $x \in F$ and $(y, \lambda, \mu) \in F^D$. Suppose to the contrary that

$$f(x) - f(y) \in -\text{int}Q. \tag{5.3}$$

If (i) holds, then

$$f(x) - f(y) - \zeta\eta(x, y) \in Q, \forall \zeta \in \partial f(y),$$

$$g(x) - g(y) - \xi\eta(x, y) \in S, \forall \xi \in \partial g(y).$$

Furthermore, we get

$$\lambda^T (f(x) - f(y)) - \lambda^T \zeta\eta(x, y) \geq 0, \forall \zeta \in \partial f(y),$$

$$\mu^T (g(x) - g(y)) - \mu^T \xi\eta(x, y) \geq 0, \forall \xi \in \partial g(y).$$

It follows from (5.2) and (5.3) that

$$\lambda^T (f(x) - f(y)) < 0, \forall \zeta \in \partial f(y),$$

$$\mu^T (g(x) - g(y)) \leq 0, \forall \xi \in \partial g(y).$$

Therefore,

$$\lambda^T \zeta \eta(x,y) < 0, \forall \zeta \in \partial f(y),$$
$$\mu^T \xi \eta(x,y) \leq 0, \forall \xi \in \partial g(y).$$

Moreover, one has

$$(\lambda^T \zeta + \mu^T \xi)\eta(x,y) < 0, \forall \zeta \in \partial f(y), \xi \in \partial g(y),$$

which contradicts (5.1).

If (ii) holds, from (5.3),

$$\zeta \eta(x,y) \in -\text{int}Q, \xi \eta(x,y) \in -S, \forall \zeta \in \partial f(y), \xi \in \partial g(y). \tag{5.4}$$

By Lemma 2.2, it follows from (5.2) and (5.3) that

$$\lambda^T \zeta \eta(x,y) < 0, \mu^T \xi \eta(x,y) \leq 0, \forall \zeta \in \partial f(y), \xi \in \partial g(y).$$

Therefore,

$$(\lambda^T \zeta + \mu^T \xi)\eta(x,y) < 0, \forall \zeta \in \partial f(y), \xi \in \partial g(y),$$

which contradicts (5.1). This completes the proof.

Theorem 5.2. (Strong duality) Let $\bar{x} \in F^w$. Assume that **Assumption A** hold at $\bar{x}$. Assume that the extended constraint qualifications hold at $\bar{x}$. Then there exist $\bar{\lambda} \in Q^* \setminus \{0\}$ and $\bar{\mu} \in S^*$ such that $(\bar{x}, \bar{\lambda}, \bar{\mu}) \in F_w^D$.

Proof. Let $\bar{x} \in F^w$. Then by Theorem 3.1, there exist $\bar{\lambda} \in Q^* \setminus \{0\}$ and $\bar{\mu} \in S^*$ such that

$$0 \in \partial f(\bar{x})^T \lambda + \partial g(\bar{x})^T \mu, \bar{\mu}^T g(\bar{x}) = 0.$$

Thus $(\bar{x}, \bar{\lambda}, \bar{\mu}) \in F^D$. It follows from Theorem 5.1 that

$$f(\bar{x}) - f(y) \notin -\text{int}Q, \forall (y, \lambda, \mu) \in F^D,$$

that is,

$$f(y) - f(\bar{x}) \notin \text{int}Q, \forall (y, \lambda, \mu) \in F^D.$$

Therefore $(\bar{x}, \bar{\lambda}, \bar{\mu}) \in F_w^D$. This completes the proof.

Theorem 5.3. (Converse duality) Let $(\bar{y}, \bar{\lambda}, \bar{\mu}) \in F_w^D$. Assume that at least one of the following statements hold:

(i) **Assumption A** hold at $\bar{y}$;

(ii) (f, g) is KT-(Q, S)-pseudoinvex with respect to $\eta : R^n \times R^n \to R^n$ at $\bar{y}$.

Then $\bar{y} \in F^w$.

Proof. Let $(\bar{y}, \bar{\lambda}, \bar{\mu}) \in F_w^D$. Suppose to the contrary that $\bar{y} \notin F^w$. Then there exists $\hat{x} \in F$ such that

$$f(\hat{x}) - f(\bar{y}) \in -\text{int}Q. \tag{5.5}$$

If (i) holds, then

$$f(\hat{x}) - f(\bar{y}) - \zeta\eta(\hat{x}, \bar{y}) \in Q, \forall \zeta \in \partial f(\bar{y}),$$
$$g(\hat{x}) - g(\bar{y}) - \xi\eta(\hat{x}, \bar{y}) \in S, \forall \xi \in \partial g(\bar{y}).$$

Furthermore, we get

$$\bar{\lambda}^T(f(\hat{x}) - f(\bar{y})) - \bar{\lambda}^T\zeta\eta(\hat{x}, \bar{y}) \geq 0, \forall \zeta \in \partial f(\bar{y}),$$
$$\bar{\mu}^T(g(\hat{x}) - g(\bar{y})) - \bar{\mu}^T\xi\eta(\hat{x}, \bar{y}) \geq 0, \forall \xi \in \partial g(\bar{y}).$$

It follows from (5.2) and (5.5) that

$$\bar{\lambda}^T(f(\hat{x}) - f(\bar{y})) < 0, \forall \zeta \in \partial f(\bar{y}),$$
$$\bar{\mu}^T(g(\hat{x}) - g(\bar{y})) \leq 0, \forall \xi \in \partial g(\bar{y}).$$

Therefore,

$$\bar{\lambda}^T\zeta\eta(\hat{x}, \bar{y}) < 0, \bar{\mu}^T\xi\eta(\hat{x}, \bar{y}) \leq 0, \forall \zeta \in \partial f(\bar{y}), \xi \in \partial g(\bar{y}).$$

Moreover, one has

$$(\bar{\lambda}^T\zeta + \bar{\mu}^T\xi)\eta(\hat{x},\bar{y}) < 0, \forall \zeta \in \partial f(\bar{y}), \xi \in \partial g(\bar{y}),$$

which contradicts (5.1).

If (ii) holds, from (5.5),

$$\zeta\eta(\hat{x},\bar{y}) \in -\text{int}Q, \forall \zeta \in \partial f(\bar{y}) \tag{5.6}$$

and

$$\xi\eta(\hat{x},\bar{y}) \in -S, \forall \xi \in \partial g(\bar{y}). \tag{5.7}$$

Consequently, from (5.6) and (5.7) we have

$$\bar{\lambda}^T\zeta\eta(\hat{x},\bar{y}) < 0, \forall \zeta \in \partial f(\bar{y}) \tag{5.8}$$

and

$$\bar{\mu}^T\xi\eta(\hat{x},\bar{y}) \leq 0, \forall \xi \in \partial g(\bar{y}). \tag{5.9}$$

Therefore, it follows from (5.8) and (5.9) that

$$(\bar{\lambda}^T\zeta + \bar{\mu}^T\xi)\eta(\hat{x},\bar{y}) < 0, \forall \zeta \in \partial f(\bar{y}), \xi \in \partial g(\bar{y}).$$

This contradicts (5.1). This completes the proof.

6 Applications

In this section, we apply the results obtained above to study the existence of solutions for Hartman-Stampacchia weak vector quasi-variational inequalities (for short, (HSVQI)) and Hartman-Stampacchia nonlinear weak vector quasi-variational inequalities (for short, (HSNVQI)). Let F be nonempty set. The (HSVQI) and (HSNVQI) are defined as follows:

(HSVQI): find $\bar{x} \in F$ such that there exists $\bar{\zeta} \in \partial f(\bar{x})$,

$$\bar{\zeta}\eta(x, \bar{x}) \notin -\text{int}Q, \forall x \in F.$$

(HSNVQI): find $\bar{x} \in F$ such that there exists $\bar{\zeta} \in \partial f(\bar{x})$,

$$\bar{\zeta}\eta(x, \bar{x}) + f(x) - f(\bar{x}) \notin -\text{int}Q, \forall x \in F.$$

Denote the solutions sets of (HSVQI) and (HSNVQI) by S_{QI} and S_{NQI}, respectively.

Theorem 6.1. Let f be generalized Q-invex with respect to $\eta : R^n \times R^n \to R^n$ at $\bar{x} \in F$. Then $\bar{x} \in S_{QI}$ or $\bar{x} \in S_{NQI}$ implies that $\bar{x} \in F^w$.

Proof. If $\bar{x} \in S_{QI}$. Suppose to the contrary that $\bar{x} \notin F^w$. Then there exists $x^0 \in F$ such that

$$f(x^0) - f(\bar{x}) \in -\text{int}Q. \tag{6.1}$$

By the generalized Q-invexity of f with respect to η at $\bar{x}$, one has

$$f(x^0) - f(\bar{x}) - \zeta\eta(x^0, \bar{x}) \in -\text{int}Q, \forall \zeta \in \partial f(\bar{x}).$$

From (6.1), we have

$$\zeta\eta(x^0, \bar{x}) \in -\text{int}Q, \forall \zeta \in \partial f(\bar{x}),$$

which contradicts $\bar{x} \in S_{QI}$.

If $\bar{x} \in S_{NQI}$. Then there exists $\bar{\zeta} \in \partial f(\bar{x})$,

$$\bar{\zeta}\eta(x, \bar{x}) + f(x) - f(\bar{x}) \notin -\text{int}Q, \forall x \in F. \tag{6.2}$$

Since f is generalized Q-invex with respect to η at $\bar{x}$, we get

$$f(x) - f(\bar{x}) - \zeta\eta(x, \bar{x}) \in Q, \forall \zeta \in \partial f(\bar{x}).$$

Moreover,

$$2(f(x) - f(\bar{x})) - (f(x) - f(\bar{x}) + \zeta\eta(x, \bar{x})) \in Q, \forall \zeta \in \partial f(\bar{x}).$$

It follows from (6.2) that

$$f(x) - f(\bar{x}) \notin -\text{int}Q, \forall x \in F,$$

that is, $\bar{x} \in F^w$. This completes the proof.

Theorem 6.2. Let f be generalized $(-Q)$-invex with respect to $\eta : R^n \times R^n \to R^n$ at $\bar{x} \in F^w$. Then $\bar{x} \in S_{QI}$. Furthermore, assume that f is generalized Q-invex with respect to η at $\bar{x}$. Then $\bar{x} \in S_{NQI}$.

Proof. Let $\bar{x} \in F^w$. Then

$$f(x) - f(\bar{x}) \notin -\text{int}Q, \forall x \in F. \tag{6.3}$$

Since f is generalized $(-Q)$-invex with respect to η at $\bar{x}$, we obtain

$$\zeta\eta(x, \bar{x}) - (f(x) - f(\bar{x})) \in Q, \forall \zeta \in \partial f(\bar{x}).$$

Thus, from (6.3), we have

$$\zeta\eta(x, \bar{x}) \notin -\text{int}Q, \forall \zeta \in \partial f(\bar{x}), x \in F.$$

Moreover, there exists $\bar{\zeta} \in \partial f(\bar{x})$ such that

$$\bar{\zeta}\eta(x, \bar{x}) \notin -\text{int}Q, \forall x \in F, \tag{6.4}$$

Therefore, $\bar{x} \in S_{QI}$.

Since f is generalized Q-invex with respect to η at $\bar{x}$, one has

$$f(x) - f(\bar{x}) - \zeta\eta(x, \bar{x}) \in Q, \forall \zeta \in \partial f(\bar{x}).$$

Furthermore, we get

$$\zeta\eta(x, \bar{x}) + f(x) - f(\bar{x}) - 2\zeta\eta(x, \bar{x}) \in Q, \forall \zeta \in \partial f(\bar{x}).$$

Consequently, there exists $\bar{\zeta} \in \partial f(\bar{x})$, by (6.4),

$$\bar{\zeta}\eta(x, \bar{x}) + f(x) - f(\bar{x}) \notin -\text{int}Q, \forall x \in F.$$

That is, $\bar{x} \in S_{NQI}$. This completes the proof.

From Theorems 3.3 and 6.2, we have the following:

Corollary 6.1. Let f be generalized Q-invex and generalized $(-Q)$-invex and g be generalized S-invex with respect to the same $\eta : R^n \times R^n \to R^n$ at $\bar{x} \in F$. Assume that there exist $\lambda \in Q^* \setminus \{0\}$ and $\mu \in S^*$ such that

$$0 \in \partial f(\bar{x})^T \lambda + \partial g(\bar{x})^T \mu; \quad \mu^T g(\bar{x}) = 0.$$

Then $\bar{x} \in S_{QI}$ and $\bar{x} \in S_{NQI}$.

7 Concluding remarks

This paper investigates the optimality conditions for nonsmooth multiobjective optimization problems involving generalized invexity with cone constraints and applications. The obtained results extended and improved corresponding results of [1, 3, 7, 12, 13, 16, 17, 19, 20] to nonsmooth case. Further research, by applying the obtained results, one can study fractional programming, set-valued optimization and so on. For instance, we can apply the obtained Kuhn-Tucker necessary and sufficient conditions to study the optimality conditions and duality for nonsmooth multiobjective fractional programming problems with generalized invexity (see, for example, [15, 22]). We can also introduce the η-approximation method and modified objective function method (see, for example, [1, 2, 3, 7]) through the relationships between weakly efficient solutions of (MOP) and solutions of (HSVQI) and (HSNVQI).

References

[1] Antczak, T.: A new approach to multiobjective programming with a modified objective function, J. Global Optim., 27(4), pp485-495(2003)

[2] Antczak, T.: An η-approximation approach for nonlinear mathematical programming problems involving invex functions, Num. Funct. Anal. Optim. 25(5-6),pp423–438(2004)

[3] Antczak, T.: An η-approximation method in nonlinear vector optimization, Nonlinear Anal. 63(2), pp236–255(2005)

[4] Arana-Jiménez, M., Rufián-Lizana, A., Osuna-Gómez, R., and Ruiz-Garzón, G.: Pseudoinvexity, optimization conditions and efficiency in multiobjective problems; duality, Nonlinear Anal. 68(1), pp24-34(2008)

[5] Chen, G. Y.: Existence of solutions for a vector variational inequality: an extension of Hartman-Stampacchia theorem, J. Optim. Theory Appl. 74(3), pp445-456(1992)

[6] Chen, G. Y., Huang, X. X., and Yang, X. Q.: Vector Optimization: Set-valued and Variational Analysis, Lecture Notes in Economics and Mathematical Systems 541, Springer-Verlag, Berlin(2005)

[7] Chen, J. W., Cho, Y. J., Kim, J. K., and Li, J.: Multiobjective optimization problems with modified objective functions and cone constraints and applications, J. Global Optim.(In press)

[8] Chen, J. W., and Zou, Y. Z.: Existence of solutions of F-implicit variational inequality problems with extended projection operators, Acta Mathematica Sinica, Chinese Series, 53(2), pp375-384(2010)

[9] Clarke, F. H.: Optimization and Nonsmooth Analysis, Wiley-Interscience, New York(1983)

[10] Craven,B. D.: Control and Optimization, Chapman & Hall(1995)

[11] Craven, B. D., and Yang, X. Q.: A nonsmooth version of alterative theorem and nonsmooth multiobjective programming, Utilitas Math., 40, pp117-128(1991)

[12] Garzon, G. R., Gomez, R. O., and Lizana, A. R.: Relationships between vector variational-like inequality and optimization problems, Euro. J. Oper. Res. 157(1), pp113-119(2004)

[13] Guu, S. M., and Li, J.: Vector variational-like inequalities with generalized bifunctions defined on nonconvex sets, Nonlinear Anal.(TMA), 71(7-8), pp2847-2855(2009)

[14] Hanson, M. A.: On sufficiency of the Kuhn-Tucker conditions, J. Math. Anal. Appl., 80, pp545-550(1981)

[15] Kim, D. S.: Nonsmooth multiobjective fractional programming with generalized invexity, Taiwanese J. Math., 10(2), pp467-478(2006)

[16] Lee, G. M., Lee, B. S., and Chang, S. S.: On vector quasivariational inequalities, J. Math. Anal. Appl. 203, pp626-638(1996)

[17] Li, L., and Li, J.: Equivalence and existence of weak Pareto optima for multiobjective optimization problems with cone constraints, Appl. Math. Lett. 21(6), pp599-606(2008)

[18] Luc, D. T.: Theory of Vector Optimization, Lecture Notes in Economics and Mathematical Systems 319, Springer-Verlag, Berlin(1989)

[19] Mishra, S. K., and Wang, S. Y.: Vector variational-like inequalities and nonsmooth vector optimization problems, Nonlinear Anal. 64(9), pp1939-1945(2006)

[20] Suneja, S. K., Khurana, S., and Vani: Generalized nonsmooth invexity over cones in vector optimization, European J. Oper. Res. 186(1), pp28-40(2008)

[21] Yang, X. M., Yang, X. Q., and Teo, K. L.: Some remarks on the Minty vector variational inequality, J. Optim. Theory Appl. 121(1), pp193-201 (2004)

[22] Yang, X. M., Yang, X. Q., and Teo, K. L.: Duality and saddle point type optimality for generalized nonlinear fractional programming, J. Math. Anal. Appl., 289(1), pp100-109(2004)