LOWER BOUNDS FOR THE CHVÁTAL-GOMORY RANK IN THE 0/1 CUBE

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ABSTRACT. We revisit the method of Chvátal, Cook, and Hartmann to establish lower bounds on the Chvátal-Gomory rank and develop a simpler method. We provide new families of polytopes in the 0/1 cube with high rank and we describe a deterministic family achieving a rank of at least $(1+1/\epsilon)n-1>n$. Finally, we show how integrality gaps lead to lower bounds.

1. INTRODUCTION

The Chvátal-Gomory procedure (see e.g., [8, 9, 5]) is a well-known cutting-plane operator to derive the integral hull of a given polyhedron. More precisely, for $P \subseteq \mathbb{R}^n$ the Chvátal-Gomory closure is defined as

$$P' := \bigcap_{\substack{(c,\delta)\in\mathbb{Z}^n\times\mathbb{Q} \\ cx\leq\delta \text{ valid for } P}} cx \leq \lfloor \delta \rfloor .$$

It is well-known that P' is a polyhedron again (cf., e.g., [12]) if P is a rational polyhedron. Clearly, $\operatorname{conv}(P\cap\mathbb{Z}^n) =: P_I \subseteq P'$ and we can iterate the operator by setting $P^{(i+1)} := (P^{(i)})'$ with $P^{(1)} := P'$ and $P^{(0)} := P$ for consistency. The *(Chvátal-Gomory) rank of a polyhedron P* is then defined to be the smallest $i \in \mathbb{N}$ such that $P^{(i)} = P_I$ holds and we denote it by $\operatorname{rk}(P)$. The rank of a polyhedron P is always finite ([5, 11]) but can be arbitrarily large, even for $n=2$. If we confine ourselves however to polytopes $P \subseteq [0,1]^n$, the rank of P is bounded by a function of n. The first known bound was exponential in the dimension n and was subsequently reduced to $O(n^3\log(n))$ (cf. [2]) and later to $O(n^2\log(n))$ (cf. [7]). Rank bounds of a related closure, the Small Chvátal operator, have been investigated in [4]. On the other hand, the best-known lower bound so far is based on the existence (non-constructive) of a family of polytopes P_n with $\operatorname{rk}(P_n) \geq (1+\epsilon)n$, for $\epsilon \leq 3.12\cdot 10^{-6}$, leaving a large relative gap of $n\log(n)$.

The later result relies on a lower bound result for the fractional stable set polytope due to [6]. Let $G=(V,E)$ be a graph on n vertices and $\mathcal{K}$ be the family of all cliques of G. We denote by $\alpha(G)$ the maximum size of a stable set in G. The stable set polytope of G (denoted by $STAB(G)$) is the convex hull of (the characteristic vectors of) all stable sets in G. The fractional stable set polytope of G (denoted by $QSTAB(G)$) is a relaxation of $STAB(G)$ defined by the following inequalities:

$$\begin{aligned} x(K) &\leq 1, && \forall K \in \mathcal{K} \\ x_v &\geq 0, && \forall v \in V \end{aligned}$$

Chvátal, Cook, and Hartmann established the following bound on the rank of this polytope: ($e := (1,\ldots,1)$ denotes the all-one vector)

Date: February 4, 2011.

Key words and phrases. Chvátal-Gomory procedure, rank lower bounds, cutting-plane procedures.

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Lemma 1.1. [6, Proof of Lemma 3.1] *Let $k < s$ be positive integers and let G be a graph with n vertices such that every subgraph of G on s vertices is k-colorable. Let P be a polyhedron that contains $STAB(G)$ and the point $\frac{1}{k}e$. Then $rk(P) \geq \frac{s}{k} \ln \frac{n}{k\alpha(G)}$.*

This result is then applied to a certain class of random graphs. More precisely, with $\mathfrak{e}$ being the Euler constant, Erdős proved that there exists $\delta > 0$ and a family of graphs $\mathcal{G}$ with arbitrarily many vertices such that for all $G \in \mathcal{G}$ we have $\alpha(G) < \frac{n}{3\mathfrak{e}}$ and every subgraph of G with at most δn vertices is 3-colorable (see e.g., [1]). Applying Lemma 1.1 to this family yields:

Corollary 1.2. *There exists $\delta > 0$ and a family of graphs $\mathcal{G}$ such that for all $n_0 > \frac{1}{\delta}$, there exists $G \in \mathcal{G}$ with $n \geq n_0$ vertices and any polytope P containing $STAB(G)$ and $\frac{1}{k}e$ satisfies $rk(P) \geq \frac{\lfloor \delta n \rfloor}{3} \geq \frac{\delta n}{6}$.*

Let $A_n \subseteq [0,1]^n$ be the polytope defined as

$$A_n := \{x \in [0,1]^n : \sum_{i \in I} x_i + \sum_{i \notin I} (1 - x_i) \geq \frac{1}{2}\}.$$

In [7] the authors considered the family of polytopes $P_G = \text{conv}\left(QSTAB(G) \cup A_n\right)$ for all $G \in \mathcal{G}$ with n vertices. Using the fact that $\frac{1}{2}e \in A_n^{(n-1)}$ and thus $\frac{1}{3}e \in P_G$ Lemma 1.1 can be applied to $P_G^{(n-1)}$. This yields $\text{rk}(P_G) \geq \frac{\delta}{6}n + n - 1$. The linear factor however is very small; a simple calculation shows that $\frac{\delta}{6} \leq 3.12 \cdot 10^{-6}$ (cf. [1, p.136]). Beyond the existence of the family of graphs provided by Erdős, this result, at its core, relies on the following lemma to establish lower bounds. Let $[n]$ denote the set $\{1, \ldots, n\}$ and $[n]_0$ denote the set $\{0, \ldots, n\}$ for $n \in \mathbb{N}$.

Lemma 1.3. [6, Lemma 2.1] *Let P be a rational polyhedron in $\mathbb{R}^n$. Further let u and v be points in $\mathbb{R}^n$ and $m_1, m_2, \ldots, m_d$ be positive numbers. Write $x^{(j)} = u - \sum_{i=1}^{j} \frac{1}{m_i} v$ for all $j \in [d]_0$. If $u \in P$ and if, for all $j \in [d]$, every inequality $ax \leq b$ valid of P_I with $a \in \mathbb{Z}^n$ and $av < m_j$ satisfies $ax^{(j)} \leq b$, then $x^{(j)} \in P^{(j)}$ for all $j \in [d]_0$.*

While this Lemma is very powerful, it is rather difficult to apply it without, *a priori*, having a precise idea of the sequence of points ones wants to consider. Furthermore, it does not provide an immediate lower bound estimate for the rank. This inconvenience motivated us to introduce a reformulation that is slightly more restricted but has certain advantages: we trade generality for simplicity. In order to apply it, no further knowledge about candidate sequences of points is needed and we readily obtain a lower bound on the rank. Furthermore, the lemma can be weakened slightly more to turn any (relative) integrality gap into a lower bound estimate for the Chvátal-Gomory rank.

The outline of the article is as follows. We introduce our new lemma in Section 2 and discuss its application to known results. In Section 3 we exploit our technique to build a deterministic family of polytopes whose rank is at least $(1 + 1/\mathfrak{e})n - 1$ and thus improve on the result given in [7]. Finally in Section 4 we show how our result can be used to estimate the rank of a polytope by examining its integrality gap.

2. A simple technique for establishing lower bounds

We will now establish a new lemma for proving lower bounds on the Chvátal-Gomory rank. It is inspired by the techniques established in [6], however we shifted the focus towards the intrinsic geometric progression in order to facilitate its application. Let $P \subseteq [0,1]^n$ be a polytope and $cx \leq \delta$ with $(c, \delta) \in \mathbb{Z}^{n+1}$ be valid for P_I. Then the *depth* of $cx \leq \delta$ (with respect to P) is the minimum number of applications ℓ of the Chvátal-Gomory procedure so that $cx \leq \delta$ is valid for $P^{(\ell)}$. The maximal depth of all facets of P_I equals the rank of P. We call a polytope $P \subseteq [0,1]^n$ *monotone* (or equivalently: *of* *anti-blocking type*) if whenever $x \in P$ and $y \in [0,1]^n$ with $y \leq x$ coordinate-wise, then $y \in P$ holds.

Lemma 2.1. *Let $P \subseteq [0,1]^n$ be a polytope, $Q_I \subseteq P_I$ be monotone and $cx \leq d$ be valid for P_I. Further, let $x^* \in P$ such that $cx^* > d$ and define $\delta := \min_{\{a \in \mathbb{N}^n : ax^* > \max_{x \in Q_I} ax\}} \left(\max_{x \in Q_I} ax\right)$. If $\delta > 0$ then the depth of $cx \leq d$ is at least*

$$\kappa = \left\lceil \frac{\ln(\frac{cx^*}{d})}{\ln((\delta+1)/\delta)} \right\rceil \geq \left\lceil \ln\left(\frac{cx^*}{d}\right) \cdot \delta \right\rceil.$$

Moreover if $x^* \leq \frac{1}{k} e$ for some $k \in \mathbb{N}$, then

$$\kappa \geq \left\lceil \ln\left(\frac{cx^*}{d}\right) \cdot \frac{1}{k} \min_{\substack{a \in \{0,1\}^n: \\ a \notin k \cdot Q_I}} (ae - 1) \right\rceil.$$

where $k \cdot Q_I$ denotes the Minkowski sum of k copies of Q_I.

Proof. Let $x_0^* = x^*$ and $x_{l+1}^* = \lambda x_l^*$ for all $l \in \mathbb{N}_+$ with $\lambda = \frac{\delta}{1+\delta}$. We prove first by induction that $x_l^* \in P^{(l)}$ for all $l \geq 0$. Clearly, the hypothesis holds for $l = 0$. Thus let $l \geq 0$ and $ax \leq b$ be a valid inequality for $P^{(l)}$ with $a \in \mathbb{Z}^n$ and let us consider the corresponding inequality $ax \leq \lfloor b \rfloor$, valid for $P^{(l+1)}$. Let a^+ be the restriction of a to its positive coefficients. Observe that since Q_I is monotone it holds $\max_{x \in Q_I} ax = \max_{x \in Q_I} a^+ x$. Suppose first that a is such that $a^+ x^* \leq \max_{x \in Q_I} ax$. Then $\lfloor b \rfloor \geq \max_{x \in Q_I} ax \geq a^+ x^* \geq a^+ x_{l+1}^* \geq a x_{l+1}^*$ and thus $x_{l+1}^* \in P^{(l+1)}$. Now suppose that a is such that $a^+ x^* > \max_{x \in Q_I} ax = \max_{x \in Q_I} a^+ x$. By definition $\max_{x \in Q_I} a^+ x \geq \delta$ and thus $\lfloor b \rfloor \geq \max_{x \in Q_I} ax = \max_{x \in Q_I} a^+ x \geq \delta$. Then $a x_{l+1}^* = \lambda a x_l^* \leq \lambda b + (1-\lambda)(\lfloor b \rfloor - \delta) \leq \lambda(\lfloor b \rfloor + 1) + (1-\lambda)(\lfloor b \rfloor - \delta) = \lfloor b \rfloor + \lambda - (1-\lambda)\delta = \lfloor b \rfloor$. Again we obtain $x_{l+1}^* \in P^{(l+1)}$.

Next we show that while $l \leq \frac{\ln(\frac{cx^*}{d})}{\ln(1/\lambda)}$ we have $x_l^* \notin P_I$. To this end it suffices to observe that since $cx_l^* = \lambda^l cx^*$ we obtain that $cx_l^* > d$ if and only if $\lambda^l cx^* > d$. We obtain κ as claimed and further we have $\kappa \geq \left\lceil \ln(\frac{cx^*}{d}) \cdot \delta \right\rceil$ since $\ln(1/\lambda) \leq \frac{1-\lambda}{\lambda} = 1/\delta$ and the first part of the result follows.

It remains to prove the second statement. Let $k \in \mathbb{N}$ be arbitrary. For $a \in \mathbb{N}^n$ let $\operatorname{supp}(a) \in \{0,1\}^n$ denote the characteristic vector of the support. We claim that $ae/k > \max_{x \in Q_I} ax$ implies that $\operatorname{supp}(a) \notin k \cdot Q_I$. For contradiction suppose that $\operatorname{supp}(a) \in k \cdot Q_I$. Then there exist $x_1, \ldots, x_k \in Q_I$ such that $\operatorname{supp}(a) = \sum_{i \in [k]} x_i$. Thus $ae = \sum_{i \in [k]} a x_i \leq k \cdot \max_{x \in Q_I} ax$ and so $ae/k \leq \max_{x \in Q_I} ax$; a contradiction. Therefore we have $\{a \in \mathbb{N}^n : ae/k > \max_{x \in Q_I} ax\} \subseteq \{a \in \mathbb{N}^n : \operatorname{supp}(a) \notin k \cdot Q_I\}$. If $x^* \leq \frac{1}{k} e$ for some $k \in \mathbb{N}$, then we have

$$\begin{aligned} \delta &\geq \min_{\substack{a \in \mathbb{N}^n: \\ \frac{1}{k} ae > \max_{x \in Q_I} ax}} \left(\max_{x \in Q_I} ax\right) \geq \min_{\substack{a \in \mathbb{N}^n: \\ \operatorname{supp}(a) \notin k \cdot Q_I}} \left(\max_{x \in Q_I} ax\right) \\ &\geq \min_{\substack{a \in \mathbb{N}^n: \\ \operatorname{supp}(a) \notin k \cdot Q_I}} \left(\max_{x \in Q_I} \operatorname{supp}(a) x\right) = \min_{\substack{a \in \{0,1\}^n: \\ a \notin k \cdot Q_I}} \left(\max_{x \in Q_I} ax\right). \end{aligned}$$

Observe that we can assume that $a \notin k \cdot Q_I$ and $a - e_i \in k \cdot Q_I$ for all i with $a_i = 1$; otherwise we could replace a with $a - e_i$. Therefore $\delta \geq \frac{1}{k} \min_{a \in \{0,1\}^n : a \notin k \cdot Q_I} (ae - 1)$. $\square$

We now demonstrate the strength of Lemma 2.1 by illustrating its application to the classical result of [5] for the rank of clique inequalities and by providing an alternative proof of Lemma 1.1. Let $\log(.)$ denote the logarithm to the basis 2.

Lemma 2.2. *Let K_n be a clique on n vertices. Let $P = \{x \in [0,1]^n : x_i + x_j \leq 1, \forall i, j \in [n]\}$. Then $rk(P) \geq \lceil \log(\frac{n}{2}) \rceil$.*

Proof. We apply Lemma 2.1 with $Q_I = P_I$ and $x^* = \frac{1}{2}e$ and we consider the inequality $ex \leq 1$. Since $e_i \in P_I$ for all $i \in [n]$ we have $ae \geq 2$ for all $a \notin P_I$. The result follows. □

Lemma 1.1. *Let $k < s$ be positive integers and let G be a graph with n vertices such that every subgraph of G on s vertices is k-colorable. Let P be a polyhedron that contains $STAB(G)$ and the point $\frac{1}{k}e$. Then $rk(P) \geq \frac{s}{k} \ln \frac{n}{k\alpha(G)}$.*

Proof. We apply Lemma 2.1 with $Q_I = P_I$ and $x^* = \frac{1}{k}e$ and we consider the inequality $ex \leq \alpha(G)$ that is valid for P_I. Since every subgraph of size s of G is k-colorable we have that $a \notin k \cdot P_I$ only if $ae > s$. The result follows. □

3. Constructing a better lower bound

As we have seen, we can use Lemma 2.1 to prove bounds of the order of ϵn (with $\epsilon \leq 3.1210^{-6}$) for the rank of polytopes in $[0,1]^n$. We will now show that we can do better by providing a new family of polytopes whose rank asymptotically equals to $n/\mathfrak{e}$.

Lemma 3.1. *Let $P = conv(\{x \in [0,1]^n : ex \leq d\} \cup \{x^*\})$ for $d \in [n]$ and $x^* = \frac{m-1}{m}e$ for $m \in \mathbb{N}_*$. Then $rk(P) \geq \ln\left(\frac{(m-1)\cdot n}{m \cdot d}\right) \cdot d$.*

Proof. It is easy to see that $P_I = \{x \in [0,1]^n : ex \leq d\}$ holds. We apply Lemma 2.1 with $Q_I = P_I$ to the inequality $ex \leq d$ and choose $k = 1$. As $\min_{a \in \{0,1\}^n : a \notin P_I} \sum_i a_i - 1 \geq d$. The result follows. □

The rank of P in Lemma 3.1, provided that m tends to ∞, is maximized by choosing d close to $n/\mathfrak{e}$. We obtain the following corollary.

Corollary 3.2. *For any $\epsilon > 0$ and any $n_0 \in \mathbb{N}_+$, there exists $n \geq n_0 \in \mathbb{N}_+$ and a polytope $P \subseteq [0,1]^n$ with $rk(P) \geq n/\mathfrak{e} - \epsilon$.*

Observe that our construction is deterministic as compared to the construction in [6] which relies on a random graph. Moreover, the split rank of P in Corollary 3.2 is 1 whereas the Chvátal-Gomory rank is $\Omega(n)$. Furthermore P_I is given by a uniform matroid and we can thus optimize over P_I in polynomial time. Last but not least, P is *almost integral*, i.e., $P \cap \{x_i = l\} = P_I \cap \{x_i = l\}$ for all $(i,l) \in [n] \times \{0,1\}$ and so we can optimize over P_I by optimizing over P with any arbitrary coordinate first being fixed to 0, and then to 1. The optimum is obtained as the min/max of the two.

It is worthwhile to note that the polytopes in Corollary 3.2 are not monotone. In fact, it can be shown that P can be described by $4n$ inequalities (see [3]).

Remark 3.3. Let $P = \operatorname{conv}(\{x \in [0,1]^n : ex \leq d\} \cup \{\lambda e\}) \subseteq [0,1]^n$ with $d \in [n]$ and $\lambda \in [\frac{d}{n}, 1)$ be defined as in Lemma 3.1. Then P is given by the following inequalities:

$$\begin{aligned} x_i &\geq 0 && \forall\, i \in [n] \\ x_i &\leq 1 && \forall\, i \in [n] \\ ex - (n - d/\lambda)x_i &\leq d && \forall\, i \in [n] \\ (1-\lambda)ex - (d - \lambda n)x_i &\leq \lambda(n-d) && \forall\, i \in [n] \end{aligned}$$

One might wonder if the lower bound provided by Lemma 2.1 when applied to our construction is a good estimate of the true rank. We use the upper bounds provided in [6, Theorem 9.1] to address this question. For $c \in \mathbb{Z}_+^n$ let $||c||_1 := ce$ be the 1-norm of c.

Lemma 3.4. [6, Theorem 9.1] *Let $P \subseteq [0,1]^n$ be a monotone polytope and let $cx \leq \delta$ be valid for P and further let $\tau = \max_{x \in P_I} cx$. If $||c||_1 \geq 2\tau + 1$ then an upper bound on the depth of $cx \leq \tau$ over P is given by*

$$\tau + 1 + \left\lceil (2\tau + 1) \ln \frac{||c||_1}{2\tau + 1} \right\rceil .$$

Since the results of [6] only applies to monotone polytopes, we consider monotone polytopes containing our family. Instead of considering conv($\{x \in [0,1]^n : ex \leq d\} \cup \{x^*\}$), we consider conv$(\{x \in [0,1]^n : ex \leq d\} \cup \{x \in [0,1]^n : x \leq x^*\})$. In this case, as $\{x \in [0,1]^n : ex \leq d\} = P_I$ and both P_I and $\{x \in [0,1]^n : x \leq x^*\}$ are monotone, it readily follows that conv$(\{x \in [0,1]^n : ex \leq d\} \cup \{x \in [0,1]^n : x \leq x^*\})$ is monotone. Applying Lemma 3.4 to this family of polytopes we obtain that rk$(P) \leq \frac{3-\ln(4)}{\mathfrak{e}} n \approx 0.594 \cdot n$. In comparison to this, our lower bound is rk$(P) \geq \frac{1}{\mathfrak{e}} \cdot n \approx 0.368 \cdot n$ leading to an overall gap of $3 - \ln(4)$. In this sense the provided lower bound is rather tight for our construction.

We are now ready to slightly improve the lower bound result of [7].

Theorem 3.5. *For any $\epsilon > 0$ and any $n_0 \in \mathbb{N}_+$, there exists $n \geq n_0 \in \mathbb{N}$ and a polytope $P \subseteq [0,1]^n$ with $rk(P) \geq (1 + 1/\mathfrak{e})n - 1 - \epsilon$.*

Proof. Let Q be the polytope defined in Corollary 3.2 with $m = 2$. Define $P :=$ conv$\left(Q \cup A_n\right)$ and note that $P_I = Q_I$ as $(A_n)_I = \emptyset$ (and no 0/1 point in the cube can be expressed as a convex combination of other points from the cube). It is well-known that $\frac{1}{2}e \in A_n^{(n-1)}$ and thus $\frac{1}{2}e \in P^{(n-1)}$. We therefore obtain that $Q \subseteq P^{(n-1)}$ and by Corollary 3.2 we know that Q has rank of at least $\frac{n}{\mathfrak{e}} - \epsilon$. Together with rk$(Q) \leq$ rk$(P^{(n-1)})$ we derive that the rank of P is at least $n - 1 + n/\mathfrak{e} - \epsilon = (1 + 1/\mathfrak{e})n - 1 - \epsilon$. ☐

We would like to close this section by pointing out that, independently, [10] have recently shown that a different family of polytopes stemming from matroid matching problems can achieve rank arbitrarily close to $n/2\mathfrak{e}$. We use our Lemma to provide an alternative proof of their result. Clearly their result can be extended in the same spirit as Theorem 3.5 to build a family of polytopes achieving rank arbitrarily close to $(1 + 1/2\mathfrak{e})n - 1$.

Corollary 3.6. *Let $P := \{y \in [0,1]^n : \sum_{i \in T} y_i \leq \frac{1}{2}(t + |T|), \forall T \subseteq [n], |T| > t\}$. Then $rk(P) \geq \ln(\frac{n/2}{t}) \cdot t$.*

Proof. We apply Lemma 2.1 with $Q_I = P_I$ and $x^* = \frac{1}{2}e$ and we consider the valid inequality $ex \leq t$ and choose $k = 1$. Together with $\min_{a \in \{0,1\}^n : a \notin P_I} (ae - 1) \geq t$. The result follows. ☐

4. Estimating rank from integrality gaps

We conclude by explaining how we can use Lemma 2.1 to establish lower bounds on the Chvátal-Gomory rank by examining the (relative) integrality gap of a polyhedral relaxation. We say that a polytope $P \subseteq [0,1]^n$ has *integrality gap* (of at least) k if there exists $c \in \mathbb{Z}_+^n$ such that

$$\max_{x \in P} cx / \max_{x \in P_I} cx \geq k.$$

Note that we consider only non-negative vectors c here; otherwise the integrality gap is not well defined. We will assume that $P \subseteq [0,1]^n$ contains the vectors e_i for all $i \in [n]$; in case of monotone polytopes the relaxation is weak otherwise and we can immediately round the particular coordinate, i.e., we have $x_i \leq \lfloor \epsilon \rfloor$ for $\epsilon < 1$. We can establish the following result:

Theorem 4.1. *Let $P \subseteq [0,1]^n$ be a polytope with $0 \in P$ and $e_i \in P$ for all $i \in [n]$. Further let the integrality gap of P be k. Then*

$$rk(P) \geq \log(k).$$

Proof. We apply Lemma 2.1 with $Q_I = \{x \in [0,1]^n : \sum_{i \in [n]} x_i \leq 1\}$ and $x^* = \frac{1}{2}e$ and we consider a valid inequality $cx \leq d$ maximizing the integrality gap. Together with $\min_{a \in \{0,1\}^n : a \notin Q_I} (ae - 1) \geq 1$ the result follows. □

We would also like to point out that the above bound is rather conservative as we assume the worst-case progression in every round. Nonetheless, whenever the integrality gap is non-constant Theorem 4.1 establishes a non-constant rank. Also note that when $c \not\geq 0$ we can apply coordinate flips. In this case however the condition $e_i \in P$ should apply to the flipped polytope.

5. Acknowledgements

We would like to thank Alexander Martin for the insightful discussions. We would also like to thank Michael Joswig for providing us with an explicit inequality description of the polytope defined in Lemma 3.1 (cf. Remark 3.3) and Santanu Dey for pointing us to [10].

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