RANDOM HALF-INTEGRAL POLYTOPES

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Abstract. We show that half-integral polytopes obtained as the convex hull of a random set of half-integral points of the 0/1 cube have rank as high as $\Omega(\log n/\log\log n)$ with positive probability — even if the size of the set relative to the total number of half-integral points of the cube tends to 0. The high rank is due to certain obstructions. We determine the exact threshold number, when these obstructions cease to exist.

1. Introduction

Given a polytope P and a cutting-plane procedure C, the rank of P with respect to C is the minimum number of rounds (equivalently, applications of C) needed in order to compute the integral hull of P. Whereas the rank problem has been studied for various polytopes in a deterministic setting, in this note we will show that the rank of random polytopes can be high (e.g., $\Omega(\log n/\log\log n)$ where n is the dimension of P) although the polytopes have only a relatively small number of vertices. In order to disentangle this effect from a possibly high facet complexity of the integral hull $P_I \coloneqq \operatorname{conv}(P \cap \mathbb{Z}^n)$ of P, we confine ourselves to polytopes P with empty integral hull.

To simultaneously study the problem for large classes of cutting-plane procedures, we use the abstract model of cutting-plane procedures introduced in Pokutta and Schulz [2010]. This model contains well-known procedures such as the Gomory-Chvátal procedure (cf., Chvátal [1973], Gomory [1958, 1960, 1963]), the lift-and-project operators (cf., Balas et al. [1993]), in particular the Lovász-Schrijver operators (cf., Lovász and Schrijver [1991]), and the split cut operator (cf., Cook et al. [1990]) and characterizes when a cutting-plane procedure has high rank. A sufficient condition is the existence of obstructing sub-polytopes that have high rank with respect to any cutting-plane procedure. We then use the probabilistic method (see e.g., Erdős [1959], Erdős and Rényi [1959], or Alon and Spencer [2000]) to infer the existence of an obstructing sub-polytope.

More precisely, we consider polytopes given as the convex hull of a random subset S of the half-integral points (i.e., with coordinates $0, 1/2, 1$) of the n-dimensional 0/1 cube except the vertices. Roughly speaking, our main result (Theorem 3.1) is that for any $k < \log n$, the rank of the convex hull of S is $\Omega(k/\log k)$ with probability at least 1/2 whenever

$$\frac{k2^k}{n-k} \log \frac{3^n - 2^n}{|S|} \leq 2,$$

as S will contain an obstruction set F_k ensuring the high rank. Here the 2 on the right-hand side is the exact threshold number: for larger numbers S will almost never contain an F_k. This implies, e.g., that whenever $\log \frac{3^n-2^n}{|S|} \sim \log n$ we have $\operatorname{rk}(\operatorname{conv}(S)) \in \Omega(\log n/\log\log n)$, see Corollary 3.2. In particular, although the relative number of points compared to the total of points $3^n - 2^n$ tends to 0, the rank of the resulting polytopes can be quite high.

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2. Preliminaries

Throughout this note, we consider polytopes $P \subseteq [0,1]^n$ with $P_I = \emptyset$. Let log denote the logarithm to the basis of 2 and ln denote the natural logarithm. For convenience we define $[n] := \{1, \dots, n\}$ for $n \in \mathbb{N}$.

2.1. **Cutting plane procedures.** We will now recall a few facts about strongly admissible cutting-plane procedure as defined in Pokutta and Schulz [2010]. Note that we consider *strongly* admissible cutting-plane procedures, i.e., admissible procedures additionally satisfying property (5) here. This however does not change the results of Pokutta and Schulz [2010]; in fact we use the lower bounds, which might only become even stronger by further restricting the model.

A cutting-plane procedure is an operator C that maps a polytope $P \subseteq [0,1]^n$ to a polytope $\mathrm{C}(P) \subseteq [0,1]^n$, which we call the C-*closure* of P. A linear inequality valid for $\mathrm{C}(P)$ is called a C-*cut*. For a k-face F of $[0,1]^n$, let $\varphi_F \colon \mathbb{R}^n \to \mathbb{R}^k$ denote the canonical projection to the subspace spanned by F identified with $\mathbb{R}^k$ via the canonical subbasis, i.e. it is merely dropping coordinates.

Definition 2.1. A cutting-plane procedure C is *strongly admissible* if it satisfies the following conditions for any polytope $P \subseteq [0,1]^n$:

(1) *Approximation of P_I:* $P_I \subseteq \mathrm{C}(P) \subseteq P$.
(2) *Monotonicity:* If $P \subseteq Q$, then $\mathrm{C}(P) \subseteq \mathrm{C}(Q)$ for all polytopes $Q \subseteq [0,1]^n$.
(3) *Homogeneity:* $\mathrm{C}(F \cap P) = F \cap \mathrm{C}(P)$ for all faces F of $[0,1]^n$.
(4) *Coordinate rounding:* If $x_i \leq \epsilon < 1$ (or $x_i \geq \epsilon > 0$) is valid for P, then $x_i \leq 0$ (or $x_i \geq 1$) is valid for $\mathrm{C}(P)$.
(5) *Substitution independence:* $\varphi_F(\mathrm{C}(P \cap F)) = \mathrm{C}(\varphi_F(P \cap F))$ for all faces F of $[0,1]^n$.
(6) *Commuting with coordinate flips:* Let $\tau_i \colon [0,1]^n \to [0,1]^n$ with $(x_1, \dots, x_n) \mapsto (x_1, \dots, x_{i-1}, 1-x_i, x_{i+1}, \dots, x_n)$ be a coordinate flip. Then $\tau_i(\mathrm{C}(P)) = \mathrm{C}(\tau_i(P))$.
(7) *Short verification:* Let $P = \{x \in [0,1]^n \mid a_i x \leq b_i, i \in [m]\}$ with $a_i \in \mathbb{Z}^n$ and $b_i \in \mathbb{Z}$. Then there exists a polynomial p such that for any inequality $cx \leq \delta$ valid for $\mathrm{C}(P)$ there is a set $I \subseteq [m]$ with $|I| \leq p(n)$ such that $cx \leq \delta$ is valid for $\mathrm{C}(\{x \in [0,1]^n : a_i x \leq b_i, i \in I\})$.

We can apply a cutting-plane operator iteratively and define $\mathrm{C}^{(i+1)}(P) := \mathrm{C}(\mathrm{C}^{(i)}(P))$. Obviously, $P_I \subseteq \mathrm{C}^{(i+1)}(P) \subseteq \mathrm{C}^{(i)}(P) \subseteq \cdots \subseteq \mathrm{C}^{(1)}(P) \subseteq P$; we put $\mathrm{C}^{(1)}(P) := \mathrm{C}(P)$ and $\mathrm{C}^{(0)}(P) := P$, for consistency. The *rank of P with respect to* C is defined as $\mathrm{rk}_{\mathrm{C}}(P) := \min\{k \in \mathbb{N} \mid P_I = \mathrm{C}^{(k)}(P)\}$; we drop the index C if it is clear from the context. In general, it is not immediate that there exists $k \in \mathbb{N}$ such that $P_I = \mathrm{C}^{(k)}(P)$. For polytopes $P \subseteq [0,1]^n$ with $P_I = \emptyset$ this follows from Definition 2.1 though as shown in Pokutta and Schulz [2010]:

Lemma 2.2. *[Pokutta and Schulz, 2010, Theorem 4] Let* C *be a strongly admissible cutting-plane procedure and let $P \subseteq [0,1]^n$ be a polytope with $P_I = \emptyset$. Then* $\mathrm{rk}(P) \leq n$.

We will later use the following polytope in our construction of random half-integral polytopes. Let A_n be the polytope

$$A_n := \left\{ x \in [0,1]^n \,\middle|\, \sum_{i \in I} x_i + \sum_{i \in [n] \setminus I} (1 - x_i) \geq \frac{1}{2} \quad \forall I \subseteq [n] \right\},$$

with $(A_n)_I = \emptyset$. It is rather easy (see e.g., Bockmayr et al. [1999]) to see that $A_n = \mathrm{conv}(F_n)$ with

$$F_n := \left\{ (a_1, \dots, a_n) \in \left\{0, \frac{1}{2}, 1\right\}^n \,\middle|\, a_i = \frac{1}{2} \text{ for exactly one entry } i \in [n] \right\}.$$

Via affine equivalence, we can also define A_k and F_k for any k-dimensional face of the cube with $k \in [n]$. The relevance of the polytopes A_n stems from the fact that they have high rank with respect to any strongly admissible cutting-plane procedure:

Lemma 2.3. *[Pokutta and Schulz, 2010, Corollary 23] For every strongly admissible cutting-plane procedure* $\mathrm{rk}(A_n) \in \Omega(n/\log n)$.

For specific, well-known cutting-plane procedures, A_n realizes the maximal rank for 0/1 polytopes (without integral points in the case of the Gomory-Chvátal procedure).

Lemma 2.4. *[Chvátal et al., 1989, Cornuéjols, 2008, Cook and Dash, 2001] The Gomory-Chvátal rank, the Lovász-Schrijver ranks, and the split rank of* A_n *are equal to* n.

2.2. **Special probabilities.** Let $\mathbb{P}[X]$ denote the probability of an event X. Our later argumentation will heavily rely on the Bonferroni inequalities, which are at the core of our proofs (see e.g., Alon and Spencer [2000]). These inequalities can be understood as an approximation of the inclusion-exclusion principle (see e.g., Mitzenmacher and Upfal [2005]) for estimating probabilities from above and below. More precisely:

Lemma 2.5 (Bonferroni inequalities)**.** *Let* $\{B_i\}_{i\in[l]}$ *be a countable sequence of events with* $l \in \mathbb{N}$. *Then for* $m \in \mathbb{N}$ *with* $2m \le l$ *the following hold:*

$$\sum_{j\in[2m]} (-1)^{j-1} \sum_{1\le i_1<i_2<\cdots<i_j\le l} \mathbb{P}\left[B_{i_1} \cap \cdots \cap B_{i_j}\right] \le \mathbb{P}\left[\bigcup_{i\in[l]} B_i\right];$$

$$\sum_{j\in[2m-1]} (-1)^{j-1} \sum_{1\le i_1<i_2<\cdots<i_j\le l} \mathbb{P}\left[B_{i_1} \cap \cdots \cap B_{i_j}\right] \ge \mathbb{P}\left[\bigcup_{i\in[l]} B_i\right].$$

In the following we will choose $m = 1$ and the B_i to be the events of having the F_k of a specific face in S, the set of random half-integral points which is chosen uniformly.

Further, let $a_n \sim b_n$ denote asymptotic equality i.e., $\lim_{n\to\infty} a_n/b_n = 1$. We shall need:

Lemma 2.6. *Let* $(a_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ *and* $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ *be sequences such that* $\lim_{n\to\infty} a_n = +\infty$ *and* $\lim_{n\to\infty} \frac{x_n^2}{a_n} = 0$. *Then:*

$$a_n(a_n-1)\cdots(a_n-x_n+1) \sim a_n^{x_n}.$$

Proof. Using $\ln(1+x) \sim x$ as $x \to 0$, we obtain

$$\ln \frac{a_n(a_n-1)\cdots(a_n-x_n+1)}{a_n^{x_n}} = \sum_{i=1}^{x_n-1} \ln\left(1-\frac{i}{a_n}\right) \sim -\sum_{i=1}^{x_n-1} \frac{i}{a_n} = -\frac{x_n(x_n-1)/2}{a_n} \to 0.$$

Note that the estimations of the summands in the middle are uniform as $\lim_{n\to\infty} \frac{x_n}{a_n} = 0$. ☐

We will now establish the probability that a random set contains an obstruction set using the hypergeometric distribution.

Lemma 2.7. *Let* S *be a random subset of fixed size* $|S|$ *of an ambient set of size* M *with uniform distribution and further let* H *be a non-random subset of the ambient set. Then:*

$$\mathbb{P}[H \subseteq S] \sim \left(\frac{|S|}{M}\right)^{|H|}$$

provided that $\lim |S| = +\infty$ *and* $\lim \frac{|H|^2}{|S|} = 0$. *Similarly,*

$$\mathbb{P}[H \cap S = \emptyset] \sim \left(1-\frac{|H|}{M}\right)^{|S|}$$

whenever $\lim |S| = +\infty$ *and* $\lim \frac{|S|^2}{M-|H|} = 0$.

Proof. Clearly,

$$\mathbb{P}\left[H \subseteq S\right] = \frac{\binom{M-|H|}{|S|-|H|}}{\binom{M}{|S|}} = \frac{|S|\left(|S|-1\right)\cdots\left(|S|-|H|+1\right)}{M\left(M-1\right)\cdots\left(M-|H|+1\right)}$$

and

$$\mathbb{P}\left[H \cap S = \emptyset\right] = \frac{\binom{M-|H|}{|S|}}{\binom{M}{|S|}} = \frac{\left(M-|H|\right)\left(M-|H|-1\right)\cdots\left(M-|H|-|S|+1\right)}{M\left(M-1\right)\cdots\left(M-|S|+1\right)}.$$

The limit conditions ensure that Lemma 2.6 applies, and the assertions follow. ☐

With $\lambda \coloneqq |F_k| = k2^{k-1} = c(n-k)/2$ we obtain the following corollary:

Corollary 2.8. *Let $F_k^{(1)}$, $F_k^{(2)}$,…,$F_k^{(l)}$ be pairwise disjoint F_k of some faces with l not depending on n, and let $\lambda^2/|S|$ tend to 0. Then*

$$\mathbb{P}\left[F_k^{(1)} \subseteq S \wedge F_k^{(2)} \subseteq S \wedge \cdots \wedge F_k^{(l)} \subseteq S\right] \sim \left(\frac{|S|}{M}\right)^{l\lambda}$$

Proof. It is sufficient to compute $\mathbb{P}\left[H \subseteq S\right]$ with $H = \dot{\bigcup}_{i\in[l]} F_k^{(i)}$ and $|H| = l\lambda$. Therefore Lemma 2.7 applies and the result is immediate. ☐

2.3. **Frequent expressions.** The following expressions occur frequently in our formulas, whose exact values are rarely needed:

$$M \coloneqq \left|\{0, \frac{1}{2}, 1\}^n \setminus \{0,1\}^n\right| = 3^n - 2^n, \qquad \text{(the number of fractional, half-integral points)}$$

$$\alpha \coloneqq \frac{|S|}{M}, \qquad \text{(the relative size of the sampled set } S\text{)}$$

$$c \coloneqq \frac{k2^k}{n-k},$$

$$\lambda \coloneqq |F_k| = k2^{k-1} = c(n-k)/2. \qquad \text{(size of } F_k\text{)}$$

3. A lower bound on the rank

Our random construction of polytopes is the following. We uniformly choose a *fixed-size* random set S of half-integral points of the n-dimensional 0/1 cube $[0,1]^n$ excluding the vertices. We consider the limit of the probability of the event '$\exists F_k \subseteq S$' that S contains the F_k of a k-dimensional face. The dimension n will tend to infinity, and k and the size $|S|$ of S may also vary. We will prove that the limit depends on the behavior of an expression of these parameters:

Theorem 3.1. *Let $n \in \mathbb{N}$ be a natural number and S be a fixed-size random subset of $\{0,1/2,1\}^n \setminus \{0,1\}^n$ with uniform distribution.*

Varying n, k and the size $|S|$ of S subject to the indicated restriction, we have the following behaviour of the event's probability as n and $|S|$ tend to infinity, $k < \log n$, and $\lambda^2/|S|$ tends to 0 (e.g., $|S| = \Omega(n^{2+\delta})$ for some $\delta > 0$):

$$\lim_{n\to\infty} \mathbb{P}\left[\exists F_k \subseteq S\right] = 0, \qquad \text{if } \frac{k2^k}{n-k}\log\frac{3^n-2^n}{|S|} = c\log(1/\alpha) \geq 2+\varepsilon, \tag{3.1}$$

$$\liminf_{n\to\infty} \mathbb{P}\left[\exists F_k \subseteq S\right] \geq \frac{1}{2}, \qquad \text{if } \frac{k2^k}{n-k}\log\frac{3^n-2^n}{|S|} = c\log(1/\alpha) \leq 2. \tag{3.2}$$

Here ε is an arbitrary small positive constant independent of n.

Proof. First, we prove sparseness, i.e., statement (3.1). This part of the proof is rather straight-forward. By applying Lemma 2.5, we obtain

$$\mathbb{P}\left[\exists F_k \subseteq S\right] \leq \sum_i \mathbb{P}\left[F_k^{(i)} \subseteq S\right],$$

where i ranges over all k-dimensional faces and F_k^i is the F_k of the face i. Note that, $\lim |S| = +\infty$ and $\lim \frac{\lambda^2}{|S|} = 0$ by assumption. Thus we can apply Lemma 2.7 and so $\mathbb{P}\left[F_k^{(i)} \subseteq S\right] \sim \left(\frac{|S|}{M}\right)^\lambda$. Since the summands are all identical, we obtain

$$\sum_i \mathbb{P}\left[F_k^{(i)} \subseteq S\right] \sim \binom{n}{k} 2^{n-k} \left(\frac{|S|}{M}\right)^\lambda.$$

Substituting values and applying straight-forward transformations together with $2^{-\varepsilon/2} < 1$ we have

$$\binom{n}{k} 2^{n-k} \left(\frac{|S|}{M}\right)^\lambda \leq n^k 2^{n-k} \cdot \alpha^{c(n-k)/2} = n^k (2\alpha^{c/2})^{n-k} < n^{\log n} (2^{-\varepsilon/2})^{n-k}.$$

The condition $c \log(1/\alpha) \geq 2 + \varepsilon$ in (3.1) is another form of saying $2\alpha^{c/2} \leq 2^{-\varepsilon/2}$. Hence the right-hand side tends to 0 as n tends to infinity. It therefore follows that $\mathbb{P}\left[\exists F_k \subseteq S\right]$ tends to 0 in this case.

We will now prove richness, i.e., statement (3.2). In a first step we will choose $|S|$ as small as possible. Obviously this does not increase the probability of containing an F_k of a face. Observe that the condition $c \log(1/\alpha) \leq 2$ in (3.2) is the same as $2^{-\frac{n-k}{\lambda}} \leq \alpha$ and therefore the minimal choice of $|S|$ is given by

$$|S| \coloneqq \left\lceil M 2^{-\frac{n-k}{\lambda}} \right\rceil.$$

In order to bound the probability $\mathbb{P}\left[\exists F_k \subseteq S\right]$ from below, we claim that

$$2^{n-k} \left(\frac{|S|}{M}\right)^\lambda \sim 1$$

with the above choice of $|S|$. Let $\tau \coloneqq M 2^{-\frac{n-k}{\lambda}}$ and observe that

$$2^{n-k} \left(\frac{|S|}{M}\right)^\lambda = \left(\frac{\lceil \tau \rceil}{\tau}\right)^\lambda.$$

Therefore

$$1 \leq 2^{n-k} \left(\frac{|S|}{M}\right)^\lambda \leq \left(1 + \frac{1}{\tau}\right)^\lambda.$$

Thus it suffices to show that $\left(1 + \frac{1}{\tau}\right)^\lambda \sim 1$. This is indeed true as $e^\tau \sim \left(1 + \frac{1}{\tau}\right)^\tau$ as τ tends to ∞ for $n \to \infty$ and

$$\frac{\lambda}{\tau} = \frac{\lambda 2^{\frac{n-k}{\lambda}}}{M} \leq \frac{n 2^{k-1} 2^{n-k}}{M} = \frac{n 2^{n-1}}{3^n - 2^n} \to 0.$$

Therefore we indeed have $2^{n-k} \left(\frac{|S|}{M}\right)^\lambda \sim 1$ and we are ready to estimate the probability $\mathbb{P}\left[\exists F_k \subseteq S\right]$.

Let I denote a set of 2^{n-k} parallel k-dimensional faces of the cube and let $F_k^{(i)}$ denote the F_k of the face $i \in I$. We obtain

$$\mathbb{P}\left[\exists F_k \subseteq S\right] \geq \mathbb{P}\left[\exists i \in I \colon F_k^{(i)} \subseteq S\right].$$

The latter is bounded from below using Lemma 2.5, i.e.,

$$\mathbb{P}\left[\exists i \in I \colon F_k^{(i)} \subseteq S\right] \geq \sum_{i \in I} \mathbb{P}\left[F_k^{(i)} \subseteq S\right] - \sum_{\substack{i<j \\ i,j \in I}} \mathbb{P}\left[F_k^{(i)} \subseteq S \wedge F_k^{(j)} \subseteq S\right].$$

As $F_k^{(i)}$ and $F_k^{(j)}$ are disjoint for all choices i, j in the latter summand, we can thus apply Corollary 2.8 and obtain

$$\sum_{i\in I} \mathbb{P}\left[F_k^{(i)} \subseteq S\right] - \sum_{\substack{i<j\\ i,j\in I}} \mathbb{P}\left[F_k^{(i)} \subseteq S \wedge F_k^{(j)} \subseteq S\right] = (1+o(1))\cdot 2^{n-k}\left(\frac{|S|}{M}\right)^{\lambda} - (1+o(1))\cdot \binom{2^{n-k}}{2}\left(\frac{|S|}{M}\right)^{2\lambda}.$$

Finally observe that $\binom{2^{n-k}}{2}\cdot\left(\frac{|S|}{M}\right)^{2\lambda} = (1/2+o(1))\left(2^{n-k}\cdot\left(\frac{|S|}{M}\right)^{\lambda}\right)^2$. Together with $2^{n-k}\cdot\left(\frac{|S|}{M}\right)^{\lambda} \sim 1$ we obtain therefore

$$(1+o(1))\cdot 2^{n-k}\left(\frac{|S|}{M}\right)^{\lambda} - \left(\frac{1}{2}+o(1)\right)\left(2^{n-k}\cdot\left(\frac{|S|}{M}\right)^{\lambda}\right)^2 = \frac{1}{2}+o(1)$$

and we conclude

$$\mathbb{P}\left[\exists F_k \subseteq S\right] \geq \frac{1}{2} + o(1)$$

which finishes the proof. □

We will now derive lower bounds on the rank using the above construction. Recall that $\mathrm{rk}_C(A_n) \in \Omega(n/\log n)$ for any strongly admissible cutting-plane procedure by Lemma 2.3.

Corollary 3.2. *Let C be a strongly admissible cutting-plane procedure with $\mathrm{rk}_C(A_m) \geq \gamma m/\log m$ for some fixed $\gamma > 0$. We choose a subset S_n of the half-integral points of the n-dimensional 0/1-cube except the vertices. The distribution of the S_n are uniform and mutually independent and we let P_n be the convex hull of S_n. Then for all $\delta > 0$ there exists $N \in \mathbb{N}$ such that for all $n > N$ (note that M depends on n)*

$$\mathbb{P}\left[\mathrm{rk}_C(P_n) \geq \gamma \frac{\log n}{\log\log n}\right] \geq \frac{1}{2} - \delta, \qquad \text{if } \left|1 - \log\frac{M}{|S_n|}\Big/\log n\right| < \delta;$$
$$\mathbb{P}\left[\mathrm{rk}_C(P_n) \geq \gamma \frac{\log\log n}{\log\log\log n}\right] \geq \frac{1}{2} - \delta, \qquad \text{if } \left|1 - \log\frac{M}{|S_n|}\Big/ n^{1-\epsilon}\right| < \delta \text{ for some } \epsilon > 0;$$

Moreover, for the Gomory-Chvátal, the Lovász-Schrijver, and the split closures we have

$$\mathbb{P}\left[\mathrm{rk}_C(P_n) \geq \log n\right] \geq \frac{1}{2} - \delta, \qquad \text{if } \left|1 - \log\frac{M}{|S_n|}\Big/\log n\right| < \delta;$$
$$\mathbb{P}\left[\mathrm{rk}_C(P_n) \geq \log\log n\right] \geq \frac{1}{2} - \delta, \qquad \text{if } \left|1 - \log\frac{M}{|S_n|}\Big/ n^{1-\epsilon}\right| < \delta \text{ for some } \epsilon > 0;$$

Proof. In order to show the first case it suffices to prove that $\lim \frac{k2^k}{n-k}\log(M/|S_n|) \leq 2$ for $k = \log n^{1-\epsilon}$ for $\epsilon > 0$. We have

$$\frac{k2^k}{n-k}\log\frac{M}{|S_n|} \sim \frac{(\log n^{1-\epsilon})\cdot n^{1-\epsilon}}{n}\log n = (1-\epsilon)\frac{\log^2 n}{n^{\epsilon}},$$

which tends to 0 for large n. Thus we can apply Theorem 3.1 and we obtain that $\mathbb{P}\left[F_k \subseteq S_n\right] \geq 1/2 - \delta$. With $\mathrm{conv}(F_k) = A_k$ and $\mathrm{rk}_C(P_n) \geq \mathrm{rk}_C(A_k)$ we finally have $\mathbb{P}\left[\mathrm{rk}_C(P_n) \geq \gamma\frac{\log n}{\log\log n}\right] \geq 1/2 - \delta$.

The second case follows similarly with the choice $k = \log\log n$. As before we have to show that $\lim \frac{k2^k}{n-k}\log(M/|S_n|) \leq 2$ which follows as

$$\frac{k2^k}{n-k}\log\frac{M}{|S_n|} \sim \frac{\log\log n\cdot\log n}{n}n^{1-\epsilon} = \frac{\log\log n\cdot\log n}{n^{\epsilon}}.$$

Finally, the proof of the assertions for the Gomory-Chvátal, the Lovász-Schrijver, and the split closure is analogous. The improvement on rank stems from the stronger lower bound for A_n, i.e., $\mathrm{rk}(A_n) = n$ by Lemma 2.4. □

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