On Computation of Performance Bounds of Optimal Index Assignment

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***Abstract*—Channel-optimized index assignment of source codewords is arguably the simplest way of improving transmission error resilience, while keeping the source and/or channel codes intact. But optimal design of index assignment is an instance of quadratic assignment problem (QAP), one of the hardest optimization problems in the NP-complete class. In this work we make a progress in the research of index assignment optimization. We apply some recent results of QAP research to compute the strongest lower bounds so far for channel distortion of BSC among all index assignments. The strength of the resulting lower bounds is validated by comparing them against the upper bounds produced by heuristic index assignment algorithms.**

***Index Terms*—Index assignment, quantization, error resilience, quadratic assignment, semidefinite programming, relaxation.**

I. Introduction

In communication systems index assignment (IA) is the problem of labeling source codewords by binary integer numbers (channel codewords). For a source code of fixed integer rate n, there are $2^n!$ orderings of $N = 2^n$ codewords. Since flip of any bits or permutation of bit indices generates identical codes, there are totally $\frac{2^n!}{2^n \times n!} = \frac{(2^n-1)!}{n!}$ distinct index assignments [17]. Over all these possible index assignments the system performance remains the same, achieving the fidelity of the source code, if the channel is error free. But in presence of channel errors the overall system performance does depend on index assignment. This is because the channel distortion is a function of the index assignment, although the source distortion is not. For this reason, one would like to optimize the index assignment with respect to channel and source statistics to lessen the damage of channel errors. Operationally speaking, channel-optimized index assignment of source codewords is the simplest way of improving the system error resilience, because it keeps the source code intact and works either with or without forward error correction coding. Also, in the event that FEC fails, optimal index assignment can be a last line of defense against bit errors. Optimal index assignment can be classified as a joint source-channel coding technique because the design objective function involves both the channel and source statistics. Intuitively, source codewords that are distant from one another in code space should be indexed by binary numbers of large Hamming distances, and vice versa. The earliest design of of index assignment is perhaps the Gray code (GC) [6], which labels two consecutive scalar source codewords by two binary numbers of Hamming distance 1.

In spite of the long history of studying the problem, optimal index assignment has remained an elusive goal. Only one special case has been solved. For uniform scalar quantization of uniform source coupled with the binary symmetric channel (BSC), the natural binary code (NBC) is optimal in the sense of minimizing the mean-squared error, which was a result in sixties [2], [9], [16]. In a variant setting, Farber and Zeger showed that for quantizers with a uniform encoder and a channel optimized decoder (which finds a weighted centroid of all possible source symbols), NBC is an optimal index assignment [3]. The challenge lies in the computational tractability of optimal index assignment. The problem is known to be NP-hard [4], and more specifically it is an instance of quadratic assignment problem (QAP). QAP is one of the hardest optimization problems in the NP-complete class. Facing the verdict of NP-completeness a large number of heuristic algorithms for optimal index assignment were published (see [7] for a survey). However, all the heuristic index assignment techniques have similar performance in mean-squared error. This begs the question: how far the channel distortions obtained by these index assignments are from the true optimum solution(s). Ben-David and Malah studied lower bounds on channel distortion of VQ systems operating in BSC, over all possible index assignments [1]. Their bound coincided with a previous projection-based lower bound for QAP [8], which is known to be rather loose in the community of operation research.

In this paper, we capitalize on some very recent progress made on the estimation of lower bounds on QAPs, and greatly sharpen the lower bounds on the performance of optimal VQ index assignment. In the optimization literature, much work has been done on computing both upper and lower bounds for QAPs (see the survey article [13]). It was not until very recent little was reported for QAPs whose size is larger than about 100 in dimension (index assignments of $N > 100$ codewords in the context of this paper). Upper bounds are usually computed with heuristic methods, and the results are typically overestimates. The larger the value of N, the looser the estimated upper bounds. Inexpensive lower bounds such as the Gilmore-Lawler bound can also easily be computed but yield underestimates, while strong lower-bounding techniques

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are too expensive to be applicable to QAP instances of $N > 60$. In [14] a relaxation technique based on semi-definite programming was proposed to compute lower bounds on QAPs. In some cases it can obtain quite strong lower bounds for dimensions up to 256. For more efficient and less memory-intensive solution of the SDPs the solver SDPT3 used in [14] was replaced by the solver SDPNAL from [18]. This new development allows us to compute sharp lower bounds on channel distortion of optimal index assignment for practically interesting cases (e.g., for 8-bit vector quantizers or $N = 256$), which was not computationally feasible in the past.

The remainder of this paper is structured as follows. In the next section we formulate optimal index assignment as a QAP problem. In section 3 we introduce the lower bounding algorithm for index assignment. The main idea is to decompose the distance matrix of codewords or the Hamming adjacency matrix into two positive semidefinite matrices, and then obtain the lower bound on the performance of optimal index assignment by relaxation and semidefinite programming (SDP). The sharpness of resulting lower bounds is assessed by comparing our results with those of other well-known QAP lower bounding algorithms, and with upper bounds obtained by the iterative local search (ILS) technique, which is one of the best heuristic QAP algorithms. The evaluation is conducted for image vector quantizers with codebook size up to $N = 256$. Encouragingly, our lower and upper bounds are much closer than previously known. These new results represent a significant progress in the design of provably good index assignments in a realistic setting of signal compression and communication.

II. Problem Statement

A basic element of a signal compression and communication system is the quantizer Q, either scalar or vector. We focus on index assignment of vector quantizers (VQ) for the superior source coding performance of VQ. All our results apply to index assignment of scalar quantizers (SQ) as well, because SQ is just a special case of VQ. A vector quantizer $Q : \Re^d \to \{\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_N\}$ maps a continuous source vector $\mathbf{x} \in \Re^d$ to a codeword $\mathbf{c}_i \in \Re^d$ in the VQ codebook $\mathbb{C} = \{\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_N\}$ by the nearest neighbor rule. The index i rather than the codeword $\mathbf{c}_i$ itself is transmitted via the channel. Upon receiving i correctly, the VQ decoder can reconstruct $\mathbf{x}$ to $\mathbf{c}_i$ by inverse quantizer mapping Q^{-1} (a simple table lookup operation). Typically, the size N of the codebook $\mathbb{C}$ is made an integer power of two, $N = 2^n$ so that the codeword index i is a binary number of n bits. An index assignment of $\mathbb{C}$ is a bijection map $\pi : \mathbb{C} \leftrightarrow \{0, 1\}^n$.

If the transmission is error free, then Q^{-1} is invariant with respect to the index assignment of the N codewords in $\mathcal{C}$, hence the overall system performance is independent of codeword index assignment. However, in the event of transmission error that an index $\pi(\mathbf{c}_i)$ is received as $\pi(\mathbf{c}_j)$, an input vector $\mathbf{x}$ such that $\mathbf{w}_i = Q(\mathbf{x})$ will be reconstructed as w_j, incurring an extra channel distortion $d(\mathbf{c}_i, \mathbf{c}_j)$ that does depend on index assignment π. Let $P(\jmath|\imath)$ be the probability of transmitting index $\imath$ but receiving index $\jmath$, and $P(\mathbf{c}_i)$ be the prior probability of codeword $\mathbf{c}_i \in \mathbb{C}$. Given an index assignment π, the expected channel distortion is

$$\bar{d}_\pi = \sum_{i=1}^{N} P(\mathbf{c}_i) \sum_{j=1}^{N} P(\pi(\mathbf{w}_j)|\pi(\mathbf{c}_i)) d(\mathbf{c}_i, \mathbf{c}_j). \quad (1)$$

Adopting the common probability model of binary symmetric channel (BSC), we have

$$P(\pi(\mathbf{w}_j)|\pi(\mathbf{c}_i)) = (1-p)^{n-h(\pi(\mathbf{w}_j),\pi(\mathbf{c}_i))} p^{h(\pi(\mathbf{w}_j),\pi(\mathbf{c}_i))} \quad (2)$$

where p is the BSC crossover probability, and $h(\cdot,\cdot)$ is the Hamming distance. To minimize the expected BSC channel distortion $\bar{d}_\pi$ one would like to find an optimal index assignment defined by the following objective function

$$\pi_* = \arg\min_{\pi} \sum_{i=1}^{N} P(\mathbf{c}_i) \sum_{j=1}^{N} (1-p)^{n-h(\pi(\mathbf{w}_j),\pi(\mathbf{c}_i))} p^{h(\pi(\mathbf{w}_j),\pi(\mathbf{c}_i)) d(\mathbf{c}_i,\mathbf{c}_j)}. \quad (3)$$

III. Lower Bounding Algorithm

For convenience, we rewrite (1) in a matrix form. Let

$$\mathbf{W} = (P(\mathbf{c}_1), P(\mathbf{c}_2), \cdots, P(\mathbf{c}_N))^T \mathbf{I} \quad (4)$$

be the diagonal matrix consisting of prior probabilities of the VQ codewords, and let

$$\mathbf{B} = \{(1-p)^{n-h(i,j)} p^{h(i,j)}\}_{1\le i\le N, 1\le j\le N} \quad (5)$$

be the symmetric matrix whose elements $B(i, j)$ are the codeword transition probabilities $P(\pi(\mathbf{w}_j)|\pi(\mathbf{c}_i))$ due to BSC bit errors of probability p. Also, denote by $\mathbf{D} = \{d(\mathbf{c}_i, \mathbf{c}_j)\}_{1\le i\le N, 1\le j\le N}$ the symmetric distance matrix between pairs of codewords, and use the $N \times N$ permutation matrix $\mathbf{X}$ to specify π. Now, the expected channel distortion of (1) has the following matrix form

$$\begin{aligned} \bar{d}_\pi &= \sum_{i=1}^{N} P(\mathbf{c}_i) \sum_{j=1}^{N} \{\mathbf{X}\mathbf{B}\mathbf{X}^T\}_{i,j} d(\mathbf{c}_i, \mathbf{c}_j) \\ &= \text{trace}(\mathbf{W}\mathbf{X}\mathbf{B}\mathbf{X}^T\mathbf{D}) \\ &= \text{trace}(\mathbf{D}\mathbf{W}\mathbf{X}\mathbf{B}\mathbf{X}^T) \end{aligned} \quad (6)$$

To apply the semidefinite programming technique to compute lower bounds on $\bar{d}_\pi$ we need to make the product $\mathbf{DW}$ an equivalent symmetric matrix, and thus define

$$\tilde{\mathbf{D}} = \mathbf{D}\mathbf{W} + \mathbf{D}^T\mathbf{W}^T. \quad (7)$$

By the symmetry of $\mathbf{B}$, we have

$$\bar{d}_\pi = \frac{1}{2}\text{trace}(\tilde{\mathbf{D}}\mathbf{X}\mathbf{B}\mathbf{X}^T) = \frac{1}{2}\text{trace}(\mathbf{B}\mathbf{X}\tilde{\mathbf{D}}\mathbf{X}^T) \quad (8)$$

from which we also appreciate, as known in the literature, that computing optimal index assignment $\bar{d}_* = \min_\pi^{-1} \bar{d}_\pi$ is a problem of QAP.

In the history of operation research QAP has a reputation of being one of the hardest problems in the NP-complete class. Even computing a good lower bound for a general instance of QAP of modest size turned out to be extremely challenging. It was only till very recent that it became feasible to compute non-trivial lower bounds for QAPs of dimension up to 256 or higher [14]. This is based on the decomposition of the matrix $\mathbf{B}$ in the QAP objective function (8) into the difference of two positive semidefinite matrices $\mathbf{B}_+$ and $\mathbf{B}_-$. With the permutation matrix $\mathbf{X}$ being the basic variable of the problem

$$\min_{\mathbf{X}} \text{trace}(\tilde{\mathbf{D}}\mathbf{X}\mathbf{B}\mathbf{X}^T), \tag{9}$$

we induce

$$\mathbf{Y}_+ = \mathbf{X}\mathbf{B}_+\mathbf{X}^T, \quad \mathbf{Y}_- = \mathbf{X}\mathbf{B}_-\mathbf{X}^T \tag{10}$$

as auxiliary variables that also have to be positive semidefinite. By relaxing the integer constraints of 0 and 1 on the elements of $\mathbf{X}$, we end up with a semidefinite program (SDP) for which many efficient solvers exist.

Specifically, the SDP splitting of $\mathbf{B}$ is obtained by the singular value decomposition of $\mathbf{B}$. Let $\mathbf{V}$ be an orthogonal matrix whose columns are the eigenvectors of the matrix $\mathbf{B}$ associated with the eigenvalues $(\lambda_1, \cdots, \lambda_N)$, namely

$$\mathbf{B} = \sum_{i=1}^{N} \lambda_i \mathbf{v}_i \mathbf{v}_i^T \tag{11}$$

where $\mathbf{v}_i$ is the i^{th} column of $\mathbf{V}$. Let

$$\mathbf{B}_+ = \sum_{i:\lambda_i \geq 0} \lambda_i \mathbf{v}_i \mathbf{v}_i^T, \quad \mathbf{B}_- = -\sum_{i:\lambda_i < 0} \lambda_i \mathbf{v}_i \mathbf{v}_i^T. \tag{12}$$

We then have

$$\mathbf{B} = \mathbf{B}_+ - \mathbf{B}_-, \quad \text{trace}(\mathbf{B}_+\mathbf{B}_-) = 0, \quad \mathbf{B}_+, \mathbf{B}_- \succeq 0. \tag{13}$$

where $\succeq$ means the matrix being semidefinite positive.

It should be noted that the ordering of the matrices $\mathbf{B}$ and $\tilde{\mathbf{D}}$ in the QAP formulation of optimal index assignment problem is immaterial. In computing the lower bound we solve the underlying SDP problem for both orderings and choose the larger bound obtained.

Using the properties of $\mathbf{X}$, we solve the following SDP, whose objective value is the lower bound.

$$\begin{aligned}
\min \quad & trace(\tilde{\mathbf{D}}(\mathbf{Y}^+ - \mathbf{Y}^-)) \\
s.t. \quad & \mathbf{Y}^+\mathbf{e} = \mathbf{X}\mathbf{B}^+\mathbf{e}, \ \mathbf{Y}^-\mathbf{e} = \mathbf{X}\mathbf{B}^-\mathbf{e}; \\
& diag(Y^+) = \mathbf{X}diag(\mathbf{B}^+), \ \mathbf{Y}^+ \geq \min(\mathbf{B}^+); \\
& diag(Y^-) = \mathbf{X}diag(\mathbf{B}^-), \ \mathbf{Y}^- \geq \min(\mathbf{B}^-); \\
& \mathbf{Y}^+ - \mathbf{X}\mathbf{B}^+\mathbf{X}^T \succeq 0, \ \mathbf{Y}^- - \mathbf{X}\mathbf{B}^-\mathbf{X}^T \succeq 0; \\
& \mathbf{X}\mathbf{e} = \mathbf{X}^T\mathbf{e} = \mathbf{e}, \ \mathbf{X} \geq 0.
\end{aligned}$$

In the above, $\mathbf{e}$ is the all 1 vector and $\min(\mathbf{B})$ is the minimum element of $\mathbf{B}$.

It is important to mention that SDPs for these large instances were not solved with interior point methods but with the method of [18]. This avoided excessive memory requirements and also reduced the CPU time needed. A further enhancement applied, as in [14], is to compute verified bounds (see Jansson *et al.* [10]). In case the SDP solver does not quite finish with good accuracy the computed bound cannot be guaranteed be a lower bound although in most cases it will. To guarantee the bound interval-arithmetic methods are employed.

Results for a number of QAP problems including those in the quadratic assignment problem library (http://www.seas.upenn.edu/qaplib/) of sizes up to $N = 256$ were given in [14]. In this paper, for the index assignment problem, we are able to solve for instances of size $N = 512$. In other words, with processing power of current computers the proposed technique can be applied to lower bound the performance of index assignment algorithms for source codes of rate up to 9 bits per index, or code books of size up to 512.

The straightforward application of the method of [14] did not produce the best results. Due to the size distribution of the elements of $\tilde{\mathbf{D}}$ it is possible to perform an additional optimization. We observe that both the adjacency matrix $\mathbf{A}$ of the hypercube and the Hamming distance matrix $\mathbf{B}$ have zero diagonals. We can thus set the diagonal of $\tilde{\mathbf{D}}$ to zero without changing the QAP. Then, we can set the diagonal of the distance matrix to a nonzero value. We maximize the computed bound with respect to this value.

IV. Results and Discussions

Having the proposed effective SDP technique of computing performance lower bounds on optimal index assignment, we now proceed to answer a long-open tantalizing question: how close the index assignments computed by existing heuristic algorithms are to the optimum. Due to the wide use of VQ in image coding, we present a case study on image VQ index assignment. A training set of 18 natural images is used to design 16-dimensional vector quantizers of various fixed integer rates n (i.e., generating VQ codebooks of size 2^n for different n). In our experiments, we assume the BSC channel crossover probability p to be sufficiently small, and ignore the events of negligible probability that more than one bit errors occur in a codeword of n bits. This simplifies the codeword transition probability expression to

$$P(\pi(\mathbf{w}_j)|\pi(\mathbf{c}_i)) = (1-p)^{n-1}p \tag{14}$$

and consequently,

$$\mathbf{B} \approx (1-p)^{n-1}p\mathbf{A} \tag{15}$$

where $\mathbf{A}$ is the adjacency matrix of the n-dimensional hypercube. The above simplification factors out p in lower bounding of the channel distortion of optimal index assignment, and unifies the resulting lower bounds up to a scaling factor $(1-p)^{n-1}p$.

Table I lists the lower bounds computed by the proposed technique of SDF splitting and relaxation (SDF-LB) on $\text{trace}(\tilde{\mathbf{D}}^{(n)}\mathbf{X}\mathbf{A}\mathbf{X}^T)$, where $\tilde{\mathbf{D}}^{(n)}$ is the symmetric matrix of weighted squared distances between codewords in the tested image VQ codebook of size $N = 2^n$. For comparison purposes two other QAP lower bounds computed for the same objective function $\text{trace}(\tilde{\mathbf{D}}^{(n)}\mathbf{X}\mathbf{A}\mathbf{X}^T)$ are also entered into the table. One is the well-known Gilmore-Lawler lower bound (GL-LB) [5], [12], and the other is the projection-based lower bound (P-LB) [8]. The P-LB bound coincides with that of [1]. In the table the values of the lower bounds are rounded to integers. One can convert these values to lower bounds for expected channel distortions

$$\bar{d}_\pi = \frac{1}{2}(1-p)^{n-1}p \cdot \text{trace}(\tilde{\mathbf{D}}^{(n)}\mathbf{X}\mathbf{A}\mathbf{X}^T) \tag{16}$$

TABLE I
DIFFERENT LOWER BOUNDS FOR TRACE($\tilde{\mathbf{D}}\mathbf{X}\mathbf{A}\mathbf{X}^T$) VS. THE PERFORMANCES OF SOME INDEX ASSIGNMENTS, WITH $\bar{d} = \frac{1}{2}(1-p)^{n-1}p \cdot$ TRACE($\tilde{\mathbf{D}}\mathbf{X}\mathbf{A}\mathbf{X}^T$).

n	P-LB	GL-LB	SDP-LB	ILS	gap for ILS	NBC	gap for NBC
5	102156	84784	304551	358984	0.71 dB	470147	1.89 dB
6	42807	58617	289883	349337	0.81 dB	530994	2.63 dB
7	< 0	45503	242657	334360	1.39 dB	592534	3.88 dB
8	< 0	43942	199959	294756	1.69 dB	650936	5.13 dB
9	< 0	38156	193271	291314	1.78 dB	719776	5.71 dB

TABLE II
DIFFERENT LOWER BOUNDS FOR TRACE($\tilde{\mathbf{D}}\mathbf{X}\mathbf{B}\mathbf{X}^T$) VS. THE PERFORMANCES OF SOME INDEX ASSIGNMENTS FOR $p_c = 0.01$.

n	P-LB	GL-LB	SDP-LB	ILS	gap for ILS	NBC	gap for NBC
5	1146	884	2891	3608	0.96 dB	4685	2.10 dB
6	614	819	2834	3734	1.20 dB	5273	2.70 dB
7	< 0	484	2412	3347	1.42 dB	5871	3.86 dB
8	< 0	465	2020	2984	1.69 dB	6424	5.02 dB
9	< 0	416	1931	2937	1.82 dB	7072	5.64 dB

for different values of p and n. As shown by table I, our new lower bounds are much improved over the previous ones.

For code designers in practice the most important information is the sharpness of a lower bound against the achievable performance of optimal index assignment. To show the role of our SDP lower bounds may play in practice, we also strive to achieve the best possible upper bounds using heuristic algorithms. We applied simulated annealing algorithm [11] and the iterative local search (ILS) algorithm [15] to search for locally optimal index assignments for the image VQ codebooks mentioned above. The ILS algorithm, which was reported highly effective on QAP in the literature, produced the best results at the time of submitting this paper. The ILS results are tabulated in Table I to be compared with the corresponding lower bounds. Also, to provide a reference, we include in the table the results of natural binary code (NBC) that labels codewords in the order of increasing energy (average intensity of the VQ block).

Encouragingly, our SDP lower bounds suggest that the performance of some good VQ index assignment algorithms is fairly close to the optimum. For instance, the gap between the ILS result and SDP-LB is as small as 0.71 dB for our image vector quantizer of $N = 32$ codewords. As bit rate n increases, the gap becomes larger. Two factors may contribute to this: the SDP-LB becomes looser for larger n since lower bounding QAP is notoriously difficult for large problem size; the performance of existing VQ index assignment algorithms might not work as well when the codebook size increases due to the exponentially increasing combinatorial complexity. Nevertheless, the proposed lower bounding technique for index assignment offers a useful tool of assessing practical methods, and it represents a significant progress in this regard, as all previous lower bounds were simply too loose to be meaningful to practitioners.

If the crossover probability p is not sufficiently small, then the true codeword transition matrix $\mathbf{B}$ of (5) cannot be satisfactorily approximated by the adjacency matrix $\mathbf{A}$ of the n-dimensional hypercube. In such cases, a lower bound for trace($\tilde{\mathbf{D}}\mathbf{X}\mathbf{B}\mathbf{X}^T$) becomes necessary. To demonstrate the performance of the proposed algorithm for the general case when multiple bit errors occur in a single source code index, we compute all the lower bounds for trace($\tilde{\mathbf{D}}\mathbf{X}\mathbf{B}\mathbf{X}^T$) when $p_c = 0.01$, and report the results in Table II.

The results of the two tables also indicate that NBC is a very poor index assignment, having large performance gaps compared with good heuristic index assignment algorithms.

V. CONCLUSION

A computational technique is proposed to lower bound channel distortion of BSC among all index assignments of source codewords. The resulting lower bounds are the strongest so far in the literature. Despite the notoriety of the hardness of QAP problems, the proposed technique can lower bound the performance of index assignment algorithms for source codes of rate up to 9 bits per index, or code books of size up to 512. This research provides a practical way of evaluating the performance of different index assignment algorithms with respect to the theoretical limit.

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