ON REFORMULATIONS OF NONCONVEX QUADRATIC PROGRAMS OVER CONVEX CONES BY SET-SEMIDEFINITE CONSTRAINTS

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Abstract. The well-known result stating that any non-convex quadratic problem over the nonnegative orthant with some additional linear and binary constraints can be rewritten as linear problem over the cone of completely positive matrices (Burer, 2009) is generalized by replacing the nonnegative orthant with an arbitrary closed convex cone. This set-semidefinite representation result implies new semidefinite lower bounds for quadratic problems over several Bishop-Phelps cones.

1. Introduction

In [3, 18, 22, 23] several hard problems from combinatorial optimization have been reformulated as linear programs over the cone of copositive or completely positive matrices. In [5] Burer generalized these results as follows: under rather weak assumptions any non-convex quadratic problem over the nonnegative orthant with some additional linear and binary constraints can be rewritten as linear problem over the cone of completely positive matrices. The main contribution of this paper consists of two results:

(i) We generalize the result from [5]. More precisely, in Section 2 we prove that we can replace in the Burer's result the nonnegativity constraint $x \in \mathbb{R}^n_+$ by a more general constraint $x \in K$ for any closed convex cone $K \subseteq \mathbb{R}^n$. The resulting optimization problem is a linear program over the cone, dual to the *set-semidefinite* cone for K, and gives the same objective value as the original problem, which is a non-convex quadratic problem with linear, binary and cone constraint.

(ii) In the second part of the paper we consider quadratic optimization problems over Bishop-Phelps cones, which are for instance subject of interest in the vector optimization area. We demonstrate how to use the representation result from Section 2 to obtain tractable relaxations for these problems.

Note that the generalization from (i) covers a really wide range of optimization problems. To be able to treat also problems with binary variables over arbitrary closed convex cones opens a new perspective for modeling and reformulation of problems.

We shall mention that at the very last stage of preparation of this paper we realized that a very similar generalization was obtained independently by Burer [6]. However, there remains a substantial difference between the papers since we focus in the sequel to the Bishop-Phelps cone while the rest of [6] is a review of the existing results. An earlier and shorter

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A nonnegativity constraint $x \in \mathbb{R}^n_+$ can be interpreted by assuming that the space $\mathbb{R}^n$ is partially ordered by the natural (or componentwise) ordering. Any partial ordering $\leq$, i.e. any reflexive and transitive binary relation which is compatible with the linear structure of the space, can be represented by a convex cone defined by $\{x \in \mathbb{R}^n : x \geq 0\}$. On the other hand, any convex cone $K \subseteq \mathbb{R}^n$ defines a partial ordering in $\mathbb{R}^n$ by $x \leq_K y$ if and only if $y - x \in K$, i.e. $x \in K$ corresponds to $x \geq_K 0$ w.r.t. the partial ordering given by the cone K. Hence $x \in K$ is just a more general nonnegativity constraint inducing wider class of optimization problems as e.g. second order cone programming.

A motivating problem for our research comes from vector optimization, i.e. from optimization with a vector valued objective function $f : \mathbb{R}^m \to \mathbb{R}^n$. Assuming the space $\mathbb{R}^n$ to be ordered by some pointed convex cone K, a point $\bar{x} \in \mathbb{R}^m$ is called a *minimal solution* of $\inf_{x \in S} f(x)$, with S a nonempty set in $\mathbb{R}^m$, if $(f(\bar{x}) - K) \cap f(S) = \{f(\bar{x})\}$. Recall that a cone K is called pointed if $K \cap (-K) = \{0\}$. Such minimal solutions can be determined by scalarization techniques which may result in an optimization problem with a cone constraint. For instance using the scalarization result introduced in [20] for two parameters $a, r \in \mathbb{R}^n$ yields the problem

$$\begin{gathered} \inf t \\ \text{such that} \\ t\,r - f(x) - y = a, \\ y \in K, \\ t \in \mathbb{R}, \\ x \in S. \end{gathered}$$

If f is a linear objective function and S is defined by linear equalities and binary constraints, then the technique presented in this paper yields a reformulation of this problem as a linear problem over a special cone eliminating the binary constraints. In vector optimization in the Euclidean space for the cone K Bishop-Phelps cones are of special interest as they are adequate for modeling preferences of decision makers [12, Remark 8]. But note, that in $\mathbb{R}^n$ any closed convex pointed cone is representable as a BP cone [21].

In [5] the reformulation is done over the cone of *completely positive* matrices which is the dual cone of the cone of *copositive* matrices defined by

$$C_{\mathbb{R}^n_+} := \{A \in \mathcal{S}^n : x^\top A x \geq 0 \ \text{ for all } \ x \in \mathbb{R}^n_+\}\ . \tag{1}$$

Here, $\mathcal{S}^n$ denotes the space of real symmetric $n \times n$ matrices equipped with the inner product defined by $\langle A, B\rangle := \text{trace}(AB)$ for all $A, B \in \mathcal{S}^n$. Recall that the dual cone of a cone C in a topological space X is in general defined by

$$C^* := \{x^* \in X^* : x^*(x) \geq 0 \text{ for all } x \in C\}\,.$$

with X^* denoting the topological dual space, i.e. the space of all continuous linear maps from X to $\mathbb{R}$.

Replacing $\mathbb{R}^n_+$ in (1) by an arbitrary nonempty set $K \subseteq \mathbb{R}^n$ (later we assume K to be a nonempty closed convex cone) we get the cone

$$C_K := \{A \in \mathcal{S}^n : x^\top A x \geq 0 \text{ for all } x \in K\}$$

which is called K-*semidefinite* (or *set-semidefinite*) cone. In opposition to [10, 11] we define here the K-semidefinite cone in the subspace of symmetric matrices instead of in the whole space of linear maps mapping from the finite dimensional Euclidean space $\mathbb{R}^n$ to $\mathbb{R}^n$. The K-semidefinite cone is a convex cone and hence defines itself a partial ordering in the space of symmetric matrices.

Under the assumptions here, i.e. $K \subseteq \mathbb{R}^n$, the dual cone of the K-semidefinite cone was given in [24] and in [15, Lemma 7.5]:

Lemma 1.1. (a) *Let $K \subseteq \mathbb{R}^n$ be a nonempty given set, then*

$$C_K^* = \text{cl cone}\,\{xx^\top : x \in K\}.$$

(b) *Let $K \subseteq \mathbb{R}^n$ be a nonempty closed convex cone, then*

$$C_K^* = \text{conv}\{xx^\top : x \in K\}$$

and C_K^* is closed.

Here, cone(Ω) for some set Ω denotes the convex cone generated by the set Ω, conv(Ω) is the convex hull and cl(Ω) is the closure of the set Ω.

We call constraints $x \in C_K$ (or $x \in C_K^*$) *set-semidefinite* constraints. Anstreicher and Burer give in [1] for low dimensions computable representations of C_K^* in terms of matrices that are positive semidefinite and componentwise nonnegative. For $n = 5$ and $K = \mathbb{R}^5_+$ examinations of the cone of completely positive matrices $C^*_{\mathbb{R}^5_+}$ are done by Burer, Anstreicher and Dür in [7]. Jarre and Schmallowsky present in [17] a numerical test for checking whether some matrix is an element of the cone of completely positive matrices $C^*_{\mathbb{R}^n_+}$. Recently very interesting approach to detect copositivity based on simplical partitions has been done by Bundfuss and Dür, see [8, 9]. We are not aware of results about separation problems for C_K or C_K^* for general K, therefore we propose for the case when K is a Bishop-Phelps cone a tractable semidefinite programming relaxations.

In the following we additionally assume the set $K \subseteq \mathbb{R}^n$ to be a nonempty closed convex cone. We start by summing up some basic properties, compare [14, 10].

Proposition 1.2. *Let $K_1, K_2, K \subseteq \mathbb{R}^n$ be closed convex nontrivial cones in $\mathbb{R}^n$.*

(i) $C^*_{K_1} + C^*_{K_2} \subseteq (C_{K_1} \cap C_{K_2})^* = (C_{K_1 \cup K_2})^*$
(ii) $K_1 \subseteq K_2$ *implies* $C_{K_2} \subseteq C_{K_1}$ *and* $C^*_{K_1} \subseteq C^*_{K_2}$.
(iii) $(C_{K_1} \cup C_{K_2})^* = C^*_{K_1} \cap C^*_{K_2}$
(iv) *For the interior of the cone C_K it holds*

$$\mathit{int}(C_K) = \{A \in \mathcal{S}^n : x^\top A x > 0 \text{ for all } x \in K \setminus \{0\}\} \neq \varnothing$$

and thus the dual cone C_K^* is pointed.

Since K is a cone in $\mathbb{R}^n$, we can use the Carathéodory theorem and represent the dual cone by

$$C_K^* = \left\{ \sum_{i=1}^{(n(n+1)/2)+1} x^i (x^i)^\top \, : \, x^i \in K, \ \forall i = 1, \ldots, \frac{n(n+1)}{2} + 1 \right\} .$$

For shortness of the representation we omit the upper limit $p := (n(n+1)/2) + 1$ in the sum above and write instead in the following $C_K^* = \left\{ \sum_i x^i (x^i)^\top \, : \, x^i \in K \right\}$. The following lemma is the base for our main result and states that the minimal value of a linear function over this dual cone is always attained in a matrix which can be written as $xx^\top$ for some $x \in K$.

Lemma 1.3. *Let a matrix $Q \in \mathcal{S}^n$ and a nonempty set $S \subseteq \mathbb{R}^n$ be given. If the matrix $\overline{Y}$ is a minimal solution of*

$$(P') \qquad \begin{array}{c} \textit{inf } \langle Q, Y\rangle \\ \textit{such that} \\ Y \in \operatorname{conv}\{xx^\top : x \in S\}, \end{array}$$

then there exists some $\bar{x} \in S$ such that $\bar{x}\bar{x}^\top$ is also a minimal solution of (P'), i.e.

$$\langle Q, \bar{x}\bar{x}^\top\rangle = \langle Q, \overline{Y}\rangle,$$

and $\bar{x}$ is also a minimal solution of

$$(P_C) \qquad \begin{array}{c} \textit{inf } \langle Q, Y\rangle \\ \textit{such that} \\ Y = xx^\top, \\ x \in S. \end{array}$$

Hence, the optimization problems (P') and (P_C) are equivalent regarding the minimal value.

Proof. Let $\overline{Y}$ be a minimal solution of (P'). Then there exists some $k \in \mathbb{N}$, some $x^i \in S$ and $\lambda_i \geq 0$ for $i = 1, \dots, k$ with $\sum_i \lambda_i = 1$ and $\overline{Y} = \sum_i \lambda_i x^i (x^i)^\top$. Let $j \in \{1, \dots, k\}$ such that $(x^j)^\top Q x^j = \min_i\{(x^i)^\top Q x^i\}$, then

$$\langle Q, \overline{Y}\rangle = \sum_i \lambda_i (x^i)^\top Q x^i \geq \left(\sum_i \lambda_i\right) (x^j)^\top Q x^j = (x^j)^\top Q x^j = \langle Q, x^j (x^j)^\top\rangle.$$

As $\overline{Y}$ is minimal for (P') and $x^j(x^j)^\top$ is also feasible for (P') we get $\langle Q, \overline{Y}\rangle = \langle Q, x^j(x^j)^\top\rangle$. Of course, x^j is then also a minimal solution of (P_C). ∎

2. Set-semidefinite reformulation of quadratic programs

In this section we examine the equivalence between a quadratic optimization problem with linear constraints, a cone constraint and binary variables, and the relaxed problem over the dual cone of set-semidefinite matrices. Let $Q \in \mathcal{S}^n$ be a symmetric matrix, $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, $K \subseteq \mathbb{R}^n$ a nonempty closed and convex cone and $B \subseteq \{1, \dots, n\}$ an index set. We study the following quadratic optimization problem

$$(\mathrm{QP}) \qquad \begin{array}{c} OPT_P \ := \ \inf x^\top Q x + 2c^\top x \\ \text{such that} \\ Ax = b, \\ x_j \in \{0, 1\} \text{ for all } j \in B, \\ x \in K. \end{array}$$

We can reformulate (QP) by introducing

$$Y = \begin{pmatrix} 1 \\ x \end{pmatrix} \cdot \begin{pmatrix} 1 \\ x \end{pmatrix}^\top \in \mathcal{S}^{n+1} \tag{2}$$

to obtain

$$\text{(QP')}\qquad \begin{array}{rcl} & \inf \left\langle \begin{pmatrix} 0 & c^\top \\ c & Q \end{pmatrix}, Y \right\rangle & \\ & \text{such that} & \\ Y &=& \begin{pmatrix} 1 \\ x \end{pmatrix} \cdot \begin{pmatrix} 1 \\ x \end{pmatrix}^\top \\ Ax &=& b, \\ x_j &=& x_j^2 \ \text{ for all } j \in B, \\ x &\in& K. \end{array}$$

A natural linearization and lifting of the problem (QP') into the dual cone of $C_{\mathbb{R}_+ \times K}$ generated by dyadic products of the type (2):

$$C^*_{\mathbb{R}_+ \times K} = \left\{ \sum_i \begin{pmatrix} \alpha_i \\ v^i \end{pmatrix} \begin{pmatrix} \alpha_i \\ v^i \end{pmatrix}^\top : \alpha_i \in \mathbb{R}_+,\ v^i \in K \right\}$$

yields the following linear problem:

$$\text{(QP}_\text{C}\text{)}\qquad \begin{array}{rcl} OPT_C &:=& \inf \left\langle \begin{pmatrix} 0 & c^\top \\ c & Q \end{pmatrix}, Y \right\rangle \\ & \text{such that} & \\ Y &=& \begin{pmatrix} 1 & x^\top \\ x & X \end{pmatrix}, \\ Y &\in& C^*_{\mathbb{R}_+ \times K}, \\ Ax &=& b, \\ x_j &=& X_{jj} \ \text{ for all } j \in B, \\ \operatorname{Diag}(AXA^\top) &=& b \circ b := \begin{pmatrix} b_1^2 \\ \vdots \\ b_n^2 \end{pmatrix}, \\ & x \in \mathbb{R},\ X \in \mathcal{S}^n. & \end{array}$$

The main difference to the problems considered in [5] is that here we replace $x \in \mathbb{R}^n_+$ by $x \in K$ for an arbitrary closed convex cone K.

We denote the feasible set of the problem (QP), which will be assumed to be nonempty, by Feas(P), and the feasible set of (QP$_\text{C}$) by

$$\operatorname{Feas}(C) := \left\{ (x,\, X) \colon \begin{pmatrix} 1 & x^\top \\ x & X \end{pmatrix} \text{ feasible for } (\text{QP}_\text{C}) \right\}.$$

Let $L := \{x \in K \colon Ax = b\}$. Then we follow the line of [5] and assume in the following:

Assumption 2.1. If $x \in L$ then $x_j \in [0, 1]$ for all $j \in B$.

Remark 2.2. Assumption 2.1 is not very restrictive. Suppose that it does not hold for some $x_j,\ j \in B$, e.g. $x_j \in L$ does not imply $x_j \in [0,\, 1]$. Then we can add two more equations $x_j + y_j = 1,\ x_j - z_j = 0$ and two sign constraints: $y_j, z_j \geq 0$. Hence using

$$L' = \{(x, y_j, z_j) \in K \times \mathbb{R}^2_+ \colon Ax = b,\ x_j + y_j = 1,\ x_j - z_j = 0\}$$

the assumption holds for x_j and $K' := K \times \mathbb{R}^2_+$ is still a closed convex cone. If the set B is empty this assumption is dispensable.

Bomze and Jarre give in [4] another proof of Burer's result by adding the constraint $x_j \leq 1$ for $j \in B$ and by imposing the weaker assumption that $x \in L$ implies that x_j is bounded for all $j \in B$. The advantage is that it then suffices to add one single slack variable and one single linear constraint.

Additionally we define

$$\begin{aligned} L_\infty &:= \{d \in K\colon Ad = 0\}, \\ L_\infty^+ &:= \operatorname{cone}\left\{ \begin{pmatrix} 0 \\ d \end{pmatrix} \begin{pmatrix} 0 \\ d \end{pmatrix}^\top : d \in L_\infty \right\}, \\ \mathrm{Feas}^+(C) &:= \left\{ \begin{pmatrix} 1 & x^\top \\ x & X \end{pmatrix} : (x, X) \in \mathrm{Feas}(C) \right\}, \\ \mathrm{Feas}^+(P) &:= \operatorname{conv}\left\{ \begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix}^\top : x \in \mathrm{Feas}(P) \right\}. \end{aligned}$$

Lemma 2.3. *Let $K \subseteq \mathbb{R}^n$ be a nonempty closed convex cone and let Assumption 2.1 be satisfied. Then*

$$\mathit{Feas}^+(P) \subseteq \mathit{Feas}^+(C) \quad \text{and} \quad \mathit{Feas}^+(P) + L_\infty^+ \subseteq \mathit{Feas}^+(C).$$

Proof. The first part of the assertion is obvious. For the second part, following the ideas given in the proof of Prop. 2.1 in [5], we consider the convex cone

$$\begin{aligned} L(C)_\infty : &= \left\{ \begin{pmatrix} 0 & x^\top \\ x & X \end{pmatrix} \in \mathcal{C}^*_{\mathbb{R}_+ \times K}\colon Ax = 0,\ \mathrm{Diag}(AXA^\top) = 0,\ x_j = X_{jj} \ \text{ for all } j \in B \right\} \\ &= \left\{ \begin{pmatrix} 0 & 0^\top \\ 0 & X \end{pmatrix} \in \mathcal{C}^*_{\mathbb{R}_+ \times K}\colon \mathrm{Diag}(AXA^\top) = 0,\ X_{jj} = 0 \ \text{ for all } j \in B \right\} \end{aligned}$$

for which it holds $\mathrm{Feas}^+(C) + L(C)_\infty \subseteq \mathrm{Feas}^+(C)$. Noting that by Assumption 2.1 $d \in L_\infty$ implies $d_j = 0$ for all $j \in B$, it can easily be seen that $L_\infty^+ \subseteq L(C)_\infty$ and thus $\mathrm{Feas}^+(P) + L_\infty^+ \subseteq \mathrm{Feas}^+(C)$. ∎

Lemma 2.4. *Let $K \subseteq \mathbb{R}^n$ be a nonempty closed convex cone and let Assumption 2.1 be satisfied. Then*

$$\mathit{Feas}^+(C) = \mathit{Feas}^+(P) + L_\infty^+.$$

Proof. Also this proof follows the ideas given in the proof of Prop. 2.1 in [5]. By Lemma 2.3, it remains to show $\mathrm{Feas}^+(C) \subseteq \mathrm{Feas}^+(P) + L_\infty^+$. Let $Y \in \mathrm{Feas}^+(C)$. Then because of $Y \in \mathcal{C}^*_{\mathbb{R}_+ \times K}$ there exists some $k \in \mathbb{N}$ and $\alpha_i \in \mathbb{R}_+$, $v^i \in K$ for $i = 1, \ldots, k$ with

$$Y = \sum_i \begin{pmatrix} \alpha_i \\ v^i \end{pmatrix} \begin{pmatrix} \alpha_i \\ v^i \end{pmatrix}^\top = \sum_i \begin{pmatrix} \alpha_i^2 & \alpha_i v^{i\top} \\ \alpha_i v^i & v^i v^{i\top} \end{pmatrix}$$

and $(\alpha_i, v^{i\top}) \neq 0$ for all $i = 1, \ldots, k$. As Y satisfies the constraints in (QP_C), it holds that

$$\sum_i \alpha_i^2 = 1, \tag{3}$$

$$\sum_i \alpha_i a^j v^i = b_j \text{ for all } j = 1, \ldots, m, \tag{4}$$

with a^j the j-th row of the matrix A, and due to $\mathrm{Diag}(AXA^\top) = b \circ b$ also

$$\sum_{i=1}^{k} (a^j v^i)^2 = b_j^2 \text{ for all } j = 1, \dots, m. \tag{5}$$

Then (3), (5) and (4) yield

$$\left(\sum_{i=1}^{k} \alpha_i^2\right) \cdot \sum_{i=1}^{k} (a^j v^i)^2 = b_j^2 = \left(\sum_{i=1}^{k} \alpha_i a^j v^i\right)^2$$

and we get

$$a^j v^i = \tau_j \alpha_i \text{ for } i = 1, \dots, k$$

for some $\tau_j \in \mathbb{R}$ and with (4) and (3) we conclude $\tau_j = b_j$, $j = 1, \dots, m$, i.e.

$$a^j v^i = b_j \alpha_i \text{ for } i = 1, \dots, k,\ j = 1, \dots, m. \tag{6}$$

Additionally for $j \in B$ it holds

$$\sum_{i=1}^{k} \alpha_i v_j^i = \sum_{i=1}^{k} (v_j^i)^2. \tag{7}$$

We define to $I := \{1, \dots, k\}$ the index sets $I^+ := \{i \in I \colon \alpha_i \neq 0\}$ and $I^0 := \{i \in I \colon \alpha_i = 0\} = I \setminus I^+$. Then we have

$$Y = \underbrace{\sum_{i \in I^+} \alpha_i^2 \begin{pmatrix} 1 \\ \frac{1}{\alpha_i} v^i \end{pmatrix} \begin{pmatrix} 1 \\ \frac{1}{\alpha_i} v^i \end{pmatrix}^\top}_{:= Y^1} + \underbrace{\sum_{i \in I^0} \begin{pmatrix} 0 \\ v^i \end{pmatrix} \begin{pmatrix} 0 \\ v^i \end{pmatrix}^\top}_{:= Y^2} \tag{8}$$

with $\sum_{i \in I^+} \alpha_i^2 = 1$, see (3). Next we show $Y^1 \in \mathrm{Feas}^+(P)$ and $Y_2 \in L_\infty^+$.

Let $i \in I^+$. For showing $\frac{1}{\alpha_i} v^i \in \mathrm{Feas}(P)$ we use that $v^i \in K$, K is a cone, $\frac{1}{\alpha_i} > 0$ and thus $\frac{1}{\alpha_i} v^i \in K$. With (6) we have $a^j \left(\frac{1}{\alpha_i} v^i\right) = b_j$ for $j = 1, \dots, m$. It remains to show

$$\frac{1}{\alpha_i} v_j^i \in \{0, 1\} \text{ for } j \in B.$$

Let $j \in B$. Based on the Assumption 2.1 $\frac{1}{\alpha_i} v_j^i \in [0, 1]$. From (7) and by setting $z_j^i := v_j^i / \alpha_i \in [0, 1]$ we get the equation

$$\sum_{i \in I^+} \alpha_i^2 \left(\frac{v_j^i}{\alpha_i} - \left(\frac{v_j^i}{\alpha_i} \right)^2 \right) = \sum_{i \in I^+} \underbrace{\alpha_i^2}_{>0} \underbrace{\left(z_j^i - (z_j^i)^2\right)}_{\geq 0} = 0$$

which implies $z_j^i - (z_j^i)^2 = 0$, i.e. $z_j^i = v_j^i / \alpha_i \in \{0, 1\}$.

Thus we have $Y^1 \in \mathrm{Feas}^+(P)$. Using (6) and $\alpha_i = 0$ for $i \in I^0$ we conclude that $Y_2 \in L_\infty^+$. ∎

Hence for $K \subseteq \mathbb{R}^n$ a nonempty closed convex cone we have

$$\mathrm{Feas}^+(P) \subseteq \mathrm{Feas}^+(C) = \mathrm{Feas}^+(P) + L_\infty^+.$$

We define for the objective functions of the problems (QP) and (QP$_{\mathrm{C}}$)

$$\begin{aligned} v_P(x) &:= x^\top Q x + 2c^\top x, \\ v_C(Y) &:= \Big\langle \underbrace{\begin{pmatrix} 0 & c^\top \\ c & Q \end{pmatrix}}_{:=\widetilde{Q}}, Y \Big\rangle. \end{aligned}$$

Instead of (x, X) a feasible element or a minimal solution of (QP$_{\mathrm{C}}$) we use the sometimes shorter notation that Y with

$$Y = \begin{pmatrix} 1 & x^\top \\ x & X \end{pmatrix}$$

is a feasible element or a minimal solution of (QP$_{\mathrm{C}}$). Thus we identify the problem

$$\inf_{Y \in \mathrm{Feas}^+(C)} \langle \widetilde{Q}, Y \rangle$$

with the problem (QP$_{\mathrm{C}}$).

Corollary 2.5. *Let $K \subseteq \mathbb{R}^n$ be a nonempty closed convex cone and let Assumption 2.1 be satisfied. Then the following holds*

$$OPT_P \geq OPT_C. \tag{9}$$

Proof. Let $\bar{x} \in \mathrm{Feas}(P)$. From Lemma 2.3 we conclude that the matrix $\bar{Y} = (1, \bar{x}^\top)^\top (1, \bar{x}^\top)$ is feasible for (QP$_{\mathrm{C}}$) and $v_P(\bar{x}) = v_C(\bar{Y})$, hence for every feasible matrix of (QP) we have a feasible matrix of (QP$_{\mathrm{C}}$) with the same objective value. Since we are looking for minimum value, the assertion follows. ■

Theorem 2.6. *Let $K \subseteq \mathbb{R}^n$ be a nonempty closed convex cone and let Assumption 2.1 be satisfied. The the following is true:*

$$OPT_P = OPT_C$$

.

Proof. The proof follows the ideas of the proof of Lemma 1.3. Due to Corollary 2.5 we need to prove only $OPT_P \leq OPT_C$.

Suppose that $Y \in \mathrm{Feas}^+(C)$. Lemma 2.4 implies that $Y = Y_1 + Y_2$ with $Y_1 \in \mathrm{Feas}^+(P)$ and $Y_2 \in L_\infty^+$. By definition we can write

$$Y_1 = \sum_{i \in I} \lambda_i \begin{pmatrix} 1 \\ z^i \end{pmatrix} \begin{pmatrix} 1 \\ z^i \end{pmatrix}^\top$$

with $z^i \in \mathrm{Feas}(P)$ for $i \in I$ and $\sum_{i \in I} \lambda_i = 1$, $\lambda_i \geq 0$ for $i \in I$, where I is some finite index set. Similarly we have

$$Y_2 = \sum_{i \in J} \begin{pmatrix} 0 \\ d^i \end{pmatrix} \begin{pmatrix} 0 \\ d^i \end{pmatrix}^\top,$$

with $d^i \in L_\infty$, and J another finite index set.

Set

$$\bar{z} \in \mathrm{argmin}\{(1, z^{i\top}) \widetilde{Q} (1, z^{i\top})^\top : i \in I\} .$$

Then

$$\begin{aligned}\left\langle \widetilde{Q}, Y_1 \right\rangle &= \sum_{i\in I} \lambda_i \left\langle \widetilde{Q}, \begin{pmatrix} 1 \\ z^i \end{pmatrix} \begin{pmatrix} 1 \\ z^i \end{pmatrix}^\top \right\rangle \\ &\geq \left\langle \widetilde{Q}, \begin{pmatrix} 1 \\ \bar{z} \end{pmatrix} \begin{pmatrix} 1 \\ \bar{z} \end{pmatrix}^\top \right\rangle .\end{aligned}$$

To finish the proof we need to consider $\langle \widetilde{Q}, Y_2 \rangle$:

- If $\langle \widetilde{Q}, Y_2 \rangle < 0$ then there exists $\bar{d} \in L_\infty$ such that $\bar{d}^\top Q \bar{d} < 0$ and for any feasible $x \in \text{Feas}(P)$ there exists $\bar{\tau} > 0$ such that $(x + \tau\bar{d})^\top Q(x + \tau\bar{d}) = x^\top Qx + 2\tau x^\top Q\bar{d} + \tau^2 \bar{d}^\top Q\bar{d} < 0$, for every $\tau > \bar{\tau}$. Since $x + \tau\bar{d} \in \text{Feas}(P)$ for every $\tau > 0$ sending τ to infinity implies that $OPT_P = -\infty$, hence by Corollary 2.5 we have $OPT_P = OPT_C = -\infty$.
- If $\langle \widetilde{Q}, Y_2 \rangle \geq 0$, then $\langle \widetilde{Q}, Y \rangle \geq \langle \widetilde{Q}, Y_1 \rangle \geq \bar{z}^\top Q\bar{z} + 2c^\top \bar{z}$. Hence we found for this particular Y a feasible point $(\bar{z})$ for (QP) with equal or smaller objective value.

Since Y was chosen arbitrary, the theorem is proven. ■

Remark 2.7. The main contribution of (QP_C) comparing to (QP) is that we got rid of the non-convexity in the objective function and the binary constraints. Problem (QP_C) is a pure conic linear problem and all difficulties are hidden in the structure of the cone $\mathcal{C}^*_{\mathbb{R}_+\times K}$.

Lemma 2.8. *Let $K \subseteq \mathbb{R}^n$ be a nonempty closed convex cone, let Assumption 2.1 be satisfied and let Feas(P) be bounded, then Feas$^+(P)$ = Feas(C).*

Proof. If Feas(P) is bounded then $L_\infty = \varnothing$ and the assertion follows by Lemma 2.4. ■

3. Optimization over Bishop-Phelps Cones

In 1962, Bishop and Phelps [2] introduced a class of ordering cones which have a rich mathematical structure and which have proven to be useful for instance in functional analysis and vector optimization. Well known cones as the nonnegative orthant or the Lorentz cone are special Bishop-Phelps cones.

Definition 3.1. For an arbitrary continuous linear functional $\phi\colon Y \to \mathbb{R}$ on the normed space $(Y, \|\cdot\|)$ the cone

$$\mathrm{K}_\phi := \{y \in Y : \|y\| \leq \phi(y)\}$$

is called *Bishop-Phelps cone.*

Note that the definition of Bishop-Phelps cone (BP cone) introduced in [2] is slightly different from the one above: Bishop and Phelps required that $\|\phi\| = 1$ and $t\|y\| \leq \phi(y)$ for some scalar $t \in (0, 1)$. Nowadays, several authors, see for instance [16], do not use the constant t and the assumption $\|\phi\| = 1$ and the Definition 3.1 follows this line.

We first collect some properties of BP cones [16]. Recall that a base B_K of a nontrivial convex cone K is a nonempty convex subset such that each element $x \in K \setminus \{0\}$ is uniquely representable as $x = \lambda b$ for some $\lambda > 0$ and some $b \in B_K$. The norm $\|\cdot\|$ on Y induces also a norm on the topological dual space Y^* by

$$\|y^*\| := \sup_{y \neq 0} \frac{|y^*(y)|}{\|y\|} \quad \text{for all} \quad y^* \in Y^*.$$

Proposition 3.2. [16] *Let $(Y, \|\cdot\|)$ be a normed space and $\phi \in Y^*$.*

(i) K_ϕ *is a closed, pointed and convex cone.*
(ii) *If $\|\phi\| > 1$ then K_ϕ is nontrivial and*

$$\mathrm{int}(\mathrm{K}_\phi) = \{y \in Y \colon \|y\| < \phi(y)\}.$$

If $\|\phi\| < 1$ then $\mathrm{K}_\phi = \{0\}$.
(iii) *If $\|\phi\| > 1$ then $B_K := \{y \in \mathrm{K}_\phi \colon \phi(y) = 1\}$ is a closed and bounded base for the cone K_ϕ.*
(iv) *The dual cone is ${\mathrm{K}_\phi}^* = cl\{\lambda z \in Y^* \colon \lambda \geq 0,\ z \in B(\phi, 1)\} \subseteq Y^*$ with $B(\phi, 1) := \{y^* \in Y^* \colon \|y^* - \phi\| \leq 1\}$.*

The following reformulation as a semidefinite condition was given in [16, Lemma 4.2].

Proposition 3.3. *Let $(Y, \|\cdot\|)$ be a normed space and $\phi \in Y^*$. Then $x \in \mathrm{K}_\phi$ if and only if the matrix*

$$M(x) := \begin{pmatrix} \phi(x) & \|x\| \\ \|x\| & \phi(x) \end{pmatrix}$$

is positive semidefinite.

In the following proposition we collect three additional semidefinite reformulations and characterizations.

Proposition 3.4. *Let $Y = \mathbb{R}^n$ and $\phi \in \mathbb{R}^n$ be given.*

(i) *Let the norm of the space be the Euclidean norm. Then $x \in \mathrm{K}_\phi = \{y \in \mathbb{R}^n \ : \ \|y\|_2 \leq \phi^\top y\}$ if and only if the matrix*

$$M(x) := \begin{pmatrix} \phi^\top x & x^\top \\ x & (\phi^\top x) I \end{pmatrix}$$

is positive semidefinite.
(ii) *Let the norm of the space be the Euclidean norm. Then $x \in \mathrm{K}_\phi = \{y \in \mathbb{R}^n \ : \ \|y\|_2 \leq \phi^\top y\}$ if $\phi^\top x \geq 0$ and there is some matrix $X \in \mathcal{S}^n$ such that $\langle I - \phi\phi^\top, X\rangle \leq 0$ and such that*

$$\begin{pmatrix} 1 & x^\top \\ x^\top & X \end{pmatrix}$$

is positive semidefinite.
(iii) *Let the norm of the space be the Maximum norm. Then $x \in \mathrm{K}_\phi = \{y \in \mathbb{R}^n \ : \ \|y\|_\infty \leq \phi^\top y\}$ if and only if the matrices*

$$\begin{pmatrix} \phi^\top x & x_i \\ x_i & \phi^\top x \end{pmatrix}, \qquad i = 1, \ldots, n$$

are positive semidefinite.

Proof. (i) This is a straightforward consequence of the well-known Schur complement (see e.g. [13, Theorem 7.7.6]). If $\phi^\top x > 0$ then $M(x)$ is positive semidefinite if and only if $\phi^\top x - \frac{1}{\phi^\top x} x^\top x \geq 0$ which is equivalent to $\|x\|_2 \leq \phi^\top x$. If $\phi^\top x < 0$ then $x \notin \mathrm{K}_\phi$ and also $M(x)$ is not positive semidefinite. If $\phi^\top x = 0$ then $x \in \mathrm{K}_\phi$ if and only if $x = 0$, and, at the same time, $M(x)$ is positive semidefinite if and only if $x = 0$.

(ii) According to the Schur complement, see e.g. [13, Theorem 7.7.6], the matrix

$$\begin{pmatrix} 1 & x^\top \\ x^\top & X \end{pmatrix}$$

is positive semidefinite if and only if $X - xx^\top$ is positive semidefinite. Therefore

$$\langle I - \phi\phi^\top,\, xx^\top \rangle \leq \langle I - \phi\phi^\top,\, X \rangle \leq 0.$$

This results in $\|x\|_2^2 = \langle I,\, xx^\top \rangle \leq \langle \phi\phi^\top,\, xx^\top \rangle = (\phi^\top x)^2$. Together with $\phi^\top x \geq 0$ we obtain $x \in \mathrm{K}_\phi$.

(iii) This result is a direct consequence of Proposition 3.3. We give the direct proof: the matrices

$$\begin{pmatrix} \phi^\top x & x_i \\ x_i & \phi^\top x \end{pmatrix}$$

are positive semidefinite for $i = 1, \ldots, n$, if and only if $\phi^\top x \geq 0$ and $(\phi^\top x)^2 - x_i^2 \geq 0$ for $i = 1, \ldots, n$. This again is equivalent to $|x_i| \leq \phi^\top x$ for $i = 1, \ldots, n$ and thus to $\|x\|_\infty \leq \phi^\top x$.

■

According to [21] every nontrivial convex cone in $\mathbb{R}^n$ is representable as a BP cone if and only if it is closed and pointed. But note that in $\mathbb{R}^n$ one might need different equivalent norms to present different nontrivial convex closed pointed cones as BP cones.

3.1. **BP cones with Euclidean norm.** Let us consider the case where $Y = \mathbb{R}^n$ with the Euclidean norm $\|\cdot\|_2$. Then $\phi \in \mathbb{R}^n$. We have

$$\mathrm{K}_\phi^2 = \{x \in \mathbb{R}^n \colon \|x\|_2 \leq \phi^\top x\} = \{x \in \mathbb{R}^n \colon x^\top(\phi\phi^\top - I)x \geq 0,\ \phi^\top x \geq 0\}. \tag{10}$$

Example 3.5. (a) The Lorentz cone (or second order cone or ice-cream cone - see e.g. [19] for definitions and applications of second order cone programming)

$$\mathrm{K}_\mathrm{L} := \{y \in \mathbb{R}^n \colon \|(y_1, \ldots, y_{n-1})\|_2 \leq y_n\}$$

is representable as a BP cone using the Euclidean norm and choosing $\phi := \sqrt{2}e_n$ with e_n denoting the nth unit vector [16, Lemma 2.4(a)], i.e.

$$\mathrm{K}_\mathrm{L} = \{y \in \mathbb{R}^n \colon \|y\|_2 \leq \sqrt{2}e_n^\top y\}.$$

(b) In the Euclidean space $\mathbb{R}^2$ Figure 1 illustrates the relation between $\phi = (\phi_1, \phi_2) \in \mathbb{R}^2$, to be more concrete between $1/\phi_1$ and $1/\phi_2$, and the represented BP cone $\mathrm{K}_\phi^2 = \{y \in \mathbb{R}^2 \colon \|y\|_2 \leq \phi^\top y\}$.

Next we illustrate how a representation as a BP cone (using the Euclidean norm) of an arbitrary closed convex pointed cone in $\mathbb{R}^2$ can be constructed. Let $a \in \mathbb{R}^2$ and $b \in \mathbb{R}^2$ denote the intersection points of the boundary of the cone and the unit ball

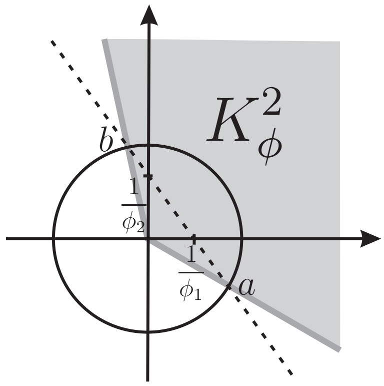

FIGURE 1. BP cone K^2_ϕ of Example 3.5(b) as well as the unit ball w.r.t. the Euclidean norm and (in dashed line) the line connecting the points $(1/\phi_1, 0)$ and $(0, 1/\phi_2)$.

w.r.t. the Euclidean norm. Assume $a_1 \neq b_1$ and $a_2 \neq b_2$. Then the line connecting a and b is given by all points (x_1, x_2) with

$$x_2 = \frac{a_2 - b_2}{a_1 - b_1} x_1 + \frac{a_1 b_2 - a_2 b_1}{a_1 - b_1}.$$

This line intersects the coordinate axes in the points

$$\left(\frac{a_2 b_1 - a_1 b_2}{a_2 - b_2}, 0\right) \quad \text{and} \quad \left(0, \frac{a_1 b_2 - a_2 b_1}{a_1 - b_1}\right).$$

Setting

$$\phi_1 := \frac{a_2 - b_2}{a_2 b_1 - a_1 b_2} \quad \text{and} \quad \phi_2 := \frac{a_1 - b_1}{a_1 b_2 - a_2 b_1}$$

the linear functional ϕ describes by $\mathrm{K}^2_\phi = \{y \in \mathbb{R}^2 \colon \|y\|_2 \leq \phi^\top y\}$ the given cone.

We can say more on the relation between BP cones with Euclidean norm and the second order cones.

Lemma 3.6. *Let $\mathrm{K}_\mathrm{L} \subseteq \mathbb{R}^n$ be the second order cone and $\mathrm{K}^2_\phi \subseteq \mathbb{R}^n$ a BP cone with respect to the Euclidian norm and the linear operator $\phi \in \mathbb{R}^n$, $\|\phi\|_2 > 1$. There exists a nonsingular linear transformation $T \colon \mathbb{R}^n \to \mathbb{R}^n$ such that $\mathrm{K}^2_\phi = T(\mathrm{K}_\mathrm{L})$.*

Proof. Note that K_L and K^2_ϕ are determined by quadratic forms

$$f_L(x) \;=\; \sum_{i=1}^{n-1} x_i^2 - x_n^2 \;=\; x^\top \underbrace{(I_n - 2E_{nn})}_{:=A_L} x \;=\; x^\top A_L x, \tag{11}$$

$$f_K(x) \;=\; \sum_{i=1}^{n} x_i^2 - x^\top \phi\phi^\top x \;=\; x^\top \underbrace{(I - \phi\phi^\top)}_{:=A_\phi} x \;=\; x^\top A_\phi x, \tag{12}$$

i.e. $\mathrm{K}_\mathrm{L} = \{x \in \mathbb{R}^n \colon x^\top A_L x \leq 0, x_n \geq 0\}$ and $\mathrm{K}^2_\phi = \{x \in \mathbb{R}^n \colon x^\top A_\phi x \leq 0, \phi^\top x \geq 0\}$.

The rank of the matrix A_L is n and its eigenvalues are 1 (with order $n-1$) and -1 (with order 1). Similarly we find out that A_K has eigenvalues 1 (with order $n-1$) and $1-\|\phi\|_2^2 < 0$ (with order 1) which corresponds to the eigenvector ϕ.

As A_ϕ is a symmetric matrix we can find a spectral decomposition of A_ϕ by $A_\phi = \widetilde{V}\widetilde{\Lambda}\widetilde{V}^\top$, where $\widetilde{\Lambda} = \mathrm{Diag}(1,\dots,1,1-\|\phi\|_2^2)$ and $\widetilde{V}$ contains an orthonormal basis of eigenvectors of A_ϕ, i.e. the last column of $\widetilde{V}$ is $\phi/\|\phi\|_2$. By setting $\Lambda := \mathrm{Diag}(1,\dots,1,-1)$ and $V := \widetilde{V}\,\mathrm{Diag}(1,\dots,1,\sqrt{\|\phi\|_2^2-1})$ we get $A_\phi = V\Lambda V^\top$ with $V^\top V = \mathrm{Diag}(1,\dots,1,\|\phi\|_2^2-1) =: D_\phi$. Then

$$V^{-1} = D_\phi^{-1}V^\top \ . \tag{13}$$

The last column of V is $(\sqrt{\|\phi\|_2^2-1}/\|\phi\|_2)\phi$. Let us take $T := V^{-\top}$. It is a nonsingular matrix, defining a bijective linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^n$. It follows

$$(\phi^\top T)^\top = T^\top\phi = V^{-1}\phi = D_\phi^{-1}V^\top\phi = D_\phi^{-1}\sqrt{(\|\phi\|_2^2-1)\|\phi\|_2^2}e_n = \sqrt{\frac{\|\phi\|_2^2}{\|\phi\|_2^2-1}}e_n \ .$$

By substitution $x = Tu$ and noting that $A_L = \Lambda$ it follows

$$\begin{aligned}
\mathrm{K}_\phi^2 &= \{x\in\mathbb{R}^n\colon x^\top A_\phi x \le 0,\ \phi^\top x \ge 0\}\\
&= \{Tu\colon u\in\mathbb{R}^n,\ u^\top T^\top A_\phi T u \le 0,\ \phi^\top T u \ge 0\}\\
&= T(\{u\in\mathbb{R}^n\colon u^\top A_L u \le 0,\ e_n^\top u \ge 0\})\\
&= T(\mathrm{K_L})
\end{aligned}$$

■

The following example shows that the assumption $\|\phi\|_2 > 1$ is essential.

Example 3.7. Consider the BP cone $\mathrm{K}_\phi^2 \subseteq \mathbb{R}^3$ for $\phi := (0,0,1)^\top$. Then $\mathrm{K}_\phi^2 = \{y\in\mathbb{R}^3\colon y_1 = y_2 = 0,\ y_3 \ge 0\}$ has an empty interior, but the Lorentz cone has a nonempty interior. So, there exists no nonsingular linear transformation $T\colon\mathbb{R}^3\to\mathbb{R}^3$ with $\mathrm{K}_\phi^2 = T(\mathrm{K_L})$. But there exists a transformation map $t\colon\mathbb{R}\to\mathbb{R}^3$, $t(y) := (0,0,y)^\top$ for all $y\in\mathbb{R}$ with $\mathrm{K}_\phi^2 = t(\mathrm{K_L})$ for the Lorentz cone $\mathrm{K_L} = \{y\in\mathbb{R}\colon y\ge 0\}$ in $\mathbb{R}$.

We will keep considering the BP cone K_ϕ^2 even though we could stick to the second order cone. The reason is that after the transformation described in Lemma 3.6 the structure of the problem gets less transparent.

Following the procedure from the previous section, we can show that under the same assumptions the optimization problem:

$$\begin{gathered}
\inf x^\top Q x + 2c^\top x\\
\text{such that}\\
Ax = b,\\
x_j\in\{0,1\}\ \text{ for all }\ j\in B,\\
x\in\mathrm{K}_\phi^2
\end{gathered}\tag{QP_{BP}}$$

has according to Theorem 2.6 the same optimal value as

$$(QP_{BP-2})\qquad \begin{array}{rcl} & \inf \left\langle \begin{pmatrix} 0 & c^\top \\ c & Q \end{pmatrix}, Y \right\rangle & \\ & \text{such that} & \\ Y &=& \begin{pmatrix} 1 & x^\top \\ x & X \end{pmatrix}, \\ Y &\in& C^*_{\mathbb{R}_+\times \mathrm{K}^2_\phi}, \\ Ax &=& b, \\ x_j &=& X_{jj} \ \text{ for all } j\in B, \\ \mathrm{Diag}(AXA^\top) &=& b\circ b, \\ & x\in\mathbb{R},\ X\in\mathcal{S}^n. & \end{array}$$

Note that

$$C^*_{\mathbb{R}_+\times \mathrm{K}^2_\phi} = \Big\{ \sum_i \begin{pmatrix} \alpha_i \\ x^i \end{pmatrix} \cdot \begin{pmatrix} \alpha_i \\ x^i \end{pmatrix}^\top : \|x^i\|_2 \le \phi^\top x^i,\ \alpha_i \ge 0 \Big\}. \tag{14}$$

Lemma 3.8. *Let $x\in\mathbb{R}^n$, $X\in\mathcal{S}^n$, $\phi\in\mathbb{R}^n$ and*

$$Y = \begin{pmatrix} 1 & x^\top \\ x & X \end{pmatrix}.$$

If $Y\in C^*_{\mathbb{R}_+\times \mathrm{K}^2_\phi}$ then

$$\mathit{trace}(X) \le \langle \phi\phi^\top,\, X\rangle,\ x\in \mathrm{K}^2_\phi \quad \textit{and } Y \textit{ is positive semidefinite.}$$

Proof. Note that if $Y\in C^*_{\mathbb{R}_+\times \mathrm{K}^2_\phi}$ then using (14) it follows $X=\sum_i x^i(x^i)^\top$ and $x=\sum_i \alpha_i x^i$, where $\alpha_i\ge 0$, $\|x^i\|_2\le \phi^\top x^i$ for all i. For $Y\in C^*_{\mathbb{R}_+\times \mathrm{K}^2_\phi}$ we conclude

$$\begin{array}{rcl} \mathrm{trace}(X) &=& \mathrm{trace}\left(\sum_i x^i(x^i)^\top\right) = \sum_i \|x^i\|_2^2 \\ &\le& \sum_i (x^i)^\top \phi\phi^\top x^i = \sum_i \langle \phi\phi^\top,\, x^i(x^i)^\top\rangle = \langle \phi\phi^\top,\, X\rangle\ . \end{array}$$

Because K^2_ϕ is a cone, $x^i\in\mathrm{K}^2_\phi$ implies $\alpha_i x^i\in\mathrm{K}^2_\phi$ and as K^2_ϕ is also convex, we get $x=\sum_i \alpha_i x^i\in\mathrm{K}^2_\phi$. ■

The following example shows that the inequality sign in $\mathrm{trace}(X)\le\langle\phi\phi^\top,\, X\rangle$ in the above lemma cannot be replaced by an equality sign in general.

Example 3.9. Consider

$$Y = \begin{pmatrix} 1 & \phi \\ \phi & \phi\phi^\top \end{pmatrix} = \begin{pmatrix} 1 \\ \phi \end{pmatrix}\begin{pmatrix} 1 \\ \phi \end{pmatrix}^\top$$

with $X=\phi\phi^\top$ and assume $\|\phi\|_2\ge 1$ (otherwise $\mathrm{K}^2_\phi=\{0\}$). Because of $\|\phi\|_2\le\|\phi\|_2^2=\phi^\top\phi$ it holds $Y\in C^*_{\mathbb{R}_+\times \mathrm{K}^2_\phi}$ and $\mathrm{trace}(X)=\mathrm{trace}(\phi\phi^\top)=\|\phi\|_2^2$ but $\langle\phi\phi^\top,\, X\rangle=\|\phi\|_2^4$.

Using the representation of the Lorentz cone $\mathrm{K_L}$ given in Example 3.5.(a) we get especially for this cone the following relaxation, which was already given in [6, Prop. 7].

Corollary 3.10. *Let $x \in \mathbb{R}^n$, $X \in \mathcal{S}^n$ and*

$$Y = \begin{pmatrix} 1 & x^\top \\ x & X \end{pmatrix}.$$

If $Y \in C^*_{\mathbb{R}_+ \times \mathrm{K_L}}$ then

$$\sum_{i=1}^{n-1} X_{ii} \leq X_{nn}, \; x \in \mathrm{K_L} \quad \text{and } Y \text{ is positive semidefinite.}$$

The following corollary follows immediately from Corollary 2.5 and the relaxation given in Lemma 3.8:

Corollary 3.11. *Let Assumption 2.1 be satisfied. The optimal value of (P_{BP}) is bounded from below by the optimal value of the following semidefinite program:*

$$(SDP_{BP-2}) \qquad \begin{array}{rcl} \textit{inf} & & \left\langle \begin{pmatrix} 0 & c^\top \\ c & Q \end{pmatrix}, Y \right\rangle \\ \textit{such that} & & \\ Y & = & \begin{pmatrix} 1 & x^\top \\ x & X \end{pmatrix} \textit{ positive semidefinite}, \\ \langle I - \phi\phi^\top, X \rangle & \leq & 0, \\ \|x\|_2 & \leq & \phi^\top x, \\ Ax & = & b, \\ x_j & = & X_{jj} \textit{ for all } j \in B, \\ \textit{Diag}(AXA^\top) & = & b \circ b, \\ x & \in & \mathbb{R}, \; X \in \mathcal{S}^n. \end{array}$$

Note that Proposition 3.4.(i) implies that the constraint $\|x\|_2 \leq \phi^\top x$ is a semidefinite programming (SDP) constraint. We may replace it by $\phi^\top x \geq 0$ according to Proposition 3.4.(ii).

3.2. **BP cones with Manhattan norm.** In the following we consider BP cones w.r.t. the 1-norm (Manhattan norm), i.e.

$$\mathrm{K}^1_\phi := \left\{ x \in \mathbb{R}^n \colon \|x\|_1 \leq \phi^\top x \right\}.$$

Example 3.12. (a) Generalizing the concept of the Lorentz cone to arbitrary p-norms, $p \in [1, \infty]$, we obtain the cone

$$K_p := \left\{ x \in \mathbb{R}^n \colon \|(x_1, \ldots, x_{n-1})\|_p \leq x_n \right\}. \tag{15}$$

For $p = 1$ we obtain $K_1 = \{x \in \mathbb{R}^n \colon \sum_{i=1}^{n-1} |x_i| \leq x_n\}$ which is a BP cone w.r.t. the 1-norm setting $\phi = 2e_n$ [16, Lemma 2.4]:

$$K_1 = \mathrm{K}^1_\phi = \{x \in \mathbb{R}^n \colon \|x\|_1 \leq 2e_n^\top x\}$$

(b) In the space $\mathbb{R}^2$ equipped with the 1-norm Figure 2 illustrates the relation between $\phi = (\phi_1, \phi_2) \in \mathbb{R}^2$ and the represented BP cone $\mathrm{K}^1_\phi = \{x \in \mathbb{R}^2 \colon \|x\|_1 \leq \phi^\top x\}$. Assume $\phi_1, \phi_2 \geq 1$. Then $\mathbb{R}^2_+ \subseteq \mathrm{K}^1_\phi$ and $\mathrm{K}^1_\phi = \mathrm{cone}\{y^A, y^B\}$ with

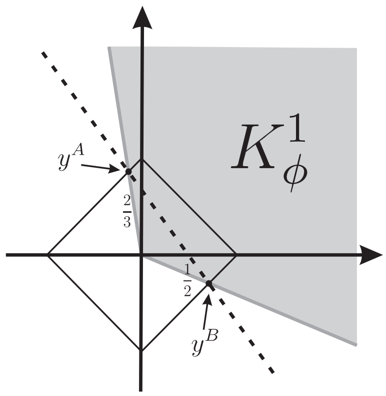

FIGURE 2. BP cone K^1_ϕ of Example 3.12.(b) for $\phi_1 = 2$ and $\phi_2 = 3/2$ as well as the unit ball w.r.t. the 1-norm and (in dashed line) the set $\{(x_1, x_2) \in \mathbb{R}^2 \colon (\phi_1, \phi_2)(x_1, x_2)^\top = 1\}$.

$$y^A := \left(\frac{1-\phi_2}{\phi_1+\phi_2}, \frac{1+\phi_1}{\phi_1+\phi_2}\right)^\top \quad \text{and} \quad y^B := \left(\frac{1+\phi_2}{\phi_1+\phi_2}, \frac{1-\phi_1}{\phi_1+\phi_2}\right)^\top,$$

compare Fig. 2. For instance for $\phi = (1,1)$ we have as a special case $\mathrm{K}^1_\phi = \mathbb{R}^2_+$.

We have

$$C^*_{\mathbb{R}_+ \times \mathrm{K}^1_\phi} = \left\{ \sum_i \begin{pmatrix} \alpha_i \\ x^i \end{pmatrix} \cdot \begin{pmatrix} \alpha_i \\ x^i \end{pmatrix}^\top : \|x^i\|_1 \leq \phi^\top x^i, \ \alpha_i \geq 0 \right\} \tag{16}$$

and obtain the following relaxation:

Lemma 3.13. *Let $x \in \mathbb{R}^n$, $X \in \mathcal{S}^n$, $\phi \in \mathbb{R}^n$ and*

$$Y = \begin{pmatrix} 1 & x^\top \\ x & X \end{pmatrix}.$$

If $Y \in C^*_{\mathbb{R}_+ \times \mathrm{K}^1_\phi}$ then

$$\sum_{i=1}^n \sum_{j=1}^n X_{i,j} \leq \langle \phi\phi^\top, \ X \rangle, \qquad x \in \mathrm{K}^1_\phi \quad \textit{and } Y \textit{ is positive semidefinite.}$$

Proof. Using (16), $Y \in C^*_{\mathbb{R}_+ \times \mathrm{K}^1_\phi}$ implies $X = \sum x^i (x^i)^\top$ with $\|x^i\|_1 \leq \phi^\top x^i$ for all i and we conclude

$$\begin{aligned} \textstyle\sum_{j=1}^n \sum_{k=1}^n X_{j,k} &= \textstyle\sum_{j=1}^n \sum_{k=1}^n \sum_i x^i_j x^i_k \\ &\leq \textstyle\sum_{j=1}^n \sum_{k=1}^n \sum_i |x^i_j|\,|x^i_k| = \sum_i \left(\sum_{j=1}^n |x^i_j|\right)^2 = \sum_i \|x^i\|_1^2 \\ &\leq \textstyle\sum_i (\phi^\top x^i)^2 = \sum_i \langle \phi\phi^\top, x^i (x^i)^\top \rangle = \langle \phi\phi^\top, X \rangle \ . \end{aligned}$$

As $x^i \in \mathrm{K}_\phi^1$ and K_ϕ^1 is a convex cone, we conclude for $x = \sum_i \alpha_i x^i$ with $\alpha_i \geq 0$ for all i that $x \in \mathrm{K}_\phi^1$. ∎

Note that $x \in \mathrm{K}_\phi^1$ implies, because of $\|x\|_2 \leq \|x\|_1$, that $\|x\|_2 \leq \phi^\top x$, which is a SDP constraint, see Proposition 3.4.(i).

Our proposal for SDP relaxation of the problem (SDP_{BP-2}), where we replace K_ϕ^2 by K_ϕ^1 is as follows:

$$
(SDP_{BP-1}) \qquad
\begin{array}{rcl}
\inf & & \left\langle \begin{pmatrix} 0 & c^\top \\ c & Q \end{pmatrix}, Y \right\rangle \\
\text{such that} & & \\
Y & = & \begin{pmatrix} 1 & x^\top \\ x & X \end{pmatrix} \text{ positive semidefinite,} \\
\langle \phi\phi^\top - J_n, X \rangle & \geq & 0, \ i = 1, \dots, n \\
\|x\|_2 & \leq & \phi^\top x, \\
Ax & = & b, \\
x_j & = & X_{jj} \ \text{ for all } j \in B, \\
\mathrm{Diag}(AXA^\top) & = & b \circ b, \\
x & \in & \mathbb{R}, \ X \in \mathcal{S}^n,
\end{array}
$$

where J_n is a matrix having all entries equal to 1. Recall that Proposition 3.4.(i) implies that the constraint $\|x\|_2 \leq \phi^\top x$ is a semidefinite programming constraint.

3.3. **BP cones with Maximum norm.**

Let us consider the BP cones for the ∞-norm (Maximum norm), i.e.

$$\mathrm{K}_\phi^\infty := \{x \in \mathbb{R}^n \colon \|x\|_\infty \leq \phi^\top x\} \ .$$

Example 3.14. Considering again the generalization of the Lorentz cone for arbitrary p-norms given in (15) we obtain for $p = \infty$

$$K_\infty := \{x \in \mathbb{R}^n \colon \|(x_1, \dots, x_{n-1})\|_\infty \leq x_n\} \ .$$

By setting $\phi := e_n$ [16, Lemma 2.4]

$$K_\infty = \mathrm{K}_\phi^\infty = \{x \in \mathbb{R}^n \colon \|x\|_\infty \leq e_n^\top x\} \ .$$

We have

$$C^*_{\mathbb{R}_+ \times \mathrm{K}_\phi^\infty} = \left\{ \sum_i \begin{pmatrix} \alpha_i \\ x^i \end{pmatrix} \cdot \begin{pmatrix} \alpha_i \\ x^i \end{pmatrix}^\top \colon \|x^i\|_\infty \leq \phi^\top x^i, \ \alpha_i \geq 0 \right\} \ . \tag{17}$$

The following lemma follows easily.

Lemma 3.15. *Let $x \in \mathbb{R}^n$, $X \in \mathcal{S}^n$, $\phi \in \mathbb{R}^n$ and*

$$Y = \begin{pmatrix} 1 & x^\top \\ x & X \end{pmatrix} .$$

If $Y \in C^*_{\mathbb{R}_+ \times \mathrm{K}^\infty_\phi}$ *then*

$$X_{ii} \le \langle \phi\phi^\top,\ X\rangle,\ i = 1,\dots,n,\ x \in \mathrm{K}^\infty_\phi \quad \textit{and } Y \textit{ is positive semidefinite.}$$

Proof. Using (17) we get for $X = \sum x^i(x^i)^\top$ with $x^i \in \mathrm{K}^\infty_\phi$ for all i, hence

$$X_{jj} = \sum_i (x^i_j)^2 \le \sum_i \|x^i\|^2_\infty \le \sum_i (\phi^\top x^i)^2 = \langle \phi\phi^\top, X\rangle\ .$$

As $x^i \in \mathrm{K}^\infty_\phi$ and K^∞_ϕ is a convex cone, we conclude for $x = \sum_i \alpha_i x^i$ with $\alpha_i \ge 0$ for all i that $x \in \mathrm{K}^\infty_\phi$. ■

Note that $x \in \mathrm{K}^\infty_\phi$ has a SDP reformulation, see Proposition 3.4.(iii).

Corollary 3.16. *Let Assumption 2.1 be satisfied. The optimal value of the problem* (QP_{BP}) *where we replace* K^2_ϕ *by* K^∞_ϕ *is bounded from below by the optimal value of the following SDP problem*

$$(SDP_{BP-\infty}) \qquad \begin{array}{rcl} \textit{inf} & & \left\langle \begin{pmatrix} 0 & c^\top \\ c & Q \end{pmatrix}, Y \right\rangle \\ \textit{such that} & & \\ Y & = & \begin{pmatrix} 1 & x^\top \\ x & X \end{pmatrix} \textit{ positive semidefinite,} \\ \langle \phi\phi^\top - E_{ii}, X\rangle & \ge & 0,\ i = 1,\dots,n, \\ \|x\|_\infty & \le & \phi^\top x \\ Ax & = & b, \\ x_j & = & X_{jj} \quad \textit{for all } j \in B, \\ \textit{Diag}(AXA^\top) & = & b \circ b, \\ x & \in & \mathbb{R},\ X \in \mathcal{S}^n, \end{array}$$

where E_{ii} *is a matrix having i-th diagonal entry equal to 1 and all other entries are 0.*

Recall that $\|x\|_\infty \le \phi^\top x$ is equivalent to a semidefinite programming constraint, compare Proposition 3.4.(iii).

4. Conclusions

We extended the result given by Burer in [5] on the reformulation of non-convex quadratic programs with linear and binary constraints over the nonnegative orthant to problems of this type over arbitrary nonempty closed convex cones. The main advantage of these reformulations is that no binary variables are necessary any more and the new objective function is linear. The reformulated problems are problems over the dual cone of the cone of set-semidefinite matrices (in the special case of Burer's result in [5] over the cone of completely positive matrices). For these dual cones hardly any numerical tests for checking, whether some matrix is an element, exist. This is an important task for future research.

In the second part of the paper we considered several special cases of closed convex cones, the so-called Bishop-Phelps cones, which appear for instance in vector optimization. For some particular norms in the underlying vector space we proposed semidefinite programming relaxations of the non-convex quadratic programs with linear and binary constraints over Bishop-Phelps cones. We have no theoretical guarantee on the quality of these lower bounds like it is the case in general with semidefinite or linear programming relaxations. However, solving these semidefinite relaxations gives us the first impression on the optimal value of the original problem and often this is also all that we can do efficiently.

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