INTEGER-EMPTY POLYTOPES IN THE 0/1-CUBE WITH MAXIMAL GOMORY-CHVÁTAL RANK

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ABSTRACT. We provide a complete characterization of all polytopes $P \subseteq [0,1]^n$ with empty integer hull whose Gomory-Chvátal rank is n (and, therefore, maximal). In particular, we show that the first Gomory-Chvátal closure of all these polytopes is identical.

1. INTRODUCTION

The Gomory-Chvátal procedure is a well-known technique to derive valid inequalities for the integral hull P_I of a polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$. It was introduced by Chvátal [2] and, implicitly, by Gomory [6, 7, 8] as a means to establish certain combinatorial properties via cutting-plane proofs. Cutting planes and Gomory-Chvátal cuts, in particular, belong to today's standard toolbox in integer programming. However, despite significant progress in recent years (see, e.g., [1, 3, 5, 9]), the Gomory-Chvátal procedure is still not fully understood from a theoretical standpoint, especially in the context of polytopes contained in the 0/1-cube. For example, the question if the currently best known upper bound of $O(n^2 \log n)$ on the Gomory-Chvátal rank, established in [5], is tight, remains open. In [5], it was also shown that there is a class of polytopes contained in the n-dimensional 0/1-cube whose rank exceeds n. (See [11] for a more explicit construction.) However, no family of polytopes in the 0/1-cube is known that realizes super-linear rank, and thus there is a large gap between the best known upper bound and the largest realized rank.

We consider the special case of $P \subseteq [0,1]^n$ with $P_I = \emptyset$ and Gomory-Chvátal rank $\text{rk}(P) = n$ (i.e., maximal rank, as $\text{rk}(P) \leq n$ holds for all $P \subseteq [0,1]^n$ with $P_I = \emptyset$; see [1]). This case is of particular interest as, so far, all known proofs of polynomial upper bounds on the rank of polytopes in the 0/1-cube (cf., [1, 5]) crucially depend on this special case. The improvement from $O(n^3 \log n)$ in [1] to $O(n^2 \log n)$ in [5] as an upper bound on the rank of polytopes in $[0,1]^n$ is a direct consequence of a better upper bound on the rank of certain polytopes in the 0/1-cube that do not contain integral points. It can actually be shown that lower bounds on the rank of polytopes $P \subseteq [0,1]^n$ with $P_I = \emptyset$ play a crucial role in understanding the rank of *any* (well-defined) cutting-plane procedure [10]. Moreover, in many cases the rank of a face $F \subseteq P$ with $F_I = \emptyset$ induces a lower bound on the rank of P itself. In fact, the construction of the aforementioned families of polytopes in $[0,1]^n$ whose rank is strictly larger than n exploits this connection.

In view of this, a thorough understanding of the Gomory-Chvátal rank of polytopes $P \subseteq [0,1]^n$ with $P_I = \emptyset$ might help to derive better upper and lower bounds for the general case. In this paper, we characterize all polytopes $P \subseteq [0,1]^n$ with $P_I = \emptyset$ and $\text{rk}(P) = n$. In particular, we show that after applying the Gomory-Chvátal procedure once one always obtains the same polytope. Furthermore, we show that $P \subseteq [0,1]^n$ with $P_I = \emptyset$ has $\text{rk}(P) = n$ if and only if $P \cap F \neq \emptyset$ for all one-dimensional faces F of the 0/1-cube $[0,1]^n$.

The paper is organized as follows. In Section 2, we introduce our notation and recall some basic facts about the Gomory-Chvátal procedure. Afterwards, in Section 3, we derive the characterization of all polytopes $P \subseteq [0,1]^n$ with $P_I = \emptyset$ and $\text{rk}(P) = n$. In particular, in Section 3.2, we relate the rank of

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a polytope $P \subseteq [0,1]^n$ with $P_I = \emptyset$ to the rank of its faces. We then prove the characterization for the two-dimensional case in Section 3.3, which is an essential ingredient for the subsequent generalization to arbitrary dimension in Section 3.4.

2. Preliminaries

Let $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ be a polytope with $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$. The *Gomory-Chvátal closure* of P is defined as

$$P' := \bigcap_{\lambda \in \mathbb{R}^m_+, \lambda A \in \mathbb{Z}^n} \{x : \lambda A x \leq \lfloor \lambda b \rfloor\}.$$

The result P' is again a polytope (see [2]), and one can apply the operator iteratively. We let $P^{(i+1)} := (P^{(i)})'$ for $i \geq 0$ and $P^{(0)} := P$. The resulting sequence $\{P^{(i)}\}_{i \geq 0}$ becomes stationary after finitely many steps [2], and the smallest k such that $P^{(k+1)} = P^{(k)}$ is the *Gomory-Chvátal rank of P* (in the following often *rank of P*), denoted by $\text{rk}(P)$. In particular, $P^{(\text{rk}(P))} = P_I$, where $P_I := \text{conv}(P \cap \mathbb{Z}^n)$ denotes the *integral hull of P*.

We will make repeated use of the following well-known lemma:

Lemma 2.1. [4, Lemma 6.33] *Let P be a rational polytope and let F be a face of P. Then $F' = P' \cap F$.*

If $P \subseteq [0,1]^n$ and $P_I = \emptyset$, Lemma 2.1 can be used to derive an upper bound on $\text{rk}(P)$:

Lemma 2.2. [1, Lemma 3] *Let $P \subseteq [0,1]^n$ be a polytope with $P_I = \emptyset$. Then $rk(P) \leq n$.*

This bound is actually tight; a family of polytopes $A_n \subseteq [0,1]^n$ with $(A_n)_I = \emptyset$ and $\text{rk}(A_n) = n$ was described in [3, p. 481].

For $i \in [n]$, the i-th coordinate flip maps $x_i \mapsto 1 - x_i$ and $x_j \mapsto x_j$ for $i \neq j$. Another property that we will extensively use is that the Gomory-Chvátal operator is commutative with unimodular transformations, in particular coordinate flips.

Lemma 2.3. [5, Lemma 4.3] *Let $P \subseteq [0,1]^n$ be a polytope and let u be a coordinate flip. Then $(u(P))' = u(P')$.*

Given polytopes $P \subseteq [0,1]^n$, $Q \subseteq [0,1]^k$, and a k-dimensional face F of $[0,1]^n$, we say that $P \cap F \cong Q$ if the canonical projection of $P \cap F$ onto $[0,1]^k$ is equal to Q. We denote the *interior of P* by $\text{Int}(P)$ and, with P, F, and Q as before, the *relative interior of P with respect to F* is defined as $\text{RInt}_F(P) := \text{Int}(Q)$. We use e to denote the all-one vector, and $\frac{1}{2}e$ to denote the all-one-half vector. If $I \subseteq [n] \times \{0,1\}$, $\frac{1}{2}e^I$ has coordinates $\frac{1}{2}e^l_i = \frac{1}{2}$ whenever $(i,l) \notin I$, and $\frac{1}{2}e^I_i = l$ for $(i,l) \in I$. Similarly, if F is a face of $[0,1]^n$ we define $\frac{1}{2}e^F \in F$ to be $\frac{1}{2}$ in those coordinates not fixed by F. Moreover, we define F_k to be the set of all vectors $x \in \{0, \frac{1}{2}, 1\}^n$ such that exactly k coordinates are equal to $\frac{1}{2}$, and the remaining coordinates are in $\{0,1\}$. For convenience, we use $[n] := \{1, \ldots, n\}$ for $n \in \mathbb{N}$.

3. Polytopes $P \subseteq [0,1]^n$ with $P_I = \emptyset$ and maximal rank

For $n \in \mathbb{N}$, we define the polytope $B_n \subseteq [0,1]^n$ by

$$B_n := \{x \in [0,1]^n \mid \sum_{i \in S} x_i + \sum_{i \in [n] \setminus S} (1 - x_i) \geq 1 \text{ for all } S \subseteq [n]\}.$$

Note that $(B_n)_I = \emptyset$. This family of polytopes will be essential to our subsequent discussion.

3.1. **Properties of B_n.** In the following section we will characterize $B_n^{(k)}$ and show, specifically, that $B_n^{(n-2)} = \{\frac{1}{2}e\}$. Moreover, we will show that $\{0, \frac{1}{2}\}$-cuts, i.e., Gomory-Chvátal cuts with $\lambda \in \{0, \frac{1}{2}\}^m$, suffice to deduce $(B_n)_I = \emptyset$, and the rank with respect to the classical Gomory-Chvátal procedure coincides with the rank if one were to use $\{0, \frac{1}{2}\}$-cuts only. Clearly, with B_n be as above and F being a k-dimensional face of $[0,1]^n$, we have $B_n \cap F \cong B_k$. As a direct consequence of the proof of [3, Lemma 7.2] one obtains:

Lemma 3.1. *Let $P \subseteq [0,1]^n$ be a polytope with $F_k \subseteq P$ for some $k < n$. Then $F_{k+1} \subseteq P'$.*

Proof. We include a proof for completeness. Let P be as above and let $ax < b+1$ with $a \in \mathbb{Z}^n$ and $b \in \mathbb{Z}$ be valid for P. We have to show that $ap \le b$ for every $p \in F_{k+1}$. Let $p \in F_{k+1}$ be arbitrary. If $ap \in \mathbb{Z}$ we are done. So assume that $ap \notin \mathbb{Z}$. Then there exists $i \in [n]$ such that $a_i \neq 0$ and $p_i = \frac{1}{2}$. We define the points p^0, p^1 by setting $p^0_j = p^1_j = p_j$ for all $j \neq i$, $p^0_i = 0$, and $p^1_i = 1$. Hence, $p = \frac{1}{2}p^0 + \frac{1}{2}p^1$. Note that $p^0, p^1 \in F_k \subseteq P$ and, therefore, $ap^l < b+1$ holds for $l \in \{0,1\}$. We derive $ap + \frac{1}{2} \le \max\{ap^0, ap^1\} < b+1$ and thus $ap < b + \frac{1}{2}$. Since $ap \in \frac{1}{2}\mathbb{Z}$ it follows that $ap \le b$, hence $p \in P'$. As the choice of $p \in F_{k+1}$ was arbitrary, we obtain $F_{k+1} \subseteq P'$. □

Note that $F_2 \subseteq B_n$. Thus, by Lemma 3.1, we have:

Corollary 3.2. $F_k \subseteq B_n^{(k-2)}$.

The following theorem specifies a family of valid inequalities for $B_n^{(k)}$.

Theorem 3.3. *Let B_n be defined as above and $k \le n$. Then*

$$\sum_{i \in I} x_i + \sum_{i \in \tilde{I} \setminus I} (1 - x_i) \ge 1$$

is valid for $B_n^{(k)}$ for all $I \subseteq \tilde{I} \subseteq [n]$ with $|\tilde{I}| = n - k$. Moreover, these inequalities can be derived as iterated $\{0, \frac{1}{2}\}$-cuts.

Proof. The proof is by induction on k. Let us first look at the case $k = 0$. By definition $\sum_{i \in I} x_i + \sum_{i \in \tilde{I} \setminus I}(1 - x_i) \ge 1$ with $\tilde{I} = [n]$ is valid for B_n. Now consider $0 < k \le n$, and assume that the claim holds for $k-1$. Let $\tilde{I} \subseteq [n]$ with $|\tilde{I}| = n-k$ be arbitrary. We have to prove that $\sum_{i \in I} x_i + \sum_{i \in \tilde{I} \setminus I}(1 - x_i) \ge 1$ with $I \subseteq \tilde{I}$ is valid for $B_n^{(k)}$. Let $I_0 = \tilde{I} \cup \{h\}$ for some $h \notin \tilde{I}$. Note that such an h exists as $k > 0$. Then

$$x_h + \sum_{i \in I} x_i + \sum_{i \in \tilde{I} \setminus I} (1 - x_i) = \sum_{i \in I \cup \{h\}} x_i + \sum_{i \in I_0 \setminus (I \cup \{h\})} (1 - x_i) \ge 1$$

and

$$(1 - x_h) + \sum_{i \in I} x_i + \sum_{i \in \tilde{I} \setminus I} (1 - x_i) = \sum_{i \in I} x_i + \sum_{i \in I_0 \setminus I} (1 - x_i) \ge 1$$

are valid for $B_n^{(k-1)}$, by induction hypothesis. By adding the two inequalities we obtain

$$2 \sum_{i \in I} x_i + 2 \sum_{i \in \tilde{I} \setminus I} (1 - x_i) \ge 1.$$

and, therefore, $\sum_{i \in I} x_i + \sum_{i \in \tilde{I} \setminus I}(1 - x_i) \ge \lceil \frac{1}{2} \rceil = 1$ is valid for $B_n^{(k)}$. □

We immediately obtain the following corollary:

Corollary 3.4. $B_n^{(n-2)} = \{\frac{1}{2}e\}$.

Proof. First note that $\frac{1}{2}e \in B_n^{(n-2)}$ by Corollary 3.2. By Theorem 3.3 we know that $\sum_{i\in I} x_i + \sum_{i\in \tilde{I}\setminus I}(1-x_i) \geq 1$ with $I \subseteq \tilde{I} = \{u,v\} \subseteq I$ is valid for $B_n^{(n-2)}$, for any pair $u,v \in [n]$, $u \neq v$. Therefore $x_u + x_v \geq 1$, $x_u + (1-x_v) \geq 1$, $(1-x_u) + x_v \geq 1$, and $(1-x_u)+(1-x_v) \geq 1$ are valid for $B_n^{(n-2)}$, which implies $x_u = x_v = \frac{1}{2}$. □

The following lemma characterizes the vertices of B_n.

Lemma 3.5. $B_n = conv(F_2)$.

Proof. Note that $\text{conv}(F_2) \subseteq B_n$. We will show that every vertex $\tilde{x}$ of B_n belongs to F_2, which would complete the proof. So let $\tilde{x}$ be an arbitrary vertex of B_n.

First, we prove that $\tilde{x}$ is half-integral. Suppose not. Let $D = \{i \in [n] \mid \tilde{x}_i \notin \{0, \frac{1}{2}, 1\}\}$. By applying appropriate coordinate flips, we may assume, without loss of generality, that $\tilde{x}_i < \frac{1}{2}$ for all $i \in D$. Since $\tilde{x}$ is a vertex of B_n there exists an index set $I \subseteq [n]$ such that $\sum_{i\in I} \tilde{x}_i + \sum_{i\in[n]\setminus I}(1-\tilde{x}_i) = 1$. Note that this implies $D \subseteq I$: If there exists $d \in D$ such that $d \notin I$, then $\sum_{i\in I\cup\{d\}} \tilde{x}_i + \sum_{i\in[n]\setminus(I\cup\{d\})}(1-\tilde{x}_i) < 1$ — a contradiction. We also obtain $|D| > 1$; otherwise the inequality cannot hold at equality. Let $s_I = \sum_{i\in I} \tilde{x}_i + \sum_{i\in[n]\setminus I}(1-\tilde{x}_i) - 1$ for all $I \subseteq [n]$. As $|D| \geq 2$, there exists $I \subseteq [n]$ with $s_I > 0$. (Just choose any I with $I \cap D = \emptyset$). Let $s = \min_{I\subseteq[n], s_I>0} s_I$, and let $j,k \in D$, $j \neq k$. For some $0 < \delta < \frac{1}{2}s$ we define $y,z \in [0,1]^n$ with $y_i = \tilde{x}_i = z_i$ for all $j \neq i \neq k$ and $y_j = \tilde{x}_j + \delta$, $y_k = \tilde{x}_k - \delta$, $z_j = \tilde{x}_j - \delta$, and $z_k = \tilde{x}_k + \delta$. Note that $\tilde{x} = \frac{1}{2}(y+z)$. It remains to show that $y,z \in B_n$, which would contradict that $\tilde{x}$ is a vertex of B_n. We have earlier seen that whenever $\sum_{i\in I} \tilde{x}_i + \sum_{i\in[n]\setminus I}(1-\tilde{x}_i) = 1$ holds for some $I \subseteq [n]$, then $D \subseteq I$. Therefore, $\sum_{i\in I} y_i + \sum_{i\in[n]\setminus I}(1-y_i) = \sum_{i\in I} \tilde{x}_i + \sum_{i\in[n]\setminus I}(1-\tilde{x}_i) + \delta - \delta = 1$ as $D \subseteq I$. Moreover, whenever $\sum_{i\in I} \tilde{x}_i + \sum_{i\in[n]\setminus I}(1-\tilde{x}_i) > 1$ holds for $I \subseteq [n]$, then

$$\sum_{i\in I} y_i + \sum_{i\in[n]\setminus I}(1-y_i) \geq \sum_{i\in I} \tilde{x}_i + \sum_{i\in[n]\setminus I}(1-\tilde{x}_i) - 2\delta \geq \sum_{i\in I} \tilde{x}_i + \sum_{i\in[n]\setminus I}(1-\tilde{x}_i) - s \geq 1.$$

Thus, $y \in B_n$, and $z \in B_n$ follows similarly. Consequently, $\tilde{x}$ is half-integral.

To finish the proof, we show that $\tilde{x}$ has exactly two coordinates that are equal to $\frac{1}{2}$. Suppose that there are more than two entries equal to $\frac{1}{2}$. Then $\sum_{i\in I} \tilde{x}_i + \sum_{i\in[n]\setminus I}(1-\tilde{x}_i) \geq \frac{3}{2}$ for all $I \subseteq [n]$. Similarly, less than two entries equal to $\frac{1}{2}$ is not possible as we would obtain $\sum_{i\in I} \tilde{x}_i + \sum_{i\in[n]\setminus I}(1-\tilde{x}_i) = \frac{1}{2} < 1$ for $I = \{i \in [n] \mid \tilde{x}_i = 0\}$. Hence, $\tilde{x} \in F_2$. □

We conclude this section by relating B_n to arbitrary polytopes $P \subseteq [0,1]^n$ with $P_I = \emptyset$.

Theorem 3.6. *Let $P \subseteq [0,1]^n$ with $P_I = \emptyset$. Then $P^{(l)} \subseteq B_n^{(l-1)}$.*

Proof. Let $p \in \{0,1\}^n$ be arbitrary, and let $I := \{i \in [n] \mid p_i = 0\}$. As $P_I = \emptyset$ we can find $\epsilon_p > 0$ such that $\sum_{i\in I} x_i + \sum_{i\in[n]\setminus I}(1-x_i) \geq \epsilon_p$ is valid for P, whereas $\sum_{i\in I} p_i + \sum_{i\in[n]\setminus I}(1-p_i) = 0$; the inequality separates p from P. In particular, we know that $\sum_{i\in I} x_i + \sum_{i\in[n]\setminus I}(1-x_i) \geq 1$ is valid for P'. Since $p \in \{0,1\}^n$ was chosen arbitrarily, we obtain that $\sum_{i\in I} x_i + \sum_{i\in[n]\setminus I}(1-x_i) \geq 1$ is valid for P' for every $I \subseteq [n]$, which implies $P' \subseteq B_n$. The claim follows from the fact that the Gomory-Chvátal procedure maintains inclusions. □

3.2. **The sandwich theorem.** In this section we will derive bounds on the growth of the rank of a polytope $P \subseteq [0,1]^n$ with $P_I = \emptyset$.

Theorem 3.7 (Sandwich Theorem)**.** *Let $P \subseteq [0,1]^n$ with $P_I = \emptyset$. Then*

$$k \leq rk(P) \leq k+1$$

where $k = \max_{(i,l)\in[n]\times\{0,1\}} rk(P \cap \{x_i = l\})$. Moreover, if there exist $i \in [n]$ and $l \in \{0,1\}$ such that $rk(P \cap \{x_i = l\}) < k$, then $rk(P) = k$.

Proof. Clearly, $k \leq \text{rk}(P)$ as there exists $(i,l) \in [n] \times \{0,1\}$ such that $\text{rk}(P \cap \{x_i = l\}) = k$. For the other inequality, observe that $P^{(k)} \cap \{x_i = l\} = (P \cap \{x_i = l\})^{(k)} = \emptyset$, by Lemma 2.1. It follows that $x_i < 1$ and $x_i > 0$ are valid for $P^{(k)}$ for all $i \in [n]$. Hence $x_i \leq 0$ and $x_i \geq 1$ are valid for $P^{(k+1)}$ for all $i \in [n]$, and, therefore, $P^{(k+1)} = \emptyset$, i.e., $\text{rk}(P) \leq k + 1$.

It remains to show that $\text{rk}(P) = k$ if there exist $i \in [n]$ and $l \in \{0,1\}$ such that $m := \text{rk}(P \cap \{x_i = l\}) < k$. Without loss of generality, we may assume that $l = 1$; otherwise we can apply the corresponding coordinate flip. Then $P^{(m)} \cap \{x_i = l\} = \emptyset$ and thus $x_i < 1$ is valid for $P^{(m)}$. Hence, $x_i \leq 0$ is valid for $P^{(k)}$. It follows that $P^{(k)} = P^{(k)} \cap \{x_i = 0\} = (P \cap \{x_i = 0\})^{(k)} = \emptyset$, which implies $\text{rk}(P) \leq k$. □

The upper bound in Theorem 3.7 is tight, as can be seen by considering the polytope A_n, introduced in [3, p. 481], whose definition is identical to that of B_n except for the right-hand side, which is $\frac{1}{2}$. Then $\text{rk}(A_n) = n$ and A_n satisfies the assumptions of the theorem. As $A_n \cap \{x_i = l\} \cong A_{n-1}$, we obtain that $\text{rk}(A_n \cap \{x_i = l\}) = n - 1$ for all $i \in [n]$ and $l \in \{0,1\}$

However, it is important to note that $\text{rk}(P \cap \{x_i = l\}) = k$ for all $(i,l) \in [n] \times \{0,1\}$ is not sufficient for $\text{rk}(P) = k + 1$. By induction, we immediately obtain a necessary condition for $\text{rk}(P) = n$.

Corollary 3.8. *Let $P \subseteq [0,1]^n$ be a polytope with $P_I = \emptyset$ and $rk(P) = n$. Then*

$$rk(P \cap F) = k$$

for all k-dimensional faces F of $[0,1]^n$, $1 \leq k \leq n$.

For the special case of $k = 1$, Corollary 3.8 was known before [5, Proof of Proposition 2.4].

3.3. **The two-dimensional case.** In this section we will provide a full characterization of polytopes $P \subseteq [0,1]^2$ with $P_I = \emptyset$ and $\text{rk}(P) = 2$. We will prove that $P \subseteq [0,1]^2$ with $P_I = \emptyset$ has rank 2 if and only if $P \cap \{x_i = l\} \neq \emptyset$ for all $(i,l) \in [2] \times \{0,1\}$, which happens if and only if $\frac{1}{2}e \in P'$. In case P is a half-integral polytope, the latter condition is equivalent to $\frac{1}{2}e \in \text{Int}(P)$. The following theorem establishes the first part:

Theorem 3.9. *Let $P \subseteq [0,1]^2$ be a polytope with $P_I = \emptyset$. Then $P \cap \{x_i = l\} \neq \emptyset$ for all $(i,l) \in [2] \times \{0,1\}$ if and only if $rk(P) = 2$.*

Proof. We first assume that P contains points $x^0 = (c_0, 0)$, $x^1 = (0, c_1)$, $x^2 = (c_2, 1)$, and $x^3 = (1, c_3)$. As the rank is monotone, we may assume that these are the only intersections of P with the boundary of the unit cube. Note that $c_i \in (0,1)$ for $0 \leq i \leq 3$. Let $ax < b + 1$ with $a \in \mathbb{Z}^2$ and $b \in \mathbb{Z}$ be valid for P. It is sufficient to prove that $a(\frac{1}{2}e) \leq b$ as this implies that $\frac{1}{2}e \in P' \neq \emptyset$. By using coordinate flips if necessary, we may assume that $a \geq 0$. Consequently, either x^2 or x^3 is maximizing a over P. We claim that $ax^m - a(\frac{1}{2}e) \geq \frac{1}{2}$ for some $m \in \{2,3\}$. This is sufficient to prove our hypothesis as $a(\frac{1}{2}e) \leq ax^m - \frac{1}{2} < b + 1 - \frac{1}{2} = b + \frac{1}{2}$ and as $a(\frac{1}{2}e) \in \frac{1}{2}\mathbb{Z}$, we obtain $a(\frac{1}{2}e) \leq b$. We distinguish three cases.

Case $a_2 = a_1$. We obtain that $a(\frac{1}{2}e) \in \mathbb{Z}$ and, therefore, $a(\frac{1}{2}e) \leq b$.

Case $a_2 \geq a_1 + 1$. It suffices to show that

$$ax^2 - a(\frac{1}{2}e) \geq \frac{1}{2} \quad \Leftrightarrow \quad a_1c_2 + a_2 - \frac{1}{2}a_1 - \frac{1}{2}a_2 \geq \frac{1}{2} \quad \Leftrightarrow \quad (c_2 - \frac{1}{2})a_1 + \frac{1}{2}a_2 \geq \frac{1}{2} \ .$$

This is true because $(c_2 - \frac{1}{2})a_1 + \frac{1}{2}a_2 \geq (c_2 - \frac{1}{2})a_1 + \frac{1}{2}(a_1 + 1) = c_2a_1 - \frac{1}{2}a_1 + \frac{1}{2}a_1 + \frac{1}{2} = c_2a_1 + \frac{1}{2} \geq \frac{1}{2}$.

Case $a_1 \geq a_2 + 1$. It suffices to show that $ax^3 - a(\frac{1}{2}e) \geq \frac{1}{2}$, which follows similarly.

For the other direction, observe that if there exists $(i,l) \in [2] \times \{0,1\}$ such that $P \cap \{x_i = l\} = \emptyset$ then $\text{rk}(P) \leq 1$ follows with Corollary 3.8. □

The following theorem is our main result for the two-dimensional case:

Theorem 3.10. *Let $P \subseteq [0,1]^2$ be a polytope with $P_I = \emptyset$. Then the following are equivalent:*
(a) *$rk(P) = 2$;*
(b) *$P \cap \{x_i = l\} \neq \emptyset$ for all $(i,l) \in [2] \times \{0,1\}$;*
(c) *$P' = \{\frac{1}{2}e\}$.*

Proof. By Theorem 3.9, (a) $\Longleftrightarrow$ (b). Clearly, if $P' = \{\frac{1}{2}e\}$, then $\text{rk}(P) = 2$. For the other direction, observe that, by Theorem 3.6, $P' \subseteq B_2 = \{\frac{1}{2}e\}$ and thus, if $\text{rk}(P) = 2$, it follows that $P' = \{\frac{1}{2}e\}$. ☐

We conclude this section with the following lemma showing that whenever $\text{rk}(P) = 2$, then $\frac{1}{2}e \in \text{Int}(P)$.

Lemma 3.11. *If $P \subseteq [0,1]^2$ is a polytope with $P_I = \emptyset$ and $\frac{1}{2}e \notin Int(P)$, then there exists $(i,l) \in [2]\times\{0,1\}$ such that $P \cap \{x_i = l\} = \emptyset$. In particular, if $rk(P) = 2$ then $\frac{1}{2}e \in Int(P)$.*

Proof. The proof of the first part is by contradiction. So let $P \subseteq [0,1]^2$ be a polytope with $P_I = \emptyset$ and $\frac{1}{2}e \notin \text{Int}(P)$. Suppose $P \cap \{x_i = l\} \neq \emptyset$ for all $(i,l) \in [2] \times \{0,1\}$. Then there exists $\tilde{x} \in P \cap \{x_{\tilde{i}} = \tilde{l}\}$ with $(\tilde{i},\tilde{l}) \in [2] \times \{0,1\}$ and $a \in \mathbb{R}^2$ such that $ax \leq a(\frac{1}{2}e)$ is valid for P and $a\tilde{x} = a(\frac{1}{2}e)$ (i.e., $a\tilde{x} = a(\frac{1}{2}e)$ is the hyperplane defined by the points $\tilde{x}$ and $\frac{1}{2}e$). Without loss of generality, we may assume that $\tilde{i} = 1$ and $\tilde{l} = 0$; otherwise we can apply coordinate permutations and flips. Then $\tilde{x}$ is of the form $\tilde{x} = (0,c)$ with $c \in (0,1)$, as $P_I = \emptyset$. It is easy to see that the hyperplanes $ax = a(\frac{1}{2}e)$ and $x_1 = 1$ intersect in the point $\tilde{y} = (1, 1-c)$. Note that $\tilde{y}$ is not necessarily in P. Let $Q = [0,1]^2 \cap \{ax \leq a(\frac{1}{2}e)\}$, and note that $P \subseteq Q$. If we maximize x_2 over P, we get $\max_{x \in P} x_2 \leq \max_{x \in Q} x_2 = \max_{x \in \{(1,1-c),(0,c)\}} x_2 < 1$, contradicting our assumption that $P \cap \{x_i = l\} \neq \emptyset$ for all $(i,l) \in [2] \times \{0,1\}$. The second claim follows from Theorem 3.10. ☐

Clearly, whenever P is half-integral, then $\frac{1}{2}e \in \text{Int}(P)$ if and only if $P \cap \{x_i = l\} \neq \emptyset$ for all $(i,l) \in [2] \times \{0,1\}$. In this case, we, therefore, obtain $\frac{1}{2}e \in \text{Int}(P)$ if and only if $\text{rk}(P) = 2$. If P is not half-integral, however, then this may not be true. Namely, consider P with $|P \cap \{x_i = l\}| = 1$ for all $(i,l) \in [2] \times \{0,1\}$, and move the vertex of the form $(p,1)$ inwards to $(p, 1-\epsilon)$, for some $\epsilon > 0$. It is easy to see that ϵ can be chosen such that $\frac{1}{2}e$ remains in the interior, however the rank of the resulting polytope is 1.

3.4. **The general case.** In this section we provide a complete characterization of all polytopes $P \subseteq [0,1]^n$ with $P_I = \emptyset$ and $\text{rk}(P) = n$. The following is the main theorem of this paper.

Theorem 3.12. *Let $P \subseteq [0,1]^n$ be a polytope with $P_I = \emptyset$. Then the following statements are equivalent:*
(a) *$rk(P) = n$;*
(b) *$P' = B_n$;*
(c) *$F \cap P \neq \emptyset$ for all one-dimensional faces F of $[0,1]^n$;*
(d) *$rk(P \cap F) = k$ for all k-dimensional faces F of $[0,1]^n$.*

Proof. First, we show that (c) implies (b). So let us assume that $H \cap P \neq \emptyset$ for all one-dimensional faces H of $[0,1]^n$. Consider $Q = P \cap F$ for some arbitrary two-dimensional face F of $[0,1]^n$. Then $F = \bigcap_{(i,l) \in I} \{x_i = l\}$ for some $I \subseteq [n] \times \{0,1\}$ with $|I| = n-2$. Let $J = [n] \setminus I$. Then $Q \cap \{x_i = l\} \neq \emptyset$ for all $(i,l) \in J \times \{0,1\}$ as $F \cap \{x_i = l\}$ is a one-dimensional face of $[0,1]^n$. Theorem 3.10 implies that $\tilde{Q}' = \{\frac{1}{2}e\}$, where $\tilde{Q} \cong Q$ and $\tilde{Q} \subseteq [0,1]^2$. Thus, $Q' = \{\frac{1}{2}e^I\}$. As the choice of I was arbitrary, we get $F_2 \subseteq P'$. By Lemma 3.5, $B_n \subseteq P'$ follows. Theorem 3.6 yields $P' \subseteq B_n$, which completes the proof of (b).

Now assume that $P' = B_n$. Corollary 3.4 gives $\{\frac{1}{2}e\} = B_n^{(n-2)} = P^{(n-1)}$. Together with Lemma 2.2, we obtain that $\text{rk}(P) = n$. So (b) implies (a).

By Corollary 3.8, $\text{rk}(P) = n$ implies $F \cap P \neq \emptyset$ for all k-dimensional faces F of $[0,1]^n$. That is, (d) follows from (a).

The missing implication, (d) to (c), is trivial. □

It is a direct consequence of Theorem 3.12 that, for any $n \in \mathbb{N}$, the only half-integral polytope $P \subseteq [0,1]^n$ with maximal rank and $P_I = \emptyset$ is A_n. Theorem 3.12 also implies that optimizing a linear function c over P' can be done in polynomial time for polytopes $P \subseteq [0,1]^n$ with $P_I = \emptyset$ and $\text{rk}(P) = n$. It suffices to apply coordinate flips so that $c \geq 0$, to then permute the coordinates such that $c_1 \geq c_2 \geq \cdots \geq c_n$, and to finally choose the optimal vertex from F_2.

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