A note on the MIR closure and basic relaxations of polyhedra

Sanjeeb Dash
IBM Research
sanjeebd@us.ibm.com

Oktay Günlük
IBM Research
gunluk@us.ibm.com

Christian Raack
ZIB
raack@zib.de

December 14, 2010

Abstract

Anderson, Cornuéjols and Li (2005) show that for a polyhedral mixed integer set defined by a constraint system $Ax \geq b$, where x is n-dimensional, along with integrality restrictions on some of the variables, any split cut is in fact a split cut for a *basic relaxation*, i.e., one defined by a subset of linearly independent constraints. This result implies that any split cut can be obtained as an intersection cut. Equivalence between split cuts obtained from simple disjunctions of the form $x_j \leq 0$ or $x_j \geq 1$ and intersection cuts was shown earlier for $0/1-$mixed integer sets by Balas and Perregaard (2002). We give a short proof of the result of Anderson, Cornuéjols and Li using the equivalence between mixed-integer rounding (MIR) cuts and split cuts.

1 Introduction and preliminaries

Consider a general mixed-integer set $P_I = P \cap Z(I)$ where

$$P = \{x \in \mathbb{R}^n \,:\, Ax \geq b\,\}, \qquad Z(I) = \{x \in \mathbb{R}^n \,:\, x_i \in \mathbb{Z}\;\forall i \in I\},$$

for given $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $I \subseteq \{1, \ldots, n\}$. Note that the variables are not explicitly defined to be nonnegative and any such variable bounds are included in the system $Ax \geq b$. Let $M = \{1, \ldots, m\}$ be the index set of the constraints, and let the jth row of $Ax \geq b$ be denoted by $a_j x \geq b_j$. Given a set $B \subseteq M$, let $P(B)$ be a relaxation of P defined as

$$P(B) = \{x \in \mathbb{R}^n \,:\, a_j x \geq b_j \; j \in B\},$$

and note that $P = P(M)$. If the vectors $\{a_j\}_{j \in B}$ are linearly independent, we call the relaxation $P(B)$, a *basic relaxation* of P. Clearly, all basic relaxations of P are translated cones. We denote the collection of subsets of M that give basic relaxations of P by $\mathcal{B}$.

Our main contribution in this note is to use the mixed integer rounding (MIR) framework to give a short proof of the result of Anderson, Cornuéjols and Li [1] that all split cuts for P_I can be generated from basic relaxations of P.

1.1 Split cuts

Consider a row vector $\alpha \in \mathbb{Z}^n$ with $\alpha_i = 0$ for $i \notin I$, and let $\gamma \in \mathbb{Z}$. For a row vector $c \in \mathbb{R}^n$ and $d \in \mathbb{R}$, the inequality $cx \geq d$ is called a *split cut* for P with respect to I if it is a valid inequality for both $P \cap \{x \in \mathbb{R}^n : \alpha x \leq \gamma\}$ and $P \cap \{x \in \mathbb{R}^n : \alpha x \geq \gamma + 1\}$. The inequality $cx \geq d$ is said to be derived using the *disjunction* $(\alpha x \leq \gamma) \vee (\alpha x \geq \gamma + 1)$. All points in P_I satisfy every split cut for P. Note that multiple split cuts can be generated using the same disjunction.

The collection of points in P that satisfy all split cuts for all $\alpha \in \mathbb{Z}^n$, $\gamma \in \mathbb{Z}$ where $\alpha_i = 0$ for $i \notin I$, is called the *split closure* of P with respect to I. We denote this set as *SplitClosure* (P). It is shown in [1] that

$$\textit{SplitClosure}\,(P) = \bigcap_{B \in \mathcal{B}} \textit{SplitClosure}\,(P(B)). \tag{1}$$

In other words, any important split cut for P can be obtained as a split cut for $P(B)$ where the set B yields a basic relaxation of P.

1.2 MIR cuts

Let $s = Ax - b$ and let $\lambda \in \mathbb{R}^m$ be a row vector. Clearly, the equation $-\lambda s + (\lambda A)x = \lambda b$ is valid for P and so is the inequality

$$(-\lambda)^+ s + (\lambda A)x \geq \lambda b \tag{2}$$

where the ith element of λ^+ is defined as $\lambda_i^+ = \max\{\lambda_i, 0\}$. Also note that $s \geq 0$ for all $x \in P$. Let

$$\Gamma = \{\lambda \in \mathbb{R}^{1\times m} : (\lambda A)_i \in \mathbb{Z} \text{ for all } i \in I,\ \ (\lambda A)_i = 0 \text{ for all } i \notin I\},$$

and notice that for $\lambda \in \Gamma$ and $x \in P_I$ we have $(-\lambda)^+ s \geq 0$ and $(\lambda A)x \in \mathbb{Z}$ in inequality (2).

Let $\beta = \lambda b - (\lceil \lambda b \rceil - 1)$. The basic mixed-integer inequality of Wolsey [8] implies that $(-\lambda)^+ s + \beta(\lambda A)x \geq \beta \lceil \lambda b \rceil$, or equivalently

$$(-\lambda)^+(Ax - b) + \beta(\lambda A)x \geq \beta \lceil \lambda b \rceil, \tag{3}$$

is a valid inequality for P_I. This inequality is called the *mixed-integer rounding (MIR)* inequality generated by λ. Inequality (2) is called the base inequality of the MIR inequality. See [6] for other ways of defining the MIR inequality, and the equivalence of (3) with the definition in [7]. It is known that when $\beta \neq 1$, the inequality (3) is a split cut for P derived using the disjunction $(\lambda A x \leq \lfloor \lambda b \rfloor) \vee (\lambda A x \geq \lceil \lambda b \rceil)$. In addition, a split cut for P derived using the disjunction $(\alpha x \leq \gamma) \vee (\alpha x \geq \gamma + 1)$ is equivalent to an MIR inequality generated by some $\lambda \in \Gamma$ with $\lambda A = \alpha$, see [7].

We define the MIR closure of P with respect to I to be the points in P that satisfy all MIR inequalities that can be generated by some $\lambda \in \Gamma$. We denote this set with *MirClosure* (P). It follows from the results in [7] that *MirClosure* (P) = *SplitClosure* (P). We will in fact show that equation (1) holds by showing that

$$\textit{MirClosure}\,(P) = \bigcap_{B \in \mathcal{B}} \textit{MirClosure}\,(P(B)).$$

One important property of the MIR inequalities that we use in this paper is the following: if a point $x^* \in P$ violates the MIR inequality (3), then $\lfloor \lambda b \rfloor < (\lambda A)x^* < \lceil \lambda b \rceil$.

2 Main result

For $\lambda \in \Gamma$, let $A(\lambda)$ and $b(\lambda)$ denote the collection of rows of A and b, respectively, for which $\lambda_i \neq 0$. Consequently, the MIR inequality generated by λ is in fact an MIR inequality for the following relaxation

$$P(\lambda) = \{x \in \mathbb{R}^n \, : \, A(\lambda)x \geq b(\lambda) \,\} \cap Z(I).$$

We next characterize an important property of such relaxations.

Theorem 1. *Let $x^* \in P \setminus MirClosure\,(P)$, then x^* violates an MIR inequality generated by $\lambda \in \Gamma$ such that $A(\lambda)$ has full-row rank.*

Proof. Let $s^* = Ax^* - b$. Some MIR inequality is violated by x^*; assume this inequality is generated by $\bar\lambda \in \Gamma$. Then, by definition, $\bar\lambda b \notin \mathbb{Z}$ and letting $\delta = \lfloor \bar\lambda b \rfloor$, the violation of the MIR inequality is

$$(\bar\lambda b - \delta)(\delta + 1 - \bar\lambda A x^*) - \sum_{\bar\lambda_i < 0} (-\bar\lambda_i) s_i^* > 0. \tag{4}$$

The expression above is simply the right hand side of (3) minus the left hand side of (3) evaluated at x^* (and s^*). Now consider the linear program MAXVIOL (here the variable is $\lambda \in \mathbb{R}^{1\times m}$)

$$\max \quad (\lambda b - \delta)(\delta + 1 - \bar\lambda A x^*) - \sum_{\bar\lambda_i < 0} (-\lambda_i) s_i^*$$

subject to

$$\lambda A = \bar\lambda A \tag{5}$$

$$\delta \leq \lambda b \leq \delta + 1 \tag{6}$$

$$\lambda_i \leq 0 \text{ if } \bar\lambda_i < 0, \quad \lambda_i \geq 0 \text{ if } \bar\lambda_i \geq 0 \; \forall i. \tag{7}$$

As $\bar\lambda \in \Gamma$, inequality (5) implies that $\lambda \in \Gamma$ for any feasible solution λ of this linear program. Further, if $\delta < \lambda b < \delta + 1$, then the objective function value equals the violation of the MIR inequality defined by λ. In addition, any optimal solution of MAXVIOL has positive objective function value, as $\bar\lambda$ is a solution of MAXVIOL, and has positive objective value by (4).

Note that MAXVIOL has $n+2$ constraints besides variable bounds and therefore any basic solution has at most $n+2$ non-zeroes. We next show that $\delta < \lambda' b < \delta + 1$ for any basic optimal solution λ'. If $\lambda' b = \delta$, then the objective function value is nonpositive as it equals $-\sum_{\bar\lambda_i<0}(-\lambda'_i)s_i^*$, where $s_i^* \geq 0$ for all $i \in M$ and $\lambda'_i \leq 0$ for all i such that $\bar\lambda_i < 0$. On the other hand, if $\lambda' b = \delta + 1$, then using the fact that $\lambda' A = \bar\lambda A$, the objective function value equals

$$(\lambda' b - \bar\lambda A x^*) - \sum_{\bar\lambda_i<0} (-\lambda'_i) s_i^* = \lambda'(b - Ax^*) - \sum_{\bar\lambda_i<0} (-\lambda'_i) s_i^* = -\lambda' s^* - \sum_{\bar\lambda_i<0} (-\lambda'_i) s_i^* = -\sum_{\bar\lambda_i \geq 0} \lambda'_i s_i^*$$

which is again nonpositive. Therefore λ' is a basic solution of the constraints (5) and (7) and consequently $A(\lambda')$ has full-row rank by standard LP theory. $\square$

3 Consequences

As discussed earlier, any MIR cut and therefore any split cut for P with respect to I can be derived using a single row relaxation of the constraint set defining P. This is one of the limitations of MIR cuts in terms of the information about the LP relaxation that is actually used to derive the cut. Theorem 1 also limits the structure (and number) of the constraints that can be simultaneously used to derive any MIR cut. A special case of this limitation is that MIR cuts do not use information about both upper and lower bounds on a variable simultaneously as these two constraints can never be a part of the same basic relaxation. Theorem 1 also gives a short proof of a result by Balas and Perregaard [4] which states that given a point x^*, a most violated lift-and-project cut with respect to a binary variable x_j (i.e., a split cut derived from a disjunction of the form $x_j \leq 0 \vee x_j \geq 1$) can be derived as a lift-and-project cut from a row of a simplex tableau of P (not necessarily feasible). If some lift-and-project cut with respect to the variable x_j is violated, then there is a violated MIR cut generated by a $\bar{\lambda} \in \Gamma$ with $\bar{\lambda} A = e_j$, where $e_j \in \mathbb{R}^n$ is a row vector with one in the jth position and zeros elsewhere. Setting $\bar{\lambda} A = e_j$ in the proof of Theorem 1, we get the result.

Theorem 1 also gives a short proof of equivalence between split cuts and intersection cuts, shown earlier in Andersen, Cornuejols and Li [1]. We refer the reader to [2, 3] for the original description of intersection cuts and to [1] for a concise description.

References

[1] K. Andersen, G. Cornuejols and Y. Li, Split Closure and Intersection Cuts, *Math. Prog.*, **102**, 457–493, (2005).

[2] E. Balas, Intersection cuts—a new type of cutting planes for integer programming. *Operations Research*, **19**, 19–39, (1971).

[3] E. Balas, Disjunctive programming, *Annuals of Discrete Mathematics*, **5**, 3–51, (1979).

[4] E. Balas, and M. Perregaard, A precise correspondence between lift-and-project cuts, simple disjunctive cuts, and mixed integer Gomory cuts for 0-1 programming, *Mathematical Programming B*, **94**, 221-245, (2002).

[5] W. J. Cook, R. Kannan, and A. Schrijver, Chvátal closures for mixed integer programming problems, *Mathematical Programming*, **47**, 155–174, (1990).

[6] S. Dash, O. Günlük and A. Lodi. MIR Closures of Polyhedral Sets, *Math. Prog.*, **121**, 33–60, (2010).

[7] G. Nemhauser and L. A. Wolsey, A recursive procedure to generate all cuts for 0-1 mixed integer programs, *Mathematical Programming*, **46**, 379–390, (1990).

[8] L.A. Wolsey, *Integer Programming*, Wiley, New York (1998).