Some Properties of Convex Hulls of Integer Points Contained in General Convex Sets

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Abstract

In this paper, we study properties of general closed convex sets that determine the closed-ness and polyhedrality of the convex hull of integer points contained in it. We first present necessary and sufficient conditions for the convex hull of integer points contained in a general convex set to be closed. This leads to useful results for special class of convex sets such as pointed cones, strictly convex sets, and sets containing integer points in their interior. We then present sufficient conditions for the convex hull of integer points in general convex sets to be polyhedron. These sufficient conditions generalize the sufficient conditions given in Meyer [8]. Under a simple technical condition, we show that these sufficient conditions are also necessary conditions for the convex hull of integer points contained in general convex sets to be polyhedra.

1 Introduction

An important goal in the study of mathematical programming is to analyze properties of the convex hull of feasible solutions. The Fundamental Theorem of Integer Programming, due to Meyer [8], states that the convex hull of feasible points in a mixed integer linear set defined by rational data is a polyhedron. The proof of this result relies on (i) the Minkowski-Weyl Representation Theorem for polyhedron and (ii) the fact that the recession cone is a rational polyhedral cone and thus generated by a finite number of integer vectors. In the world of mixed integer linear programming (MILP) problems, these sufficient conditions for polyhedrality of the convex hull of feasible solutions are reasonable since we expect most instances to be described using rational data.

A convex integer program is an optimization problem where the feasible region is of the form $K \cap \mathbb{Z}^n$ where $K \subseteq \mathbb{R}^n$ is a closed convex set. Let $\text{conv}(K \cap \mathbb{Z}^n)$ represent the convex hull of $K \cap \mathbb{Z}^n$. In this setting typically we do not have Minkowski-Weyl type Representation Theorem for K or nice properties of recession cone of K. Therefore a natural question is to generalize Meyer's Theorem, in order to understand properties of the set K that lead to $\text{conv}(K \cap \mathbb{Z}^n)$ being a polyhedron. Note that [3] presents condition about the set $K \cap \mathbb{Z}^n$ (and more generally any subset of $\mathbb{Z}^n$) such that elements of $K \cap \mathbb{Z}^n$ have a finite integral generating set. In contrast, here we are interested in properties of the set K that allow us to deduce that $\text{conv}(K \cap \mathbb{Z}^n)$ is a polyhedron.

Observe that if $\text{conv}(K \cap \mathbb{Z}^n)$ is a polyhedron, then $\text{conv}(K \cap \mathbb{Z}^n)$ is a closed set. To the best of our knowledge, even the basic question of conditions that lead to $\text{conv}(K \cap \mathbb{Z}^n)$ being closed is not well-understood. (See [9] for some sufficient conditions for closed-ness of $\text{conv}(K \cap \mathbb{Z}^n)$ when K is a polyhedron that is not necessarily described by rational data). We therefore divide this paper into two parts.

In the first part of this paper (Section 2), we present necessary and sufficient conditions for $\text{conv}(K \cap \mathbb{Z}^n)$ to be closed when K contains no lines (Theorem 2.1). This characterization also leads to useful results for special classes of convex sets such as sets containing integer points in their interior (Theorem 2.3), strictly convex sets (Theorem 2.4), and pointed cones (Theorem 2.5). The necessary and sufficient conditions for sets containing integer points in their interior generalize the sufficient conditions presented in [9]. The case where K contains lines is then dealt separately (Theorem 2.6).

In the second part of this paper (Section 3), we present sufficient conditions for the convex hull of integer points contained in general convex sets to be a polyhedron (Theorem 3.1). These sufficient conditions generalize the sufficient conditions presented in [8]. For a general convex set K containing at least one integer point in its interior, we show that these sufficient conditions are also necessary conditions for $\text{conv}(K \cap \mathbb{Z}^n)$ to be a polyhedron (Theorem 3.1).

We conclude with some remarks in Section 4.

2 Closed-ness of conv$(K \cap \mathbb{Z}^n)$

For $u \in \mathbb{R}^n$ and $\epsilon > 0$, we use the notation $B(u, \epsilon)$ to denote the set $\{x \in \mathbb{R}^n \,|\, \|x - u\| \leq \epsilon\}$. Let $K \subseteq \mathbb{R}^n$. In this paper $\overline{K}$ represents the closure of K, $\text{int}(K)$ represents the interior of K, $\text{bd}(K)$ represents the boundary of K, $\text{rel.int}(K)$ represents the relative interior of K, $\dim(K)$ represents the dimension of K, $\text{rec.cone}(K)$ represents the recession cone of K, $\text{lin.space}(K)$ represents the lineality space of K and $\text{aff}(K)$ represents the affine hull of K. Note that

$$\text{rec.cone}(K) = \{d \in \mathbb{R}^n \,|\, x + \lambda d \in K \;\forall x \in K, \forall \lambda \geq 0\},$$

is defined for all sets and not just for closed sets. For example, consider the set $A = \text{conv}(\{(0,1)\} \cup \{x \in \mathbb{R}^2 \,|\, x_2 = 0\})$. Then $\text{rec.cone}(A) = \{0\}$ and $\text{rec.cone}(\text{int}(A)) = \text{rec.cone}(\overline{A}) = \{\lambda_1(1,0) + \lambda_2(-1,0) \,|\, \lambda_1, \lambda_2 \geq 0\}$. In general, the following result is true; see [11] for a proof.

Lemma 2.1 ([11]). *Let $K \subseteq \mathbb{R}^n$ be a convex set. Then*

$$\text{rec.cone}(\text{rel.int}(K)) = \text{rec.cone}(\overline{K}) \supseteq \text{rec.cone}(K).$$

Before presenting the results of this section, we develop some intuition by examining some examples.

Example 2.1. *If K is a bounded convex set, then $\text{conv}(K \cap \mathbb{Z}^n)$ is a polytope. Therefore properties of the recession cone play an important role in determining the closed-ness of $\text{conv}(K \cap \mathbb{Z}^n)$. Intuitively, it appears that irrational extreme recession directions of K may cause $\text{conv}(K \cap \mathbb{Z}^n)$ to be not closed. However this is not entirely true as illustrated in the next few examples.*

1. *First consider the set $K^1 = \{x \in \mathbb{R}^2 \,|\, x_2 - \sqrt{2}x_1 \leq 0, x_2 \geq 0\}$. It is easily verified that in this case $\text{conv}(K^1 \cap \mathbb{Z}^2)$ is not closed. In particular, the half-line $\{x \in \mathbb{R}^2 \,|\, x_2 - \sqrt{2}x_1 = 0, x_2 > 0\}$, belongs to $\overline{\text{conv}}(K \cap \mathbb{Z}^2)$ but not to $\text{conv}(K \cap \mathbb{Z}^2)$. In this case it is clear that the irrational data describing the polyhedron causes $\text{conv}(K^1 \cap \mathbb{Z}^2)$ to be not closed.*

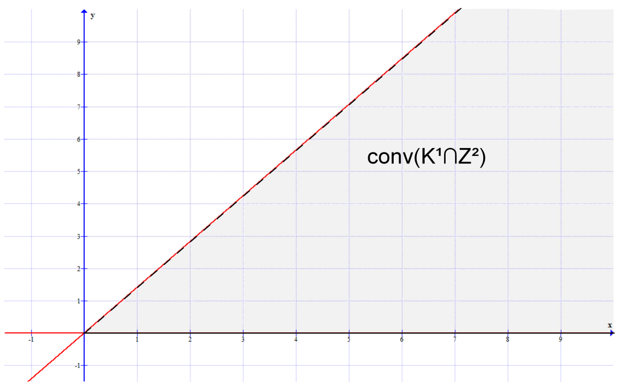

Figure 1: K^1 and $\text{conv}(K^1 \cap \mathbb{Z}^2)$.

2. *Now consider the set* $K^2 = \{x \in \mathbb{R}^2 \,|\, x_2 - \sqrt{2}x_1 \leq 0, x_2 \geq 0, x_1 \geq 1\}$*. Notice that the recession cone of* K^1 *and* K^2 *are the same. In fact* $(K^1 \cap \mathbb{Z}^2) = (K^2 \cap \mathbb{Z}^2) \cup \{(0,0)\}$*. However, we will verify (also see Figure 2) that* $\mathrm{conv}(K^2 \cap \mathbb{Z}^2)$ *is closed.*

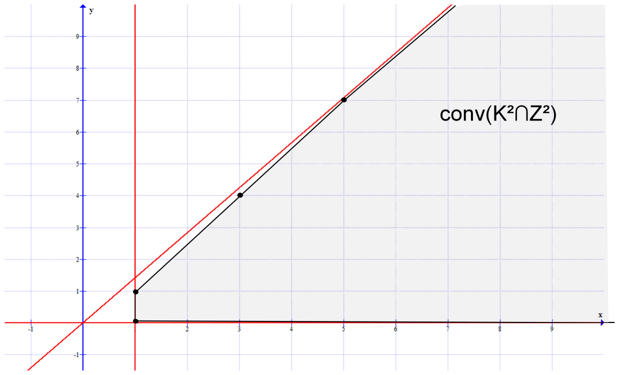

Figure 2: K^2 and $\mathrm{conv}(K^2 \cap \mathbb{Z}^2)$.

We next illustrate a similar observation (i.e. recession cone of K *has irrational extreme ray, but* $\mathrm{conv}(K \cap \mathbb{Z}^n)$ *is closed) using non-polyhedral sets.*

3. *Let* $K^3 = \{x \in \mathbb{R}^2 \,|\, x_2 \geq x_1^2\}$*. The recession cone of* K^3 *is* $\{x \in \mathbb{R}^2 \,|\, x_1 = 0, x_2 \geq 0\}$*. It can be shown that* $\mathrm{conv}(K^3 \cap \mathbb{Z}^2)$ *is closed.*

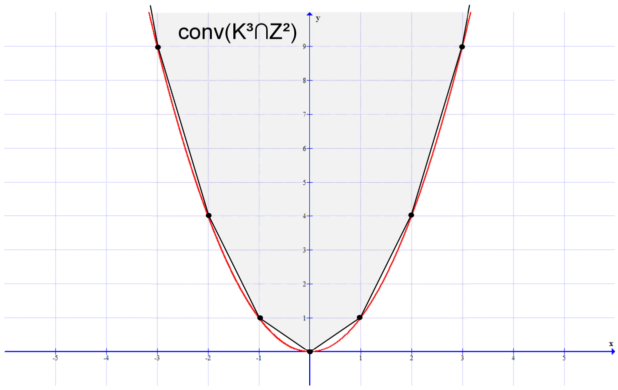

Figure 3: K^3 and $\mathrm{conv}(K^3 \cap \mathbb{Z}^2)$.

4. *Now consider the set where we rotate the parabola* K^3 *such that the new recession cone is* $\{x \in \mathbb{R}^2 \,|\, \sqrt{2}x_1 = x_2, x_2 \geq 0\}$*, i.e., consider the set* $K^4 = \{x \in \mathbb{R}^2 \,|\, \frac{1}{\sqrt{3}} \begin{bmatrix} -1 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix} x \in K^3\}$*. In this case, even though the recession cone is a non-rational polyhedral set, it can be verified that* $\mathrm{conv}(K^4 \cap \mathbb{Z}^2)$ *is closed.*

Observe that all the sets discussed above have polyhedral recession cones. However, sets whose recession cone are non-polyhedral can also lead to $\mathrm{conv}(K \cap \mathbb{Z}^n)$ *being closed.*

5. *Consider the set* $K^5 = \{(0,0,1)\} \cup \{(0,1,1)\} \cup \{(\frac{1}{n}, \frac{1}{n^2}, 1)\} \; \forall n \in \mathbb{Z}, n \geq 1$*. Then* K^5 *is closed, since it contains all its limit points. Therefore* K^5 *is a compact set and thus* $\mathrm{conv}(K^5)$ *is compact (Theorem 17.2 [11]). Therefore,* $K^6 = \mathrm{conv}\left(\{\sum_{u \in K^5} \lambda_u u \,|\, \lambda_u \geq 0 \; \forall u \in R^5\}\right)$ *is a closed convex cone. Finally, it can be verified that* $\mathrm{conv}(K^6 \cap \mathbb{Z}^3) = K^6$ *is closed.*

2.1 Necessary and sufficient conditions for closed-ness of conv$(K \cap \mathbb{Z}^n)$ for sets with no lines

Definition 2.1 ($u(K)$). *Given a convex set $K \subseteq \mathbb{R}^n$ and $u \in K \cap \mathbb{Z}^n$, we define $u(K) = \{d \in \mathbb{R}^n \mid u + \lambda d \in \mathrm{conv}(K \cap \mathbb{Z}^n)\ \forall \lambda \geq 0\}$.*

In this section we will prove the following result.

Theorem 2.1. *Let $K \subseteq \mathbb{R}^n$ be a closed convex set not containing a line. Then $\mathrm{conv}(K \cap \mathbb{Z}^n)$ is closed if and only if $u(K)$ is identical for every $u \in K \cap \mathbb{Z}^n$.*

Note here that when $u(K)$ is identical for every $u \in K \cap \mathbb{Z}^n$, Theorem 2.1 implies that $\mathrm{conv}(K \cap \mathbb{Z}^n)$ is closed and therefore we obtain $u(K) = \mathrm{rec.cone}(\mathrm{conv}(K \cap \mathbb{Z}^n))$ is closed for every $u \in K \cap \mathbb{Z}^n$.

It is not difficult to verify that,

$$\mathrm{conv}(K \cap \mathbb{Z}^n) = \mathrm{conv}\left(\bigcup_{u \in K \cap \mathbb{Z}^n} (u + u(K))\right). \tag{1}$$

Hence Theorem 2.1 states that if the recession cone of each $u + u(K)$ is identical, then the convex hull of their union is closed. Therefore Theorem 2.1 is very similar in flavor to the following result.

Lemma 2.2 ([11]). *If K_1, ..., K_m are non-empty closed convex sets in $\mathbb{R}^n$ all having the same recession cone, then $\mathrm{conv}(K_1 \cup \ldots \cup K_m)$ is closed.*

Note however that Lemma 2.2 is not directly useful in verifying the 'sufficient part' of Theorem 2.1 since the number of integer points in a general convex set in not necessarily finite and thus the union in the right-hand-side of equation (1) is possibly over a countably infinite number of sets. Lemma 2.2 does not extend to infinite unions, in fact it does not hold even if the individual sets are polyhedra with same recession cone. (Consider for example $\mathrm{conv}(\cup_{i \in \mathbb{Z}, i \geq 1} K_i)$ where $K_i = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = \frac{1}{i}, x_2 \geq 0\}$.) However, we note here that the proof of Theorem 1 presented here will eventually use Lemma 2.2 is some cases, by suitably converting the set $\mathrm{conv}\left(\cup_{u \in K \cap \mathbb{Z}^n} (u + u(K))\right)$ to the convex hull of the union of a finite number of appropriate sets.

We begin by presenting some results that are required for the proof of Theorem 2.1. The following crucial result is from [5]. Also see [6].

Lemma 2.3 ([5]). *Let $K \subseteq \mathbb{R}^n$ be a nonempty closed set. Then every extreme point of $\overline{\mathrm{conv}}(K)$ belongs to K.*

Lemma 2.4. *Let A be a $n \times n$ matrix and let $K \subseteq \mathbb{R}^n$ be a closed convex set. Then $\mathrm{conv}(AK) = A\mathrm{conv}(K)$.*

Proof. Let $\lambda_1, \ldots, \lambda_m \in \mathbb{R}_+$ such that $\sum_{i=1}^m \lambda_i = 1$ and let $x_1, \ldots, x_m \in K$. By linearity of A, we have the following identity $A\left(\sum_{i=1}^m \lambda_i x_i\right) = \sum_{i=1}^m \lambda_i A x_i$. This identity implies the result. □

Lemma 2.5. *Let U be a $n \times n$ unimodular matrix and let $K \subseteq \mathbb{R}^n$ be a closed convex set. Then $\mathrm{conv}(K \cap \mathbb{Z}^n)$ is closed iff $\mathrm{conv}((UK) \cap \mathbb{Z}^n)$ is closed.*

Proof. Since the linear transformation defined by $U : \mathbb{R}^n \to \mathbb{R}^n$ is an homeomorphism, we conclude $\mathrm{conv}(K \cap \mathbb{Z}^n)$ is closed iff $U\mathrm{conv}(K \cap \mathbb{Z}^n)$ is closed. It remains to show that $U\mathrm{conv}(K \cap \mathbb{Z}^n) = \mathrm{conv}((UK) \cap \mathbb{Z}^n)$. Since U is unimodular, we obtain that $U(K \cap \mathbb{Z}^n) = U(K) \cap U(\mathbb{Z}^n) = U(K) \cap \mathbb{Z}^n$. Therefore, by Lemma 2.4, $U(\mathrm{conv}(K \cap \mathbb{Z}^n)) = \mathrm{conv}(U(K \cap \mathbb{Z}^n)) = \mathrm{conv}((UK) \cap \mathbb{Z}^n)$. □

Lemma 2.6 ([11]). *Let $K \subseteq \mathbb{R}^n$ be a closed convex set not containing a line. Let S be the set of extreme points of K. Then $K = \mathrm{conv}(S) + \mathrm{rec.cone}(K)$.*

A convex set $K \subseteq \mathbb{R}^n$ is called *lattice-free*, if $\mathrm{int}(K) \cap \mathbb{Z}^n = \emptyset$. A lattice-free convex set $K \subseteq \mathbb{R}^n$ is called *maximal lattice-free convex set* if there does not exist a lattice-free convex set $K' \subseteq \mathbb{R}^n$ satisfying $K' \supsetneq K$.

We note here that every lattice-free convex set is contained in a maximal lattice-free convex set. The following characterization of maximal lattice-free convex set is from [7]. See also [1] for related result.

Theorem 2.2 ([7])**.** *A full-dimensional lattice-free convex set $K \subseteq \mathbb{R}^n$ is a maximal lattice-free convex set is and only if K is a polyhedron of the form $K = P + L$, where P is a polytope and L is a rational linear subspace and every facet of K contains a point of $\mathbb{Z}^n$ in its relative interior.*

We now present the proof of the main result of this section.

Proof. of Theorem 1 If $K \cap \mathbb{Z}^n = \emptyset$, then the result is trivial. Therefore, we will assume that $K \cap \mathbb{Z}^n \neq \emptyset$.
$\Rightarrow$ If conv$(K \cap \mathbb{Z}^n)$ is closed, then $\forall\, u \in K \cap \mathbb{Z}^n$, $u(K) = \text{rec.cone}(\overline{\text{conv}}(K \cap \mathbb{Z}^n))$. Thus $u(K)$ is identical for all $u \in K \cap \mathbb{Z}^n$.
$\Leftarrow$ By definition of $u(K)$, we have that

$$\text{rec.cone}(\text{conv}(K \cap \mathbb{Z}^n)) \subseteq u(K) \subseteq \text{rec.cone}(\overline{\text{conv}}(K \cap \mathbb{Z}^n)) \;\forall\, u \in K \cap \mathbb{Z}^n. \tag{2}$$

Assume now that $u(K)$ is identical for every $u \in K \cap \mathbb{Z}^n$. We first claim that $u(K) = \text{rec.cone}(\text{conv}(K \cap \mathbb{Z}^n))$ $\forall\, u \in K \cap \mathbb{Z}^n$. Let $r \in u(K)$ and $x \in \text{conv}(K \cap \mathbb{Z}^n)$. We can write $x = \sum_{i=1}^{N} \alpha_i z_i$, where $z_i \in K \cap \mathbb{Z}^n$, $\alpha_i \geq 0$ for all $i = 1, \dots, N$ and $\sum_{i=1}^{N} \alpha_i = 1$. Since $r \in z_i(K)$, $\forall\, i = 1, \dots, N$, we have $z_i + \lambda r \in \text{conv}(K \cap \mathbb{Z}^n)$ for all $\lambda \geq 0$. Since $x + \lambda r = \sum_{i=1}^{N} \alpha_i (z_i + \lambda r)$, we obtain that $x + \lambda r \in \text{conv}(K \cap \mathbb{Z}^n)$. Thus, $u(K) \subseteq \text{rec.cone}(\text{conv}(K \cap \mathbb{Z}^n))$ and by (2) we obtain that

$$u(K) = \text{rec.cone}(\text{conv}(K \cap \mathbb{Z}^n)) \;\forall\, u \in K \cap \mathbb{Z}^n. \tag{3}$$

We will now show that conv$(K \cap \mathbb{Z}^n)$ is closed. There are two cases:

- **Case 1:** rel.int$(\text{conv}(K \cap \mathbb{Z}^n)) \cap \mathbb{Z}^n \neq \emptyset$. We will verify that $\text{conv}(K \cap \mathbb{Z}^n) \supseteq \overline{\text{conv}}(K \cap \mathbb{Z}^n)$. Let $u \in \text{rel.int}(\text{conv}(K \cap \mathbb{Z}^n)) \cap \mathbb{Z}^n$. By definition of $u(K)$, rec.cone(rel.int$(\text{conv}(K \cap \mathbb{Z}^n)) \subseteq u(K)$. Since $\text{rec.cone}(\overline{\text{conv}}(K \cap \mathbb{Z}^n)) = \text{rec.cone}(\text{rel.int}(\text{conv}(K \cap \mathbb{Z}^n))$ (Lemma 2.1), by using (2) we conclude that $u(K) = \text{rec.cone}(\overline{\text{conv}}(K \cap \mathbb{Z}^n))$. Therefore by using (3) we obtain that $\text{rec.cone}(\text{conv}(K \cap \mathbb{Z}^n)) = \text{rec.cone}(\overline{\text{conv}}(K \cap \mathbb{Z}^n))$. Finally by Lemma 2.3, the extreme points of $\overline{\text{conv}}(K \cap \mathbb{Z}^n)$ belong to conv$(K \cap \mathbb{Z}^n)$. Since $\overline{\text{conv}}(K \cap \mathbb{Z}^n) \subseteq K$, it does not contain any lines. Thus, by Lemma 2.6, $\overline{\text{conv}}(K \cap \mathbb{Z}^n)$ is given by the convex hull of its extreme points plus its recession cone. Since the extreme points of $\overline{\text{conv}}(K \cap \mathbb{Z}^n)$ belongs to conv$(K \cap \mathbb{Z}^n)$ and $\text{rec.cone}(\text{conv}(K \cap \mathbb{Z}^n)) = \text{rec.cone}(\overline{\text{conv}}(K \cap \mathbb{Z}^n))$, we obtain that $\text{conv}(K \cap \mathbb{Z}^n) \supseteq \overline{\text{conv}}(K \cap \mathbb{Z}^n)$. Therefore, conv$(K \cap \mathbb{Z}^n)$ is closed.

- **Case 2:** rel.int$(\text{conv}(K \cap \mathbb{Z}^n)) \cap \mathbb{Z}^n = \emptyset$. We will use induction on the dimension of conv$(K \cap \mathbb{Z}^n)$. The base case, $\dim(\text{conv}(K \cap \mathbb{Z}^n)) = 0, 1$ is straightforward to verify.

  Suppose now the property is true for every closed convex set K' such that $\dim(\text{conv}(K' \cap \mathbb{Z}^n)) < \dim(\text{conv}(K \cap \mathbb{Z}^n))$.

  First for convenience, we redefine $K := K \cap \text{aff}(K \cap \mathbb{Z}^n)$. Therefore $\dim(K \cap \mathbb{Z}^n) = \dim(K)$. Let $z \in K \cap \mathbb{Z}^n$. We now translate K as $K - \{z\}$ and note that it is sufficient to show that conv$(K \cap \mathbb{Z}^n)$ is closed for this new set K. Observe that aff$(K \cap \mathbb{Z}^n)$ is a rational linear subspace, since it is generated by integer vector. Now by selecting a suitable unimodular matrix (see [12]) and by the application of Lemma 2.5, we may assume that aff$(K \cap \mathbb{Z}^n)$ is of the form $\{x \,|\, x_i = 0 \;\forall i = k+1, ..., n\}$. Finally, we can project out the last $n-k$ components (every point in K has zero in these components) and note that it is sufficient to show that conv$(K \cap \mathbb{Z}^k)$ is closed for this new set $K \subseteq \mathbb{R}^k$.

  In particular, without loss of generality, we may assume that conv$(K \cap \mathbb{Z}^n)$ is full-dimensional.

  Note that conv$(K \cap \mathbb{Z}^n)$ is lattice-free, and therefore there exists a full-dimensional maximal lattice-free polyhedron $Q \subseteq \mathbb{R}^n$ such that conv$(K \cap \mathbb{Z}^n) \subseteq Q$ and $Q = P + L$, where P is a polytope and L is a rational linear subspace.

  Let F_i, $i = 1, \dots, N$ be the facets of Q such that $K \cap F_i \cap \mathbb{Z}^n \neq \emptyset$. We will verify that

$$\text{conv}(K \cap \mathbb{Z}^n) \cap F_i = \text{conv}(K \cap F_i \cap \mathbb{Z}^n). \tag{4}$$

  Since conv$(K \cap \mathbb{Z}^n) \cap F_i$ is a convex set and contains $K \cap F_i \cap \mathbb{Z}^n$ we have $\text{conv}(K \cap \mathbb{Z}^n) \cap F_i \supseteq \text{conv}(K \cap F_i \cap \mathbb{Z}^n)$. On the other hand, let $x \in \text{conv}(K \cap \mathbb{Z}^n) \cap F_i$. Therefore $x = \sum_{j=1}^{M} \alpha_j z_j$, where

$z_j \in K \cap \mathbb{Z}^n$, $\alpha_j \geq 0$ for all $j = 1, \ldots, M$, and $\sum_{j=1}^{M} \alpha_j = 1$. Since $K \cap \mathbb{Z}^n \subseteq Q$ and $x \in F_i$, we must have $z_j \in F_i$, $\forall$, $j = 1, \ldots, M$, so $x \in \text{conv}(K \cap F_i \cap \mathbb{Z}^n)$.

Next we verify that

$$u(K \cap F_i) = u(K) \cap L \ \forall \ u \in K \cap F_i \cap \mathbb{Z}^n, \ \forall \ i = 1, \ldots, N. \tag{5}$$

Let $r \in u(K \cap F_i)$. Therefore, by definition we have that $u + \lambda r \in \text{conv}(K \cap F_i \cap \mathbb{Z}^n) \ \forall \ \lambda \geq 0$. By (4), this is equivalent to $u + \lambda r \in \text{conv}(K \cap \mathbb{Z}^n) \cap F_i \ \forall \ \lambda \geq 0$. This is also equivalent to $u + \lambda r \in \text{conv}(K \cap \mathbb{Z}^n) \ \forall \ \lambda \geq 0$ and $u + \lambda r \in F_i \ \forall \ \lambda \geq 0$. Thus equivalently we obtain that $r \in u(K)$ and $r \in \text{rec.cone}(F_i) = L$, i.e., $r \in u(K) \cap L$. Thus, $u(K \cap F_i) = u(K) \cap L$.

Since $u(K)$ is identical for all $u \in K \cap \mathbb{Z}^n$, (5) implies that $u(K \cap F_i)$ is identical for every $u \in K \cap F_i \cap \mathbb{Z}^n$ and $\forall \ i = 1, \ldots, N$. Moreover, since $\text{conv}(K \cap F_i \cap \mathbb{Z}^n) \subseteq F_i$, $\dim(\text{conv}(K \cap F_i \cap \mathbb{Z}^n)) < \dim(\text{conv}(K \cap \mathbb{Z}^n))$. So we can apply the induction hypothesis to conclude that $\text{conv}(K \cap F_i \cap \mathbb{Z}^n)$ is a closed set. (Note that if $\text{rel.int}(\text{conv}(K \cap F_i \cap \mathbb{Z}^n)) \cap \mathbb{Z}^n \neq \emptyset$, then this follows from case 1.)

We now have that $u(K \cap F_i) = \text{rec.cone}(\text{conv}(K \cap F_i \cap \mathbb{Z}^n)) = u(K) \cap L$ for all $i = 1, \ldots, N$. So the recession cone of $\text{conv}(K \cap F_i \cap \mathbb{Z}^n)$ is the same for all $i = 1, \ldots, N$. Observe that,

$$\text{conv}(K \cap \mathbb{Z}^n) = \text{conv}\left[\bigcup_{i \in \{1, \ldots, N\}} \text{conv}(K \cap F_i \cap \mathbb{Z}^n)\right].$$

Since the convex hull of a finite union of closed convex sets with the same recession cone is closed (Lemma 2.2), we conclude that $\text{conv}(K \cap \mathbb{Z}^n)$ is closed.

□

We note here that the condition that K contains no line in the statement of Theorem 2.1 is not superfluous. This is illustrated in the next Example.

Example 2.2. *Consider the set* $K^7 := \{(x_1, x_2) \in \mathbb{R}^2 \,|\, 0.5 \leq x_2 - \sqrt{2}x_1 \leq 0.7\}$ *which contains a line and let* $u \in K^7 \cap \mathbb{Z}^2$. *Since* $\text{bd}(K^7) \cap \mathbb{Z}^2 = \emptyset$, *we have that* $u \in \text{int}(K^7)$. *Since* $u \in \text{int}(K^7)$, *it can be verified that* $u(K^7) = \text{rec.cone}(K^7)$ *(see Lemma 2.10 in the next Section). Thus* $u(K^7)$ *is closed and identical for all* $u \in K^7 \cap \mathbb{Z}^2$. *However,* $\text{conv}(K^7 \cap \mathbb{Z}^2)$ *is not closed, since* $\text{conv}(K^7 \cap \mathbb{Z}^2) = \text{int}(K^7)$.

2.2 Closed-ness of conv$(K \cap \mathbb{Z}^n)$ where int$(K) \cap \mathbb{Z}^n \neq \emptyset$

In this section, we simplify the conditions of Theorem 2.1 for the case where $\text{int}(\text{conv}(K \cap \mathbb{Z}^n)) \cap \mathbb{Z}^n \neq \emptyset$. We will assume that K and $K \cap \mathbb{Z}^n$ are full-dimensional throughout this section. In particular if K and $K \cap \mathbb{Z}^n$ are not full-dimensional, then by application of Lemma 2.5 as in the proof of Theorem 2.1, we can modify K and subsequently apply projection to achieve full-dimensionality of K and $K \cap \mathbb{Z}^n$.

In this section, we prove the following result.

Theorem 2.3. *Let* $K \subseteq \mathbb{R}^n$ *be a closed convex set not containing a line and containing an integer point in its interior and assume that* $K \cap \mathbb{Z}^n$ *is also full-dimensional. Then the following are equivalent.*

1. $\text{conv}(K \cap \mathbb{Z}^n)$ *is closed.*
2. $u(K) = \text{rec.cone}(K) \ \forall u \in K \cap \mathbb{Z}^n$.
3. *The following property holds for every proper exposed face* F *of* K*: If* $F \cap \mathbb{Z}^n \neq \emptyset$, *then for all* $u \in F \cap \mathbb{Z}^n$ *and for all* $r \in \text{rec.cone}(F)$, $\{u + \lambda r \,|\, \lambda \geq 0\} \subseteq \text{conv}(F \cap \mathbb{Z}^n)$.

Theorem 2.3 converts the question of verification of closed-ness of $\text{conv}(K \cap \mathbb{Z}^n)$ to the verification of a somewhat simpler property of the faces of the set K. To see a simple application of Theorem 2.3 consider the cases (1.) and (2.) presented in Example 2.1. Note that both K^1 and K^2 contain integer points in their interior and $\text{conv}(K \cap \mathbb{Z}^n)$ is full-dimensional. In (1.), the facet $F := \{x \in \mathbb{R}^2 \,|\, x_2 = \sqrt{2}x_1, x_2 \geq 0\}$ contains only the point $(0, 0)$ and thus does not satisfy the property presented in Theorem 2.3. Hence we deduce that conv$(K^1 \cap \mathbb{Z}^n)$ is not closed. On the other hand, since the facet $\{x \in \mathbb{R}^2 \mid x_2 = \sqrt{2}x_1, x_2 \geq 0, x_1 \geq 1\}$ contains no integer point and all other faces of K^2 also satisfy the property presented in Theorem 2.3, we can deduce that conv$(K^2 \cap \mathbb{Z}^n)$ is closed.

We note that Theorem 2.3 generalizes sufficient conditions for closed-ness of conv$(K \cap \mathbb{Z}^n)$ presented in [9]. [9] shows that conv$(K \cap \mathbb{Z}^n)$ is closed if K is a polyhedron that contains no lines, rec.cone(K) is full-dimensional and every face of K satisfies the conditions described in the statement of Theorem 2.3.

Before we present the proof of Theorem 2.3, we first present a sequence of preliminary Lemmas. The following Lemma is a consequence of the Dirichlet Diophantine Approximation and was proven in this form in [1].

Lemma 2.7 ([1]). *If $x \in \mathbb{Z}^n$ and $r \in \mathbb{R}^n$, then for all $\epsilon > 0$ and $\gamma \geq 0$, there exists a point of $\mathbb{Z}^n$ at a distance less than ϵ from the half line $\{x + \lambda r \mid \lambda \geq \gamma\}$.*

We will call a vector $r \in \mathbb{R}^n$ *rational scalable* if there exists $\lambda \in \mathbb{R} \setminus \{0\}$ such that $\lambda r \in \mathbb{Z}^n$. The proof of the next lemma is similar to the proof of a related result in [1]

Lemma 2.8. *Let r be a vector that is not rational scalable. Let $\gamma \geq 0$ and let Λ be the projection of $P := \{x \in \mathbb{Z}^n \mid \langle r, x\rangle \geq \gamma\}$ on the linear subspace $\{x \in \mathbb{R}^n \mid \langle r, x\rangle = 0\}$. Then there exists a non-trivial subspace V of $\mathbb{R}^n$ such that $\Lambda \cap V$ is dense in V.*

Proof: By Lemma 2.7, for every $\epsilon > 0$ there exists a point $y \in \Lambda$, $y \neq 0$ such that $\|y\| \leq \epsilon$. Let V_ϵ be the linear subspace generated by all $y \in \Lambda$ such that $\|y\| \leq \epsilon$. Observe that $\epsilon^1 > \epsilon^2$ implies that $V_{\epsilon^1} \supseteq V_{\epsilon^2}$. Therefore, there exists ϵ_0 such that $V_\epsilon = V_{\epsilon_0} \neq \{0\}$ for all $0 < \epsilon \leq \epsilon_0$. Now we claim that $\Lambda \cap U$ is dense in $U = V_{\epsilon_0}$.

First we claim that for any $\epsilon > 0$ and $u \in \mathbb{Z}^n$ s.t. $\langle u, r\rangle < 0$, there exists $v \in P$ (i.e. $v \in \mathbb{Z}^n$ and $\langle v, r\rangle \geq \gamma$) such that the projection of u and v onto the subspace $\{x \in \mathbb{R}^n \mid \langle r, x\rangle = 0\}$ are at a distance less than ϵ. Let $u = a - \alpha r$ where $\langle a, r\rangle = 0$ and $\alpha > 0$. Since $u \in \mathbb{Z}^n$, by Lemma 2.7 we obtain that there exists $v \in \mathbb{Z}^n$ so that the distance between the half-line $\{u + \lambda r \mid \lambda \geq \frac{\gamma}{\|r\|^2} + \alpha\}$ and v is less than ϵ. This completes the proof of the claim.

Let $x \in U$, $\epsilon > 0$, $m = \dim(U)$. Choose any $\delta > 0$ s.t. $\delta < \min\{\epsilon_0, \frac{\epsilon}{m}\}$. By definition of U, $x \in V_\delta$. So, $x = \sum_{i=1}^m \lambda_i y_i$ where $\lambda_i \in \mathbb{R}$ and $y_i \in \Lambda \cap U$ and $\|y_i\| < \delta$ for $i = 1, \ldots, m$. Choose $n_i \in \mathbb{Z}$ such that $|n_i - \lambda_i| \leq \frac{1}{2}$. Let y_i be the projection of $p_i \in P$. If $n_i \geq 1$, then $n_i y_i$ is the projection of the point $n_i p_i \in P$. If $n_i \leq 0$, by the use of previous claim, let $q_i \in P$ such that distance between $n_i y_i$ and w_i, the projection of q_i on the linear subspace $\{x \in \mathbb{R}^n \mid \langle r, x\rangle = 0\}$, is less than $\frac{\delta}{2}$. Let $y = \sum_{n_i > 0} n_i y_i + \sum_{n_i \leq 0} w_i$. Then,

$$\begin{aligned}
\|x - y\| &= \left\| \sum_{n_i > 0} (\lambda_i - n_i) y_i + \sum_{n_i \leq 0} (\lambda_i y_i - w_i) \right\| \\
&= \left\| \sum_{n_i > 0} (\lambda_i - n_i) y_i + \sum_{n_i \leq 0} (\lambda_i y_i - n_i y_i + n_i y_i - w_i) \right\| \\
&= \left\| \sum_{i=1}^m (\lambda_i - n_i) y_i + \sum_{n_i \leq 0} (n_i y_i - w_i) \right\| \\
&\leq \left\| \sum_{i=1}^m (\lambda_i - n_i) y_i \right\| + \sum_{n_i \leq 0} \|(n_i y_i - w_i)\| \\
&< m\delta < \epsilon.
\end{aligned}$$

Finally, since y is an integer non-negative linear combination of elements of $\Lambda \cap U$, we conclude that $y \in \Lambda \cap U$, completing the proof. □

Lemma 2.9. *Let $V \subseteq \mathbb{R}^n$ be a linear subspace and let $\Lambda \subseteq V$ be dense in V. Then for all $\epsilon > 0$, $0 \in \text{conv}(B(0, \epsilon) \cap \Lambda)$.*

Proof. For clarity, we use the notation 0^t to represent the t-dimensional vector of zeros and use 0 for 0^1 in this proof.

Observe that $\Lambda \cap B(0^n, \epsilon)$ is dense in $\Psi := V \cap B(0^n, \epsilon)$. For ease in computation, without loss of generality, we apply an invertible linear transformation to V such that $V = (\mathbb{R}^k, 0^{n-k})$. Let $\delta > 0$ such that $\delta < \frac{\epsilon}{2}$. For $B \subseteq \{1, \dots, k\}$ let

$$\tilde{v}_B = \begin{cases} \delta & i \in B \\ -\delta & i \notin B. \end{cases}$$

Observe that since $\Lambda \cap B(0^n, \epsilon)$ is dense in Ψ, there exists $v_B \in \Lambda \cap \Psi$ such that $\|v_B - \tilde{v}_B\| < \frac{\delta}{3}$ for all $B \subseteq \{1, \dots, k\}$. Now we claim that 0^n belongs to $\text{conv}(\cup_{B \subset \{1,\dots,k\}} v_B)$. The proof is by induction on the dimension of Ψ, i.e., on k. It is straightforward to verify that this result is true for $k = 1$. Now by induction hypothesis, assume that the result is true when $k = t - 1$. Let $\mathcal{B}^- = \{B \subseteq \{1, \dots, t\} \,|\, t \notin B\}$ and $\mathcal{B}^+ = \{B \subseteq \{1, \dots, t\} \,|\, t \in B\}$. By the induction hypothesis, there exists a point of the form $(0^{t-1}, \sum_{B \in \mathcal{B}^-} \sigma_B (v_B)_t, 0^{n-t}) =: \sum_{B \in \mathcal{B}^-} \sigma_B v_B$ that is a convex combination of v_B, $B \in \mathcal{B}^-$. Note that $-\delta - \frac{\delta}{3} \leq \sum_{B \in \mathcal{B}^-} \sigma_B (v_B)_t \leq -\delta + \frac{\delta}{3}$. Similarly, by the induction hypothesis, there exists a point of the form $(0^{t-1}, \sum_{B \in \mathcal{B}^+} \rho_B (v_B)_t, 0^{n-t})$ that is a convex combination of v_B, $B \in \mathcal{B}^+$ and $\delta - \frac{\delta}{3} \leq \sum_{B \in \mathcal{B}^-} \rho_B (v_B)_t \leq \delta + \frac{\delta}{3}$. Now observe that 0^n is a convex combination of $(0^{t-1}, \sum_{B \in \mathcal{B}^-} \gamma_B (v_B)_t, 0^{n-t})$ and $(0^{t-1}, \sum_{B \in \mathcal{B}^+} \rho_B (v_B)_t, 0^{n-t})$, completing the proof. □

Lemma 2.10. *Let $K \subseteq \mathbb{R}^n$ be a closed convex set, let $u \in K \cap \mathbb{Z}^n$ and let $d = \{u + \lambda r \,|\, \lambda > 0\} \subseteq \text{int}(K)$. Then $\{u\} \cup d \subseteq \text{conv}(K \cap \mathbb{Z}^n)$.*

Proof. If r is rational scalable, then the result is straightforward. Suppose therefore that r is not rational scalable.

Without loss of generality we may assume that $u = 0$. Observe that $u \in \text{conv}(K \cap \mathbb{Z}^n)$. Therefore it is sufficient to show that for $\gamma > 0$, $u + \gamma r \in \text{conv}(K \cap \mathbb{Z}^n)$. Let Λ be the projection of $\{x \in \mathbb{Z}^n \,|\, \langle r, x \rangle \geq \gamma\}$ on the linear subspace $\{x \in \mathbb{R}^n \,|\, \langle r, x \rangle = 0\}$. Then by Lemma 2.8, there exists a linear subspace, say V, such that Λ is dense in V and $V \neq \{0\}$. Note now that $d = \{u + \lambda r \,|\, \lambda > 0\} \subseteq \text{int}(K)$, is equivalent to $\exists \epsilon > 0$ such that $B(u + \lambda r, \epsilon) \subseteq K \; \forall \lambda > \gamma$. Let $\Psi := V \cap B(0, \epsilon)$. Then note that $\Lambda \cap B(0, \epsilon)$ is dense in Ψ.

By Lemma 2.9, we have that $0 = \sum_{i=1}^p \lambda_i v^i$ where $v^1, v^2, \dots, v^p \in \Lambda \cap B(0, \epsilon)$, $0 < \lambda_i \leq 1$, $\sum_{1=1}^p \lambda_i = 1$. Let $v^1, \dots, v^p$ be the projection of the integer points $u^1, \dots, u^p$ where $u^i = v^i + \mu_i r \in \mathbb{Z}^n$ and $\mu_i \geq \gamma$. Since distance between v^i and the half-line $\{u + \lambda r \,|\, \lambda \geq 0\}$ is less than ϵ and $\mu_i \geq \gamma$, we obtain that $u^i \in \{u + \lambda r \,|\, \lambda \geq \gamma\} + B(0, \epsilon) \subseteq K$. Therefore $u^i \in K \cap \mathbb{Z}^n$.

Now observe that

$$\sum_{i=1}^p \lambda_i (v^i + \mu_i r) = \sum_{i=1}^p \lambda_i v^i + r \sum_{i=1}^p \lambda_i \mu_i = r \sum_{i=1}^p \lambda_i \mu_i.$$

Since $\sum_{i=1}^p \lambda_i = 1$, we obtain that $\sum_{i=1}^p \lambda_i \mu_i \geq \gamma$. Thus, a point of the form μr where $\mu \geq \gamma$ belongs to $\text{conv}(K \cap \mathbb{Z}^n)$, completing the proof. □

Now we have all the tools needed to verify Theorem 2.3.

Proof. of Theorem 2.3. Let $u \in \text{int}(K) \cap \mathbb{Z}^n$. Then we claim that $u(K) = \text{rec.cone}(K)$. Observe first that $u(K) \subseteq \text{rec.cone}(K)$. Let $r \in \text{rec.cone}(K)$. Now observe that since $u \in \text{int}(K) \cap \mathbb{Z}^n$, $\{u + \lambda r \,|\, \lambda > 0\} \subseteq \text{int}(K)$. Thus by Lemma 2.10, the half-line line $\{u + \lambda r \,|\, \lambda \geq 0\} \subseteq \text{conv}(K \cap \mathbb{Z}^n)$. Thus, $r \in u(K)$, completing the proof of the claim.

Now observe Theorem 2.1 implies *(2.)* $\Rightarrow$ *(1.)* and the above claim together with Theorem 2.1 implies *(1.)* $\Rightarrow$ *(2.)*. We now verify *(1.)* $\iff$ *(3.)*.
$\Leftarrow$ Assume that every exposed face of K satisfies the condition. We will verify that $\text{conv}(K \cap \mathbb{Z}^n)$ is closed. By Theorem 2.1, it is sufficient to show that $u(K)$ is closed and identical for every $u \in K \cap \mathbb{Z}^n$. Observe that we have verified that $u(K) = \text{rec.cone}(K) \; \forall u \in \text{int}(K) \cap \mathbb{Z}^n$. Therefore, it remains to be shown that $u(K) = \text{rec.cone}(K)$ for all $u \in \text{bd}(K)$. Consider any $u \in \text{bd}(K)$ and let $r \in \text{rec.cone}(K)$. Then either

$u + r\lambda \in \text{int}(K)$ for all $\lambda > 0$ or $r \in \text{rec.cone}(F)$ for some exposed face F. In the first case by Lemma 2.10, the half-line line $\{u + \lambda r \,|\, \lambda \geq 0\} \subseteq \text{conv}(K \cap \mathbb{Z}^n)$. In the second case, by the condition, we have that $\{u + \lambda r \,|\, \lambda \geq 0\} \subseteq \text{conv}(F \cap \mathbb{Z}^n) \subseteq \text{conv}(K \cap \mathbb{Z}^n)$. Thus $u(K) = \text{rec.cone}(K)$, completing the proof.
$\Rightarrow$ Let $\text{conv}(K \cap \mathbb{Z}^n)$ be closed. Then by Theorem 2.1, we know that $u(K)$ is closed and identical for all $u \in K \cap \mathbb{Z}^n$. Thus $u(K) = \text{rec.cone}(K)$ for all $u \in K \cap \mathbb{Z}^n$. Now examine any exposed face of F. If $F \cap \mathbb{Z}^n \neq \emptyset$, $u \in F \cap \mathbb{Z}^n$ and $r \in \text{rec.cone}(F)$, then we have that $r \in \text{rec.cone}(K)$ and thus $\{u + \lambda r \,|\, \lambda \geq 0\} \subseteq \text{conv}(K \cap \mathbb{Z}^n)$. Therefore, it remains to verify that $\text{conv}(K \cap \mathbb{Z}^n) \cap F = \text{conv}(F \cap \mathbb{Z}^n)$ to complete the proof. Clearly $\text{conv}(F \cap \mathbb{Z}^n) \subseteq \text{conv}(K \cap \mathbb{Z}^n) \cap F$. If $x \in \text{conv}(K \cap \mathbb{Z}^n) \cap F$, then x is a convex combination of $z^1, \ldots, z^p$ where $z^i \in K \cap \mathbb{Z}^n$ for $i \in \{1, \ldots, p\}$. However, since $x \in F$, $z^i \in F$ for all $i \in \{1, ..., p\}$. Thus, $x \in \text{conv}(F \cap \mathbb{Z}^n)$, completing the proof. □

2.3 Closed-ness of conv$(K \cap \mathbb{Z}^n)$ where K is strictly convex set

A set $K \subseteq \mathbb{R}^n$ is called a *strictly convex set*, if K is a convex set and for all $x, y \in K$, $\lambda x + (1 - \lambda)y \in \text{rel.int}(K)$ for $\lambda \in (0, 1)$.

Theorem 2.4. *If $K \subseteq \mathbb{R}^n$ is a full-dimensional closed strictly convex set, then* $\text{conv}(K \cap \mathbb{Z}^n)$ *is closed.*

Proof. First note that if K is bounded or if $K \cap \mathbb{Z}^n = \emptyset$, then $\text{conv}(K \cap \mathbb{Z}^n)$ is closed. Therefore we assume that K is unbounded and $K \cap \mathbb{Z}^n \neq \emptyset$.

We first claim that K does not contain a line. Assume by contradiction that K contains a line in the direction $r \neq 0$. Examine $x \in \text{bd}(K)$. Then points of the form $x + \lambda r$ and $x - \lambda r$ belong to K, where $\lambda > 0$. In particular, $x + \lambda r, x - \lambda r \in \text{bd}(K)$ since $x \in \text{bd}(K)$. However this contradicts the fact that K is strictly convex.

Consider a point $u \in K \cap \mathbb{Z}^n$. Let $r \in \text{rec.cone}(K)$. Since K is strictly convex, we obtain that that set $\{u + \lambda r \,|\, \lambda > 0\}$ belongs to the interior of K. Therefore, by Lemma 2.10 we obtain that that the set $\{u + \lambda r \,|\, \lambda \geq 0\}$ belongs to $\text{conv}(K \cap \mathbb{Z}^n)$. Thus, $u(K) = \text{rec.cone}(K)$ for all $u \in K \cap \mathbb{Z}^n$. Therefore, by Theorem 2.1 we obtain that $\text{conv}(K \cap \mathbb{Z}^n)$ is closed. □

Thus in the case of full-dimensional closed strictly convex set K, $\text{conv}(K \cap \mathbb{Z}^n)$ is closed independent of the recession cone. The sets K^3 and K^4 in Example 2.1 are examples of this fact.

It is easily verified that every face of K is zero-dimensional, i.e. a single point. Therefore in fact the statement of Theorem 2.4 follows straightforwardly from Theorem 2.3 in the case when K is not lattice-free. It turns out that if $K \subseteq \mathbb{R}^n$ is a full-dimensional unbounded closed strictly convex set and $K \cap \mathbb{Z}^n \neq \emptyset$, then K is not lattice-free. The proof would follow from a variant of Lemma 2.10.

2.4 Closed-ness of conv$(K \cap \mathbb{Z}^n)$ where K is full-dimensional pointed closed convex cone

In this section we prove the following result.

Theorem 2.5. *Let K be a full-dimensional pointed closed convex cone in $\mathbb{R}^n$. Then* $\overline{\text{conv}}(K \cap \mathbb{Z}^n) = K$. *In particular,* $\text{conv}(K \cap \mathbb{Z}^n)$ *is closed if and only if every extreme ray of K is rational scalable.*

We begin with a few Lemma's before presenting the proof of Theorem 2.5.

Lemma 2.11. *If $A, B \subseteq \mathbb{R}^n$ are closed full dimensional convex sets such that $A \subsetneq B$, then* $\text{int}(B) \setminus A \neq \emptyset$.

Proof. Assume by contradiction that $\text{int}(B) \setminus A = \emptyset$. Equivalently we have that $\text{int}(B) \subseteq A$. Since A is closed, $\overline{\text{int}(B)} \subseteq A$. However since B is convex and full-dimensional, $B = \overline{\text{int}(B)} \subseteq A$, a contradiction (See [4] for a proof of the first equality.) □

Lemma 2.12. *Let $K \subseteq \mathbb{R}^n$ be a convex cone. Then $\overline{K}$ is a convex cone.*

Proof. Let $u \in \overline{K} \setminus K$ and let $\lambda > 0$. Consider the open ball of radius ϵ around λu, i.e. $B(\lambda u, \epsilon)$. We shown that $B(\lambda u, \epsilon) \cap K \neq \emptyset$ for all $\epsilon > 0$. Since $u \in \overline{K}$, we obtain that $B(u, \frac{\epsilon}{\lambda}) \cap K \neq 0$. This implies that $B(\lambda u, \epsilon) \cap K \neq \emptyset$. Therefore $\overline{K}$ is a cone. Moreover $\overline{K}$ is convex, since the closure of a convex set is convex. □

Lemma 2.13. *Let K be a full-dimensional, pointed closed convex cone in $\mathbb{R}^n$. Then $\overline{\text{conv}}(K \cap \mathbb{Z}^n)$ is a cone.*

Proof. Let $Q := \text{conv}(K \cap \mathbb{Z}^n)$. If $u \in Q$, then there exists a finite number of points $p^1, ..., p^k \in K \cap \mathbb{Z}^n$, such that u is a convex combination of $p^1, ..., p^k$. Let $\lambda \geq 0$. We show that $\lambda u \in Q$. If $\lambda \geq 1$, then the points $\lceil \lambda \rceil p^1, ..., \lceil \lambda \rceil p^k \in K \cap \mathbb{Z}^n$ and therefore λu is a convex combination of the points $p^1, ..., p^k$ and $\lceil \lambda \rceil p^1, ..., \lceil \lambda \rceil p^k$. If $\lambda \leq 1$, then u is a convex combination of 0 and $p^1, ..., p^k$. Thus, Q is a cone. Now by Lemma 2.12, we obtain that $\overline{\text{conv}}(K \cap \mathbb{Z}^n)$ is a cone. □

Proof. of Theorem 2.5 We first verify that $\overline{\text{conv}}(K \cap \mathbb{Z}^n) = K$.
$\subseteq$ By convexity of K we obtain that $\text{conv}(K \cap \mathbb{Z}^n) \subseteq K$. Since K is also closed, we obtain that $\overline{\text{conv}}(K \cap \mathbb{Z}^n) \subseteq K$.
$\supseteq$ Assume by contradiction that $\overline{\text{conv}}(K \cap \mathbb{Z}^n) \subsetneq K$. Then by Lemma 2.11, we obtain that there exists $u \in \text{int}(K)$ such that $u \notin \overline{\text{conv}}(K \cap \mathbb{Z}^n)$. Clearly $u \neq 0$. Moreover since $u \in \text{int}(K)$ and $\overline{\text{conv}}(K \cap \mathbb{Z}^n)$ is closed, there exists $\epsilon > 0$ such that the set $B(\epsilon, u) \in K \setminus \overline{\text{conv}}(K \cap \mathbb{Z}^n)$. Since K and $\overline{\text{conv}}(K \cap \mathbb{Z}^n)$ are cones (Lemma 2.13), we now obtain that the set $B(\epsilon, u) + \{\lambda u \,|\, \lambda \geq 0\}$ is a subset of K and is not contained in $\overline{\text{conv}}(K \cap \mathbb{Z}^n)$. However, by Lemma 2.7 the set $B(\epsilon, u) + \{\lambda u \,|\, \lambda \geq 0\}$ contains an integer point. Since this integer points belongs to K and not to $\overline{\text{conv}}(K \cap \mathbb{Z}^n)$, we obtain a contradiction.

We now verify that $\text{conv}(K \cap \mathbb{Z}^n)$ is closed if and only if all the extreme rays of K are rational scalable rays. Suppose $\text{conv}(K \cap \mathbb{Z}^n)$ is closed. Then $\text{conv}(K \cap \mathbb{Z}^n) = K$. If r is any extreme ray of K, then observe that $K \setminus \{\lambda r \,|\, \lambda > 0\}$ is a convex set. Since $\{\lambda r \,|\, \lambda > 0\} \subseteq \text{conv}(K \cap \mathbb{Z}^n)$, there must be an integer point in the set $\{\lambda r \,|\, \lambda > 0\}$. In other words, r is rational scalable.

Now assume that every extreme ray of K is rational scalable. Let R be the set of all extreme rays. Then observe that
$$K = \text{cone}(R) \subseteq \text{conv}(K \cap \mathbb{Z}^n) \subseteq K,$$
where the first equality follows from Lemma 2.6. Thus, $\text{conv}(K \cap \mathbb{Z}^n) = K$ or equivalently $\text{conv}(K \cap \mathbb{Z}^n)$ is closed. □

We note here that K^6 in Example 2.1 is an example for a non-polyhedral cone where each extreme ray is rational scalable. Therefore $\text{conv}(K^6 \cap \mathbb{Z}^3) = K^6$.

2.5 Closed-ness of conv$(K \cap \mathbb{Z}^n)$ where K contains lines

Given $V \subseteq \mathbb{R}^n$ a linear subspace, we denote by $V^\perp$ the linear subspace orthogonal to V and we denote by $P_{V^\perp}$ the projection on to the set $V^\perp$. Given a set K and a half-line $d := \{u + \lambda r \,|\, \lambda \geq 0\}$ we say K is *coterminal* with d if
$$\sup\{\mu \,|\, \mu > 0, u + \mu r \in K\} = \infty.$$
This definition is originally presented in [6]. Given a closed convex set K, a face F of K is called extreme facial ray of K if F is a closed half-line.

In this section, we will verify the following result.

Theorem 2.6. *Let $K \subseteq \mathbb{R}^n$ be a closed convex set such that the lineality space $L = \text{lin.space}(\overline{\text{conv}}(K \cap \mathbb{Z}^n))$ is not trivial. Then, $\text{conv}(K \cap \mathbb{Z}^n)$ is closed if and only if*

1. *The set $K \cap L^\perp$ is coterminal with every extreme facial ray of $\overline{\text{conv}}(K \cap L^\perp \cap P_{L^\perp}(\mathbb{Z}^n))$.*
2. *L is a rational subspace.*

Note that if the set $K \cap L^\perp$ does not contain any lines, (1.) of Theorem 2.6 is equivalent to saying that $\text{conv}(K \cap L^\perp \cap P_{L^\perp}(\mathbb{Z}^n))$ is closed. This is due to the following result from [5]: If A is a closed convex set not containing a line, then $\text{conv}(A)$ is closed if and only if A is coterminal with all the extreme facial rays of $\overline{\text{conv}}(A)$. (See Theorem 2.7 below for the general version of this result from [5].) Since L is a rational subspace (otherwise we already know that $\text{conv}(K \cap \mathbb{Z}^n)$ is not closed), we obtain that $P_{L^\perp}(\mathbb{Z}^n)$ is a Lattice. Therefore we can characterize the closed-ness of $\text{conv}(K \cap L^\perp \cap P_{L^\perp}(\mathbb{Z}^n))$ using the properties we have for convex sets not containing lines.

Before presenting the proof of Theorem 2.6, we describe some useful corollaries based on the above discussion.

Corollary 2.1. *Let K be a closed convex set and let* $\text{rec.cone}(K)$ *be a rational polyhedral cone. Then* $\text{conv}(K \cap \mathbb{Z}^n)$ *is closed.*

Proof. Let $L = \text{lin.space}(K)$. Since L is rational, $\text{lin.space}(\overline{\text{conv}}(K \cap \mathbb{Z}^n)) = L$. Now observe that $K \cap L^\perp$ contains no line. Therefore, we need to verify that $\text{conv}(K \cap L^\perp \cap P_{L^\perp}(\mathbb{Z}^n))$ is closed.

To simplify the proof, we may assume by using Lemma 2.5 that $L^\perp = \{x \in \mathbb{R}^n \,|\, x_i = 0 \ \forall i = k+1, \ldots, n\}$. Thus, it is sufficient (after projecting out the last $n-k$ components) to show that $\text{conv}(K' \cap \mathbb{Z}^k)$ is closed, where $K' \subseteq \mathbb{R}^k$ is a closed convex set not containing any line and $\text{rec.cone}(K')$ is a rational polyhedral cone. However note now that $u(K') \supseteq \text{rec.cone}(K') \supseteq \text{rec.cone}(\overline{\text{conv}}(K' \cap \mathbb{Z}^k)) \supseteq u(K')$ for all $u \in K' \cap \mathbb{Z}^n$, where the first inclusion is due to the fact that $\text{rec.cone}(K')$ is a rational polyhedral cone, the second inclusion is due to the fact that K' is closed and the last inclusion is the same as (2). Thus $u(K')$ is closed and identical for all $u \in K' \cap \mathbb{Z}^k$. Therefore by Theorem 2.1 we conclude that K' is closed which completes the proof. □

Note that maximal lattice-free convex sets have rational lineality space but are not necessary rational polyhedron. Corollary 2.1 yields the following corollary.

Corollary 2.2. *If K is a maximal lattice-free convex set, then* $\text{conv}(K \cap \mathbb{Z}^n)$ *is closed.*

Corollary 2.3. *If* $\text{lin.space}(K)$ *is not a rational subspace and* $\text{int}(K) \cap \mathbb{Z}^n \neq \emptyset$*, then* $\text{conv}(K \cap \mathbb{Z}^n)$ *is not closed.*

Proof. By Lemma 2.10, we conclude $\text{lin.space}(\overline{\text{conv}}(K \cap \mathbb{Z}^n)) = L$, which completes the proof. □

Next we present some results needed to verify Theorem 2.6. The crucial result needed is the following Theorem from [5].

Theorem 2.7 ([5])**.** *Let $A \subseteq \mathbb{R}^n$ such that* $L = \text{lin.space}(\overline{\text{conv}}(A))$ *is not trivial. Then,* $\text{conv}(A)$ *is closed if and only if*

1. *The set $P_{L^\perp}(A)$ is coterminal with every extreme facial ray of $\overline{\text{conv}}(A) \cap L^\perp$.*
2. *For every extreme point z of $\overline{\text{conv}}(A) \cap L^\perp$,* $\text{conv}(A \cap (z + L)) = z + L$.

Lemma 2.14. *Let $A, B \subseteq \mathbb{R}^n$ and denote* $L = \text{lin.space}(\overline{\text{conv}}(A))$*. We have the following:*

1. $P_{L^\perp}(\overline{B}) \subseteq \overline{P_{L^\perp}(B)}$.
2. $P_{L^\perp}(\text{conv}(B)) = \text{conv}(P_{L^\perp}(B))$.
3. $P_{L^\perp}(\overline{\text{conv}}(A)) = \overline{\text{conv}}(A) \cap L^\perp$.
4. $P_{L^\perp}(\overline{\text{conv}}(A)) = \overline{\text{conv}}(P_{L^\perp}(A))$.

Proof. Lets prove all the assertions:

1. This follows from the fact that $P_{L^\perp}$ is a continuous function and another equivalent definition of continuity is $\forall\, C \subseteq \mathbb{R}^n,\ P_{L^\perp}(\overline{C}) \subseteq \overline{P_{L^\perp}(C)}$ (see for example [10].)
2. This follows from Lemma 2.4.
3. The inclusion $P_{L^\perp}(\overline{\text{conv}}(A)) \supseteq \overline{\text{conv}}(A) \cap L^\perp$ is straightforward. Let $x \in P_{L^\perp}(\overline{\text{conv}}(A))$, there exists $l \in L$ such that $x + l \in \overline{\text{conv}}(A)$. Since $-l \in L$, $(x + l) - l \in \overline{\text{conv}}(A)$. Therefore $x \in \overline{\text{conv}}(A)$. Since $x \in L^\perp$, we conclude $x \in \overline{\text{conv}}(A) \cap L^\perp$.
4. We have $P_{L^\perp}(\text{conv}(A)) \subseteq P_{L^\perp}(\overline{\text{conv}}(A))$. By *3.*, $P_{L^\perp}(\overline{\text{conv}}(A))$ is a closed set and therefore $\overline{P_{L^\perp}(\text{conv}(A))} \subseteq P_{L^\perp}(\overline{\text{conv}}(A))$. By *2.*, we conclude that $\overline{\text{conv}}(P_{L^\perp}(A)) \subseteq P_{L^\perp}(\overline{\text{conv}}(A))$. On the other hand by *1.* and *2.* we obtain that $P_{L^\perp}(\overline{\text{conv}}(A)) \subseteq \overline{P_{L^\perp}(\text{conv}(A))} = \overline{\text{conv}}(P_{L^\perp}(A))$. We therefore conclude that $P_{L^\perp}(\overline{\text{conv}}(A)) = \overline{\text{conv}}(P_{L^\perp}(A))$. □

□

Lemma 2.15. *Let $K \subseteq \mathbb{R}^n$ be a closed convex set. Denote $L = \text{lin.space}(\overline{\text{conv}}(K \cap \mathbb{Z}^n))$. Then $P_{L^\perp}(K \cap \mathbb{Z}^n) = P_{L^\perp}(K) \cap P_{L^\perp}(\mathbb{Z}^n)$. In particular, $P_{L^\perp}(K \cap \mathbb{Z}^n) = K \cap L^\perp \cap P_{L^\perp}(\mathbb{Z}^n)$.*

Proof. The inclusion $P_{L^\perp}(K \cap \mathbb{Z}^n) \subseteq P_{L^\perp}(K) \cap P_{L^\perp}(\mathbb{Z}^n)$ is straightforward. Let $x \in P_{L^\perp}(K) \cap P_{L^\perp}(\mathbb{Z}^n)$. There exist $l_1, l_2 \in L$ such that $x + l_1 \in K$ and $x + l_2 \in \mathbb{Z}^n$. Notice that since $\overline{\text{conv}}(K \cap \mathbb{Z}^n) \subseteq K$ is a closed set, then $L \subseteq \text{lin.space}(K)$. Hence, since $l_2 - l_1 \in L$ we have $x + l_1 + (l_2 - l_1) \in K$. Therefore, $x + l_2 \in K \cap \mathbb{Z}^n$. In conclusion, since $x \in L^\perp$, we conclude $x \in P_{L^\perp}(K \cap \mathbb{Z}^n)$.

Finally, since $L \subseteq \text{lin.space}(K)$, we have $P_{L^\perp}(K) = K \cap L^\perp$. The proof is the same as that of *3.* of Lemma 2.14. □

We now have all the tools for proving Proposition 2.6.

Proof. of Theorem 2.6 Note that by Lemma 2.14 and Lemma 2.15 we have $\overline{\text{conv}}(K \cap \mathbb{Z}^n) \cap L^\perp = P_{L^\perp}(\overline{\text{conv}}(K \cap \mathbb{Z}^n)) = \overline{\text{conv}}(P_{L^\perp}(K \cap \mathbb{Z}^n)) = \overline{\text{conv}}(K \cap L^\perp \cap P_{L^\perp}(\mathbb{Z}^n))$. Therefore (1.) of Theorem 2.6 is just a restatement of (1.) of Theorem 2.7 with $A = K \cap \mathbb{Z}^n$.

Observe that since the set $\overline{\text{conv}}(K \cap \mathbb{Z}^n) \cap L^\perp$ does not contain any lines, it must have at least one extreme point.

$\Rightarrow$ Suppose $\text{conv}(K \cap \mathbb{Z}^n)$ is closed. Then, by Theorem 2.7, we have (1.) of Theorem 2.6 and also have that for every extreme point z of $\overline{\text{conv}}(K \cap \mathbb{Z}^n) \cap L^\perp$, $\text{conv}(K \cap \mathbb{Z}^n \cap (z + L)) = z + L$. Since $z + L \neq \emptyset$, $\emptyset \neq K \cap \mathbb{Z}^n \cap (z + L) \subseteq \mathbb{Z}^n$. Thus $z + L$ is the convex hull of some non-empty subset of integer points and therefore L is a rational subspace.

$\Leftarrow$ Now suppose (1.) and (2.) of Theorem 2.6. Then we have (1.) of Theorem 2.7. We will prove (2.) of Theorem 2.7, that is for every extreme point z of $\overline{\text{conv}}(K \cap \mathbb{Z}^n) \cap L^\perp$, $\text{conv}(K \cap \mathbb{Z}^n \cap (z + L)) = z + L$. Let z be an extreme point of $\overline{\text{conv}}(K \cap \mathbb{Z}^n) \cap L^\perp$. We will first prove that $(z + L) \cap K \cap \mathbb{Z}^n \neq \emptyset$. Since $\overline{\text{conv}}(K \cap \mathbb{Z}^n) \cap L^\perp = \overline{\text{conv}}(P_{L^\perp}(K \cap \mathbb{Z}^n))$ by Lemma 2.3 we have that $z \in P_{L^\perp}(K \cap \mathbb{Z}^n)$, therefore there exists $l \in L$ such that $z + l \in K \cap \mathbb{Z}^n$. Hence, $(z + L) \cap K \cap \mathbb{Z}^n \neq \emptyset$. Now let $\{l_1, \ldots, l_p\} \subseteq \mathbb{Z}^n$ be a basis of L and since $(z + L) \cap K \cap \mathbb{Z}^n \neq \emptyset$, let $w \in (z + L) \cap K \cap \mathbb{Z}^n$. Since $L \subseteq \text{lin.space}(K)$ for all $\lambda_1, \ldots, \lambda_p \in \mathbb{Z}$, the points $w, w + \lambda_1 l_1, \ldots, w + \lambda_p l_p$ belongs to $(z + L) \cap K \cap \mathbb{Z}^n$. Thus, by convexity of $\text{conv}((z + L) \cap K \cap \mathbb{Z}^n)$, for all $\lambda_1, \ldots, \lambda_p \in \mathbb{R}$, the points $w, w + \lambda_1 l_1, \ldots, w + \lambda_p l_p$ belongs to $\text{conv}((z + L) \cap K \cap \mathbb{Z}^n)$. Thus, $\text{conv}((z + L) \cap K \cap \mathbb{Z}^n)$ contains a affine subspace whose dimension is the same as that of $z + L$. Since $\text{conv}(K \cap \mathbb{Z}^n \cap (z + L)) \subseteq z + L$, we therefore obtain that $\text{conv}(K \cap \mathbb{Z}^n \cap (z + L)) = z + L$. Thus we obtain (2.) of Theorem 2.7 and hence $\text{conv}(K \cap \mathbb{Z}^n)$ is closed. □

3 Polyhedrality of $\text{conv}(K \cap \mathbb{Z}^n)$

Let us develop some intuition regarding the question of polyhedrality of $\text{conv}(K \cap \mathbb{Z}^n)$. Suppose for simplicity that K contains no lines, $K \cap \mathbb{Z}^n$ is full-dimensional and $\text{int}(K) \cap \mathbb{Z}^n$ is non-empty. Then by Theorem 2.3, we obtain that a necessary condition for $\text{conv}(K \cap \mathbb{Z}^n)$ to be closed is that $\text{rec.cone}(\text{conv}(K \cap \mathbb{Z}^n)) = \text{rec.cone}(K)$. Therefore in this setting if we require $\text{rec.cone}(K \cap \mathbb{Z}^n)$ to be polyhedron, it is necessary that K has a rational polyhedral recession cone. However this is not sufficient. Consider the case of the parabola K^3 presented in Example 2.1. It is easy to verify that $\text{conv}(K^3 \cap \mathbb{Z}^2)$ is not a polyhedron. To see what is 'going wrong', observe that $\min\{x_1 \,|\, x \in K^3\} = -\infty$ even though $(-1, 0)$ is orthogonal to the all vectors in the recession cone. Intuitively, this causes $\text{conv}(K^3 \cap \mathbb{Z}^2)$ to have an infinite number of extreme points. This motivates the following definition.

Definition 3.1 (Thin Convex set)**.** *Let $K \subseteq \mathbb{R}^n$ be a closed convex set. We say K is* thin *if the following holds:* $\min\{\langle c, x\rangle \,|\, x \in K\} = -\infty$ *if and only if there exist* $d \in \text{rec.cone}(K)$ *such that* $\langle d, c\rangle < 0$.

In this section we verify the following result.

Theorem 3.1. *If $K \subseteq \mathbb{R}^n$ is thin and recession cone of K is a rational polyhedral cone, then* $\text{conv}(K \cap \mathbb{Z}^n)$ *is a polyhedron. Moreover, if* $\text{int}(K) \cap \mathbb{Z}^n \neq \emptyset$ *and* $\text{conv}(K \cap \mathbb{Z}^n)$ *is a polyhedron, then K is thin and* $\text{rec.cone}(K)$ *is a rational polyhedral cone.*

Since every polyhedron is a thin set, Theorem 3.1 generalizes the result in [8]. We present a simple example illustrating Theorem 3.1 when K is not a polyhedral set.

Example 3.1. *Consider the set* $K^8 = \{(x_1, x_2) \in \mathbb{R}^2_+ \,|\, x_1x_2 \geq 1\}$*. It is straightforward to verify that* K^8 *is thin and* $\text{rec.cone}(K^8) = \{(y_1, y_2) \in \mathbb{R}^2 \,|\, y_1 \geq 0, y_2 \geq 0\}$ *is a rational polyhedron. Thus,* $\text{conv}(K^8 \cap \mathbb{Z}^2) = \{(x_1, x_2) \,|\, x_1 \geq 1, x_2 \geq 1\}$ *is a polyhedron. On the other hand observe that while each of the sets* K^1*,* K^2*,* K^3*,* K^4*,* K^6 *in Example 2.1 contains integer points in its interior, none of them are both thin and have rational polyhedral recession cone. Thus by Theorem 3.1, the convex hull of integer points in all these sets is non-polyhedral.*

3.1 Sufficient conditions for conv($K \cap \mathbb{Z}^n$) to be polyhedral

We begin with a few Lemmas before presenting the 'sufficiency direction' of Theorem 3.1.

Lemma 3.1. *If* $K \subseteq \mathbb{R}^n$ *is thin and* $T \subseteq \mathbb{R}^n$ *is a closed subset of* K *such that* $\text{rec.cone}(T) = \text{rec.cone}(K)$*, then* T *is thin.*

Proof. Suppose $\min\{\langle c, x\rangle \,|\, x \in T\}$ is unbounded. Then, $\min\{\langle c, x\rangle \,|\, x \in K\}$ is unbounded. Since K is thin, there exists $d \in \text{rec.cone}(K) = \text{rec.cone}(T)$ such that $\langle d, c\rangle < 0$. If $\min\{\langle c, x\rangle \,|\, x \in T\}$ is bounded, then $\langle d, c\rangle \geq 0$ for all $d \in \text{rec.cone}(T)$. □

Lemma 3.2. *Let* $K \subseteq \mathbb{R}^n$ *be a thin set whose recession cone is polyhedral. Let* $v \in \mathbb{R}^n$*,* $b \in \mathbb{R}$ *and let* $T := K \cap \{x \,|\, \langle v, x\rangle \geq b\}$*. Then* T *is a thin set.*

Proof. If $T = \emptyset$, then the result is trivial. Therefore we assume that $T \neq \emptyset$. Note first that

$$\text{rec.cone}(T) = \{u \in \text{rec.cone}(K) \,|\, \langle v, u\rangle \geq 0\}. \tag{6}$$

Assume by contradiction that T is not thin. Then there exists $c \in \mathbb{R}^n$ such that $\sup\{\langle c, x\rangle \,|\, x \in T\}$ is unbounded but there does not exist a vector u in rec.cone(T) such that $\langle u, c\rangle > 0$. Let $r^1, ..., r^k$ generate rec.cone(K). Note now that if $\langle r^i, c\rangle > 0$ (and there exists one such r^i since K is thin and $T \subseteq K$), then we must have (by assumption) that $r^i \notin \text{rec.cone}(T)$ and therefore using (6) we obtain that $\langle r^i, v\rangle < 0$. Let

$$\mu = \max\left\{-\frac{\langle r^i, c\rangle}{\langle r^i, v\rangle} \,|\, \langle r^i, c\rangle > 0 \, i \in \{1, \dots, k\}\right\},$$

and let

$$\lambda = \min\left\{-\frac{\langle r^i, c\rangle}{\langle r^i, v\rangle} \,|\, \langle r^i, c\rangle \leq 0, \langle r^i, v\rangle > 0, \, i \in \{1, \dots, k\}\right\}.$$

($\lambda = +\infty$ if $\langle r^i, c\rangle \leq 0, \langle r^i, v\rangle > 0$ does not hold for any i).
Observe that $\mu > 0$ and $\lambda \geq 0$. Now examine the following cases:

1. $\mu \leq \lambda$: In this case consider the vector $c + \mu v$ and examine the product $\langle r^i, c + \mu v\rangle$:
   (a) $\langle r^i, c\rangle > 0$. Then $\langle r^i, v\rangle < 0$ and $-\frac{\langle r^i, c\rangle}{\langle r^i, v\rangle} \leq \mu$ or equivalently $\langle r^i, c + \mu v\rangle \leq 0$.
   (b) $\langle r^i, c\rangle = 0$. If $\langle r^i, v\rangle \leq 0$, then $\langle r^i, c + \mu v\rangle \leq 0$. Note that $\langle r^i, v\rangle > 0$ is not possible since $\lambda \geq \mu > 0$.
   (c) $\langle r^i, c\rangle < 0$. If $\langle r^i, v\rangle \leq 0$, then $\langle r^i, c + \mu v\rangle \leq 0$. If $\langle r^i, v\rangle > 0$, then $-\frac{\langle r^i, c\rangle}{\langle r^i, v\rangle} \geq \lambda \geq \mu$ or $\langle r^i, c + \mu v\rangle \leq 0$.

   Therefore, $\langle r^i, c + \mu v\rangle \leq 0$ for all $i \in \{1, ..., k\}$. Let $\{x^i\}_{i=1}^{\infty}$ be the sequence of points in T such that $\lim_{i\to\infty}\langle c, x^i\rangle = +\infty$. Now observe that

   $$\langle c + \mu v, x^i\rangle \geq \langle c, x^i\rangle + \mu b.$$

   Therefore $\lim_{i\to\infty}\langle c + \mu v, x^i\rangle = +\infty$, which contradicts the thinness of K.

2. $\mu > \lambda$. Without loss of generality, let $\mu = -\frac{\langle r^1,c\rangle}{\langle r^1,v\rangle}$ and $\lambda = -\frac{\langle r^2,c\rangle}{\langle r^2,v\rangle}$, where $\langle r^1, c\rangle > 0$, $\langle r^1, v\rangle < 0$, $\langle r^2, c\rangle \leq 0$ and $\langle r^2, v\rangle > 0$. Therefore $\delta := -\frac{\langle r^1,v\rangle}{\langle r^2,v\rangle} > 0$. Now observe that $\langle r^1 + \delta r^2, v\rangle = 0$ and therefore $r^1 + \delta r^2 \in \text{rec.cone}(T)$. On the other hand

$$\begin{aligned} \langle r^1 + \delta r^2, c\rangle &> 0 \\ \iff \langle r^1, c\rangle &> -\delta\langle r^2, c\rangle \\ \iff \langle r^1, c\rangle &> \frac{\langle r^1, v\rangle}{\langle r^2, v\rangle}\langle r^2, c\rangle \\ \iff \frac{\langle r^1, c\rangle}{\langle r^1, v\rangle} &< \frac{\langle r^2, c\rangle}{\langle r^2, v\rangle} \\ \iff -\frac{\langle r^1, c\rangle}{\langle r^1, v\rangle} &> -\frac{\langle r^2, c\rangle}{\langle r^2, v\rangle} \\ \iff \mu &> \lambda. \end{aligned}$$

Thus, we obtain a vector $r \in \text{rec.cone}(T)$ such that $\langle r, c\rangle > 0$ which is in contradiction to our assumption.

□

Corollary 3.1. *If K is thin, recession cone of K is a polyhedral cone and F is an exposed face of K, then F is thin.*

Corollary 3.2. *If K is thin and recession cone of K is a polyhedral cone, then the intersection of K with a affine space is a thin set.*

Lemma 3.3. *Let A be a $m \times n$ matrix and let $K \subseteq \mathbb{R}^n$ be a thin convex set. Then $AK \subseteq \mathbb{R}^m$ is a thin convex set.*

Proof. We first claim that $\text{rec.cone}(AK) \supseteq A(\text{rec.cone}(K))$. Consider any $r \in \text{rec.cone}(K)$. Let $Ax \in AK$. Then observe that $Ax + \lambda Ar = A(x + \lambda r) \in AK$ for all $\lambda \geq 0$. Thus, $Ar \in \text{rec.cone}(AK)$.

Now suppose there exists $c \in \mathbb{R}^m$ such that $\sup\{\langle c, x\rangle \,|\, x \in AK\} = \infty$. Observe that,

$$\sup\{\langle A^T c, x\rangle \,|\, x \in K\} = \sup\{\langle c, Ax\rangle \,|\, x \in K\} = \sup\{\langle c, x\rangle \,|\, x \in Ak\} = \infty. \tag{7}$$

Since K is thin, (7) implies that there exists $d \in \text{rec.cone}(K)$ such that $\langle A^T c, d\rangle > 0$ or equivalently, $\langle c, Ad\rangle > 0$. Since $Ad \in \text{rec.cone}(Ak)$, this completes the proof. □

Corollary 3.3. *Let $K \subseteq \mathbb{R}^n$ be a thin convex set. Then $K' \subseteq \mathbb{R}^k$, the projection of K onto the first k coordinates is thin.*

Proof. Let A the matrix defining the projection of $\mathbb{R}^n$ onto the first k coordinates. Then $K' = AK$, so we are done. □

We use the following notation in this section. Let $K \subseteq \mathbb{R}^n$ be a closed convex set. Then $\sigma_K : \mathbb{R}^n \to \mathbb{R}$ defined as $\sigma_K(a) = \sup\{\langle a, x\rangle \,|\, x \in K\}$ is the support function of K. Given a cone T, we represent its polar by T^*. In particular, $(\text{rec.cone}(K))^* = \{d \in \mathbb{R}^n \,|\, \langle d, u\rangle \leq 0 \ \forall u \in \text{rec.cone}(K)\}$.

Lemma 3.4. *Let K be a thin set and let $\text{rec.cone}(K)$ be polyhedral. Then $\sigma_K : (\text{rec.cone}(K))^* \to \mathbb{R}$ is a continuous function.*

Proof. The support function is sublinear and therefore lower semi-continuous (see Theorem 13.2, Theorem 7.1 [11]). Next observe that $(\text{rec.cone}(K))^*$ is a polyhedral cone and since K is thin we obtain that $\sigma_K(d) < +\infty$ for all $d \in (\text{rec.cone}(K))^*$. Then by Theorem 10.2 [11], σ_K is an upper semi-continuous function on $(\text{rec.cone}(K))^*$. Therefore the result follows. □

Lemma 3.5. *Let P is a rational polyhedron, and F be a face of P. Then there exists $c \in \mathbb{Z}^n$ and $d \in \mathbb{Z}$ such that $P \subseteq \{x \in \mathbb{R}^n \,|\, \langle c, x\rangle \leq d\}$ and $F = \{x \in P \,|\, \langle c, x\rangle = d\}$.*

Proof. Let $P = \{x \in \mathbb{R}^n \,|\, \langle a^i, x\rangle \leq b^i, \forall\ i \in U \cup V\}$ and $F = \{x \in \mathbb{R}^n \,|\, \langle a^i, x\rangle \leq b^i, \forall\ i \in U, \langle a^i, x\rangle = b^i,\ \forall\ i \in V\}$. Without loss of generality, we can assume all data is integral.

Consider $c = \sum_{i\in V} a^i$ and $d = \sum_{i\in V} b^i$. Then observe that if $x \in F$, then

$$\langle c, x\rangle = \langle \sum_{i\in V} a^i, x\rangle = \sum_{i\in V} \langle a^i, x\rangle = \sum_{i\in V} b^i = d.$$

On the other hand, if $x \in P \setminus F$, then there exists $i \in V$ such that $\langle a^i, x\rangle < b^i$. Thus,

$$\langle c, x\rangle = \langle \sum_{i\in V} a^i, x\rangle = \sum_{i\in V} \langle a^i, x\rangle < \sum_{i\in V} b^i = d.$$

□

The next lemma essentially is a restatement of result from [8].

Lemma 3.6. *Let $P \subseteq \mathbb{R}^n$ be a polyhedron such that* rec.cone(P) *is a rational polyhedral cone. Then* conv$(P \cap \mathbb{Z}^n)$ *is a rational polyhedron.*

Proof. By Minkowski-Weyl Theorem, let P be generated by the convex combination of the points $u^1, ..., u^s$ and the conic combination of $r^1, ..., r^k$ which are scaled to be integers (since rec.cone(P) is a rational polyhedral cone).

Define T as

$$T = \text{conv}\left(\left\{ \begin{array}{ccc} x \in \mathbb{Z}^n & | & x = \sum_{i\in\{1,\ldots,s\}} \lambda_i u^i + \sum_{i\in\{1,\ldots,t\}} \gamma_i r^i \\ & & \lambda_i \geq 0,\ \sum_{i\in\{1,\ldots,s\}} \lambda_i = 1,\ 0 \leq \gamma_i \leq 1. \end{array} \right\}\right)$$

To complete the proof of this lemma, we verify that conv$(P \cap \mathbb{Z}^n) = T +$ rec.cone(P). Now clearly conv$(P \cap \mathbb{Z}^n) \supseteq T +$ rec.cone(P). On the other hand, if $x \in (P \cap \mathbb{Z}^n)$, then $x = \sum_{i\in\{1,\ldots,s\}} \lambda_i u^i + \sum_{i\in\{1,\ldots,t\}} \gamma_i r^i = \sum_{i\in\{1,\ldots,s\}} \lambda_i u^i + \sum_{i\in\{1,\ldots,t\}} (\gamma_i - \lfloor \gamma_i \rfloor) r^i + \sum_{i\in\{1,\ldots,t\}} \lfloor \gamma_i \rfloor r^i \in T +$ rec.cone(P). □

Proposition 3.1 (Sufficient Condition)**.** *If K is thin and recession cone of K is a rational polyhedral cone, then* conv$(K \cap \mathbb{Z}^n)$ *is a polyhedron.*

Proof. If $K \cap \mathbb{Z}^n = \emptyset$, then the result is straightforward. So we assume that $K \cap \mathbb{Z}^n \neq \emptyset$.

Since the recession cone of K is a rational polyhedral cone, by Corollary 2.1 we obtain that conv$(K \cap \mathbb{Z}^n)$ is closed. We prove this statement by induction on the dimension of K. Note that for dimensions 0 and 1, the statement is true. Assume that the statement is true for all dimensions less than the dimension of K.

We first illustrate that we may assume that K is full-dimensional. Observe that since aff$(K \cap \mathbb{Z}^n)$ is a rational affine subspace, we have that $K \cap$ aff$(K \cap \mathbb{Z}^n)$ is a thin set (Corollary 3.2) with a recession cone that is a rational polyhedron. Let $z \in K \cap \mathbb{Z}^n$. We now translate K as $K - \{z\}$ and note that it is sufficient to show that conv$(K \cap \mathbb{Z}^n)$ is polyhedral for this new set K, which is a thin set with a recession cone that is a rational polyhedron. Now by selecting a suitable unimodular matrix (see [12]) and by the application linear map corresponding to this matrix to $\mathbb{R}^n$, we may assume that aff$(K \cap \mathbb{Z}^n)$ is of the form $\{x \,|\, x_i = 0\ \forall i = k+1, ..., n\}$ and K is a thin set (Lemma 3.3) with a recession cone that is a rational polyhedron. (Note that linear maps preserve polyhedrality.) Finally, we can project out the last $n-k$ components (every point in K has zero in these components) and note that it is sufficient to show that conv$(K \cap \mathbb{Z}^k)$ is polyhedron where K is a full-dimensional thin set (Lemma 3.3) with a recession cone that is a rational polyhedron.

Therefore, we assume hence forth that $K \subseteq \mathbb{R}^n$ is full-dimensional thin set with a recession cone that is a rational polyhedron.

Since the recession cone of K is a rational polyhedral, rec.cone(conv$(K \cap \mathbb{Z}^n)$) = rec.cone(K). Thus by Corollary 3.1, we obtain that conv$(K \cap \mathbb{Z}^n)$ is thin. Define $D = \{d \in \mathbb{R}^n \,|\, \|d\| = 1, \langle d, u\rangle \leq 0\ \forall\ u \in$ rec.cone$(K)\}$.

Let $v \in D$ and let $F_v = \{x \in K \,|\, \langle v, x\rangle = \sigma_K(v)\}$ be the proper exposed face of K (since K is full-dimensional set and $v \neq 0$, F_v is a proper face of K) corresponding to the vector v. We will show first that $\text{conv}(F_v \cap \mathbb{Z}^n)$ is a polyhedron. If $F_v \cap \mathbb{Z}^n = \emptyset$, then there remains nothing to verify. So assume that $F_v \cap \mathbb{Z}^n \neq \emptyset$. Let $L = \text{aff}\{x \in \mathbb{Z}^n \,|\, \langle v, x\rangle = \sigma_K(v)\}$. Since L is generated by integer vectors it is a rational affine subspace. By Corollary 3.1 we obtain that F_v is a thin set. Now by Corollary 3.2 we obtain that $F_v \cap L$ is thin. Now observe that

$$\begin{aligned}
\text{rec.cone}(F_v \cap L) &= \text{rec.cone}(K \cap \{x \in \mathbb{R}^n \,|\, \langle v, x\rangle = \sigma_K(v)\} \cap L) \\
&= \text{rec.cone}(K) \cap \text{rec.cone}(\{x \in \mathbb{R}^n \,|\, \langle v, x\rangle = \sigma_K(v)\}) \\
&\quad \cap \text{rec.cone}(L) \\
&= \text{rec.cone}(K) \cap \text{rec.cone}(L),
\end{aligned}$$

where the last equality follows from the fact that $\text{rec.cone}(L) \subseteq \text{rec.cone}(\{x \in \mathbb{R}^n \,|\, \langle v, x\rangle = \sigma_K(v)\})$. Thus, $\text{rec.cone}(F_v \cap L)$ is a rational polyhedral cone. Moreover, $\dim(F_v \cap L) < \dim(K)$. Therefore by the induction hypothesis, $\text{conv}(F_v \cap L \cap \mathbb{Z}^n)$ is a polyhedron. Now the result follows from the fact that $\text{conv}(F_v \cap \mathbb{Z}^n) = \text{conv}(F_v \cap L \cap \mathbb{Z}^n)$.

For any $v \in D$, we now verify that there exists a polyhedron $P_v \subseteq \mathbb{R}^n$ such that

1. $P_v \cap \mathbb{Z}^n \supseteq K \cap \mathbb{Z}^n$
2. Either $\sigma_{P_v}(v) < \sigma_K(v)$ or $u \in P_v$ s.t. $\langle u, v\rangle = \sigma_K(v) \Rightarrow u \in K$.

Let $\text{rec.cone}(K) := \{x \in \mathbb{R}^n \,|\, \langle a_i, x\rangle \leq 0 \;\forall i \in \{1, ..., t\}\} = \{\sum_{i=1}^k \theta_i r^i \,|\, \theta_i \geq 0\}$, where $r^i \in \mathbb{Z}^n$ generate $\text{rec.cone}(K)$. Let $\tilde{P} = \{x \in \mathbb{R}^n \,|\, \langle a_i, x\rangle \leq \sigma_{\text{conv}(K\cap\mathbb{Z}^n)}(a_i) \;\forall i \in \{1, ..., t\}\}$. Note that $\sigma_{\text{conv}(K\cap\mathbb{Z}^n)}$ is finite because $\text{conv}(K \cap \mathbb{Z}^n)$ is thin. Observe that $\text{rec.cone}(\tilde{P}) = \text{rec.cone}(K)$ and $\tilde{P} \cap \mathbb{Z}^n \supseteq K \cap \mathbb{Z}^n$. Let

$$\tilde{P}_v = \{x \in \tilde{P} \,|\, \langle v, x\rangle \leq \sigma_{\text{conv}(K\cap\mathbb{Z}^n)}\}.$$

Since $\tilde{P} \cap \mathbb{Z}^n \supseteq K \cap \mathbb{Z}^n$, we obtain that $\tilde{P}_v \cap \mathbb{Z}^n \supseteq K \cap \mathbb{Z}^n$ Observe also that $\text{rec.cone}(\tilde{P}_v) = \text{rec.cone}(\tilde{P}) = \text{rec.cone}(K)$ and since $\text{rec.cone}(K)$ is a rational polyhedral cone, by Lemma 3.6 we obtain that

$$Q_v := \text{conv}(\tilde{P}_v \cap \mathbb{Z}^n)$$

is a rational polyhedron. There are two cases to consider:

1. $\sigma_{Q_v}(v) < \sigma_K(v)$: In this case, we define $P_v := Q_v$. Then observe that $P_v \cap \mathbb{Z}^n = \tilde{P}_v \cap \mathbb{Z}^n \supseteq K \cap \mathbb{Z}^n$ and $\sigma_{P_v}(v) < \sigma_K(v)$.
2. $\sigma_{Q_v}(v) = \sigma_K(v)$: Now let G be the face of Q_v defined as $G := \{x \in Q_v \,|\, \langle v, x\rangle = \sigma_K(v)\}$. Since Q_v is rational polyhedron, by Lemma 3.5, let $c \in \mathbb{Z}^n$ and $d \in \mathbb{Z}$ such that $G = \{x \in Q_v \,|\, \langle c, x\rangle = d\}$. Now we consider two cases:

   (a) $F_v \cap \mathbb{Z}^n = \emptyset$. In this case let

   $$P_v = \{x \in Q_v \,|\, \langle c, x\rangle \leq d - 1\}.$$

   We first claim that $P_v \cap \mathbb{Z}^n \supseteq (K \cap \mathbb{Z}^n)$. Let $x \in K \cap \mathbb{Z}^n$. Then $x \in Q_v \cap \mathbb{Z}^n$ since $Q_v \cap \mathbb{Z}^n = \tilde{P}_v \cap \mathbb{Z}^n \supseteq K \cap \mathbb{Z}^n$. Since $F_v \cap \mathbb{Z}^n = \emptyset$, we have that $\langle v, x\rangle < \sigma_K(v)$. Thus, $x \notin G$. Therefore, $x \in (Q_v \cap \mathbb{Z}^n) \setminus G$ or equivalently $x \in P_v$.
   Next we claim that $\sigma_{P_v}(v) < \sigma_K(v)$. Assume by contradiction that $\sigma_{P_v}(v) = \sigma_K(v)$. Since P_v is a polyhedron, there exists $x \in P_v$ such that $\langle v, x\rangle = \sigma_K(v)$. Since $P_v \subseteq Q_v$, we obtain that $x \in \{y \in Q_v \,|\, \langle v, y\rangle = \sigma_K(v)\}$. However, this implies that $x \in G$ or equivalently $\langle c, x\rangle = d$, a contradiction since $\sigma_{P_v}(c) \leq d - 1$.

   (b) $F_v \cap \mathbb{Z}^n \neq \emptyset$. Let G, c, d be defined as in the previous case. Let

   $$P_v^1 = \{x \in Q_v \,|\, \langle c, x\rangle \leq d - 1\}.$$

   We first claim that $P_v^1 \cap \mathbb{Z}^n \supseteq (K \cap \mathbb{Z}^n) \setminus (F_v \cap \mathbb{Z}^n)$. Let $x \in (K \cap \mathbb{Z}^n) \setminus (F \cap \mathbb{Z}^n)$. Then as before we obtain that $x \in Q_v \cap \mathbb{Z}^n$. Moreover, since $x \notin F_v$, we obtain that $\langle v, x\rangle < \sigma_K(v)$. Thus, $x \notin G$. Therefore, $x \in (Q_v \cap \mathbb{Z}^n) \setminus G$ or equivalently $x \in P_v^1$.

Next we claim that $\sigma_{P^1_v}(v) < \sigma_K(v)$. Assume by contradiction that $\sigma_{P^1_v}(v) = \sigma_K(v)$. Since P^1_v is a polyhedron, there exists $x \in P^1_v$ such that $\langle v, x\rangle = \sigma_K(v)$. Since $P^1_v \subseteq Q_v$, we obtain that $x \in \{y \in Q_v \,|\, \langle v, y\rangle = \sigma_K(v)\}$. However, this implies that $x \in G$ or equivalently $\langle c, x\rangle = d$, a contradiction since $\sigma_{P^1_v}(c) \leq d - 1$.
Now if $P^1_v \neq \emptyset$, then define

$$P_v := \text{conv}\left((P^1_v + \text{rec.cone}(K)) \cup (\text{conv}(F_v \cap \mathbb{Z}^n) + \text{rec.cone}(K))\right),$$

else define

$$P_v := \text{conv}(F \cap \mathbb{Z}^n) + \text{rec.cone}(K).$$

We first verify that P_v is a polyhedron. Observe first that, by previous claim, $\text{conv}(F_v \cap \mathbb{Z}^n)$ is a polyhedron. Therefore $(P^1_v + \text{rec.cone}(K))$ and $(\text{conv}(F_v \cap \mathbb{Z}^n) + \text{rec.cone}(K))$ are two polyhedra with the same recession cone. Thus, P_v is a polyhedron.
Now we verify that $P_v \cap \mathbb{Z}^n \supseteq K \cap \mathbb{Z}^n$. If $x \in (K \cap \mathbb{Z}^n) \setminus (F_v \cap \mathbb{Z}^n)$, then as verified before $x \in P^1_v \cap \mathbb{Z}^n$ or equivalently $x \in P_v$. If $x \in F_v \cap \mathbb{Z}^n$, then by definition of P_v, $x \in P_v$.
Finally we verify that

$$u \in P_v \text{ s.t. } \langle u, v\rangle = \sigma_K(v) \Rightarrow u \in K. \tag{8}$$

Observe that if $u \in P_v$ such that $\langle u, v\rangle = \sigma_K(v)$, then $u \in \text{conv}(F_v \cap \mathbb{Z}^n) + \text{rec.cone}(K)$ since $\sigma_{P^1_v + \text{rec.cone}(K)}(v) = \sigma_{P^1_v}(v) < \sigma_K(v)$. Since $\text{conv}(F_v \cap \mathbb{Z}^n) + \text{rec.cone}(K) \subseteq K$, the result follows.

Claim 1: For any $v \in D$, there exists a neighborhood N_v of v (wrt D) such that $\sigma_{P_v}(v') \leq \sigma_K(v')$ for all $v' \in N_v$. There are two cases:

1. $\sigma_{P_v}(v) < \sigma_K(v)$. By Lemma 3.4, both σ_{P_v} and σ_K are continuous functions over their domain $(\text{rec.cone}(K))^*$. In particular, σ_{P_v} and σ_K are continuous over $D \subsetneq (\text{rec.cone}(K))^*$. Therefore, $\sigma_{P_v} - \sigma_K$ is a continuous function over D. This implies that there is a neighborhood N_v of v such that $\sigma_{P_v}(v') - \sigma_K(v') < 0$ for all $v' \in N_v$.

2. $\sigma_{P_v}(v) = \sigma_K(v)$. By Minkowski-Weyl Theorem, let $P_v = \text{conv}\{u^1, ..., u^s\} + \text{cone}\{r^1, ..., r^k\}$. Let $O \subseteq \{u^1, ..., u^s\}$ such that $u^j \in O$ implies $\langle u^j, v\rangle = \sigma_K(v)$. We note here that $O \neq \emptyset$, since $\sigma_{P_v}(v) = \sigma_K(v)$. By (8), we obtain that $O \subseteq K$. Let $N = \{u^1, ..., u^s\} \setminus O$. Note that $\langle r^i, v\rangle \leq 0$ $i \in \{1, ..., k\}$. Let $\delta = \sigma_K(v) - \max_{u^i \in N}\{\langle u^i, v\rangle\} > 0$. Let $\eta = \max_{u^i \in O}\{\|u^i\|\}$. Let $N_v = \{d \in D \,|\, \|d - v\| < \frac{\delta}{2\eta}\}$.

   (a) For $u^j \in O$ and $v' \in N_v$, observe that

   $$\langle u^j, v'\rangle = \langle u^j, v\rangle + \langle u^j, v' - v\rangle \geq \sigma_K(v) - \|u^j\|\|v' - v\| > \sigma_K(v) - \frac{\delta}{2}.$$

   (b) For $u^j \in N$ and $v' \in N_v$, observe that

   $$\langle u^j, v'\rangle = \langle u^j, v\rangle + \langle u^j, v' - v\rangle \leq \sigma_K(v) - \delta + \|u^j\|\|v' - v\| < \sigma_K(v) - \frac{\delta}{2}.$$

   Therefore for any $v' \in N_v$ we obtain that,

   $$\sigma_{P_v}(v') = \max_{1 \leq j \leq s}\{\langle v', u^j\rangle\} = \max_{u^j \in O}\{\langle v', u^j\rangle\} \leq \sigma_K(v'), \tag{9}$$

   where the last inequality is a consequence of the fact that $O \subseteq K$.

This completes the proof of the claim.

Now observe that the sets N_v for all $v \in D$ represent an open cover of D. Since D is compact, there exists a finite set of vectors $v^1, ..., v^l$, such that $\cup_{i=1}^l N_{v^i} = D$.
Claim 2: $P' := \cap_{i=1}^l P_{v^i} \subseteq K$:

Assume by contradiction that there exists $x \in \cap_{i=1}^{l} P_{v^i} \setminus K$. Then by separation theorem, there exists d such that

$$\sigma_K(d) < \langle x, d \rangle \leq \sigma_{P'}(d). \tag{10}$$

Since $\sigma_K(d)$ is finite and $d \neq 0$, we obtain (by possibly scaling d) that $d \in D$ and therefore $d \in N_{v^i}$ for some $i \in \{1, ...k\}$. Therefore,

$$\sigma_{P'}(d) \leq \sigma_{P_{v^i}}(d) \leq \sigma_K(d), \tag{11}$$

where the last inequality follows from Claim 1. This is the required contradiction to (10).

Since $P_{v^i} \supseteq \text{conv}(K \cap \mathbb{Z}^n)$, we have that $P' := \cap_{i=1}^{l} P_{v^i} \supseteq \text{conv}(K \cap \mathbb{Z}^n)$. On the other hand by Claim 2, $P' \cap \mathbb{Z}^n \subseteq K \cap \mathbb{Z}^n$. In other words,

$$\text{conv}(K \cap \mathbb{Z}^n) = \text{conv}(P' \cap \mathbb{Z}^n). \tag{12}$$

Moreover note that $\text{rec.cone}(P') = \text{rec.cone}(K)$. Now we arrive at the conclusion that $\text{conv}(P' \cap \mathbb{Z}^n)$ is a rational polyhedron by using Lemma 3.6. □

We note here that argument based on the compactness of D in Proposition 3.1 is similar to an argument used in [2].

3.2 Necessary conditions for conv$(K \cap \mathbb{Z}^n)$ to be polyhedral

Lemma 3.7. *Let $Q \subseteq \mathbb{R}^n$ be a full-dimensional maximal lattice free convex set and let $\langle c, x \rangle \leq d$ be a valid inequality for Q. Then there exists $\delta > 0$ such that $\langle c, x \rangle \geq d - \delta$ is a valid inequality for Q.*

Proof. Assume by contradiction that $\inf\{\langle c, x \rangle \mid x \in Q\} = -\infty$. Since Q is polyhedron (Theorem 2.2), we obtain there exists a recession direction r of Q such that $\langle r, c \rangle < 0$. However because $\text{rec.cone}(Q) = \text{lin.space}(Q)$, we have that $-r$ is a recession direction of Q. Then $\sup\{\langle c, x \rangle \mid x \in Q\} = +\infty$, contradicting the assumption that $\langle c, x \rangle \leq d$ is a valid inequality for Q. □

Proposition 3.2 (Necessary Condition)**.** *Let $K \subseteq \mathbb{R}^n$ be a convex set such that* $\text{int}(K) \cap \mathbb{Z}^n \neq \emptyset$*. If* $\text{conv}(K \cap \mathbb{Z}^n)$ *is a polyhedron, then K is thin and* $\text{rec.cone}(K)$ *is a rational polyhedral cone.*

Proof. Let $P = \{x \in \mathbb{R}^n \mid \langle a_i, x \rangle \leq b_i,\ i \in \{1, ..., m\}\}$ be a description of $\text{conv}(K \cap \mathbb{Z}^n)$. Note that P is a rational polyhedron. We will show first that for all $i \in \{1, ..., m\}$, $\sup\{\langle a_i, x \rangle \mid x \in K\} < \infty$. Let $i \in \{1, ..., m\}$ and assume by contradiction that $\sup\{\langle a_i, x \rangle \mid x \in K\} = \infty$. Consider the set $K_i = K \cap \{x \in \mathbb{R}^n \mid \langle a_i, x \rangle \geq b_i\}$. Notice that $\text{int}(K) \cap \mathbb{Z}^n \neq \emptyset$, so K must be a full dimensional set. Also, by assumption, we have $K \subsetneq \{x \in \mathbb{R}^n \mid \langle a_i, x \rangle \leq b_i\}$. Therefore it can be verified that $\text{int}(K) \cap \{x \in \mathbb{R}^n \mid \langle a_i, x \rangle > b_i\} \neq \emptyset$. This implies $\text{int}(K_i) = \text{int}(K) \cap \{x \in \mathbb{R}^n \mid \langle a_i, x \rangle > b_i\} \neq \emptyset$ and thus K_i is of full dimension.

Moreover, we have $\text{int}(K_i) \cap \mathbb{Z}^n = (\text{int}(K_i) \cap K) \cap \mathbb{Z}^n \subseteq \text{int}(K_i) \cap P = \emptyset$, so K_i is a lattice-free set. Hence, there exists a full dimensional maximal lattice-free polyhedron $Q = \{x \in \mathbb{R}^n \mid \langle c_j, x \rangle \leq d_j,\ j \in \{1, \ldots, q\}\}$ such that $K_i \subseteq Q$.

Since K is not lattice-free we obtain that $K \subsetneq Q$. Therefore there exists $x_0 \in K$ and $j \in \{1, \ldots, q\}$ such that $\langle a_i, x_0 \rangle < b_i$ and $\langle c_j, x_0 \rangle > d_j$. By Lemma 3.7, there exists $\delta > 0$ such that $x \in Q$ implies $\langle c_j, x \rangle \geq d_j - \delta$.

Let $\{x_n\}_{n \geq 1} \subseteq K_i$ such that $\lim_{n \to \infty} \langle a_i, x_n \rangle = \infty$ and $\lambda_n \in (0, 1)$ such that the point $y_n = (1 - \lambda_n)x_0 + \lambda_n x_n$ satisfies $\langle a_i, y_n \rangle = b_i$. Since $x_0, x_n \in K$, by convexity of K, we have $y_n \in K$. Therefore we obtain that $y_n \in K_i$.

On the other hand,

$$\begin{aligned} \langle c_j, y_n \rangle - d_j &= (1 - \lambda_n)\langle c_j, x_0 \rangle + \lambda_n \langle c_j, x_n \rangle - d_j \\ &\geq (1 - \lambda_n)(\langle c_j, x_0 \rangle - d_j) - \lambda_n \delta \\ &= (\langle c_j, x_0 \rangle - d_j) - \lambda_n[(\langle c_j, x_0 \rangle - d_j) + \delta]. \end{aligned}$$

Notice that, by definition, $\lambda_n = \frac{b_i - \langle a_i, x_0 \rangle}{\langle a_i, x_n \rangle - \langle a_i, x_0 \rangle}$ and thus $\lim_{n\to\infty} \lambda_n = 0$. Hence, for sufficiently large n, we have $\langle c_j, y_n \rangle > d_j$, a contradiction with the fact $y_n \in K_i \subseteq Q$. So, we must have $\sup\{\langle a_i, x\rangle \mid x \in K\} < \infty$, for all $i \in \{1, ..., m\}$.

We conclude that there exist numbers $\bar{b}_i,\ i \in \{1, ..., m\}$, with $b_i \leq \bar{b}_i < \infty$ such that $K \subseteq P' := \{x \mid \langle a_i, x\rangle \leq \bar{b}_i,\ i \in \{1, ..., m\}\}$. Hence, since $P \subseteq K \subseteq P'$, we have $\text{rec.cone}(K) = \{x \mid \langle a_i, x\rangle \leq 0,\ i \in \{1, ..., m\}\}$, so rec.cone($K$) is a rational polyhedral cone. Moreover, every polyhedron is thin, so by Lemma 3.1, we conclude K is also thin, as desired. □

We note here that the addition technical condition that $\text{int}(K) \cap \mathbb{Z}^n \neq \emptyset$ in Proposition 3.2 is not artificial. We illustrate this with examples next.

Example 3.2. *1. Here is an example that shows that* $\text{conv}(K \cap \mathbb{Z}^n)$ *can be a polyhedron and yet it is not thin, since it is lattice-free. Consider the set*

$$\begin{aligned} K^9 \quad := \quad & \text{conv}(\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 = 0, x_1 = 0, x_2 \geq 0\} \\ \cup \quad & \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 = 0.5, x_2 \geq x_1^2\} \\ \cup \quad & \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 = 1, x_1 = 0, x_2 \geq 0\}). \end{aligned}$$

Observe that $\text{conv}(K^9 \cap \mathbb{Z}^3) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 = 0, x_2 \geq 0, 0 \leq x_3 \leq 1\}$ *is a polyhedron. However note that* K^9 *is not thin since* $\text{rec.cone}(K^9) = \{\lambda(0,1,0) \mid \lambda \geq 0\}$ *and* $\inf\{\langle(-1,0,0), x\rangle \mid x \in K^9\} = -\infty$ *but* $\langle(0,1,0), (-1,$
$0,0)\rangle = 0$. *Finally note that* K^9 *is lattice-free.*

2. Here is an example that shows that $\text{conv}(K \cap \mathbb{Z}^n)$ *can be a polyhedron and yet* rec.cone(K) *does not have a polyhedral recession cone, since it is lattice-free. Consider the set*

$$\begin{aligned} K^{10} \quad := \quad & \text{conv}(\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_2 - \sqrt{2}x_1 = 0, x_3 = 0\} \\ \cup \quad & \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_2 - \sqrt{2}x_1 = 1, x_3 = 0.5\} \\ \cup \quad & \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_2 - \sqrt{2}x_1 = -1, x_3 = 0.5\} \\ \cup \quad & \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_2 - \sqrt{2}x_1 = 0, x_3 = 1\}). \end{aligned}$$

Then $K^{10} \cap \mathbb{Z}^3 = \{(0,0,0), (0,0,1)\}$ *and thus* $\text{conv}(K^{10} \cap \mathbb{Z}^3)$ *is a polyhedron. However note that* $\text{rec.cone}(K^{10})$ *is not a rational polyhedral cone. Also observe that* K^{10} *is lattice-free.*

4 Remarks

We first remark that all the key results in this paper (Theorem 2.1, Theorem 2.3, Theorem 2.4, Theorem 2.5, Theorem 2.6, Theorem 3.1) hold if we replace $\mathbb{Z}^n$ by any general lattice $\Gamma \subseteq \mathbb{R}^n$ and investigate the closed-ness and polyhedrality of $\text{conv}(K \cap \Gamma)$.

It is possible to relax the requirement of $\text{conv}(K \cap \mathbb{Z}^n)$ being a polyhedron and ask the question when $\text{conv}(K \cap \mathbb{Z}^n)$ is *locally polyhedron*, i.e., the intersection of $\text{conv}(K \cap \mathbb{Z}^n)$ with any polytope is also a polytope. To the best of our knowledge the most general sufficient conditions known for $\text{conv}(K \cap \mathbb{Z}^n)$ to be locally polyhedral are presented in [9] for the case where K is general polyhedron (not necessary rational). Coming up with necessary and sufficient conditions for $\text{conv}(K \cap \mathbb{Z}^n)$ to be locally polyhedron in the case where K is general convex sets is an interesting open question.

Another important question is determining necessary and sufficient conditions for the following optimization problem

$$z^* = \min \quad \langle d, x\rangle \tag{13}$$
$$s.t. \quad x \in K \cap \mathbb{Z}^n \tag{14}$$

to be solvable, i.e., if z^* is bounded and $K \cap \mathbb{Z}^n \neq \emptyset$ implies there exists $x^* \in K \cap \mathbb{Z}^n$ such that $\langle d, x^*\rangle \leq \langle d, x\rangle\ \forall x \in K \cap \mathbb{Z}^n$. Clearly if $\text{conv}(K \cap \mathbb{Z}^n)$ is a polyhedron, then the optimization problem is solvable for any d. Another sufficient condition that can be easily verified is that d is a rational vector. However, finding general necessary and sufficient conditions for (13)-(14) to be solvable is a challenging question.

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