On n-step MIR and Partition Inequalities for Integer Knapsack and Single-node Capacitated Flow Sets

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Abstract

Pochet and Wolsey [Y. Pochet, L.A. Wolsey, Integer knapsack and flow covers with divisible coefficients: polyhedra, optimization and separation. Discrete Applied Mathematics 59(1995) 57–74] introduced *partition inequalities* for three substructures arising in various mixed integer programs, namely the integer knapsack set with nonnegative divisible/arbitrary coefficients and two forms of single-node capacitated flow set with divisible coefficients. They developed the partition inequalities by proving properties of the optimal solution in optimizing a linear function over these sets. More recently, the author and Fathi [K. Kianfar, Y. Fathi: Generalized mixed integer rounding inequalities: facets for infinite group polyhedra. Mathematical Programming 120(2009) 313–346] introduced the *n-step mixed integer rounding (MIR)* inequalities for the mixed-integer knapsack set with arbitrary coefficients through a generalization of MIR. In this paper, we show that the n-step MIR generates facet-defining inequalities not only for the three sets considered by Pochet and Wolsey but also for their generalization to the case where coefficients are not necessarily divisible. In the case of divisible coefficients, n-step MIR directly generates the partition inequalities for all three sets (and in some cases stronger inequalities for one of the sets). We show that n-step MIR gives facets for the integer knapsack set with arbitrary coefficients that either dominate or are not obtainable by the partition inequalities. We also derive new (the first) facets for the two capacitated flow sets with arbitrary coefficients using n-step MIR. Our results provide a new perspective based on n-step MIR into the polyhedral properties of these three substructures, extend them to the case of arbitrary coefficients, and underscore the power of n-step MIR to easily generate strong valid inequalities.

Key words: n-step mixed integer rounding, partition inequality, integer knapsack set, capacitated flow set, valid inequality, facet

1. Introduction

Understanding the polyhedral structure of (mixed) integer sets that frequently arise as substructures in larger mixed integer programs (MIPs) and developing valid inequalities and facets for them are always of general interest. Three of such sets are the integer knapsack set

$$X = \left\{x \in \mathbb{Z}_+^N : \sum\nolimits_{j=1}^{N} C_j x_j \geq b\right\},$$

the single-node capacitated flow set

$$Y = \left\{(x, y) \in \mathbb{Z}_+^N \times \mathbb{R}_+ : y \leq \sum\nolimits_{j=1}^{N} C_j x_j, y \leq b\right\},$$

and its extension to

$$Z = \left\{(x, y) \in \mathbb{Z}_+^N \times \mathbb{R}_+^N : \sum\nolimits_{j=1}^{N} y_j \leq b, y_j \leq C_j x_j \text{ for } j = 1, \ldots, N\right\},$$

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in all of which we assume $0 < C_1 \leq C_2 \leq \ldots \leq C_N$. These sets arise as substructures in integer programming formulations of several network design problems [2, 4], coin-changing [11], and cutting stock [12] problems. Wolsey [16] has surveyed valid inequalities for these sets alongside several other simple MIP structures.

Pochet and Wolsey [14] studied special cases of these sets where the coefficients are divisible, i.e. C_{j+1} is an integer multiple of C_j ($C_j | C_{j+1}$) for all j. Here we denote these special cases by X_d, Y_d, and Z_d, respectively. They showed that the convex hulls of X_d and Y_d can be described by inequalities that they called *partition inequalities* and presented optimization and separation algorithms over these sets. They also presented partition inequalities for Z_d and showed that optimization over X_d, Y_d, and Z_d can be done in polynomial time. Moreover, they showed that valid inequalities very similar to the partition inequalities for X_d can be generated for X too if certain conditions on coefficients C_j and the right-hand side b are satisfied. The approach in [14] to derive all the aforementioned results is based on characterization of the optimal solution in optimizing an arbitrary linear objective function over X_d, Y_d, Z_d, and X, which proves to have nice decomposition properties.

In a more general setting, Kianfar and Fathi [9] recently introduced the *n-step mixed integer rounding (MIR) inequalities* for a general mixed-integer knapsack set

$$K = \left\{(x, s) \in \mathbb{Z}_+^{|I|} \times \mathbb{R}_+ : \sum\nolimits_{i \in I} a_i x_i + s \geq b\right\},$$

where I is the index set and $a_i, b \in \mathbb{R}$, using a completely different approach which is based on a generalization of the concept of MIR [3, 13, 15]. An n-step MIR inequality is generated by simply applying an *n-step MIR function* [9] on the coefficients $a_i, i \in I$, and the right-hand side b. While theoretical derivation of these functions is rather involved, they can be presented in a simple compact form and calculated very fast at any given point. As a result, n-step MIR inequalities can be generated very fast. It is shown in [9, 10] that n-step MIR inequalities are facet-defining for the infinite and finite group problems [5, 6, 7, 8] and also for certain mixed-integer knapsack sets. Later Atamtürk and Kianfar [1] showed that n-step MIR inequalities define facets for the general mixed-integer knapsack set K under certain conditions. More specifically, they introduced n-step mingling inequalities and proved their facet-defining properties, and showed that these properties also hold for n-step MIR as a special case.

In this paper, we show that the n-step MIR directly generates facet-defining inequalities not only for the three sets X_d, Y_d and Z_d considered by Pochet and Wolsey, but also for their generalization to the case where coefficients are not necessarily divisible, i.e. X, Y, and Z. In the case of X_d, Y_d and Z_d, we show that the n-step MIR directly generates the partition inequalities for all three sets. For Y_d, we will see that there are cases in which n-step MIR gives facet-defining inequalities that are stronger than partition inequalities implying that those partition inequalities can be dropped from the convex hull description. In the case of sets with arbitrary coefficients, we show that the n-step MIR gives facets for X that either dominate or are not obtainable by the partition inequalities. We also derive new facets for Y and Z using n-step MIR. Our results provide a new perspective based on the n-step MIR into the polyhedral properties of these substructures. They underscore the power of n-step MIR (as a valid inequality method for general MIPs) to directly generate facet-defining inequalities that encompass or dominate inequalities previously developed for special sets using customized approaches, and extend them to more general sets.

After a brief review of background in Section 2, we present the results related to the sets X, Y and Z in Sections 3, 4, and 5, respectively. We conclude in Section 6.

2. Brief review

In this section we briefly review the partition inequalities of [14] and the n-step MIR inequality and n-step MIR functions of [9] as required for our developments in the next sections.

2.1. Partition inequalities for X and X_d

Pochet and Wolsey [14] introduced the so-called *partition inequalities* for the set X through an approach based on characterization of the optimal solution in minimizing an arbitrary linear objective function over X. Let $J :=$

$\{1,\ldots,N\}$. Assuming $C_r \leq b$ and $C_{r+1} > b$ for some $r \in J$, these inequalities are generated as follows: consider a partitioning of J into blocks $\{i_1,\ldots,j_1\},\{i_2,\ldots,j_2\},\ldots,\{i_p,\ldots,j_p\}$ with $i_1 = 1$, $j_p = N$, $i_p \leq r$, $i_t = j_{t-1}+1$ for $t = 2,\ldots,p$. For blocks $t = p,\ldots,1$ define β_t and κ_t as follows $\beta_p = b$, $\kappa_t = \lceil \beta_t/C_{i_t} \rceil$, $\beta_{t-1} = \beta_t - C_{i_t}(\kappa_t - 1)$. Now the corresponding partition inequality is

$$\sum_{t=1}^{p}\left(\prod_{s=1}^{t-1}\kappa_s\right)\sum_{j=i_t}^{j_t}\min\left\{\left\lceil C_j/C_{i_t}\right\rceil,\kappa_t\right\}x_j \geq \prod_{t=1}^{p}\kappa_t, \tag{1}$$

and it is valid for X if

$$\kappa_t \leq C_{i_{t+1}}/C_{i_t} \quad \text{for } t = 1,\ldots,p-1. \tag{2}$$

For the special case of X_d, conditions (2) are always satisfied because for any $t = 1,\ldots,p-1$ we have $\beta_t \leq C_{i_{t+1}}$ by definition and so $\beta_t/C_{i_t} \leq C_{i_{t+1}}/C_{i_t}$, which since $C_{i_{t+1}}/C_{i_t}$ is an integer, means $\kappa_t \leq C_{i_{t+1}}/C_{i_t}$. Therefore for X_d inequality (1) is valid and reduces to $\sum_{t=1}^{p}\left(\prod_{s=1}^{t-1}\kappa_s\right)\sum_{j=i_t}^{j_t}\min\{C_j/C_{i_t},\kappa_t\}x_j \geq \prod_{t=1}^{p}\kappa_t$, which is the partition inequality (as defined in [14]) for X_d. As a result, we will use (1) for both X and X_d. Pochet and Wolsey proved in [14] that the convex hull of X_d is described by the non-negativity constraints and partition inequalities (1). However, for X they only show the validity of (1).

2.2. *n-step MIR inequalities for K*

Kianfar and Fathi [9] introduced the n-step MIR inequality for the set K. An n-step MIR inequality is generated by applying an n-step MIR function on the coefficients of the defining inequality of K: Let $\alpha = \{\alpha_1,\alpha_2,\ldots\}$ be a sequence in $\mathbb{R}_{>0}$. For $u \in \mathbb{R}$ we define $u^{(0)} := u$ and $u^{(k)} := u^{(k-1)} - \alpha_k \left\lfloor u^{(k-1)}/\alpha_k \right\rfloor$ for $k \geq 1$. Based on this definition, for any $n \geq 1$ and $b \in \mathbb{R}$ we can define a partitioning of $\mathbb{R}$ as follows:

$$\mathcal{I}_m^n := \left\{u \in \mathbb{R} : u^{(k)} < b^{(k)},\ k = 1,\ldots,m,\ u^{(m+1)} \geq b^{(m+1)}\right\} \quad \text{for } m = 0,\ldots,n-1;$$
$$\mathcal{I}_n^n := \left\{u \in \mathbb{R} : u^{(k)} < b^{(k)},\ k = 1,\ldots,n\right\}.$$

Based on [9] for $\alpha = (\alpha_1,\ldots,\alpha_n)$ and b, where the conditions

$$\left\lceil b^{(k-1)}/\alpha_k \right\rceil \leq \alpha_{k-1}/\alpha_k \quad \text{for } k = 2,\ldots,n \tag{3}$$

are satisfied, the n-step MIR function is defined as

$$\mu_{\alpha,b}^n(u) = \begin{cases} b^{(n)}\sum\limits_{k=1}^{m}\prod\limits_{l=k+1}^{n}\left\lceil\frac{b^{(l-1)}}{\alpha_l}\right\rceil\left\lfloor\frac{u^{(k-1)}}{\alpha_k}\right\rfloor + b^{(n)}\prod\limits_{l=m+2}^{n}\left\lceil\frac{b^{(l-1)}}{\alpha_l}\right\rceil\left\lceil\frac{u^{(m)}}{\alpha_{m+1}}\right\rceil & \text{if } u \in \mathcal{I}_m^n;\ m = 0,1,\ldots,n-1 \\ b^{(n)}\sum\limits_{k=1}^{n}\prod\limits_{l=k+1}^{n}\left\lceil\frac{b^{(l-1)}}{\alpha_l}\right\rceil\left\lfloor\frac{u^{(k-1)}}{\alpha_k}\right\rfloor + u^{(n)} & \text{if } u \in \mathcal{I}_n^n. \end{cases} \tag{4}$$

(note that throughout the paper if $a > b$, then $\sum_a^b(.) = 0$ and $\prod_a^b(.) = 1$). Kianfar and Fathi [9] then showed that the n-step MIR inequality generated using this function, i.e.

$$\sum\nolimits_{i\in I}\mu_{\alpha,b}^n(a_i)\,x_i + s \geq \mu_{\alpha,b}^n(b), \tag{5}$$

is valid for K. Note that the n-step MIR function $\mu_{\alpha,b}^n$ is a subadditive, nondecreasing, piecewise linear, and continuous.

3. *n*-step MIR for *X* and its relationship with partition inequalities

We first present a few helpful lemmas in Section 3.1, and then present the n-step MIR for X and establish its relationship with partition inequalities in Section 3.2. We address the facet-defining properties in Section 3.3.

3.1. Preliminary Lemmas

The first lemma is regarding the value of the n-step MIR function in several special cases and will be very useful in proving many results of the paper.

Lemma 1. *Let $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_1 > \alpha_2 > \ldots > \alpha_n$, and $b^{(n)} > 0$. Then*

(i). *If $b \le u \le \alpha_1 \lceil b/\alpha_1 \rceil$, then $\mu^n_{\alpha,b}(u) = b^{(n)} \prod_{l=2}^n \left\lceil \frac{b^{(l-1)}}{\alpha_l} \right\rceil \left\lceil \frac{u}{\alpha_1} \right\rceil$.*

(ii). *If $b^{(s)} \le u < \alpha_s$ for some $s \in \{1, \ldots, n\}$, then $\mu^n_{\alpha,b}(u) = b^{(n)} \prod_{l=s+1}^n \left\lceil \frac{b^{(l-1)}}{\alpha_l} \right\rceil$.*

(iii). *If $\alpha_{s+1} | u < b^{(s)}$ for some $s \in \{0, \ldots, n-1\}$, , then $\mu^n_{\alpha,b}(u) = b^{(n)} \prod_{l=s+2}^n \left\lceil \frac{b^{(l-1)}}{\alpha_l} \right\rceil \frac{u}{\alpha_{s+1}}$.*

(iv). *If $u = \alpha_s$ for some $s \in \{1, \ldots, n\}$, then $\mu^n_{\alpha,b}(u) = b^{(n)} \prod_{l=s+1}^n \left\lceil \frac{b^{(l-1)}}{\alpha_l} \right\rceil$.*

(v). *If $u \le b^{(n)}$ then $\mu^n_{\alpha,b}(u) = u$.*

Proof. (i). In this case $u^{(1)} \ge b^{(1)}$ so $u \in \mathcal{I}^n_0$. Formulation (4) gives the result.

(ii). The value of $\mu^n_{\alpha,b}(u)$ depends on which set among $\mathcal{I}^n_m$, $m = 0, \ldots, n$, u belongs to. To determine, observe that since $u < \alpha_s < \ldots < \alpha_1$, we have

$$u^{(t)} = u \quad \text{for } t = 1, \ldots, s. \tag{6}$$

Now let q be the smallest nonnegative integer less than n for which $u^{(q+1)} \ge b^{(q+1)}$ if it exists, and let $q = n$ otherwise. Therefore $u \in \mathcal{I}^n_q$. By definition $b^{(1)} \ge b^{(2)} \ge \ldots \ge b^{(n)}$; therefore since $u \ge b^{(s)}$, based on (6), we will have $q \le s - 1$. This implies $q \le n - 1$ so according to formulation (4), we have

$$\mu^n_{\alpha,b}(u) = b^{(n)} \sum_{k=1}^{q} \prod_{l=k+1}^{n} \left\lceil \frac{b^{(l-1)}}{\alpha_l} \right\rceil \left\lfloor \frac{u^{(k-1)}}{\alpha_k} \right\rfloor + b^{(n)} \prod_{l=q+2}^{n} \left\lceil \frac{b^{(l-1)}}{\alpha_l} \right\rceil \left\lceil \frac{u^{(q)}}{\alpha_{q+1}} \right\rceil = b^{(n)} \prod_{l=q+2}^{n} \left\lceil \frac{b^{(l-1)}}{\alpha_l} \right\rceil = b^{(n)} \prod_{l=s+1}^{n} \left\lceil \frac{b^{(l-1)}}{\alpha_l} \right\rceil \tag{7}$$

The second identity holds because we have $u^{(1)} = \ldots = u^{(q)} = u < \alpha_s < \alpha_{q+1} < \alpha_q < \ldots < \alpha_1$ (since $q \le s - 1$). Thus $\left\lfloor u^{(k-1)}/\alpha_k \right\rfloor = 0$, $k = 1, \ldots, q$ and $\left\lceil u^{(q)}/\alpha_{q+1} \right\rceil = 1$. The third identity holds because if $q \le s - 2$ then we can write $b^{(s-1)} \le \ldots \le b^{(q+1)} \le u < \alpha_s < \ldots < \alpha_{q+2}$, which means $\left\lceil b^{(l-1)}/\alpha_l \right\rceil = 1$, $l = q + 2, \ldots, s$.

(iii). Note that $u < b^{(s)} < \alpha_s < \ldots < \alpha_1$. Therefore

$$u^{(t)} = u \quad \text{for } t = 1, \ldots, s, \tag{8}$$

and since $\alpha_{s+1} | u$, we also have

$$u^{(t)} = 0 \quad \text{for } t = s + 1, \ldots, n. \tag{9}$$

Now defining q the same way as in (ii), by (8) and (9) we have $q = n$ because $0 < b^{(n)} \le \ldots \le b^{(s)}$; so according to formulation (4)

$$\mu^n_{\alpha,b}(u) = b^{(n)} \sum_{k=1}^{n} \prod_{l=k+1}^{n} \left\lceil \frac{b^{(l-1)}}{\alpha_l} \right\rceil \left\lfloor \frac{u^{(k-1)}}{\alpha_k} \right\rfloor + u^{(n)} = b^{(n)} \prod_{l=s+2}^{n} \left\lceil \frac{b^{(l-1)}}{\alpha_l} \right\rceil \frac{u}{\alpha_{s+1}} \tag{10}$$

The second identity holds because $u^{(1)} = \ldots = u^{(s-1)} = u < \alpha_s < \ldots < \alpha_1$, $\left\lfloor u^{(s)}/\alpha_{s+1} \right\rfloor = \lfloor u/\alpha_{s+1} \rfloor = u/\alpha_{s+1}$, and identities (9) hold.

(iv). In this case we have $u^{(t)} = u$ for $t = 1, \ldots, s - 1$ and $u^{(t)} = 0$ for $t = s, \ldots, n$. Defining q the same way as in (ii), we either have $q \le s - 2$ in which case, the result can be proved very similar to (ii), or have $q = n$, in which case the result can be proved similar to (iii).

(v). Notice that we have $u \le b^{(n)} < \alpha_n < \alpha_{n-1} < \ldots < \alpha_1$, therefore

$$u^{(t)} = u \quad \text{for } t = 1, \ldots, n \tag{11}$$

Define q the same way as in (ii). We consider two cases: First, if $u = b^{(n)}$: In this case because of (11) and the fact that

$$b^{(n)} \leq \ldots \leq b^{(2)} \leq b^{(1)}, \tag{12}$$

we will have $q \leq n-1$ and

$$b^{(q+1)} = b^{(q+2)} = \ldots = b^{(n)} < \alpha_n < \ldots < \alpha_1. \tag{13}$$

Therefore according to formulation (4), we have

$$\mu^n_{\alpha,b}(u) = \mu^n_{\alpha,b}\left(b^{(n)}\right) = b^{(n)} \sum_{k=1}^{q} \prod_{l=k+1}^{n} \left\lceil \frac{b^{(l-1)}}{\alpha_l} \right\rceil \left\lfloor \frac{(b^{(n)})^{(k-1)}}{\alpha_k} \right\rfloor + b^{(n)} \prod_{l=q+2}^{n} \left\lceil \frac{b^{(l-1)}}{\alpha_l} \right\rceil \left\lceil \frac{(b^{(n)})^{(q)}}{\alpha_{q+1}} \right\rceil = b^{(n)} = u \tag{14}$$

The third identity is true because of (13) and the fact that $(b^{(n)})^{(k)} = b^{(n)}$ for $k = 1, \ldots, n$. Second, if $u < b^{(n)}$, by (11) and (12), we will have $q = n$. So according to formulation (4)

$$\mu^n_{\alpha,b}(u) = b^{(n)} \sum_{k=1}^{n} \prod_{l=k+1}^{n} \left\lceil \frac{b^{(l-1)}}{\alpha_l} \right\rceil \left\lfloor \frac{u^{(k-1)}}{\alpha_k} \right\rfloor + u^{(n)} = u \tag{15}$$

The second identity holds because of (11) and the fact that $u < b^{(n)} < \alpha_n < \ldots < \alpha_1$. Identities (14) and (15) complete the proof. □

The next lemma is easy to verify (we skip the proof) and reduces the number of distinct partition inequalities.

Lemma 2. *The partition inequality corresponding to the partitioning* $\{i_1, \ldots, j_1\}, \ldots, \{i_t, \ldots, j_t\}, \{i_{t+1}, \ldots, j_{t+1}\}, \ldots, \{i_p, \ldots, j_p\}$ *of* J *is the same as the partition inequality corresponding to the partitioning* $\{i_1, \ldots, j_1\}, \ldots, \{i_t, \ldots, j-1\}\{j, \ldots, j_t, i_{t+1}, \ldots, j_{t+1}\}, \ldots, \{i_p, \ldots, j_p\}$ *if* $C_j = C_{i_{t+1}}$ *for* j *in block* t *(in case of* $j = i_t$ *the number of blocks reduces by one).* □

Based on Lemma 2, without loss of generality, throughout the rest of the paper we only consider the partitions where $C_{j_t} < C_{i_{t+1}}$ for $t = 1, \ldots, p-1$.

3.2. n-step MIR inequality for X dominates partition inequality

We can generate n-step MIR inequalities for the set X as follows: Since $X = \{x \in \mathbb{Z}^n_+ : \sum_{j=1}^{N} C_j x_j \geq b\} = \{x \in \mathbb{Z}^n_+ : \sum_{j=1}^{r} C_j x_j + \sum_{j=r+1}^{N} b x_j \geq b\}$, the defining inequality of X can be strengthened to

$$\sum\nolimits_{j=1}^{r} C_j x_j + \sum\nolimits_{j=r+1}^{N} b x_j \geq b. \tag{16}$$

Now consider a partitioning of J such as $\{i_1, \ldots, j_1\}, \{i_2, \ldots, j_2\}, \ldots, \{i_p, \ldots, j_p\}$ with exactly the same conditions used in generating a partition inequality (1). Let

$$\alpha_t := C_{i_{p+1-t}} \quad \text{for} \quad t = 1, \ldots, p. \tag{17}$$

Now choose some $n \in \{1, ..., p\}$ and define $\alpha = (\alpha_1, \ldots, \alpha_n)$. In general, the relationship between the blocks and α_t, $t = 1, \ldots, n$, is as follows:

$$\{i_1, \ldots, j_1\}, \ldots, \{i_{p-n}, \ldots, j_{p-n}\}, \{\underset{\alpha_n = C_{i_{p-n+1}}}{\underset{\downarrow}{i_{p-n+1}}}, \ldots, j_{p-n+1}\}, \ldots, \{\underset{\alpha_2 = C_{i_{p-1}}}{\underset{\downarrow}{i_{p-1}}}, \ldots, j_{p-1}\}, \{\underset{\alpha_1 = C_{i_p}}{\underset{\downarrow}{i_p}}, \ldots, r, r+1, \ldots, j_p\}.$$

Then if conditions (3) are satisfied, the n-step MIR inequality

$$\sum\nolimits_{j=1}^{r} \mu^n_{\alpha,b}\left(C_j\right) x_j + \sum\nolimits_{j=r+1}^{N} \mu^n_{\alpha,b}(b)\, x_j \geq \mu^n_{\alpha,b}(b) \tag{18}$$

is valid for X.

Any $n \in \{1,\ldots,p\}$ results in an inequality of the form (18) provided that conditions (3) are satisfied. However, as we will see, only one certain value of n will be enough to generate an inequality (18) which dominates the partition inequality (1) corresponding to the same partitioning. This value is $n = n'$, where n' is the smallest nonnegative integer less than p for which $b^{(n'+1)} = 0$, if any, and $n' = p$ if $b^{(p)} \neq 0$. Other values of n in (18) can yield other facet-defining inequalities for X which are not obtainable by partition inequalities (see Example 1).

In the rest of this section, we assume $n := n'$. Now note that if $n > 0$, we have $b^{(t)} > 0$ for $t = 1,\ldots,n$, so

$$b^{(t)} = \beta_{p-t} \quad \text{for } t = 1,\ldots,n. \tag{19}$$

Hence

$$\left\lceil b^{(t-1)}/\alpha_t \right\rceil = \left\lceil \beta_{p-t+1}/C_{p-t+1} \right\rceil = \kappa_{p-t+1} \quad \text{for } t = 2,\ldots,n+1 \tag{20}$$

Based on (20), conditions (3) for validity of (18) can be written as

$$\kappa_t \leq C_{i_{t+1}}/C_{i_t} \quad \text{for } t = p-n+1,\ldots,p-1, \tag{21}$$

which are the same as (or a subset of) conditions (2) for validity of partition inequality (1) for X. Therefore for any given partitioning satisfying conditions (2), partition inequality (1) and n-step MIR inequality (18) both are valid. We will prove in Corollary 8 that in general n-step MIR inequality (18) dominates the partition inequality (1) for X. This will be based on Theorem 7, in which we show that the partition inequality (1) is in fact an n-step MIR inequality obtained from a relaxation of (16) instead of (16) itself. This relaxation is constructed as follows:

To simplify notation, define $j'_t = j_t$ for $t = 1,\ldots,p-1$ and $j'_p = r$. For the block $\{i_t,\ldots,j_t\}$, $t = 1,\ldots,p$, define $l_t = \max\{k \in \{i_t,\ldots,j'_t\} : \lceil C_k/C_{i_t} \rceil \leq \kappa_t\}$ (note that for block p we always have $l_p = j'_p$). We relax (16) to

$$\sum\nolimits_{t=1}^{p} \left(\sum\nolimits_{j=i_t}^{l_t} \left\lceil C_j/C_{i_t} \right\rceil C_{i_t} x_j + \sum\nolimits_{j=l_t+1}^{j'_t} C_j x_j \right) + \sum\nolimits_{j=r+1}^{N} b x_j \geq b. \tag{22}$$

By applying the same n-step MIR function used in (18) on the relaxed inequality (22), we get the following n-step MIR inequality for X:

$$\sum\nolimits_{t=1}^{p} \left(\sum\nolimits_{j=i_t}^{l_t} \mu^n_{\alpha,b}\left(\left\lceil C_j/C_{i_t} \right\rceil C_{i_t}\right) x_j + \sum\nolimits_{j=l_t+1}^{j'_t} \mu^n_{\alpha,b}\left(C_j\right) x_j \right) + \sum\nolimits_{j=r+1}^{N} \mu^n_{\alpha,b}(b)\, x_j \geq \mu^n_{\alpha,b}(b). \tag{23}$$

Comparing the two n-step MIR inequalities (18) and (23), note that for any $j \in \{i_t,\ldots,l_t\}$, $t = 1,\ldots,p$, we have $C_j \leq \left\lceil C_j/C_{i_t} \right\rceil C_{i_t}$ and so $\mu^n_{\alpha,b}\left(C_j\right) \leq \mu^n_{\alpha,b}\left(\left\lceil C_j/C_{i_t} \right\rceil C_{i_t}\right)$ because $\mu^n_{\alpha,b}$ is a non-decreasing function. All other coefficients and the right-hand sides are the same. Hence (18) dominates (23). In Corollary 8 we show that this domination is strict in many cases.

Remark 1. Note that if $n = p$, conditions (21) are the same as conditions (2) required for validity of (1) for X [14], but if $n < p$ then conditions (21) are only a subset of conditions (2). This means that if $n < p$, the n-step MIR inequalities (18) and (23) are valid even if the conditions $\kappa_t \leq C_{i_{t+1}}/C_{i_t}$ are not satisfied for $t = 1,\ldots,p-n$, however the n-step MIR inequality obtained in this case will not be a partition inequality. In this sense, partition inequalities may be only a subset of all possible inequalities (23).

To prove Theorem 7, we will need the following lemmas:

Lemma 3. *If $n = n' > 0$ and conditions* (21) *are satisfied, then for any $i_h \leq j \leq j'_h$, where $h \geq p-n+1$, the coefficient of x_j in inequality* (23) *is equal to $b^{(n)}\left(\prod_{s=p-n+1}^{h-1} \kappa_s\right)\min\{\left\lceil C_j/C_{i_h} \right\rceil, \kappa_h\}$.*

Proof. We show the lemma is true for all possible cases:

(I). $j > l_h$: In this case the coefficient of x_j is $\mu^n_{\alpha,b}\left(C_j\right)$. By definition of l_h, we have $\left\lceil C_j/C_{i_h} \right\rceil > \kappa_h = \left\lceil b^{(p-h)}/C_{i_h} \right\rceil$, which means $C_j > b^{(p-h)}$. Since $C_{j'_p} \leq b$, this means $p-h \neq 0$ so $p-h \geq 1$. Also observe that $C_j < C_{i_{h+1}} = \alpha_{p-h} < \ldots < \alpha_1$.

So altogether we have $1 \leq p-h \leq n-1$ and $b^{(p-h)} < C_j < \alpha_{p-h}$. Therefore by Lemma 1(ii), we have

$$\mu_{\alpha,b}^n\left(C_j\right) = b^{(n)} \prod_{l=p-h+1}^{n} \left\lceil \frac{b^{(l-1)}}{\alpha_l} \right\rceil = b^{(n)} \prod_{l=p-h+1}^{n} \kappa_{p-l+1} = b^{(n)} \prod_{s=p-n+1}^{h} \kappa_s = b^{(n)} \left(\prod_{s=p-n+1}^{h-1} \kappa_s \right) \min\{\left\lceil \frac{C_j}{C_{i_h}} \right\rceil, \kappa_h\} \tag{24}$$

The second identity is based on (20). The third identity is a simple change of index and the fourth identity is true because $j > l_h$ and hence $\kappa_h < \left\lceil C_j/C_{i_h} \right\rceil$.

(II). $j \leq l_h$: In this case the coefficient of x_j is $\mu_{\alpha,b}^n\left(\left\lceil C_j/C_{i_h} \right\rceil C_{i_h}\right)$. If $h = p$ two cases are possible: First, if $\left\lceil C_j/C_{i_p} \right\rceil C_{i_p} \geq b$, since we also have $\left\lceil C_j/C_{i_p} \right\rceil C_{i_p} \leq \left\lceil b/C_{i_p} \right\rceil C_{i_p}$, by Lemma 1(i) we have

$$\mu_{\alpha,b}^n\left(\left\lceil C_j/C_{i_p} \right\rceil C_{i_p}\right) = b^{(n)} \prod_{l=2}^{n} \left\lceil \frac{b^{(l-1)}}{\alpha_l} \right\rceil \left\lceil \frac{C_j}{C_{i_p}} \right\rceil = b^{(n)} \left(\prod_{s=p-n+1}^{p-1} \kappa_s \right) \min\{\left\lceil \frac{C_j}{C_{i_p}} \right\rceil, \kappa_p\}. \tag{25}$$

The third identity in (25) is true by the same change of index as in (24) and the fact that $\left\lceil C_j/C_{i_p} \right\rceil \leq \kappa_p$ because $j \leq l_p$. Second, if $\left\lceil C_j/C_{i_p} \right\rceil C_{i_p} < b$, then since $\alpha_1 = C_{i_p} \mid \left\lceil C_j/C_{i_p} \right\rceil C_{i_p}$, by Lemma 1(iii) a series of identities exactly like (25) is true.

If $h < p$, then observe that $j \leq l_h$ means

$$\left\lceil C_j/C_{i_h} \right\rceil \leq \kappa_h = \left\lceil b^{(p-h)}/C_{i_h} \right\rceil. \tag{26}$$

On the other hand, based on condition (21) for $t = h$ (note that $h \geq p-n+1$), we have

$$\kappa_h \leq C_{i_{h+1}}/C_{i_h}. \tag{27}$$

Inequalities (26) and (27) imply

$$\left\lceil C_j/C_{i_h} \right\rceil C_{i_h} \leq C_{i_{h+1}}. \tag{28}$$

Now based on (28) we consider two cases for $\left\lceil C_j/C_{i_h} \right\rceil C_{i_h}$:

(II-A). $\left\lceil C_j/C_{i_h} \right\rceil C_{i_h} < C_{i_{h+1}} = \alpha_{p-h}$: In this case two possibilities exist: First, if $\left\lceil C_j/C_{i_h} \right\rceil C_{i_h} \geq b^{(p-h)}$, then by Lemma 1(ii) similar to (24) we get

$$\mu_{\alpha,b}^n\left(\left\lceil C_j/C_{i_h} \right\rceil C_{i_h}\right) = b^{(n)} \prod_{s=p-n+1}^{h} \kappa_s = b^{(n)} \left(\prod_{s=p-n+1}^{h-1} \kappa_s \right) \min\{\left\lceil \frac{C_j}{C_{i_h}} \right\rceil, \kappa_h\} \tag{29}$$

The last identity in (29) is true because (26) along with $\left\lceil C_j/C_{i_h} \right\rceil C_{i_h} \geq b^{(p-h)}$ imply $\left\lceil C_j/C_{i_h} \right\rceil = \left\lceil b^{(p-h)}/C_{i_h} \right\rceil = \kappa_h$. Second, if $\left\lceil C_j/C_{i_h} \right\rceil C_{i_h} < b^{(p-h)}$, since we also have $\alpha_{p-h+1} = C_{i_h} \mid \left\lceil C_j/C_{i_h} \right\rceil C_{i_h}$, by Lemma 1(iii) we get

$$\mu_{\alpha,b}^n\left(\left\lceil C_j/C_{i_h} \right\rceil C_{i_h}\right) = b^{(n)} \prod_{l=p-h+2}^{n} \left\lceil \frac{b^{(l-1)}}{\alpha_l} \right\rceil \left\lceil \frac{C_j}{C_{i_h}} \right\rceil = b^{(n)} \left(\prod_{s=p-n+1}^{h-1} \kappa_s \right) \min\{\left\lceil \frac{C_j}{C_{i_h}} \right\rceil, \kappa_h\} \tag{30}$$

The last identity in (30) is true by a change of index similar to (24) and since $\left\lceil C_j/C_{i_h} \right\rceil < \left\lceil b^{(p-h)}/C_{i_h} \right\rceil = \kappa_h$.

(II-B). $\left\lceil C_j/C_{i_h} \right\rceil C_{i_h} = C_{i_{h+1}} = \alpha_{p-h}$: In this case observe that by (26) and (27), we have

$$\left\lceil C_j/C_{i_h} \right\rceil = \left\lceil b^{(p-h)}/C_{i_h} \right\rceil = \kappa_h. \tag{31}$$

Now based on Lemma 1(iv), a series of identities exactly like (29) is true. The last identity is true because of (31). This exhausts all the cases and completes the proof.

□

Lemma 4. *If* $0 < n = n' \leq p-1$ *and conditions* (21) *are satisfied, then for any* $i_{p-n} \leq j \leq j'_{p-n}$*, the coefficient of* x_j *in inequality* (23) *is equal to* $C_{i_{p-n}} \min\{\lceil C_j/C_{i_{p-n}} \rceil, \kappa_{p-n}\}$.

Proof. We know $C_{i_{p-n}} = \alpha_{n+1}|b^{(n)} = \beta_{p-n}$. This implies

$$\kappa_{p-n} = \lceil \beta_{p-n}/C_{i_{p-n}} \rceil = b^{(n)}/C_{i_{p-n}}. \tag{32}$$

We will have two cases: First, if $j > l_{p-n}$, following a line of argument similar to case I of Lemma 3 only setting $h = p-n$, we get

$$\mu^n_{\alpha,b}\left(C_j\right) = b^{(n)} = C_{i_{p-n}} \min\{\lceil C_j/C_{i_{p-n}} \rceil, \kappa_{p-n}\}. \tag{33}$$

The second identity in (33) is true because of (32) and the fact that $\kappa_{p-n} < \lceil C_j/C_{i_{p-n}} \rceil$ since $j > l_{p-n}$. Second, if $j \leq l_{p-n}$, we have $\lceil C_j/C_{i_{p-n}} \rceil \leq \kappa_{p-n}$, which along with (32) implies $\lceil C_j/C_{i_{p-n}} \rceil C_{i_{p-n}} \leq b^{(n)}$. Therefore based on Lemma 1(v), we have

$$\mu^n_{\alpha,b}\left(\lceil C_j/C_{i_{p-n}} \rceil C_{i_{p-n}}\right) = \lceil C_j/C_{i_{p-n}} \rceil C_{i_{p-n}} = C_{i_{p-n}} \min\{\lceil C_j/C_{i_{p-n}} \rceil, \kappa_{p-n}\} \tag{34}$$

This completes the proof. □

Lemma 5. *If* $0 \leq n = n' \leq p-2$ *and conditions* (2) *are satisfied, then for* $t = 1, \ldots, p-n-1$ *we have*

$$C_{i_t}|C_{i_{t+1}}, \beta_t = C_{i_{t+1}}, \textit{and } \kappa_t = C_{i_{t+1}}/C_{i_t}. \tag{35}$$

Also $l_h = j'_h$ *for every* $h \leq p-n-1$.

Proof. According to the definition of n, $b^{(n+1)} = 0$. Therefore we have

$$C_{i_{p-n}} = \alpha_{n+1}|b^{(n)} = \beta_{p-n}. \tag{36}$$

This implies that $\kappa_{p-n} = \beta_{p-n}/C_{i_{p-n}}$ and therefore $\beta_{p-n-1} = C_{i_{p-n}}$, so $\kappa_{p-n-1} = \lceil C_{i_{p-n}}/C_{i_{p-n-1}} \rceil$. But according to conditions (2), we also have $\kappa_{p-n-1} \leq C_{i_{p-n}}/C_{i_{p-n-1}}$. These together imply that $C_{i_{p-n-1}}|C_{i_{p-n}}$ and $\kappa_{p-n-1} = C_{i_{p-n}}/C_{i_{p-n-1}}$, which in turn imply $\beta_{p-n-2} = C_{i_{p-n-1}}$. Repeating this argument for $\kappa_t, t = p-n-2, ..., 1$ proves (35). Now observe that since $h \leq p-n-1$, by (35), $\kappa_h = C_{i_{h+1}}/C_{i_h}$, which is an integer. Therefore $\lceil C_j/C_{i_h} \rceil \leq C_{i_{h+1}}/C_{i_h} = \kappa_h$ for every $i_h \leq j \leq j'_h$. Therefore $l_h = j'_h$. □

Lemma 6. *If* $0 < n = n' \leq p-2$ *and conditions* (2) *are satisfied, then for any* $i_h \leq j \leq j'_h$*, where* $h \leq p-n-1$*, the coefficient of* x_j *in inequality* (23) *is equal to* $\lceil C_j/C_{i_h} \rceil C_{i_h}$.

Proof. Since $h \leq p-n-1$, by Lemma 5, $l_h = j'_h$. So for any given j where $i_h \leq j \leq j'_h$, the coefficient of x_j in (23) is $\mu^n_{\alpha,b}\left(\lceil C_j/C_{i_h} \rceil C_{i_h}\right)$. Now observe that by (36), $C_{i_{p-n}} \leq b^{(n)}$, and by Lemma 5, $\lceil C_j/C_{i_h} \rceil \leq \kappa_h = C_{i_{h+1}}/C_{i_h}$. Therefore since $h \leq p-n-1$, we have $\lceil C_j/C_{i_h} \rceil C_{i_h} \leq C_{i_{h+1}} \leq C_{i_{p-n}} \leq b^{(n)}$. Therefore by Lemma 1(v), we get $\mu^n_{\alpha,b}\left(\lceil C_j/C_{i_h} \rceil C_{i_h}\right)$ $= \lceil C_j/C_{i_h} \rceil C_{i_h}$. This completes the proof. □

Now we are ready to prove our main result in this section:

Theorem 7. *Assuming conditions* (2) *are satisfied and* $n = n'$*, the following is true: If* $0 < n \leq p$*, then partition inequality* (1) *for* X *is the same as inequality* (23) *divided by* γ*, where* $\gamma = C_{i_1}$ *if* $n < p$ *and* $\gamma = b^{(p)}$ *if* $n = p$*. If* $n = 0$*, then partition inequality* (1) *is simply inequality* (22) *divided by* C_{i_1}.

Proof. First consider the case where $0 < n \leq p$. Notice that by Lemma 1(i)

$$\mu^{n}_{\alpha,b}(b) = b^{(n)} \prod_{l=1}^{n} \left\lceil \frac{b^{(l-1)}}{\alpha_l} \right\rceil = b^{(n)} \prod_{l=1}^{n} \kappa_{p-l+1} = b^{(n)} \prod_{s=p-n+1}^{p} \kappa_s. \tag{37}$$

Now if conditions (2) are satisfied, based on (37) and Lemmas 3, 4, and 6, inequality (23) reduces to

$$\begin{aligned} &\sum_{t=1}^{p-n-1} \sum_{j=i_t}^{j'_t} \left\lceil \frac{C_j}{C_{i_t}} \right\rceil C_{i_t} x_j + \sum_{j=i_{p-n}}^{j'_{p-n}} C_{i_{p-n}} \min\{\left\lceil \frac{C_j}{C_{i_{p-n}}} \right\rceil, \kappa_{p-n}\} x_j \\ &+ \sum_{t=p-n+1}^{p} \sum_{j=i_t}^{j'_t} b^{(n)} \left(\prod_{s=p-n+1}^{t-1} \kappa_s \right) \min\{\left\lceil \frac{C_j}{C_{i_t}} \right\rceil, \kappa_t\} x_j + \sum_{j=r+1}^{N} b^{(n)} \left(\prod_{s=p-n+1}^{p} \kappa_s \right) x_j \geq b^{(n)} \prod_{s=p-n+1}^{p} \kappa_s. \end{aligned} \tag{38}$$

On the other hand consider partition inequality (1). Note that if $0 < n \leq p-1$, from (35) in Lemma 6 and (32) in Lemma 4, we have $\kappa_t = C_{i_{t+1}}/C_{i_t}$ for $t = 1, \ldots, p-n-1$ and $\kappa_{p-n} = b^{(n)}/C_{i_{p-n}}$. Let $\gamma_1 = b^{(n)}/C_{i_1}$ if $n < p$ and $\gamma_1 = 1$ if $n = p$. Then for any $0 < n \leq p$, we can write

$$\prod\nolimits_{s=1}^{t-1} \kappa_s = C_{i_t}/C_{i_1} \quad \text{for } t = 1, \ldots, p-n, \tag{39}$$

and

$$\prod\nolimits_{s=1}^{t-1} \kappa_s = \gamma_1 \prod\nolimits_{s=p-n+1}^{t-1} \kappa_s \quad \text{for } t = p-n+1, \ldots, p+1. \tag{40}$$

Also note that for $t = 1, \ldots, p-n-1$ and every $i_t \leq j \leq j'_t$, we have

$$\min\{\lceil C_j/C_{i_t} \rceil, \kappa_t\} = \lceil C_j/C_{i_t} \rceil \tag{41}$$

because $\lceil C_j/C_{i_t} \rceil \leq \kappa_t$ as argued in Lemma 6. Moreover for $j \geq r+1$, clearly

$$\min\{\lceil C_j/C_{i_p} \rceil, \kappa_p\} = \kappa_p. \tag{42}$$

Using (39), (40), (41), and (42), the partition inequality (1) easily reduces to

$$\begin{aligned} &\sum_{t=1}^{p-n-1} \sum_{j=i_t}^{j'_t} \left\lceil \frac{C_j}{C_{i_t}} \right\rceil \frac{C_{i_t}}{C_{i_1}} x_j + \sum_{j=i_{p-n}}^{j'_{p-n}} \frac{C_{i_{p-n}}}{C_{i_1}} \min\{\left\lceil \frac{C_j}{C_{i_{p-n}}} \right\rceil, \kappa_{p-n}\} x_j \\ &+ \sum_{t=p-n+1}^{p} \sum_{j=i_t}^{j'_t} \gamma_1 \left(\prod_{s=p-n+1}^{t-1} \kappa_s \right) \min\{\left\lceil \frac{C_j}{C_{i_t}} \right\rceil, \kappa_t\} x_j + \sum_{j=r+1}^{N} \gamma_1 \left(\prod_{s=p-n+1}^{p} \kappa_s \right) x_j \geq \gamma_1 \prod_{s=p-n+1}^{p} \kappa_s, \end{aligned} \tag{43}$$

which is inequality (38) divided by γ.

Now consider the case where $n = 0$. For block p we always have $l_p = j'_p$, and by Lemma 5, we also have $l_h = j'_h$ for $h = 1, \ldots, p-1$. Therefore inequality (22) reduces to

$$\sum_{t=1}^{p} \sum_{j=i_t}^{j'_t} \lceil C_j/C_{i_t} \rceil C_{i_t} x_j + \sum_{j=r+1}^{N} b x_j \geq b. \tag{44}$$

On the other hand, consider inequality (1). Again since $l_h = j'_h$ for $h = 1, \ldots, p$, we have

$$\min\{\lceil C_j/C_{i_h} \rceil, \kappa_h\} = \lceil C_j/C_{i_h} \rceil \quad \text{for } i_h \leq j \leq j'_h,\ h = 1, \ldots, p. \tag{45}$$

Also for $j > r$,

$$\min\{\lceil C_j/C_{i_p} \rceil, \kappa_p\} = \kappa_p = b/C_{i_p} \tag{46}$$

The second identity is true because $b^{(1)} = 0$. Now using (45) and (46) and Lemma 5, partition inequality (1) reduces to

$$\sum_{t=1}^{p} \sum_{j=i_t}^{j'_t} (\lceil C_j/C_{i_t} \rceil C_{i_t}/C_{i_1}) x_j + \sum_{j=r+1}^{N} (b/C_{i_1}) x_j \geq b/C_{i_1}, \tag{47}$$

which is the same as (44) divided by C_{i_1}. This concludes the proof. □

Corollary 8. *Assuming $n = n' > 0$ and conditions* (2) *are satisfied, n-step MIR inequality* (18) *for X dominates partition inequality* (1)*. This domination is strict unless $\mu^n_{\alpha,b}\left(\lceil C_j/C_{i_t} \rceil C_{i_t}\right) = \mu^n_{\alpha,b}\left(C_j\right)$ for every block t and all $i_t \leq j \leq l_t$. As a result in the particular case of X_d, the two inequalities are equivalent.*

Proof. By Theorem 7 partition inequality (1) is equivalent to inequality (23). Since the n-step MIR function $\mu^n_{\alpha,n}$ is non-decreasing, we have $\mu^n_{\alpha,b}\left(C_j\right) \leq \mu^n_{\alpha,b}\left(\lceil C_j/C_{i_t} \rceil C_{i_t}\right)$. Therefore inequality (23) (and hence partition inequality (1)) is dominated by n-step MIR inequality (18). Moreover, this domination is strict unless $\mu^n_{\alpha,b}\left(\lceil C_j/C_{i_t} \rceil C_{i_t}\right) = \mu^n_{\alpha,b}\left(C_j\right)$ for every block t and all $i_t \leq j \leq l_t$. One such particular case is X_d, for which because of divisibility of the coefficients, we have $\lceil C_j/C_{i_t} \rceil C_{i_t} = C_j$ and hence $\mu^n_{\alpha,b}\left(\lceil C_j/C_{i_t} \rceil C_{i_t}\right) = \mu^n_{\alpha,b}\left(C_j\right)$ for every block t and all $i_t \leq j \leq l_t$. Therefore inequality (23) for X_d reduces to inequality (18). Thus partition inequality (1) and (18) will be equivalent for this set. □

Example 1. *Consider the set $X = \{x \in \mathbb{Z}^8_+ : 3x_1 + 6x_2 + 7x_3 + 13x_4 + 17x_5 + 20x_6 + 25x_7 + 48x_8 \geq 56\}$ and the partitioning $\{3\}, \{6, 7, 13, 17\}, \{20, 25, 48\}$ of the coefficients. Notice that the coefficients are not divisible. We have $\beta_3 = 56$, $\kappa_3 = 3$, $\beta_2 = 16$, $\kappa_2 = 3$, $\beta_1 = 4$, and $\kappa_1 = 2$. These values satisfy conditions* (2)*, so the partition inequality of Pochet and Wolsey [14], i.e. inequality* (1) *for X, will be*

$$x_1 + 2x_2 + 4x_3 + 6x_4 + 6x_5 + 6x_6 + 12x_7 + 18x_8 \geq 18. \tag{48}$$

Now if we let $\alpha_1 = C_6 = 20$ and $\alpha_2 = C_2 = 6$, and $\alpha_3 = C_1 = 3$, then $n' = p = 3$ because $b^{(3)} = 1 \neq 0$. So letting $n = n' = 3$, conditions (3) *are satisfied and the n-step MIR inequality* (18) *will be the 3-step MIR inequality obtained by applying $\mu^3_{(20,6,3),56}$ on the defining inequality of X. This inequality is*

$$x_1 + 2x_2 + 3x_3 + 5x_4 + 6x_5 + 6x_6 + 8x_7 + 15x_8 \geq 18. \tag{49}$$

We see that 3-step MIR inequality (49) *strictly dominates partition inequality* (48)*, which is what we expect according to Corollary 8. In fact, inequality* (49) *defines a facet for conv*(X) *as we will see in Theorem 10.*

Moreover notice that we can set n to values other that n' to get other facet-defining inequalities for X which are not obtainable by partition inequalities (1) *using any partitioning. For $n = 2$, the 2-step MIR inequality using $\mu^2_{(20,6),56}$ is*

$$3x_1 + 4x_2 + 5x_3 + 9x_4 + 12x_5 + 12x_6 + 16x_7 + 30x_8 \geq 36, \tag{50}$$

and for $n = 1$, the 1-step MIR inequality using $\mu^1_{20,56}$ is

$$3x_1 + 6x_2 + 7x_3 + 13x_4 + 16x_5 + 16x_6 + 21x_7 + 40x_8 \geq 48. \tag{51}$$

Based on Theorem 10, both (50) *and* (51) *are facet-defining for conv*(X) *too.* □

3.3. *n-step MIR defines facets for conv*(X)

Atamtürk and Kianfar [1] presented sufficient conditions under which n-step MIR inequality (5) defines a facet for conv(K) (they proved these conditions as a special case of the facet-defining conditions for the so-called *n-step mingling* inequalities). For later use, here we restate their result only for the case where all the coefficients in K are positive :

Theorem 9 (Corollary 1 of [1]). *The n-step MIR inequality* (5) *is facet-defining for conv*(K) *if the following conditions are satisfied: (i) $b^{(n)} > 0$, (ii) $\alpha_k = a_{i_k}$, where $i_k \in I$ and $a_{i_k} > 0$ for $k = 1, \ldots, n$, (iii) $a_i \leq \alpha_1 \lceil b/\alpha_1 \rceil$ for all $i \in I$, and (iv) $\alpha_{k-1} \geq \alpha_k \left\lceil b^{(k-1)}/\alpha_k \right\rceil$ for $k = 2, \ldots, n$.*

Based on Theorem 9 we can prove the following:

Theorem 10. *For $n > 0$, assuming $b^{(n)} > 0$ and conditions* (3) *are satisfied, n-step MIR inequality* (18) *defines a facet for conv*(X).

Proof. We verify that the conditions of Theorem 9 hold: Condition (i) is satisfied and also by definition of α in (17), we have $\alpha_k = C_{i_{p-k+1}} > 0$ so condition (ii) is also satisfied. Moreover we have $C_j \leq C_r \leq b \leq C_{i_p}\left\lceil b/C_{i_p}\right\rceil = \alpha_1 \lceil b/\alpha_1 \rceil$ for all $j \in \{1,\ldots,r\}$ so condition (iii) holds. Condition (iv) is the same as conditions (3). This proves the result based on Theorem 9. □

Moreover, we can state the following result on the facet-defining property of partition inequality (1) for X:

Corollary 11. *Assuming conditions* (2) *are satisfied and partition inequality* (1) *is not equivalent to* (22)*, partition inequality* (1) *defines a facet for conv*(X) *if and only if*

$$\mu^{n'}_{\alpha,b}(\left\lceil C_j/C_{i_t}\right\rceil C_{i_t}) = \mu^{n'}_{\alpha,b}(C_j) \quad \text{for } i_t \leq j \leq l_t,\ t = 1,\ldots,p, \tag{52}$$

where α and n' are defined for the partitioning used in partition inequality (1).

Proof. Since (1) is not equivalent to (22), by Theorem 7, $n' > 0$. Let $n = n'$. If conditions (52) are satisfied, then inequality (23) reduces to (18) and by Theorems 7 and 10, partition inequality (1) is facet-defining for conv(X). Conversely, if (1) is a facet, by Theorem 7, inequality (23) is a facet and hence not dominated by any valid inequality. The function $\mu^n_{\alpha,b}$ is non-decreasing so $\mu^n_{\alpha,b}(\left\lceil C_j/C_{i_t}\right\rceil C_{i_t}) \geq \mu^n_{\alpha,b}(C_j)$, which means for (23) not to be dominated by (18), conditions (52) must be satisfied. □

In [14], as mentioned before, the facet-defining property of partition inequality (1) was proved for the special case X_d through characterization of the optimal solution in optimizing a linear function over X_d. Notice that Theorem 10 also gives an alternative proof based on n-step MIR for the facet-defining property of (1) for the set X_d:

Corollary 12. *If $n = n' > 0$, n-step MIR inequality* (18)*, or equivalently partition inequality* (1)*, defines a facet for conv*(X_d).

Proof. Conditions (2) are satisfied for X_d and the proof is the direct result of Theorem 10 and Corollary 8. □

Remark 2. It was proved in [14] that the partition inequalities (1) along with the non-negativity constraints define the convex hull of X_d if $C_r \nmid b$ (otherwise inequality (16) and non-negativity constraints are enough to define the convex hull [14]). As a result, based on Corollary 8, we can conclude that n-step MIR inequalities (18) along with the non-negativity constraints define the convex hull of X_d if $C_r \nmid b$. This result underscores the strength of the n-step MIR inequalities.

4. *n*-step MIR for *Y*

We can also use the n-step MIR to generate valid inequalities for the set Y. Note that the inequality

$$\sum_{j=1}^{N} C_j x_j + (b - y) \geq b \tag{53}$$

is valid for Y and we have $b - y \geq 0$. Like the case of X, we assume $C_r \leq b, C_{r+1} > b$ for some $r \in J$, so (53) can be strengthened to

$$\sum_{j=1}^{r} C_j x_j + \sum_{j=r+1}^{N} b x_j + (b - y) \geq b \tag{54}$$

Now consider a partition of J such as $\{i_1,\ldots,j_1\},\ldots,\{i_p,\ldots,j_p\}$ where $i_p \leq C_r$, and like the case of X let $\alpha_t = C_{i_{p-t+1}}$ for $t = 1,\ldots,p$. Choose some $n \in \{1,...,p\}$ and define $\alpha = (\alpha_1,\ldots,\alpha_n)$, and accordingly $J_\alpha = \{i_p, i_{p-1},\ldots,i_{p-n+1}\}$.

Then if conditions (3) are satisfied, since $b - y \geq 0$, we can treat $b - y$ as s in the set K, and based on (5), apply the n-step MIR function $\mu^n_{\alpha,b}$ on (54) to get the n-step MIR inequality

$$\sum_{j=1}^{r} \mu^n_{\alpha,b}\left(C_j\right) x_j + \sum_{j=r+1}^{N} \mu^n_{\alpha,b}(b)\, x_j + b - y \geq \mu^n_{\alpha,b}(b), \tag{55}$$

which is valid for Y.

4.1. Special case of Y_d: relationship with partition inequalities

In [14] Pochet and Wolsey presented a set of partition inequalities as follows for Y_d (Theorem 16 of [14]):

$$\sum_{j=1}^{q-1} C_j x_j + \tau_{q-1} \sum_{t=1}^{p} \left(\prod_{s=1}^{t-1} \kappa_s \right) \sum_{j=i_t}^{j_t} \min\{C_j/C_{i_t}, \kappa_t\} x_j + b - y \geq \tau_{q-1} \prod_{t=1}^{p} \kappa_t, \tag{56}$$

where $1 \leq q \leq r, \tau_{q-1} = b - \lfloor b/C_q \rfloor C_q, \{i_1, \ldots, j_1\}, \ldots, \{i_p, \ldots, j_p\}$ is a partitioning of $\{q, \ldots, N\}$, and κ_t's are as defined in a partition inequality (1) for

$$\sum_{j=q}^{N} C_j x_j \geq b. \tag{57}$$

We note that in [14] instead of (57), the inequality

$$\sum_{j=q}^{N} (C_j/C_q) x_j \geq \lceil b/C_q \rceil \tag{58}$$

is used, however it can be easily verified that the κ_t's obtained from (58) are the same as those obtained from (57) (more specifically, it can be easily verified that because of the divisibility of coefficients in Y_d, changing the right-hand side of (58) to b/C_q does not change κ_t's. Furthermore, multiplying the inequality by C_q clearly does not change κ_t's either). In [14] it is proved that inequalities (56) along with the strengthened inequality (54) and the bounds on variables define the convex hull of Y_d (Theorem 16 of [14]). We prove the following result on the relationship of n-step MIR and partition inequalities for Y_d:

Theorem 13. *Partition inequality* (56) *written for q and partitioning* $\{i_1, \ldots, j_1\}, \ldots, \{i_p, \ldots, j_p\}$ *of* $\{q, \ldots, N\}$ *(assuming* $\tau_{q-1} > 0$*) is the same as inequality* (55) *written for* $n = p$ *and* $\alpha_t = C_{i_{p-t+1}}$ *for* $t = 1, ..., p$ *if* $C_{q-1} \leq \tau_{q-1}$*. Otherwise, the former is dominated by the latter.*

Proof. In (56), the coefficients of x_j for $j \geq q$ and the right-hand side are τ_{q-1} times the coefficients of x_j and the right-hand side in the partition inequality written for (57), respectively. Now based on Theorem 7, the partition inequality for (57) is the same as the inequality

$$\sum_{j=q}^{r} \mu^n_{\alpha,b}\left(C_j\right) x_j + \sum_{j=r+1}^{N} \mu^n_{\alpha,b}(b)\, x_j \geq \mu^n_{\alpha,b}(b) \tag{59}$$

divided by $b^{(p)} (= \tau_{q-1})$, where $\alpha_t = C_{i_{p-t+1}}$ for $t = 1, ..., p$ and $n = n' = p$ (note that due to the divisibility of coefficients we have $b^{(p)} = \tau_{q-1} > 0$ and so $n' = p$). Therefore using the coefficients and right-hand side of (59), inequality (56) becomes

$$\sum_{j=1}^{q-1} C_j x_j + \sum_{j=q}^{r} \mu^n_{\alpha,b}\left(C_j\right) x_j + \sum_{j=r+1}^{N} \mu^n_{\alpha,b}(b)\, x_j + b - y \geq \mu^n_{\alpha,b}(b). \tag{60}$$

Comparing (60) with inequality (55) for $n = p$ and $\alpha_t = C_{i_{p-t+1}}$ for $t = 1, ..., p$, the coefficients of all x_j's are the same except for $j < q$. Now based on Lemma 1(v), if $C_{q-1} \leq b^{(p)}$, i.e. $C_{q-1} \leq \tau_{q-1}$, then $\mu^n_{\alpha,b}\left(C_j\right) = C_j$ for all $j < q$ so both inequalities become the same. Otherwise there exists a $k \in \{1, \ldots, q-2\}$ such that $\mu^n_{\alpha,b}\left(C_j\right) < C_j$ for $k \leq j < q$ (it is a well-known property of $\mu^n_{\alpha,b}$ that $\mu^n_{\alpha,b}\left(C_j\right) < C_j$ for $C_j > b^{(n)}$). Therefore (60) is dominated. □

Theorem 13 shows that (56) is not facet-defining for Y_d if $C_{q-1} > \tau_{q-1}$. As a result, we can strengthen the convex hull result of [14]:

Corollary 14. *The convex hull of Y_d is described by the inequalities $0 \le y \le b$, $x \ge 0$, (54), and in addition if $C_r \nmid b$ inequalities (56) in which $C_{q-1} \le \tau_{q-1}$.*

Proof. Theorem 16 of [14] proves that inequalities $0 \le y \le b$, $x \ge 0$, (54), and in addition if $C_r \nmid b$ inequalities (56) describe the convex hull of Y_d. Based on Theorem 13, we can remove any inequality (56) in which $C_{q-1} > \tau_{q-1}$ from this set because it is not facet-defining. □

Remark 3. As mentioned Theorem 13 shows that (56) with $q = q_1$ and partitioning $\{i_1, \ldots, j_1\}, \ldots, \{i_p, \ldots, j_p\}$ of $\{q_1, \ldots, N\}$ is dominated by inequality (55) with $n = p$ and $\alpha = (C_{i_p}, \ldots, C_{i_1})$ if $C_{q_1-1} > \tau_{q_1-1}$. Note that in this case the latter inequality is the same as a partition inequality generated by a different q and partitioning, i.e. $q = q_2$ where $q_2 = \min\{j : C_j > \tau_{q_1-1}\}$ and partitioning $\{q_2, \ldots, q_1 - 1\}\{i_1, \ldots, j_1\}, \ldots, \{i_p, \ldots, j_p\}$ of $\{q_2, \ldots, N\}$. This is true because based on Theorem 13 such a partition inequality is equivalent to

$$\sum\nolimits_{j=1}^{r} \mu^{p+1}_{\alpha',b}(C_j)x_j + \sum\nolimits_{j=r+1}^{N} \mu^{p+1}_{\alpha',b}(b)x_j + b - y \ge \mu^{p+1}_{\alpha',b}(b), \tag{61}$$

where $\alpha' = (\alpha, C_{q_2})$. But according to formulation (4), (61) reduces to (55) with $n = p$ because $\alpha'_{p+1} = C_{q_2} > \tau_{q_1-1} = b^{(p)}$. □

We can state the following result too:

Corollary 15. *The convex hull of Y_d is described by the inequalities $0 \le y \le b$, $x \ge 0$, (54), and in addition if $C_r \nmid b$, the n-step MIR inequalities (55).*

Proof. This is the direct result of Theorem 13 and Corollary 14. □

Example 2. *Consider the set $Y_d = \{(x, y) \in \mathbb{Z}^5_+ \times \mathbb{R}_+ : y \le 3x_1 + 6x_2 + 12x_3 + 36x_4 + 72x_5, y \le 56\}$. Let $q = 2$ and consider the partitioning $\{6, 12\}, \{36, 72\}$ of the coefficients greater than $C_1 = 3$. Notice that the coefficients are divisible. Inequality (57) in this case is $6x_2 + 12x_3 + 36x_4 + 72x_5 \ge 56$. So we will have $\beta_2 = 56$, $\kappa_2 = 2$, $\beta_1 = 20$, $\kappa_1 = 4$, and $\tau_1 = 2$. Therefore the partition inequality of Pochet and Wolsey [14], i.e. inequality (56) for Y_d, will be*

$$3x_1 + 2x_2 + 4x_3 + 8x_4 + 16x_5 + 56 - y \ge 16. \tag{62}$$

Now if we let $\alpha_1 = C_4 = 36$ and $\alpha_2 = C_2 = 6$, then $n = n' = p = 3$ because $b^{(2)} = \tau_1 = 2 \ne 0$. So the n-step MIR inequality (55) will be the 2-step MIR inequality obtained by $\mu^2_{(36,6),56}$. This inequality is

$$2x_1 + 2x_2 + 4x_3 + 8x_4 + 16x_5 + 56 - y \ge 16. \tag{63}$$

We see that 2-step MIR inequality (63) strictly dominates partition inequality (62), which is what we expect according to Theorem 13 because $C_{q-1} = C_1 = 3 > 2 = \tau_1$. This shows that partition inequality (62) is not facet-defining for conv(Y_d). In fact, inequality (63) defines a facet for conv(Y_d) as we will see in Theorem 16. Note that as explained in Remark 3, inequality (63) can be obtained using a partition inequality with $q = 1$ and partitioning $\{3\}, \{6, 12\}, \{36, 72\}$. □

The more general set Y has not been studied in [14]. We note that n-step MIR can be used to develop partition inequalities for the set Y too. This would be by relaxing (54) similar to the relaxation in (22) and then applying the n-step MIR function. However, there is no point in constructing such inequalities because, by a reasoning similar to Theorem 7 and Corollary 8, they will be dominated by the corresponding n-step MIR inequality (55).

4.2. *n-step MIR defines facets for conv(Y)*

The inequalities (55) are also facet-defining for conv(Y) as proved below.

Theorem 16. *For $n > 0$, assuming $b^{(n)} > 0$ and conditions (3) are satisfied, the n-step MIR inequality (55) defines a facet for conv(Y).*

Proof. Consider the set $Y_s = \{(x,s) \in \mathbb{Z}_+^N \times \mathbb{R}_+ : \sum_{j=1}^r C_j x_j + \sum_{j=r+1}^N b x_j + s \geq b\}$. If $b^{(n)} > 0$ and conditions (3) are satisfied, based on Theorem 9 and by an argument similar to the proof of Theorem 10, inequality

$$\sum\nolimits_{j=1}^{r} \mu_{\alpha,b}^{n}\left(C_j\right) x_j + \sum\nolimits_{j=r+1}^{N} \mu_{\alpha,b}^{n}(b)\, x_j + s \geq \mu_{\alpha,b}^{n}(b) \tag{64}$$

defines a facet for conv(Y_s). Let $a_j = C_j$ for $1 \leq j \leq r$, $a_j = b$ for $r < j \leq N$. According to the proof of Theorem 9 (refer to Theorem 2 in [1] for this proof), the reason why (64) is facet-defining is because the $N+1$ points $P_0, P_1, \ldots, P_N$ listed below are in Y_s, satisfy (64) at equality, and are affinely independent (only nonzero coordinates are presented):

- The point $P_0 = (x,s)$ such that $s = b^{(n)}$, $x_{i_{p-t+1}} = \left\lfloor b^{(t-1)}/\alpha_t \right\rfloor$ for $t = 1, \ldots, n$;
- For each $m \in \{1, \ldots, n\}$, the point $P_{i_{p-m+1}} = (x,s)$ such that $x_{i_{p-t+1}} = \left\lfloor b^{(t-1)}/\alpha_t \right\rfloor$ for $t = 1, \ldots, m-1$, $x_{i_{p-m+1}} = \left\lceil b^{(m-1)}/\alpha_m \right\rceil$;
- For each $j \in J \backslash J_\alpha$ where $a_j \in \mathcal{I}_m^n$ for some $m \in \{0, \ldots, n-1\}$, the point $P_j = (x,s)$ such that $x_j = 1$, $x_{i_{p-t+1}} = \left\lfloor b^{(t-1)}/\alpha_t \right\rfloor - \left\lfloor {a_j}^{(t-1)}/\alpha_t \right\rfloor$ for $t = 1, \ldots, m+1$;
- For each $j \in J \backslash J_\alpha$ where $a_j \in \mathcal{I}_n^n$, the point $P_j = (x,s)$ such that $s = b^{(n)} - {a_j}^{(n)}$, $x_j = 1$, $x_{i_{p-t+1}} = \left\lfloor b^{(t-1)}/\alpha_t \right\rfloor - \left\lfloor {a_j}^{(t-1)}/\alpha_t \right\rfloor$ for $t = 1, \ldots, n$.

We see that $s \leq b$ in all points $P_0, \ldots, P_N$. That means (64) is also facet-defining for the convex hull of the set

$$\bar{Y}_s = \{(x,s) \in \mathbb{Z}_+^N \times \mathbb{R}_+ : \sum\nolimits_{j=1}^{r} C_j x_j + \sum\nolimits_{j=r+1}^{N} b x_j + s \geq b;\, s \leq b\} \tag{65}$$

Now notice that the points in $\bar{Y}_s$ have one-to-one correspondence with the points in Y ($(x,s) \in \bar{Y}_s$ corresponds to $(x,y) \in Y$, where $s = b - y$). So conv$(\bar{Y}_s)$ and conv(Y) are isomorphic and there is a one-to-one correspondence between their facets. Replacing s in $\bar{Y}_s$ and (64) with $b - y$ we get Y and (55), respectively. That means (55) is also facet-defining for conv(Y). □

Note that Theorem 16 along with Theorem 13 gives a proof based on n-step MIR for the fact that partition inequalities (56) define facets for conv(Y_d) if $C_{q-1} < \tau_{q-1}$.

5. *n*-step MIR for *Z*

The n-step MIR can also be used to generate valid inequalities for the set Z. Take an arbitrary $S \subseteq J$. Note that the inequality

$$\sum\nolimits_{j\in S} C_j x_j + b - \sum\nolimits_{j\in S} y_j \geq b \tag{66}$$

is valid for Z and we have $b - \sum_{j\in S} y_j \geq 0$. Assuming $C_r \leq b, C_{r+1} > b$ for some $r \in S$, (66) can be strengthened to

$$\sum\nolimits_{j\in S, j\leq r} C_j x_j + \sum\nolimits_{j\in S, j>r} b x_j + b - \sum\nolimits_{j\in S} y_j \geq b. \tag{67}$$

Now consider a partition of S such as $\{i_1, \ldots, j_1\}, \ldots, \{i_p, \ldots, j_p\}$, and like the case of X let $\alpha_t = C_{i_{p-t+1}}$ for $t = 1, \ldots, p$. Choose some $n \in \{1, ..., p\}$ and define $\alpha = (\alpha_1, \ldots, \alpha_n)$, and accordingly $J_\alpha = \{i_p, i_{p-1}, \ldots, i_{p-n+1}\}$. Then if conditions (3) are satisfied, since $b - \sum_{j\in S} y_j \geq 0$, we can treat it as s in the set K, and based on (5), apply the n-step MIR function $\mu_{\alpha,b}^n$ on (67) to get the n-step MIR inequality

$$\sum\nolimits_{j\in S, j\leq r} \mu_{\alpha,b}^{n}\left(C_j\right) x_j + \sum\nolimits_{j\in S, j>r} \mu_{\alpha,b}^{n}(b)\, x_j + b - \sum\nolimits_{j\in S} y_j \geq \mu_{\alpha,b}^{n}(b)\,. \tag{68}$$

which is valid for Z.

5.1. *Special case of Z_d: relationship with partition inequalities*

In [14] Pochet and Wolsey presented a set of partition inequalities as follows for the special case Z_d (Proposition 20 of [14]):

$$\tau\left(\sum_{t=1}^{p}\left(\prod_{s=1}^{t-1}\kappa_s\right)\sum_{j=i_t,j\in S}^{j_t}\min\{C_j/C_{i_t},\kappa_t\}x_j\right)+b-\sum_{j\in S}y_j\geq\tau\prod_{t=1}^{p}\kappa_t, \tag{69}$$

where $S \subseteq J$, $\{i_1,\ldots,j_1\},\ldots,\{i_p,\ldots,j_p\}$ is a partitioning of S, $\tau = b - (\lceil b/C_{i_1}\rceil - 1)C_{i_1}$, and κ_t's are as defined for a partition inequality (1) for

$$\sum\nolimits_{j\in S} C_j x_j \geq b. \tag{70}$$

We note that in [14] instead of (70), the inequality $\sum_{j\in S}(C_j/C_{i_1})x_j \geq \lceil b/C_{i_1}\rceil$ is used, however by reasons similar to those presented for (58), we can also use (70). We prove the following result:

Theorem 17. *Partition inequality* (69) *written for partitioning* $\{i_1,\ldots,j_1\},\ldots,\{i_p,\ldots,j_p\}$ *of* S *is the same as inequality* (68) *written for* S *with* $n = n'$ *and* $\alpha_t = C_{i_{p-t+1}}$ *for* $t = 1,...,p$.

Proof. In (69), the coefficients of x_j and the right-hand side are τ times the coefficients of x_j and the right-hand side in the partition inequality written for (70), respectively. Now based on Theorem 7, the partition inequality for (70) is the same as the inequality

$$\sum\nolimits_{j\in S,j\leq r}\mu^n_{\alpha,b}\left(C_j\right)x_j+\sum\nolimits_{j\in S,j>r}\mu^n_{\alpha,b}(b)\,x_j\geq\mu^n_{\alpha,b}(b) \tag{71}$$

divided by γ, where $\alpha_t = C_{i_{p-t+1}}$ for $t = 1,...,p$, $n = n'$, $\gamma = b^{(p)}$ if $n = p$, and $\gamma = C_{i_1}$ if $n < p$. Now notice that since $n = n'$ and the coefficients in Z_d are divisible, if $n = p$ we have $b^{(p)} > 0$ which means $\tau = b^{(p)}$, and if $n < p$ we have $b^{(p)} = 0$ which means $C_{i_1} \mid b$ and $\tau = C_{i_1}$. As a result, $\tau = \gamma$. Therefore using the coefficients and right-hand side of (71), inequality (69) becomes the same as inequality (68). □

The more general set Z has not been studied in [14]. We note that n-step MIR can be used to develop partition inequalities for the set Z too. This would be by relaxing (67) similar to the relaxation in (22) and then applying the n-step MIR function. However, there is no point in constructing such inequalities because, by a reasoning similar to Theorem 7 and Corollary 8, they will be dominated by the corresponding n-step MIR inequality (68).

5.2. *n-step MIR defines facets for conv(Z)*

The structure of Z is more complicated than Y in the sense that it has N continuous variables whereas Y has only one. Nevertheless, the n-step MIR gives facet-defining inequalities for conv(Z) under some conditions as proved below. Let $a_j = C_j$ for $1 \leq j \leq r$, $a_j = b$ for $r < j \leq N$, and $J_\alpha = \{i_p, i_{p-1},\ldots,i_{p-n+1}\}$.

Theorem 18. *For* $n > 0$, *assuming* $b^{(n)} > 0$ *and conditions* (3) *are satisfied, the n-step MIR inequality* (68) *defines a facet for conv*(Z) *if*

$$\text{for every } j \in S\backslash J_\alpha,\ a_j \in \mathcal{I}^n_m \text{ for some } m \in \{0,\ldots,n-1\} \text{ and } a_j^{(m+1)} \neq b^{(m+1)}. \tag{72}$$

Proof. Based on the last part of (72), we must have $a_j < b$ for all $j \in S$. Therefore inequality (68) reduces to

$$\sum\nolimits_{j\in S}\mu^n_{\alpha,b}\left(C_j\right)x_j+b-\sum\nolimits_{j\in S}y_j\geq\mu^n_{\alpha,b}(b) \tag{73}$$

Consider the set $Z_s = \{(x,s) \in \mathbb{Z}^N_+ \times \mathbb{R}_+ : \sum_{j\in S} C_j x_j + s \geq b\}$. Since $b^{(n)} > 0$ and conditions (3) are satisfied, based on Theorem 9 and by an argument similar to the proof of Theorem 10, inequality

$$\sum\nolimits_{j\in S}\mu^n_{\alpha,b}\left(C_j\right)x_j+s\geq\mu^n_{\alpha,b}(b) \tag{74}$$

defines a facet for conv(Z_s). Again according to the proof of Theorem 9 (see Theorem 2 in [1] for this proof), the reason why (74) is facet-defining is because the $|S| + 1$ points P_0, P_j, $j \in S$ in Z_s satisfy (74) at equality, and are affinely independent, where $P_0, P_j, j \in S$ are described just like the points in the proof of Theorem 16 except that S replaces J. The points $P_0, P_j, j \in S$ can be used to construct a set of $2N$ affinely-independent points which are in Z and satisfy (73) at equality. These points are listed below (only nonzero values for the coordinates are presented). Each of these points is constructed based on one of the points $P_0, \ldots, P_{|S|}$ by using the same values for $x_j, j \in S$ and a set of values for $y_j, j \in S$ such that $s = b - \sum_{j\in S} y_j$. Therefore since $P_0, \ldots, P_{|S|}$ satisfy (74) at equality, these points satisfy (73) at equality. We constructed the points $Q_j^1, Q_j^2, j \in J\backslash S$ and $Q_{i_p}^1$ based on P_0, the point $Q_{i_{p-t+1}}^1$ based on $P_{i_{p-t+1}}$ for $t = 2, \ldots, n$, the point $Q_{i_{p-t+1}}^2$ based on $P_{i_{p-t+1}}$ for $t = 1, \ldots, n$, and the points Q_j^1 and Q_j^2 based on P_j for $j \in S\backslash J_\alpha$:

- For each $j \in J\backslash S$, the point $Q_j^1 = (x, y)$ such that $x_{i_{p-t+1}} = \left\lfloor b^{(t-1)}/\alpha_t \right\rfloor$ for $t = 1, \ldots, n$, $y_{i_{p-t+1}} = \alpha_t \left\lfloor b^{(t-1)}/\alpha_t \right\rfloor$ for $t = 1, \ldots, n$, $x_j = 1$;
- For each $j \in J\backslash S$, the point $Q_j^2 = (x, y)$ such that $x_{i_{p-t+1}} = \left\lfloor b^{(t-1)}/\alpha_t \right\rfloor$ for $t = 1, \ldots, n$, $y_{i_{p-t+1}} = \alpha_t \left\lfloor b^{(t-1)}/\alpha_t \right\rfloor$ for $t = 1, \ldots, n$, $x_j = 1$, $y_j = \min\{b^{(n)}, C_j\}$;
- The point $Q_{i_p}^1 = (x, y)$ such that $x_{i_{p-t+1}} = \left\lfloor b^{(t-1)}/\alpha_t \right\rfloor$ for $t = 1, \ldots, n$, $y_{i_{p-t+1}} = \alpha_t \left\lfloor b^{(t-1)}/\alpha_t \right\rfloor$ for $t = 1, \ldots, n$;
- For each $m \in \{2, \ldots, n\}$, the point $Q_{i_{p-m+1}}^1 = (x, y)$ such that $x_{i_{p-t+1}} = \left\lfloor b^{(t-1)}/\alpha_t \right\rfloor$ for $t = 1, \ldots, m-1$, $y_{i_{p-t+1}} = \alpha_t \left\lfloor b^{(t-1)}/\alpha_t \right\rfloor - \gamma_t$ for some $\gamma_t \geq 0$, $t = 1, \ldots, m-1$, where $\sum_{t=1}^{m-1} \gamma_t = \alpha_m \left\lceil b^{(m-1)}/\alpha_m \right\rceil - b^{(m-1)}$, $x_{i_{p-m+1}} = \left\lceil b^{(m-1)}/\alpha_m \right\rceil$, $y_{i_{p-m+1}} = \alpha_m \left\lceil b^{(m-1)}/\alpha_m \right\rceil$;
- For each $m \in \{1, \ldots, n\}$, the point $Q_{i_{p-m+1}}^2 = (x, y)$ such that $x_{i_{p-t+1}} = \left\lfloor b^{(t-1)}/\alpha_t \right\rfloor$ for $t = 1, \ldots, m-1$, $y_{i_{p-t+1}} = \alpha_t \left\lfloor b^{(t-1)}/\alpha_t \right\rfloor$ for $t = 1, \ldots, m-1$, $x_{i_{p-m+1}} = \left\lceil b^{(m-1)}/\alpha_m \right\rceil$, $y_{i_{p-m+1}} = b^{(m-1)}$;
- For each $j \in S\backslash J_\alpha$ where $a_j \in \mathcal{I}_m^n$ for some $m \in \{0, \ldots, n-1\}$, the point $Q_j^1 = (x, s)$ such that $x_{i_{p-t+1}} = \left\lfloor b^{(t-1)}/\alpha_t \right\rfloor - \left\lfloor a_j^{(t-1)}/\alpha_t \right\rfloor$ for $t = 1, \ldots, m+1$, $y_{i_{p-t+1}} = \alpha_t \left\lfloor b^{(t-1)}/\alpha_t \right\rfloor - \alpha_t \left\lfloor a_j^{(t-1)}/\alpha_t \right\rfloor - \gamma_t$ for some $\gamma_t \geq 0$, $t = 1, \ldots, m+1$, where $\sum_{t=1}^{m+1} \gamma_t = a_j^{(m+1)} - b^{(m+1)}$, $x_j = 1$, $y_j = a_j$.
- For each $j \in S\backslash J_\alpha$ where $a_j \in \mathcal{I}_m^n$ for some $m \in \{0, \ldots, n-1\}$, the point $Q_j^2 = (x, s)$ such that $x_{i_{p-t+1}} = \left\lfloor b^{(t-1)}/\alpha_t \right\rfloor - \left\lfloor a_j^{(t-1)}/\alpha_t \right\rfloor$ for $t = 1, \ldots, m+1$, $y_{i_{p-t+1}} = \alpha_t \left\lfloor b^{(t-1)}/\alpha_t \right\rfloor - \alpha_t \left\lfloor a_j^{(t-1)}/\alpha_t \right\rfloor$ for $t = 1, \ldots, m+1$, $x_j = 1$, $y_j = a_j + b^{(m+1)} - a_j^{(m+1)}$;

The feasibility of these points in Z can be easily verified (note that the condition of the theorem on $a_j, j \in S\backslash J_\alpha$ makes sure Q_j^1 and Q_j^2 are feasible and distinct for all $j \in S\backslash J_\alpha$). The affine independence of the $2N$ points above can be verified as follows: Form a $2N \times 2N$ matrix whose rows from top to bottom are the following points: The pairs $Q_{i_{p-t+1}}^1, Q_{i_{p-t+1}}^2$ for $t = 1, \ldots, n$, followed by the pairs Q_j^1, Q_j^2 for $j \in S\backslash J_\alpha$ in decreasing order of j followed by the pairs Q_j^1, Q_j^2 for $j \in J\backslash S$ in decreasing order of j. Rearrange the columns from left to right in the following order: $x_{i_{p-t+1}}, y_{i_{p-t+1}}$ pairs for $t = 1, \ldots, n$ followed by x_j, y_j pairs for $j \in S\backslash J_\alpha$ in decreasing order of j followed by x_j, y_j pairs for $j \in J\backslash S$ in decreasing order of j. Denote this matrix by R, its k'th row by r_k, and its (k, l) entry by r_{kl}. It can be verified that the rows of R are affinely independent. Notice that: (i) $r_2, \ldots, r_{2N}$ are lower triangular except for the elements right next to the diagonal on r_{2k-1}, $k = 2, \ldots, N$; (ii) for each $k = 2, \ldots, N$, the pair r_{2k-1} and r_{2k} are clearly linearly independent due the values in columns $2k-1$ and $2k$. (i) and (ii) imply $r_2, \ldots, r_{2N}$ are linearly independent. Therefore, it remains to show that r_1 cannot be written as an affine combination of rows $r_2, \ldots, r_{2N}$. This can be shown by reaching a contradiction: Let $r_1 = \sum_{k=2}^{2N} \lambda_k r_k$ such that $\sum_{k=2}^{2N} \lambda_k = 1$. Then (i) and (ii) and the fact that $r_{1k} = 0$ for $k = 2n+1, \ldots, 2N$ imply $\lambda_k = 0$ for $k = 2n+1, \ldots, 2N$. Now notice that since $b^{(n)} > 0$, we have

$$\left\lfloor b^{(t-1)}/\alpha_t \right\rfloor \neq \left\lceil b^{(t-1)}/\alpha_t \right\rceil \quad \text{for } t = 1, \ldots, n. \tag{75}$$

Now in column 1 we have $r_{11} = r_{31} = r_{41} = \cdots = r_{2n-1,1} = r_{2n,1} = \lfloor b/\alpha_1 \rfloor$ and $r_{21} = \lceil b/\alpha_1 \rceil$. This along with (75) and $\sum_{k=2}^{2n} \lambda_k = 1$ implies $\lambda_2 = 0$. Then in column 3 we have $r_{13} = r_{53} = r_{63} = \cdots = r_{2n-1,1} = r_{2n,1} = \left\lfloor b^{(1)}/\alpha_2 \right\rfloor$ and $r_{23} = r_{24} = \left\lceil b^{(1)}/\alpha_2 \right\rceil$, which along with (75) and $\sum_{k=3}^{2n} \lambda_k = 1$ implies $\lambda_3 + \lambda_4 = 0$. Repeating this arguments for the columns $5, \ldots, 2n-1$, we further get $\lambda_{2k-1} + \lambda_{2k} = 0$ for $k = 3, \ldots, n$, which contradicts $\sum_{k=2}^{2N} \lambda_k = 1$. This completes the proof. □

As a special case, based on Theorems 17 and 18, partition inequalities (69) define facets for conv(Z_d) if conditions (72) are satisfied.

Example 3. *Consider the set* $Z = \{(x, y) \in \mathbb{Z}^5_+ \times \mathbb{R}^5_+ : \sum_{j=1}^{5} y_j \leq 56, y_1 \leq 6x_1, y_2 \leq 17x_2, y_3 \leq 20x_3, y_4 \leq 25x_4, y_5 \leq 48x_5\}$. *We choose* $S = \{6, 17, 20, 25\}$, *so* $J\backslash S = \{48\}$. *By choosing* $\alpha_1 = C_3 = 20$ *and* $\alpha_2 = C_1 = 6$, *we will have* $b = 56$, $b^{(1)} = 16$, $b^{(2)} = 4$, $\lfloor b/\alpha_1 \rfloor = \left\lfloor b^{(1)}/\alpha_2 \right\rfloor = 2$, *and* $\lceil b/\alpha_1 \rceil = \left\lceil b^{(1)}/\alpha_2 \right\rceil = 3$. *Notice that all conditions of Theorem 18 are satisfied. Inequality* (68) *(or* (73)*) in this case is generated by the 2-step MIR function* $\mu^2_{(20,6),56}$ *and is as follows:*

$$4x_1 + 12x_2 + 12x_3 + 16x_4 + 56 - y_1 - y_2 - y_3 - y_4 \geq 36. \tag{76}$$

Based on Theorem (18)*, inequality* (76) *defines a facet for conv(Z). The matrix R of the points generated in the proof of Theorem* (18) *in this case is as follows:*

	x_3	y_3	x_1	y_1	x_4	y_4	x_2	y_2	x_5	y_5
Q_3^1	2	40	2	12	0	0	0	0	0	0
Q_3^2	3	56	0	0	0	0	0	0	0	0
Q_1^1	2	38	3	18	0	0	0	0	0	0
Q_1^2	2	40	3	16	0	0	0	0	0	0
Q_4^1	1	19	2	12	1	25	0	0	0	0
Q_4^2	1	20	2	12	1	24	0	0	0	0
Q_2^1	2	39	0	0	0	0	1	17	0	0
Q_2^2	2	40	0	0	0	0	1	16	0	0
Q_5^1	2	40	2	12	0	0	0	0	1	0
Q_5^2	2	40	2	12	0	0	0	0	1	4

It can be seen that, as proved in Theorem 18, all these points belong to Z, satisfy (76) *at equality, and are affinely independent.* □

6. Conclusion

We showed that n-step MIR generates facet-defining inequalities not only for the three substructures X_d, Y_d, and Z_d considered in [14] but also for their generalization to the case where the coefficients are not necessarily divisible, i.e. the sets X, Y, and Z. In the case of divisible coefficients, n-step MIR directly generates inequalities that encompass all the partition inequalities presented in [14] (for Y_d in particular, n-step MIR in some cases gives inequalities that dominate the partition inequalities). In the case of arbitrary coefficients, n-step MIR gives facet-defining inequalities that either dominate or are not obtainable by the partition inequalities for X. It also gives new facets for Y and Z. Our results underscore the power of n-step MIR as a valid inequality method for general MIPs to directly generate facet-defining inequalities that encompass or dominate inequalities previously developed for special sets using customized approaches, and extend them to more general sets.

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References

[1] A. Atamtürk, K. Kianfar, n-step mingling inequalities: new facets for the mixed-integer knapsack set, Mathematical Programming , In press, Available Online First, DOI: 10.1007/s10107-010-0382-6 (2010).

[2] W.A. Cook, Integer programming solutions for capacity expansion of the local access network, Technical Report TM-ARH-017914, Bell Communication Research, 1990.
[3] S. Dash, O. Günlük, Valid inequalities based on simple mixed-integer sets, Mathematical Programming 105 (2006) 29–53.
[4] B. Gailly, L.A. Wolsey, Capacity Expansion of the Local Access Network, Technical Report, Universite Catholique de Louvain, Louvain-la-Neuve, 1991.
[5] R.E. Gomory, Some polyhedra related to combinatorial problems, Linear Algebra and its Applications 2 (1969) 451–558.
[6] R.E. Gomory, E.L. Johnson, Some continuous functions related to corner polyhedra, Mathematical Programming 3 (1972) 23–85.
[7] R.E. Gomory, E.L. Johnson, Some continuous functions related to corner polyhedra, II, Mathematical Programming 3 (1972) 359–389.
[8] R.E. Gomory, E.L. Johnson, T-space and cutting planes, Mathematical Programming 96 (2003) 341–375.
[9] K. Kianfar, Y. Fathi, Generalized mixed integer rounding inequalities: facets for infinite group polyhedra, Mathematical Programming 120 (2009) 313–346.
[10] K. Kianfar, Y. Fathi, Generating facets for finite master cyclic group polyhedra using n-step mixed integer rounding functions, European Journal of Operational Research 207 (2010) 105–109.
[11] M.J. Magazine, G.L. Nemhauser, L.E. Trotter, When the greedy solution solves a class of knapsack problems, Operations Research 23 (1975) 207–217.
[12] O. Marcotte, The cutting stock problem and integer rounding, Mathematical Programming 33 (1985) 82–92.
[13] G.L. Nemhauser, L.A. Wolsey, Integer and Combinatorial Optimization, Wiley, New York, USA, 1988.
[14] Y. Pochet, L.A. Wolsey, Integer knapsack and flow covers with divisible coefficients: polyhedra, optimization and separation, Discrete Applied Mathematics 59 (1995) 57–74.
[15] L.A. Wolsey, Integer Programming, Wiley, New York, USA, 1998.
[16] L.A. Wolsey, Strong formulations for mixed integer programs: valid inequalities and extended formulations, Mathematical Programming B 97 (2003) 423–447.