Design and Verify: A New Scheme for Generating Cutting-Planes

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Abstract

A cutting-plane procedure for integer programming (IP) problems usually involves invoking a black-box procedure (such as the Gomory-Chvátal (GC) procedure) to *compute* a cutting-plane. In this paper, we describe an alternative paradigm of using the same cutting-plane black-box. This involves two steps. In the first step, we *design* an inequality $cx \leq d$ where c and d are integral, *independent* of the cutting-plane black-box. In the second step, we *verify* that the designed inequality is a valid inequality by verifying that the set $P \cap \{x \in \mathbb{R}^n \mid cx \geq d+1\} \cap \mathbb{Z}^n$ is empty using cutting-planes from the black-box. Here P is the feasible region of the linear-programming relaxation of the IP. We refer to the closure of all cutting-planes that can be verified to be valid using a specific cutting-plane black-box as the *verification closure* of the considered cutting-plane black-box. This paper undertakes a systematic study of properties of verification closures of various cutting-plane black-box procedures.

1 Introduction

Cutting-planes are indispensable for solving Integer Programs (IPs). When using generic cutting-planes (like Gomory-Chvátal or split cuts), often the only guiding principal used is that the incumbent fractional point must be separated. In a way, cutting-planes are generated 'almost blindly', where we apply some black-box method to constructively compute valid cutting-planes and hope for the right set of cuts to appear that helps in proving optimality or close significant portion of the integrality gap. One possible approach to improve such a scheme would therefore be if we were somehow able to deliberately design strong cutting-planes that were tailor-made, for example, to prove the optimality of known good candidate solutions. This motivates a different paradigm to generate valid cutting-planes for integer programs: First we design cutting-planes which we believe will be useful without considering their validity. Then, once the cutting-planes are designed, we verify that it is valid.

For $n \in \mathbb{N}$, let $[n] = \{1, ..., n\}$ and for a rational polytope $P \subseteq \mathbb{R}^n$ denote its *integral hull* by $P_I := \text{conv}\,(P \cap \mathbb{Z}^n)$. We now precisely describe the verification scheme (abbreviated as: $\mathbb{V}$-scheme). Let M be an *admissible* cutting-plane procedure (i.e., a valid and 'reasonable' cutting-plane system - we will formally define these) and let $\text{M}(P)$ be the closure with respect to the family of cutting-planes obtained using M. For example, M could represent split cuts and then $\text{M}(P)$ represents the split closure of P. Usually using cutting-planes from a cutting-plane procedure M, implies using valid inequalities for $\text{M}(P)$ as cutting-planes. In the $\mathbb{V}$-scheme, we apply the following procedure: We design or guess the inequality $cx \leq d$ where $(c, d) \in \mathbb{Z}^n \times \mathbb{Z}$. To verify that this inequality is valid for P_I, we apply M to $P \cap \{x \in \mathbb{R}^n \mid cx \geq d+1\}$ and check whether $\text{M}(P \cap \{x \in \mathbb{R}^n \mid cx \geq d+1\}) = \emptyset$. If $\text{M}(P \cap \{x \in \mathbb{R}^n \mid cx \geq d+1\}) = \emptyset$, then $cx \leq d$ is a valid inequality for P_I. This leads us to the following definition.

Definition 1. *We say that the inequality $cx \leq d$ is* verifiable *using a cutting plane operator* M *for a rational polytope $P \subseteq \mathbb{R}^n$ if $c \in \mathbb{Z}^n$, $d \in \mathbb{Z}$ and $\text{M}(P \cap \{x \in \mathbb{R}^n \mid cx \geq d+1\}) = \emptyset$.*

We might wonder how much we gain from having to only *verify* that a given inequality $cx \leq d$ is valid for P_I, rather than actually *computing* it. In fact at a first glance, it is not even clear that there would

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be any difference between computing and verifying. The strength of the verification scheme lies in the following inclusion that can be readily verified for admissible cutting-plane procedures:

$$\mathrm{M}(P \cap \{x \in \mathbb{R}^n \mid cx \geq d+1\}) \subseteq \mathrm{M}(P) \cap \{x \in \mathbb{R}^n \mid cx \geq d+1\}. \tag{1}$$

The interpretation of this inclusion is that an additional inequality $cx \geq d+1$ appended to the description of P can provide us with crucial extra information when deriving new cutting-planes by using M that is not available when considering P alone. In other words, (1) can potentially be a strict inclusion such that $\mathrm{M}(P \cap \{x \in \mathbb{R}^n \mid cx \geq d+1\}) = \emptyset$ while $\mathrm{M}(P) \cap \{x \in \mathbb{R}^n \mid cx \geq d+1\} \neq \emptyset$. This is equivalent to saying that we can *verify* the validity of $cx \leq d$, however we are not able to *compute* $cx \leq d$. To the best of our knowledge, the only paper discussing a related idea is [4], but theoretical and computational potential of this approach has not been further investigated.

The set obtained by intersecting all cutting-planes verifiable using M will be called the verification closure (abbreviated as: $\mathbb{V}$-closure) of M and denoted by $\partial\mathrm{M}(P)$, i.e.,

Definition 2. *Let* M *be a cutting plane operator. Then*

$$\partial\mathrm{M}(P) := \bigcap_{\substack{(c,d)\in\mathbb{Z}^n\times\mathbb{Z} \\ \text{s.t. } \mathrm{M}(P\cap\{x\in\mathbb{R}^n \mid cx\geq d+1\})=\emptyset}} \{x \in \mathbb{R}^n \mid cx \leq d\}. \tag{2}$$

Under mild conditions, (1) implies $\partial\mathrm{M}(P) \subseteq \mathrm{M}(P)$ for all rational polytopes P. (We formally verify this later.) Since there exist inequalities that can be verified but not computed, this inclusion can be proper. We illustrate this in the next example.

Example 1. *Let* $\mathrm{SC}^i(P)$ *denote the i-th split closure of a polytope* P*. Also we denote* $\mathrm{SC}^1(P)$ *as* $\mathrm{SC}(P)$*. Consider the following family of polytopes [3] for* $n \in \mathbb{N}$*:*

$$A_n := \left\{ x \in [0,1]^n \mid \sum_{i\in I} x_i + \sum_{i\notin I}(1-x_i) \geq \frac{1}{2} \quad \forall\, I \subseteq [n] \right\}. \tag{3}$$

Note that $(A_n)_I = \emptyset$ *and recall that it takes* n *rounds of split cuts to establish that* A_n *is infeasible [6]. For simplicity, consider the instance* $P := A_3$*. Then* $\mathrm{SC}^2(A_3) \neq \emptyset$ *and* $\mathrm{SC}^3(A_3) = \emptyset$*.*

We will show that the $\mathbb{V}$*-split closure of* A_3 *is the empty set, i.e.,* $\partial\mathrm{SC}(A_3) = \emptyset$*. We first design the inequality* $x_1+x_2+x_3 \geq 2$*. In order to show that the inequality* $x_1+x_2+x_3 \geq 2$ *is verifiable for* $\partial\mathrm{SC}(A_3)$ *we will establish that* $\mathrm{SC}(Q) = \emptyset$ *where* $Q := A_3 \cap \{x \in \mathbb{R}^3 \mid x_1+x_2+x_3 \leq 1\}$*. It is easy to see that* $\max\{x_i \mid x \in Q\} < 1$ *for* $i \in [3]$ *and so we obtain that the split cuts* $x_i \leq 0$ *for* $i \in [3]$ *are valid for* $\mathrm{SC}(Q)$*. However,* $x_1+x_2+x_3 \geq \frac{1}{2}$ *is in the description of* Q*. Thus,* $\mathrm{SC}(Q) = \emptyset$*, and so* $x_1+x_2+x_3 \geq 2$ *can be obtained via the* $\mathbb{V}$*-split closure, i.e., it is valid for* $\partial\mathrm{SC}(A_3)$*. By symmetry, we also obtain that* $\partial\mathrm{SC}(A_3) \subseteq \{x \in \mathbb{R}^3 \mid x_1+x_2+x_3 \leq 1\}$ *and so it follows that* $\partial\mathrm{SC}(A_3) = \emptyset$*.* □

We note that rank of A_3 with respect to Gomory-Chvátal (GC) cuts [15, 2], Lift-and-project (LP) cuts [1], and Matrix cone cuts ($\mathrm{N}_0, \mathrm{N}, \mathrm{N}_+$) [16] is also 3 but the $\mathbb{V}$-rank is 1 for any of these operators.

Outline and contribution. This paper undertakes a systematic study of the strengths and weaknesses of the $\mathbb{V}$-closures. In Section 2, we prove basic properties of the $\mathbb{V}$-closure. In order to present these results, we first describe general classes of reasonable cutting-planes, the so called *admissible cutting-plane procedures*, a machinery developed in [19]. We prove that $\partial\mathrm{M}$ is *almost admissible*, i.e. the $\mathbb{V}$-schemes satisfy many important properties that all known classes of admissible cutting-plane procedures including GC cuts, lift-and-project cuts, split cuts (SC), and $\mathrm{N}, \mathrm{N}_0, \mathrm{N}_+$ cuts satisfy.

In Section 3, we show first that $\mathbb{V}$-schemes have natural inherent strength, i.e., even if M is an arbitrarily weak admissible cutting-plane procedure, $\partial\mathrm{M}$ is at least as strong as the GC and the N_0 closures. We then compare the strength of various regular closures (GC cuts, split cuts, and $\mathrm{N}_0, \mathrm{N}, \mathrm{N}_+$ cuts) with their $\mathbb{V}$-versions and with each other. For example, we show that $\partial\mathrm{GC}(P) \subseteq \mathrm{SC}(P)$ and $\partial\mathrm{N}_0(P) \subseteq \mathrm{SC}(P)$ for every rational polytope P. The complete list of these results is illustrated in Figure 1.

In Section 4, we present upper and lower bounds on the rank of valid inequalities with respect to the $\mathbb{V}$-closures for a large class of 0/1 problems. These results show that while the $\mathbb{V}$-closures are strong compared to the regular closures, they not unrealistically so.

In Section 5, we illustrate the strength of the $\mathbb{V}$-schemes when applied on specific structured problems. We show that facet-defining inequalities of *monotone polytopes* contained in $[0,1]^n$ have low rank with respect to any ∂M operator. We show that numerous families of inequalities with high GC, N_0, or N rank [16] (such as clique inequalities) for the *stable set polytope* have a rank of 1 with respect to any ∂M with M being arbitrarily weak and admissible. We will also show that for the subtour elimination relaxation of the *traveling salesman problem* the rank for ∂M with $\mathrm{M} \in \{\mathrm{GC}, \mathrm{SC}, \mathrm{N}_0, \mathrm{N}, \mathrm{N}_+\}$ is in $\Theta(n)$ where n is the number of nodes, i.e., the rank is $\Theta(\sqrt{\dim(P)})$ with P being the TSP-polytope. It is well-known that for the case of *rational polytopes in* $\mathbb{R}^2$ the GC rank can be arbitrarily large. In contrast, we establish that the rank of rational polytopes in $\mathbb{R}^2$ with respect to ∂GC is 1.

An extended abstract of the results in this paper is presented in [11].

2 General properties of the $\mathbb{V}$-closure.

Definition 3 ([19])**.** *A cutting-plane procedure* M *defined for a rational polytope* $P := \{x \in [0,1]^n \mid Ax \leq b\}$ *is* admissible *if the following holds:*

1. Validity*:* $P_I \subseteq \mathrm{M}(P) \subseteq P$*.*
2. Inclusion preservation*: If* $P \subseteq Q$*, then* $\mathrm{M}(P) \subseteq \mathrm{M}(Q)$ *for all polytopes* $P, Q \subseteq [0,1]^n$*.*
3. Homogeneity*:* $\mathrm{M}(F \cap P) = F \cap \mathrm{M}(P)$*, for all faces* F *of* $[0,1]^n$*.*
4. Single coordinate rounding*: If* $x_i \leq \epsilon < 1$ *(or* $x_i \geq \epsilon > 0$*) is valid for* P*, then* $x_i \leq 0$ *(or* $x_i \geq 1$*) is valid for* $\mathrm{M}(P)$*.*
5. Commuting with coordinate flips and duplications*:* $\tau_i(\mathrm{M}(P)) = \mathrm{M}(\tau_i(P))$*, where* τ_i *is either one of the following two operations: (i) Coordinate flip:* $\tau_i : [0,1]^n \to [0,1]^n$ *with* $(\tau_i(x))_i = (1 - x_i)$ *and* $(\tau_i(x))_j = x_j$ *for* $j \in [n] \setminus \{i\}$*; (ii) Coordinate Duplication:* $\tau_i : [0,1]^n \to [0,1]^{n+1}$ *with* $(\tau_i(x))_{n+1} = x_i$ *and* $(\tau_i(x))_j = x_j$ *for* $j \in [n]$*.*
6. Substitution independence*: Let* φ_F *be the projection onto the face* F *of* $[0,1]^n$*. Then* $\varphi_F(\mathrm{M}(P \cap F)) = \mathrm{M}(\varphi_F(P \cap F))$*.*
7. Short verification*: There exists a polynomial* p *such that for any inequality* $cx \leq d$ *that is valid for* $\mathrm{M}(P)$ *there is a set* $I \subseteq [m]$ *with* $|I| \leq p(n)$ *such that* $cx \leq d$ *is valid for* $\mathrm{M}(\{x \in \mathbb{R}^n \mid a_i x \leq b_i, i \in I\})$*. We call* $p(n)$ *the* verification degree *of* M*.*

If M *is defined for general rational polytopes* $P \subseteq \mathbb{R}^n$*, then we say* M *is admissible if (A.)* M *satisfies (1.)-(7.) when restricted to polytopes contained in* $[0,1]^n$ *and (B.) for general polytopes* $P \subseteq \mathbb{R}^n$*,* M *satisfies (1.), (2.), (7.) and Homogeneity is replaced by*

8. Strong Homogeneity: *If* $P \subseteq F^{\leq} := \{x \in \mathbb{R}^n \mid ax \leq b\}$ *and* $F = \{x \in \mathbb{R}^n \mid ax = b\}$ *where* $(a,b) \in \mathbb{Z}^n \times \mathbb{Z}$*, then* $\mathrm{M}(F \cap P) = \mathrm{M}(P) \cap F$*.*

In the following, we assume that $\mathrm{M}(P)$ *is a closed convex set. If* M *satisfies all required properties for being admissible except (7.), then we say* M *is* almost admissible*.*

Requiring strong homogeneity in the general case leads to a slightly more restricted class than the requirement of homogeneity in the 0/1 case. We note here that almost all known classes of cutting-plane schemes such as GC cuts, lift-and-project cuts, split cuts, and $\mathrm{N}, \mathrm{N}_0, \mathrm{N}_+$ are admissible (cf. [19] for more details). Observe that (1) in Section 1 follows from inclusion preservation.

All polytopes are assumed to be rational polytopes in this paper. We will use e^n to represent the vector of all ones in $\mathbb{R}^n$. If the dimension of the vector is obvious from context, then we will use e instead of e^n. Recall that $A_n := \left\{x \in [0,1]^n \mid \sum_{i \in I} x_i + \sum_{i \notin I}(1 - x_i) \geq \frac{1}{2} \quad \forall\, I \subseteq [n]\right\}$; this set is referred regularly in the rest of the paper. We will use $\{\alpha x \leq \beta\}$ as a shorthand for $\{x \in \mathbb{R}^n \mid \alpha x \leq \beta\}$ whenever the ambient dimension n is understood from context. Let φ_F be the projection onto the face F of $[0,1]^n$ and $Q = \varphi_F(P \cap F)$. Then instead of the cumbersome notation $\varphi_F(\mathrm{M}(P \cap F)) = \mathrm{M}(\varphi_F(P \cap F))$ for substitution independence, we will simply say $\mathrm{M}(Q) \cong \mathrm{M}(P \cap F)$. Note that if $\mathrm{M}(Q) \cong \mathrm{M}(P \cap F)$, then $\mathrm{M}(P \cap F) = \emptyset$ if and only if $\mathrm{M}(Q) = \emptyset$.

Next we present a technical lemma that we require for the main result of this section.

Lemma 2. *Let $Q \subseteq \mathbb{R}^n$ be a compact set contained in the interior of the set $\{\beta x \leq \zeta\}$ with $(\beta, \zeta) \in \mathbb{Z}^n \times \mathbb{Z}$ and let $(\alpha, \eta) \in \mathbb{Z}^n \times \mathbb{Z}$. Then there exists a positive integer τ such that Q is strictly contained in the set $\{(\alpha + \tau\beta)x \leq \eta + \tau\zeta\}$.*

Proof. Since Q is a bounded set, $\alpha x \leq \eta + M$ for all $x \in Q$. Also since Q is contained in the interior of the set $\{\beta x \leq \zeta\}$, there exists an $\epsilon > 0$ such that $\beta x \leq \zeta - \epsilon$ for all $x \in Q$. Therefore for a suitably large $\tau \in \mathbb{Z}_+$ such that $M - \tau\epsilon < 0$, we obtain that $(\alpha + \tau\beta)x \leq \eta + M + \tau\zeta - \tau\epsilon < \eta + \tau\zeta \ \forall x \in Q$. □

We next show that $\partial\mathrm{M}$ satisfies almost all properties that we observe in most well-known cutting-plane procedures.

Theorem 1. *Let M be an admissible cutting-plane procedure. Then $\partial\mathrm{M}$ is almost admissible. In particular,*

1. *For 0/1 polytopes, $\partial\mathrm{M}$ satisfies properties (1.) to (6.).*
2. *If M is defined for general polytopes, then $\partial\mathrm{M}$ satisfies property (8.).*

Proof. It is straightforward to verify (1.), (2.), and (4.) - (6.). The non-trivial part is property (8.) (or (3.) respectively). In fact it follows from the original operator M having this property. We will prove (8.); property (3.) in the case of $P \subseteq [0,1]^n$ follows *mutatis mutandis.*

First observe that $\partial\mathrm{M}(P \cap F) \subseteq \partial\mathrm{M}(P)$ and $\partial\mathrm{M}(P \cap F) \subseteq F$. Therefore, $\partial\mathrm{M}(P \cap F) \subseteq \partial\mathrm{M}(P) \cap F$. To verify $\partial\mathrm{M}(P \cap F) \supseteq \partial\mathrm{M}(P) \cap F$, we show that if $\hat{x} \notin \partial\mathrm{M}(P \cap F)$, then $\hat{x} \notin \partial\mathrm{M}(P) \cap F$. Observe first that if $\hat{x} \notin P \cap F$, then $\hat{x} \notin \partial\mathrm{M}(P) \cap F$. Therefore, we assume that $\hat{x} \in P \cap F$. Hence we need to prove that if $\hat{x} \notin \partial\mathrm{M}(P \cap F)$ and $\hat{x} \in P \cap F$, then $\hat{x} \notin \partial\mathrm{M}(P)$. Since $\hat{x} \notin \partial\mathrm{M}(P \cap F)$, there exists $c \in \mathbb{Z}^n$ and $d \in \mathbb{Z}$ such that $c\hat{x} > d$ and $\mathrm{M}(P \cap F \cap \{cx \geq d+1\}) = \emptyset$. By strong homogeneity of M, we obtain

$$\mathrm{M}(P \cap \{cx \geq d+1\}) \cap F = \emptyset. \tag{4}$$

Let $F^{\leq} = \{ax \leq b\}$ and $F = \{ax = b\}$ with $P \subseteq F^{\leq}$. Now observe that (4) is equivalent to saying that $\mathrm{M}(P \cap \{cx \geq d+1\})$ is contained in the interior of the set $\{ax \leq b\}$. Therefore by Lemma 2, there exists a $\tau \in \mathbb{Z}_+$ such that $\mathrm{M}(P \cap \{cx \geq d+1\})$ is contained in the interior of $\{(c + \tau a)x \leq d + 1 + \tau b\}$. Equivalently, $\mathrm{M}(P \cap \{cx \geq d+1\}) \cap \{(c + \tau a)x \geq d + 1 + \tau b\} = \emptyset$ which implies

$$\mathrm{M}(P \cap \{cx \geq d+1\}) \cap (P \cap \{(c + \tau a)x \geq d + 1 + \tau b\}) = \emptyset. \tag{5}$$

Since $P \subseteq F^{\leq}$, we obtain that

$$P \cap \{(c + \tau a)x \geq d + 1 + \tau b\} \subseteq P \cap \{cx \geq d+1\}. \tag{6}$$

Now using (5), (6) and the inclusion preservation property of M it follows that $\mathrm{M}(P \cap \{(c + \tau a)x \geq d + 1 + \tau b\}) = \emptyset$. Thus $(c + \tau a)x \leq d + \tau b$ is a verifiable inequality for $\partial\mathrm{M}(P)$. Moreover note that since $\hat{x} \in P \cap F$, we have that $a\hat{x} = b$. Therefore, $(c + \tau a)\hat{x} = c\hat{x} + \tau b > d + \tau b$, where the last inequality follows from the fact that $c\hat{x} > d$. □

It can be shown that short verification, i.e., property (7.) of admissible systems follows whenever $\partial\mathrm{M}(P)$ is a rational polyhedron. However, we do not need this property for the results in this paper.

3 Strength and comparisons of $\mathbb{V}$-closures.

In this section, we compare various regular closures and their verification counterparts with each other. We first formally define possible relations between admissible closures and the notation we use.

Definition 4. *Let L, M be almost admissible. Then*

1. L refines M, *if for all rational polytopes* P *we have* $\mathrm{L}(P) \subseteq \mathrm{M}(P)$*. We write:* $\mathrm{L} \subseteq \mathrm{M}$*. It is indicated by empty arrow heads in Figure 1.*

2. L strictly refines M, *if* L *refines* M *and there exists a rational polytope* P *such that* $\mathrm{L}(P) \subsetneq \mathrm{M}(P)$*. We write:* $\mathrm{L} \subsetneq \mathrm{M}$*. It is indicated by a filled arrow heads in Figure 1.*

3. L is incompatible with M, *if there exist rational polytopes* P *and* Q *such that* $\mathrm{M}(P) \subsetneq \mathrm{L}(P)$ *and* $\mathrm{M}(Q) \supsetneq \mathrm{L}(Q)$*. We write:* $\mathrm{L} \perp \mathrm{M}$*. It is indicated with an arrow with circle head and tail in Figure 1.*

In each of the above definitions, if either one of L *or* M *is defined only for polytopes* $P \subseteq [0,1]^n$*, then we confine the comparison to this class of polytopes.*

In Section 3.1, we will establish the following result.

Theorem 2. *Let* M *be an admissible cutting plane operator. Then*

1. $\partial\mathrm{M} \subsetneq \mathrm{M}$.
2. $\partial\mathrm{M} \subseteq \mathrm{GC}$ *and* $\partial\mathrm{M} \subseteq \mathrm{N}_0$.

In Section 3.2, we will establish the following result.

Theorem 3. *Let* L *and* M *be admissible cutting plane operators such that* $\mathrm{L} \subseteq \mathrm{M}$*. Then* $\partial\mathrm{L} \subseteq \partial\mathrm{M}$*. Moreover,*

1. $\partial\mathrm{GC} \subsetneq \mathrm{SC}$.
2. $\partial\mathrm{N}_0 \perp \partial\mathrm{GC}$
3. $\partial\mathrm{N}_0 \perp \mathrm{SC}$
4. $\partial\mathrm{N} \subsetneq \partial\mathrm{N}_0$.

Well-known relations between the operators $\{\mathrm{GC}, \mathrm{SC}, \mathrm{N}_0, \mathrm{N}, \mathrm{N}_+\}$ and those presented in Theorem 2 and Theorem 3 are depicted in Figure 1.

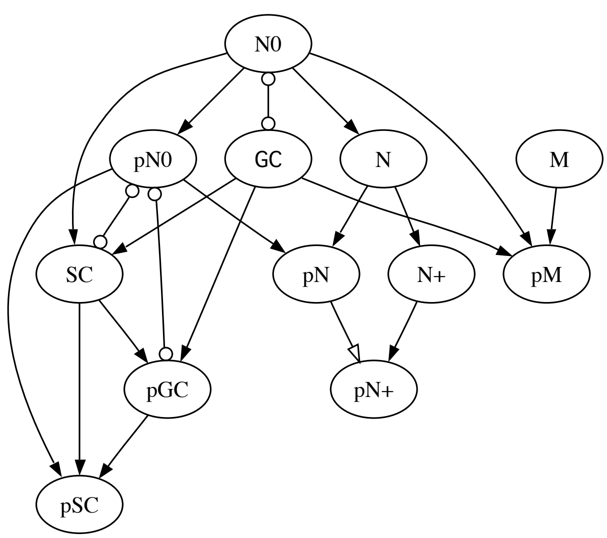

Figure 1: Direct and $\mathbb{V}$-closures and their relations. pL in the figure represents ∂L and M is an arbitrarily weak admissible system. In order to simplify the figure, we have removed the arcs corresponding to $\mathrm{GC} \perp \mathrm{N}$, $\mathrm{GC} \perp \mathrm{N}_+$, $\mathrm{SC} \perp \mathrm{N}$, $\mathrm{SC} \perp \mathrm{N}_+$.

3.1 Strength of ∂M for arbitrary admissible cutting-plane procedures M

In order to show that ∂M refines M, we require the following technical lemma; see [8] for a similar result. We use the notation $\sigma_P(\cdot)$ to refer to the support function of a set P, i.e., $\sigma_P(c) = \sup\{cx \mid x \in P\}$.

Lemma 3. *Let $P, Q \subseteq \mathbb{R}^n$ be compact convex sets. If $\sigma_P(c) \leq \sigma_Q(c)$ for all $c \in \mathbb{Z}^n$, then $P \subseteq Q$.*

Proof. For a compact convex set T, we have that $T = \cap_{c\in\mathbb{Z}^n}\{x \in \mathbb{R}^n \mid cx \leq \sigma_T(c)\}$. (See [8] for a proof.) Therefore, if $\hat{x} \in P$, then $c\hat{x} \leq \sigma_P(c)$ for all $c \in \mathbb{Z}^n$. By assumption $\sigma_P(c) \leq \sigma_Q(c)$, we obtain that $c\hat{x} \leq \sigma_Q(c)$ for all $c \in \mathbb{Z}^n$. However since $Q = \cap_{c\in\mathbb{Z}^n}\{x \in \mathbb{R}^n \mid cx \leq \sigma_Q(c)\}$, we obtain that $\hat{x} \in Q$. □

Proposition 4. *Let* M *be admissible. Then* ∂M $\subsetneq$ M.

Proof. We first verify that ∂M $\subseteq$ M. Let P be a rational polytope. Since $\mathrm{M}(P) \subseteq P$ and $\partial\mathrm{M}(P) \subseteq P$, both $\mathrm{M}(P)$ and $\partial\mathrm{M}(P)$ are bounded. Moreover since $\mathrm{M}(P)$ is closed by definition, and $\partial\mathrm{M}(P)$ is defined as the intersection of halfspaces (thus a closed set), we obtain that $\mathrm{M}(P)$ and $\partial\mathrm{M}(P)$ are both compact convex sets. Thus, by Lemma 3, it is sufficient to compare the support functions of $\mathrm{M}(P)$ and $\partial\mathrm{M}(P)$ with respect to integer vectors only. Let $\sigma_{\mathrm{M}(P)}(c) = d$ for $c \in \mathbb{Z}^n$. We verify that $\sigma_{\partial\mathrm{M}(P)}(c) \leq \lfloor d \rfloor$. Observe that, $\mathrm{M}(P \cap \{cx \geq \lfloor d \rfloor + 1\}) \subseteq \mathrm{M}(P) \cap \{cx \geq \lfloor d \rfloor + 1\}$, where the inclusion follows from the inclusion preservation property of M. However note that since $cx \leq d$ is a valid inequality for $\mathrm{M}(P)$, we obtain that $\mathrm{M}(P) \cap \{cx \geq \lfloor d \rfloor + 1\} = \emptyset$. Thus, $\mathrm{M}(P \cap \{cx \geq \lfloor d \rfloor + 1\}) = \emptyset$ and so $cx \leq \lfloor d \rfloor$ is a valid inequality for $\partial\mathrm{M}(P)$. Equivalently we have $\sigma_{\partial\mathrm{M}(P)}(c) \leq \lfloor d \rfloor \leq d = \sigma_{\mathrm{M}(P)}(c)$, completing the proof.

Now we verify ∂M $\subsetneq$ M. Let $n \in \mathbb{N}$ be such that $\mathrm{M}(A_n) \neq \emptyset$ and $\mathrm{M}(A_{n-1}) = \emptyset$; such an n exists (due to the coordinate rounding property of M we have that $\mathrm{M}(A_1) = \emptyset$ and since the rank of A_n with respect to M is in $\Omega(n/\log(n))$ [19], there exists $t \in \mathbb{N}$ such that $\mathrm{M}(A_t) \neq \emptyset$). We claim that $\partial\mathrm{M}(A_n) = \emptyset$ which implies that ∂M $\subsetneq$ M follows.

In order to establish the claim, observe that $\mathrm{M}(A_n \cap \{x_n \leq 0\}) \cong \mathrm{M}(A_{n-1}) = \emptyset$, where the last equality is due to the choice of n. Therefore $x_n \geq 1$ is valid for $\partial\mathrm{M}(A_n)$. Similarly, we can derive the validity of $x_n \leq 0$ for $\partial\mathrm{M}(A_n)$. We therefore conclude that $\partial\mathrm{M}(A_n) = \emptyset$. □

We next show that even if M is chosen arbitrarily, ∂M is at least as strong as the GC closure and the N_0 closure.

Proposition 5. *Let* M *be admissible. Then* ∂M $\subseteq$ GC *and* ∂M $\subseteq$ N_0 *(the latter holding for rational polytopes $P \subseteq [0,1]^n$).*

Proof. Let $P \subseteq \mathbb{R}^n$ be a rational polytope. First let $cx < d + 1$ with $c \in \mathbb{Z}^n$ and $d \in \mathbb{Z}$ be valid for P. Then $cx \leq d$ is valid for $\mathrm{GC}(P)$. It suffices to consider inequalities of this type. Observe that $P \cap \{cx \geq d+1\} = \emptyset$ and so clearly, $\mathrm{M}(P \cap \{cx \geq d+1\}) = \emptyset$. It follows that $cx \leq d$ is valid for $\partial\mathrm{M}(P)$ and thus $\partial\mathrm{M}(P) \subseteq \partial\mathrm{GC}(P)$.

Now let P be a rational polytope with $P \subseteq [0,1]^n$. For proving $\partial\mathrm{M}(P) \subseteq \mathrm{N}_0(P)$, recall that $\mathrm{N}_0 = \bigcap_{i\in[n]} P_i$ where $P_i := \mathrm{conv}\{(P \cap \{x_i = 0\}) \cup (P \cap \{x_i = 1\})\}$. We will show that $\partial\mathrm{M}(P) \subseteq P_i \ \forall\ i \in [n]$. Therefore let $cx \leq d$ with $c \in \mathbb{Z}^n$ and $d \in \mathbb{Z}$ be valid for P_i with $i \in [n]$ arbitrary. (Note that it is sufficient to consider only inequalities with integer coefficients since P_i is a rational polytope.) In particular, $cx \leq d$ is valid for $P \cap \{x_i = l\}$ with $l \in \{0,1\}$. Thus we can conclude that $P \cap \{cx \geq d+1\} \cap \{x_i = l\} = \emptyset$ for $i \in \{0,1\}$. Therefore $x_i > 0$ and $x_i < 1$ are valid for $P \cap \{cx \geq d+1\}$ and so by coordinate rounding (Property (4.) of Definition 3), $x_i \leq 0$ and $x_i \geq 1$ are valid $\mathrm{M}(P \cap \{cx \geq d+1\})$. We obtain $\mathrm{M}(P \cap \{cx \geq d+1\}) = \emptyset$ and thus $cx \leq d$ is valid for $\partial\mathrm{M}(P)$. □

3.2 Comparing M and ∂M for M being GC, SC, N_0, N, or N_+

We now compare various closures and their associated $\mathbb{V}$-closures. The first result shows that the verification scheme of the Gomory-Chvátal procedure is at least as strong as split cuts.

Proposition 6. ∂GC $\subsetneq$ SC.

Proof. We first verify that $\partial\text{GC} \subseteq \text{SC}$. Consider $cx \leq d$ being valid for $P \cap \{\pi x \leq \pi_0\}$ and $P \cap \{\pi x \geq \pi_0 + 1\}$ with $c, \pi \in \mathbb{Z}^n$ and $d, \pi_0 \in \mathbb{Z}$. Clearly, $cx \leq d$ is valid for $SC(P)$ and it suffices to consider inequalities $cx \leq d$ with this property; all others are dominated by positive combinations of these. Therefore consider $P \cap \{cx \geq d+1\}$. By $cx \leq d$ being valid for the disjunction $\pi x \leq \pi_0$ and $\pi x \geq \pi_0 + 1$ we obtain that $P \cap \{cx \geq d+1\} \cap \{\pi x \leq \pi_0\} = \emptyset$ and $P \cap \{cx \geq d+1\} \cap \{\pi x \geq \pi_0 + 1\} = \emptyset$. This implies that $P \cap \{cx \geq d+1\} \subseteq \{\pi x > \pi_0\}$ and similarly $P \cap \{cx \geq d+1\} \subseteq \{\pi x < \pi_0 + 1\}$. We thus obtain that $\pi x \geq \pi_0 + 1$ and $\pi x \leq \pi_0$ are valid for $\text{GC}(P \cap \{cx \geq d+1\})$. It follows $\text{GC}(P \cap \{cx \geq d+1\}) = \emptyset$. Thus $cx \leq d$ is valid for $\partial\text{GC}(P)$.

To see that $\partial\text{GC} \subsetneq \text{SC}$, observe that $\partial\text{GC}(A_2) = \emptyset$ and $\text{SC}(A_2) \neq \emptyset$. □

Next we compare $\mathbb{V}$-schemes of two closures that are comparable. Before we present these results, we clarify the difference between the notion of *verifiable inequalities* against the notion of *valid inequalities for* $\mathbb{V}$-*closure of* M. Recall that given a rational polytope, $P \subseteq \mathbb{R}^n$, we say $cx \leq d$ is a verifiable inequality if $c \in \mathbb{Z}^n$, $d \in \mathbb{Z}$ and $\text{M}(P \cap \{cx \geq d+1\}) = \emptyset$. Thus the $\mathbb{V}$-closure of M is the intersection of all verifiable inequalities. On the other hand, there may be a valid inequality for $\partial\text{M}(P)$ that are not verifiable. A trivial example of such as a valid inequality $cx \leq d$ for $\partial\text{M}(P)$ is when c is not a rational vector. The following example illustrates this difference more explicitly.

Example 7. *Consider the set* $P = \{x \in [0,1]^2 \mid x_1 + x_2 \geq \frac{1}{2}, x_1 - x_2 \leq \frac{1}{2}, -x_1 + x_2 \leq \frac{1}{2}\}$. *Observe that* $\partial\text{N}_0(P) = P_I = \{(1,1)\}$. *This can be obtained by observing the the inequalities* $x_1 \geq 1$ *and* $x_2 \geq 1$ *are verifiable using* N_0. *Now consider the inequality* $2x_1 + 3x_2 \geq 5$. *Clearly* $2x_1 + 3x_2 \geq 5$ *is valid for* $\partial\text{N}_0(P)$ *but is not verifiable since* $\text{N}_0(P \cap \{2x_1 + 3x_2 \leq 4\}) \supseteq \text{N}_0(A_2) = \frac{1}{2}e$.

The next result shows that switching to the verification schemes preserves inclusion.

Proposition 8. *Let* L, M *be admissible such that* $\text{L} \subseteq \text{M}$. *Then* $\partial\text{L} \subseteq \partial\text{M}$.

Proof. Let $P \subseteq \mathbb{R}^n$ be a rational polytope. By the definition of ∂M, it is sufficient to show that every inequality $cx \leq d$ verifiable by using M is valid for $\partial\text{L}(P)$. Now observe that since $cx \leq d$ is verifiable by using M, we have that $\text{M}(P \cap \{cx \geq d+1\}) = \emptyset$. Thus, $\text{L}(P \cap \{cx \geq d+1\}) = \emptyset$ since $\text{L} \subseteq \text{M}$ and therefore $cx \leq d$ is verifiable using L. Equivalently $cx \leq d$ is valid for $\partial\text{L}(P)$, completing the proof. □

In order to prove strict refinement or incompatibility between $\mathbb{V}$-closures the following proposition is helpful. It establishes when strict refinement carries over to the $\mathbb{V}$-schemes.

Proposition 9. *Let* L, M *be admissible. If* $P \subseteq [0,1]^n$ *is a polytope with* $P_I = \emptyset$ *such that* $\text{M}(P) = \emptyset$ *and* $\text{L}(P) \neq \emptyset$, *then* ∂L *does not refine* ∂M.

Before presenting the proof of Proposition 9, we first present a lemma.

Lemma 10. *Let* $P \subseteq [0,1]^n$ *be a polytope,* $P \neq \emptyset$ *and* $P_I = \emptyset$. *Define* $Q \subseteq [0,1]^{n+1}$ *as* $Q = \text{conv}(\{(x,1) \in \mathbb{R}^{n+1} \mid x \in P\} \cup \{(y,0) \in \mathbb{R}^{n+1} \mid y \in [0,1]^n\})$. *Then* $\partial\text{L}(Q) = Q_I$ *iff* $\text{L}(P) = \emptyset$.

Proof. ($\Rightarrow$) If $\text{L}(P) = \emptyset$, then observe that $\text{L}(Q \cap \{x \in \mathbb{R}^{n+1} \mid x_{n+1} \geq 1\}) \cong L(P) = \emptyset$. Therefore $x_{n+1} \leq 0$ is valid for $\partial\text{L}(Q)$. Thus $\partial\text{L}(Q) = Q_I$.
($\Leftarrow$) Suppose $\partial\text{L}(Q) = Q_I$. Clearly $x_{n+1} \leq 0$ is a valid inequality for $\partial\text{L}(Q)$. If this inequality is also a verifiable inequality for $\partial\text{L}(P)$, then we have that $\text{L}(P) \cong \text{L}(Q \cap \{x_{n+1} \geq 1\}) = \emptyset$, where the last equality is due to verifiability of $x_{n+1} \leq 0$. However the validity of $x_{n+1} \leq 0$ does not imply its verifiability. Hence we proceed as follows. Note that since

$$Q_I = \partial\text{L}(Q) = \bigcap_{\substack{c \in \mathbb{Z}^{n+1}, d \in \mathbb{Z}, \\ \text{s.t. } \text{L}(Q \cap \{x \mid cx \geq d+1\}) = \emptyset}} \{cx \leq d\},$$

there exists a verifiable cut $cx \leq d$, (i.e. $\text{L}(Q \cap \{cx \geq d+1\}) = \emptyset$) which separates the point $\left(\frac{1}{2}e^n, \varepsilon\right) \in [0,1]^{n+1}$ from $\partial\text{L}(Q)$ where e^n is the vector of all ones in $\mathbb{R}^n$ and $1 > \varepsilon > 0$.
Claim. $d \geq 0$. Since $\mathbf{0} \in Q_I$, we obtain that $0 = c\mathbf{0} \leq d$.
Claim. If $(c_1, \ldots, c_n, c_{n+1}) = (0, \ldots, 0, 1)$, then $\text{L}(P) = \emptyset$. Observe that if $(c_1, \ldots, c_n, c_{n+1}) = (0, \ldots, 0, 1)$, then the inequality $cx \leq d$ reduces to $x_{n+1} \leq d$ which separates $(\frac{1}{2}e^n, \varepsilon)$ if and only if $d \leq 0$. Since

$d \geq 0$ by the previous claim, we obtain that $d = 0$. Now by assumption $cx \leq d$ is verifiable, i.e., $\mathrm{L}(Q \cap \{x_{n+1} \geq 1\}) = \emptyset$ and since $\mathrm{L}(P) \cong \mathrm{L}(Q \cap \{x_{n+1} \geq 1))$, we obtain that $\mathrm{L}(P) = \emptyset$.

Therefore, henceforth we assume that $(c_1, ...c_n, c_{n+1}) \neq (0, ..., 0, 1)$.

Claim. If $\varepsilon < \frac{1}{4}$, then $c_{n+1} \geq 2$. If $c_i = 0$ for all $i \in [n]$ and $c_{n+1} = 1$, then we are in the case discussed in the previous claim. Also if $c_i = 0$ for all $i \in [n]$ and $c_{n+1} \leq 0$, then the point $(\frac{1}{2}e^n, \varepsilon)$ cannot be separated by $cx \leq d$ (since $d \geq 0$). Thus we have that if $(c_1, \ldots, c_n) = (0, ..., 0)$, then $c_{n+1} \geq 2$. Therefore we may assume that $(c_1, \ldots, c_n) \neq (0, \ldots, 0)$. Examine the following two cases.

1. $\sum_{i=1}^n c_i \leq -1$. In this case observe that since the inequality $cx \leq d$ is a separating hyperplane for the point $(\frac{1}{2}e^n, \varepsilon)$ we obtain that

$$\frac{1}{2}\sum_{i=1}^{n} c_i + c_{n+1}\varepsilon > d. \tag{7}$$

   or equivalently

$$c_{n+1}\varepsilon > \frac{1}{2}.$$

   Since $0 < \varepsilon < \frac{1}{4}$, we have that $c_{n+1} \geq 2$.

2. $\sum_{i=1}^n c_i \geq 0$. Since $(c_1, \ldots, c_n) \neq (0, \ldots, 0)$, in this case, there exists a subset $S' \subseteq [n]$ such that $\sum_{i \in S'} c_i \geq 1$. Since $\{(y, 0) \mid y \in [0, 1]^n\} \subseteq Q_I$, we obtain that

$$\sum_{i \in S} c_i \leq d \qquad \forall\ S \subseteq [n]. \tag{8}$$

   By examining the case where $S = S'$, we obtain that

$$d \geq 1.$$

   Now by combining (8) for the case of $S = [n]$ and (7), we obtain that

$$c_{n+1}\varepsilon > \frac{1}{2}d \geq \frac{1}{2}.$$

   Since $0 < \varepsilon < \frac{1}{4}$ it follows we have that $c_{n+1} \geq 2$.

Claim. If $\varepsilon \leq \frac{1}{8}$ and $cx \leq d$ separates $\left(\frac{1}{2}e^n, \varepsilon\right)$, then $\{(x, 1) \in \mathbb{R}^{n+1} \mid x \in [0, 1]^n\} \subseteq \{cx \geq d + 2\}$. Let $T \subseteq [n]$ and examine the point $(\hat{x}, 1)$ where $\hat{x}_i = 1$ if and only if $i \in T$. We show that $(\hat{x}, 1) \in \{cx \geq d+2\}$. By combining (8) for the case of $S = [n] \setminus T$ and (7) we obtain that

$$\frac{1}{2}\sum_{i \in T} c_i + c_{n+1}\varepsilon > \frac{1}{2}d \tag{9}$$

or equivalently,

$$\sum_{i \in T} c_i + c_{n+1}(2\varepsilon) > d. \tag{10}$$

By previous claim, we have $c_{n+1} \geq 2$. Moreover since $1 - 2\varepsilon \geq \frac{3}{4}$, we obtain that $c_{n+1}(1 - 2\varepsilon) \geq \frac{3}{2}$. Equivalently,

$$\sum_{i \in T} c_i + c_{n+1} \geq \sum_{i \in T} c_i + c_{n+1}(2\varepsilon) + \frac{3}{2} > d + \frac{3}{2}, \tag{11}$$

where the last inequality follows from (10). Now note that since $c \in \mathbb{Z}^{n+1}$ and $d \in \mathbb{Z}$, we obtain that

$$\sum_{i \in T} c_i + c_{n+1} \geq d + 2, \tag{12}$$

or equivalently $\sum_{i\in[n]} \hat{x}_i c_i + c_{n+1} = \sum_{i\in T} c_i + c_{n+1} \geq d + 2$.

Now we complete the proof. Let ε be any positive number less than $\frac{1}{8}$. Since $\mathrm{L}(Q) = Q_I$, there exists $c \in \mathbb{Z}^{n+1}$ and $d \in \mathbb{Z}$ such that $cx \leq d$ separates the point $(\frac{1}{2}e^n, \varepsilon)$ and $\mathrm{L}(Q \cap \{cx \geq d+1\}) = \emptyset$. Now observe that $\mathrm{L}(P) \cong \mathrm{L}(\{(x,1) \mid x \in P\}) \subseteq \mathrm{L}(Q \cap \{cx \geq d+2\}) \subseteq \mathrm{L}(Q \cap \{cx \geq d+1\}) = \emptyset$, where the first inclusion follows from the previous claim. □ □

We will use the following notation in the remainder of this section. Let $G \subseteq [0,1]^n$ be a closed convex set. For $l \in [0,1]$, by $G_{x_{n+1}=l}$ we denote the set $S \subseteq [0,1]^{n+1}$ such that $S \cap \{x_{n+1} = l\} \cong G$ and S does not contain any other points. We can think of S arising from G by padding the coordinates of the vertices with l to the right. If G is the singleton $\{p\}$, then we write $\{p\}_{x_{n+1}=l}$ as $p_{x_{n+1}=l}$.

Proof. of Proposition 9 Consider the auxiliary polytope Q given as $Q := \mathrm{conv}\left(P_{x_{n+1}=1} \cup [0,1]^n_{x_{n+1}=0}\right)$. By Lemma 10, $\partial\mathrm{L}(Q) = \emptyset$ if and only if $\mathrm{L}(P) \cong \mathrm{L}(Q \cap \{x_{n+1} \geq 1\}) = \emptyset$ (and similarly for M). Since we have $\mathrm{M}(P) = \emptyset$ but $\mathrm{L}(P) \neq \emptyset$, we obtain $Q_I = \partial\mathrm{M}(Q) \not\supseteq \partial\mathrm{L}(Q)$. □

In the following propositions, polytopes are presented that help establish the strict inclusion or incompatibility depicted in Figure 1, via Proposition 9.

Proposition 11. $\partial\mathrm{N}_0 \perp \partial\mathrm{GC}$ *via the two polytopes* $P_1 := \mathrm{conv}([0,1]^3 \cap \{x_1 + x_2 + x_3 = 3/2\}) \subseteq [0,1]^3$ *and* $P_2 := \mathrm{conv}(\{(\frac{1}{4}, \frac{1}{4}, 0), (\frac{1}{4}, \frac{1}{4}, 1), (\frac{1}{2}, 0, \frac{1}{2}), (\frac{1}{2}, 1, \frac{1}{2}),$
$(0, \frac{1}{2}, \frac{1}{2}), (1, \frac{1}{2}, \frac{1}{2})\}) \subseteq [0,1]^3$.

Proof. By Proposition 9 it suffices to show that $\mathrm{GC}(P_1) = \emptyset \neq \mathrm{N}_0(P_1)$ and, vice versa, $\mathrm{GC}(P_2) \neq \emptyset = \mathrm{N}_0(P_2)$.

For the first case, clearly $\mathrm{GC}(P_1) = \emptyset$. For proving that $\mathrm{N}_0(P_1) \neq \emptyset$ it suffices to show that $\frac{1}{2}e$ is contained in $\mathrm{conv}((P_1 \cap \{x_i = 0\}) \cup (P_1 \cap \{x_i = 1\}))$ for all $i \in [3]$. By symmetry, it suffices to show this for $i = 1$. This is true as $\frac{1}{2}e$ is the convex combination of the points $(0, 1, 1/2)$ and $(1, 0, 1/2)$.

For the second case, we first show that $\mathrm{N}_0(P_2) = \emptyset$. For this observe that $\mathrm{conv}((P_2 \cap \{x_3 = 0\}) \cup (P_2 \cap \{x_3 = 1\}))$ contains only points those first two coordinates are equal to $1/4$. On the other hand

$$\begin{aligned} &\mathrm{conv}((P_2 \cap \{x_1 = 0\}) \cup (P_2 \cap \{x_1 = 1\})) \\ \cap&\mathrm{conv}((P_2 \cap \{x_2 = 0\}) \cup (P_2 \cap \{x_2 = 1\})) = \frac{1}{2}e, \end{aligned}$$

as $P_2 \cap \{x_3 = 1/2\} \cong A_2$ and thus $\mathrm{N}_0(P_2) = \emptyset$. It thus remains to show that $\mathrm{GC}(P_2) \neq \emptyset$. We will show that $\frac{1}{2}e \in P_2$. Let $cx \leq d$ with $c \in \mathbb{Z}^n$ be valid for P_2. We divide the proof into two cases:

1. Either c_1 or c_2 are non-zero. In this case observe that

$$\begin{aligned} d \;&\geq\; d_0 := \max\left\{ c\left(\frac{1}{2}, 0, \frac{1}{2}\right), c\left(\frac{1}{2}, 1, \frac{1}{2}\right), c\left(0, \frac{1}{2}, \frac{1}{2}\right), c\left(1, \frac{1}{2}, \frac{1}{2}\right) \right\} \\ &>\; d_1 := c\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), \end{aligned}$$

   where the second inequality follows from the fact that $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ lies in the relative interior of the convex hull of $\left(\frac{1}{2}, 0, \frac{1}{2}\right), \left(\frac{1}{2}, 1, \frac{1}{2}\right), \left(0, \frac{1}{2}, \frac{1}{2}\right), \left(1, \frac{1}{2}, \frac{1}{2}\right)$. Now observe that since $d_0, d_1 \in \frac{1}{2}\mathbb{Z}$, we obtain that the interval $[d_1, d_0]$ contains at least one integer number. Thus, $\lfloor d \rfloor \geq \lfloor d_0 \rfloor \geq d_1 = c\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$.

2. $c_1 = c_2 = 0$. If $c_3 > 0$, then $d \geq c_3$ (since $(\frac{1}{4}, \frac{1}{4}, 1) \in P_2$) and we obtain the GC inequality $c_3 x_3 \leq \lfloor c_3 \rfloor$ where $\lfloor c_3 \rfloor \geq 1$. Thus this inequality cannot separate $\frac{1}{2}e$. Similarly if $c_3 \leq -1$, it can the verified that the resulting inequality cannot separate $\frac{1}{2}e$.

□

Proposition 12. $\partial\mathrm{N}_0 \perp \mathrm{SC}$ *via* $P_1 := A_3 \subseteq [0,1]^3$ *and* $P_2 := \mathrm{conv}([0,1]^3 \cap \{x_1 + x_2 + x_3 = 3/2\})$.

Proof. Clearly SC $\not\subseteq \partial\mathrm{N}_0$ as $\partial\mathrm{N}_0(P_1) = \emptyset$ (proof similar to Example 1) but $\mathrm{SC}(P_1) \neq \emptyset$ (cf. Lemma 3.3 in [7]).

For the converse, by Proposition 11 we have $\mathrm{N}_0(P_2) \neq \emptyset$. However, $\mathrm{SC}(Q) = Q_I$ by observing that the split $x_1 + x_2 + x_3 \leq 1$ and $x_1 + x_2 + x_3 \geq 2$ derives Q_I. Now the result follows from Proposition 9. □

Proposition 13. $\partial\mathrm{N} \subsetneq \partial\mathrm{N}_0$.

Proof. We will show that there exists a polytope Q contained in the 0/1 cube such that $Q_I = \emptyset$ and $\emptyset = \mathrm{N}(Q) \subsetneq \mathrm{N}_0(Q)$. Then the result follows by the use of Proposition 9.

Let $P \subseteq [0,1]^n$ such that $\mathrm{N}(P) \subsetneq \mathrm{N}_0(P)$, for example as discussed in page 171 of [16], for some $n \in \mathbb{N}$. Let $p \in \mathrm{N}_0(P) \setminus \mathrm{N}(P)$ and define

$$Q := \operatorname{conv}\left(P_{x_{n+1}=1/2} \cup \left\{p_{x_{n+1}=1}, p_{x_{n+1}=0}\right\}\right).$$

Clearly, $Q_I = \emptyset$.

We first verify that $\mathrm{N}(Q) = \emptyset$. Observe first that

$$\mathrm{N}(Q) \subseteq \mathrm{N}_0(Q) \subseteq \operatorname{conv}\left(Q \cap \{x_{n+1} = 0\} \cup Q \cap \{x_{n+1} = 1\}\right) = \operatorname{conv}(p_{x_{n+1}=0}, p_{x_{n+1}=1}).$$

On the other hand, it is easily verified that if $\sum_{i=1}^n c_i x_i \leq d$ is a valid inequality for P, then it is also a valid inequality for Q. Therefore we obtain that $\mathrm{N}(Q) \subseteq \operatorname{conv}(\mathrm{N}(P)_{x_{n+1}=0}, \mathrm{N}(P)_{x_{n+1}=1})$. Now since $\operatorname{conv}(\mathrm{N}(P)_{x_{n+1}=0}, \mathrm{N}(P)_{x_{n+1}=1}) \cap \operatorname{conv}(p_{x_{n+1}=0}, p_{x_{n+1}=1}) = \emptyset$, we obtain that $N(Q) = \emptyset$.

Next we verify that $\mathrm{N}_0(Q) \neq \emptyset$. As $p \in \mathrm{N}_0(P)$ we can conclude that

$$p_{x_{n+1}=1/2} \in \bigcap_{i \in [n]} \operatorname{conv}\left(Q \cap \{x_i = 0\} \cup Q \cap \{x_i = 1\}\right).$$

Thus we have to show that $p_{x_{n+1}=1/2} \in \operatorname{conv}\left(Q \cap \{x_{n+1} = 0\} \cup Q \cap \{x_{n+1} = 1\}\right)$. This is clear though as $\left\{p_{x_{n+1}=1}, p_{x_{n+1}=0}\right\} \subseteq Q$. □

4 Rank of valid inequalities with respect to $\mathbb{V}$-closures.

In this section, we establish several bounds on the rank of $\partial\mathrm{M}$ for the case of polytopes $P \subseteq [0,1]^n$. Given a natural number k, we use the notation $\mathrm{M}^k(P)$ and $\mathrm{rk}_{\mathrm{M}}(P)$ to be denote that k^{th} closure of P with respect to M and the rank of P with respect to M respectively. As $\partial\mathrm{M} \subseteq \mathrm{N}_0$ we obtain:

Proposition 14 (Upper bound in $[0,1]^n$)**.** *Let* M *be admissible and* $P \subseteq [0,1]^n$ *be a polytope. Then* $rk_{\partial\mathrm{M}}(P) \leq n$.

Proof. As $\partial\mathrm{M} \subseteq \mathrm{N}_0$ and $\mathrm{rk}_{\mathrm{N}_0}(P) \leq n$ the result follows. □

Note that in general the property of M being admissible, does not guarantee that the upper bound on rank is n. For example, the GC closure can have a rank strictly higher than n (cf. [14, 20]).

4.1 Rank of A_n

In quest for lower bounds on the rank of 0/1 polytopes, we note that among polytopes $P \subseteq [0,1]^n$ that have $P_I = \emptyset$, the polytope $A_n = \{x \in [0,1]^n \mid \sum_{i \in I} x_i + \sum_{i \notin I}(1 - x_i) \geq \frac{1}{2} \quad \forall\, I \subseteq [n]\}$ has maximal rank (of n) for many admissible systems [18]. We will now establish that $\partial\mathrm{M}$ is not unrealistically strong by showing that it is subject to similar limitations. Recall that we do not prove *short verification* (property (7.)) for $\partial\mathrm{M}$ which is the basis for the lower bound in [19, Corollary 23] for admissible systems. We will show that the lower bound for $\partial\mathrm{M}$ is *inherited* from the original operator M. Let

$$F_n^k := \{x \in \{0, 1/2, 1\}^n \mid \text{ exactly } k \text{ entries equal to } 1/2\},$$

and let $A_n^k := \operatorname{conv}\left(F_n^k\right)$ be the convex hull of F_n^k. (Note $A_n^1 = A_n$.) With F being a face of $[0,1]^n$ let $I(F)$ denote the index set of those coordinate that are fixed by F. We begin with a crucial lemma.

Lemma 15. *Let* M *be admissible and let* $\ell \in \mathbb{N}$ *such that* $A_n^{k+\ell} \subseteq \mathrm{M}(A_n^k)$ *for all* $n, k \in \mathbb{N}$ *with* $k + \ell \leq n$. *If* $n \geq k + 2\ell + 1$, *then* $A_n^{k+2\ell+1} \subseteq \partial\mathrm{M}(A_n^k)$.

Proof. Let $P := A_n^k$ and let $cx \leq d$ with $c \in \mathbb{Z}^n$ and $d \in \mathbb{Z}$ be verifiable for $\partial\mathrm{M}(P)$, i.e. $\mathrm{M}(P \cap \{cx \geq d+1\}) = \emptyset$. To prove this result, it is sufficient to prove that $A_n^{k+2\ell+1} \subseteq P \cap \{cx \leq d\}$.

We first claim that

$$A_{k+\ell}^k \cong A_n^k \cap F \nsubseteq P \cap \{cx \geq d+1\} \tag{13}$$

for all $(k+\ell)$-dimensional faces F of $[0,1]^n$. Assume by contradiction that $A_n^k \cap F \subseteq P \cap \{cx \geq d+1\}$. As $A_{k+\ell}^{k+\ell} \subseteq \mathrm{M}(A_{k+\ell}^k)$ by assumption, we obtain $\emptyset \neq A_{k+\ell}^{k+\ell} \subseteq \mathrm{M}(A_{k+\ell}^k) \subseteq \mathrm{M}(P \cap \{cx \geq d+1\})$ which contradicts the verifiability of $cx \leq d$ over $\partial\mathrm{M}(P)$.

Without loss of generality we can further assume that $c \geq 0$ and $c_i \geq c_j$ whenever $i \leq j$ by applying coordinate flips and permutations.

Next we claim that for all $(k+\ell)$-dimensional faces F of $[0,1]^n$, the point v^F defined as

$$v_i^F := \begin{cases} \in \{0,1\} \text{ according to } F, \text{ for all } i \in I(F) \\ 0, \text{ if } c_i \text{ is one of the } \ell \text{ largest coefficients of } c \text{ with } i \notin I(F) \\ 1/2, \text{ otherwise} \end{cases} \tag{14}$$

for $i \in [n]$ is not contained in $P \cap \{cx \geq d+1\}$, i.e., $cv^F < d+1$ and so $cv^F \leq d + 1/2$. Note that $v^F \in P$ and observe that $v^F := \mathrm{argmin}_{x \in F_n^k \cap F}\, cx$. Therefore, if $v^F \in P \cap \{cx \geq d+1\}$, then $A_n^k \cap F \subseteq P \cap \{cx \geq d+1\}$ which in turn contradicts (13). This claim holds in particular for those faces F fixing coordinates to 1.

Finally, we claim that $A_n^{k+2\ell+1} \subseteq P \cap \{cx \leq d\}$. It suffices to show that $cv \leq d$ for all $v \in F_n^{k+2\ell+1}$ and we can confine ourselves to the worst case v given by

$$v_i := \begin{cases} 1, \text{ if } i \in [n-(k+2\ell+1)] \\ 1/2, \text{ otherwise.} \end{cases}$$

Observe that $cv \geq cw$ holds for all $w \in F_n^{k+2\ell+1}$. Let F be the $(k+\ell)$-dimensional face of $[0,1]^n$ obtained by fixing the first $n-(k+\ell)$ coordinates to 1. Then

$$\begin{aligned} cv &= \sum_{i=1}^{n-(k+2\ell+1)} c_i + \frac{1}{2} \sum_{i=n-(k+2\ell+1)+1}^{n} c_i \\ &\leq \sum_{i=1}^{n-(k+\ell)} c_i - \frac{1}{2} c_{n-(k+\ell)} + \sum_{i=n-(k+\ell)+1}^{n-k} 0 + \frac{1}{2} \sum_{i=(n-k)+1}^{n} c_i \\ &= cv^F - \frac{1}{2} c_{n-(k+\ell)} \leq d + \frac{1}{2} - \frac{1}{2} c_{n-(k+\ell)}. \end{aligned}$$

In case $c_{n-(k+\ell)} \geq 1$ it follows that $cv \leq d$. Therefore consider the case $c_{n-(k+\ell)} = 0$. Then we have that $c_i = 0$ for all $i \geq n-(k+\ell)$. In this case cv^F is integral and $cv^F < d+1$ implies $cv^F \leq d$. So $cv \leq cv^F \leq d$ follows, which completes the proof. ☐

Using Lemma 15 we can establish the following lower bound on the rank of $\partial\mathrm{M}$ for A_n.

Theorem 4 (Lower bound for A_n)**.** *Let* M *be admissible and let* $\ell \in \mathbb{N}$ *such that* $A_n^{k+\ell} \subseteq \mathrm{M}(A_n^k)$ *for all* $n, k \in \mathbb{N}$ *with* $k + \ell \leq n$. *If* $n \geq k + 2\ell + 1$, *then* $rk_{\partial\mathrm{M}}(A_n) \geq \left\lfloor \frac{n-1}{2\ell+1} \right\rfloor$.

Proof. We will show the $A_n^{1+k(2\ell+1)} \subseteq (\partial\mathrm{M})^k(A_n)$ as long as $n \geq k + 2\ell + 1$. The proof is by induction on k. Let $k = 1$, then $A_n^{1+2\ell+1} \subseteq \partial\mathrm{M}(A_n^1) = \partial\mathrm{M}(A_n)$ by Lemma 15. Therefore consider $k > 1$. Now $(\partial\mathrm{M})^k(A_n) = \partial\mathrm{M}((\partial\mathrm{M})^{k-1}(A_n)) \supseteq \partial\mathrm{M}(A_n^{1+(k-1)(2\ell+1)}) \supseteq A_n^{1+k(2\ell+1)}$, where the first inclusion follows by induction and the second inclusion by Lemma 15 again. Thus $(\partial\mathrm{M})^k(A_n) \neq \emptyset$ as long as $1 + k(2\ell+1) \leq n$, which is the case as long as $k \leq \left\lfloor \frac{n-1}{2\ell+1} \right\rfloor$ and we can thus conclude $\mathrm{rk}_{\partial\mathrm{M}}(A_n) \geq \left\lfloor \frac{n-1}{2\ell+1} \right\rfloor$. ☐

For $\mathrm{M} \in \{\mathrm{GC}, \mathrm{SC}, \mathrm{N}_0, \mathrm{N}, \mathrm{N}_+\}$ we have that $\ell = 1$ (see [19]) and therefore we obtain the following corollary.

Corollary 1. *Let* $\mathrm{M} \in \{\mathrm{GC}, \mathrm{N}_0, \mathrm{N}, \mathrm{N}_+, \mathrm{SC}\}$ *and* $n \in \mathbb{N}$ *with* $n \geq 4$. *Then* $rk_{\partial\mathrm{M}}(A_n) \geq \left\lfloor \frac{n-1}{3} \right\rfloor$.

We can also derive an upper bound on the rank of A_n as follows.

Proposition 16 (Upper bound for A_n)**.** *Let* M *be admissible and* $n \in \mathbb{N}$. *Then* $rk_{\partial\mathrm{M}}(A_n) \leq n - 2$.

Proof. For $n \leq 3$, observe that the arguments presented in Example 1 for the case of ∂SC would be valid for any admissible cutting plane operator. Thus, the result holds for $n = 1$.

For $n \geq 4$, the proof is by induction on n. Consider $A_n \cap \{x_i = l\} \cong A_{n-1}$ for $(i, l) \in [n] \times \{0, 1\}$. Then after $n-3$ applications of ∂M, by induction we have $(\partial\mathrm{M})^{(n-3)}(A_n \cap \{x_i = l\}) = \emptyset$. As $(i, l) \in [n] \times \{0, 1\}$ was arbitrary we obtain that $x_i < 1$ and $x_i > 0$ are valid for $(\partial\mathrm{M})^{(n-3)}(A_n)$. Another application of ∂M suffices to derive $x_i \leq 0$ and $x_i \geq 1$ and thus $(\partial\mathrm{M})^{(n-2)}(A_n) = \emptyset$ follows. □

5 $\mathbb{V}$-closures for well-known and structured problems.

We first establish a useful lemma which holds for any ∂M with M being admissible. The lemma is analogous to Lemma 1.5 in [16].

Lemma 17. *Let* M *be admissible and let* $P \subseteq [0, 1]^n$ *be a closed convex set with* $(c, d) \in \mathbb{Z}_+^{n+1}$. *If* $cx \leq d$ *is valid for* $P \cap \{x_i = 1\}$ *for every* $i \in [n]$ *with* $c_i > 0$, *then* $cx \leq d$ *is valid for* $\partial\mathrm{M}(P)$.

Proof. Clearly, $cx \leq d$ is valid for P_I: if $x \in P \cap \mathbb{Z}^n$ is non-zero, then there exists an $i \in [n]$ with $x_i = 1$, otherwise $cx \leq d$ is trivially satisfied.

We claim that $cx \leq d$ is valid for $\partial\mathrm{M}(P)$. Let $Q := P \cap \{cx \geq d + 1\}$ and observe that $Q \cap \{x_i = 1\} = \emptyset$ for any $i \in [n]$ with $c_i > 0$. Therefore by the coordinating rounding property of admissible operators, we have that $\mathrm{M}(Q) \subseteq \bigcap_{i \in [n] : c_i > 0} \{x_i = 0\}$. By definition of Q we also have that $\mathrm{M}(Q) \subseteq \{cx \geq d + 1\}$. Since $c \geq 0$ and $d \geq 0$ we deduce $\mathrm{M}(Q) = \emptyset$ and the claim follows. □

5.1 Monotone polytopes

The following theorem is a direct consequence of Lemma 17 and follows in a similar fashion as Lemma 2.7 in [5] or Lemma 2.14 in [16].

Theorem 5. *Let* M *be admissible. Further, let* $P \subseteq [0, 1]^n$ *be a polytope and* $(c, d) \in \mathbb{Z}_+^{n+1}$ *such that* $cx \leq d$ *is valid for* $P \cap F$ *whenever* F *is an* $(n - k)$*-dimensional face of* $[0, 1]^n$ *obtained by fixing coordinates to* 1. *Then* $cx \leq d$ *is valid* $(\partial\mathrm{M})^k(P)$.

Proof. The proof is by induction on k, the number of coordinates fixed to obtain a $n - k$ dimensional face. For $k = 1$ the assertion follows with Lemma 17. Therefore let $k > 1$. Define $Q_i = P \cap \{x_i = 1\}$ for all $i \in [n]$. Then $cx \leq d$ is valid for $Q_i \cap \tilde{F}$ whenever $\tilde{F}$ is an $(n - 1) - (k - 1)$-dimensional face of $[0, 1]^{n-1}$ fixing $k - 1$ coordinates to 1 and i is not one of those coordinates. We can apply the induction hypothesis obtaining that $cx \leq d$ is valid for $(\partial\mathrm{M})^{k-1}(Q_i)$ for all $i \in [n]$. By homogeneity of ∂M we obtain $(\partial\mathrm{M})^{k-1}(Q_i) = (\partial\mathrm{M})^{k-1}(P) \cap \{x_i = 1\}$ for all $i \in [n]$. Applying Lemma 17 once more yields that $cx \leq d$ is valid for $(\partial\mathrm{M})^k(P)$. □

We call a polytope $P \subseteq [0, 1]^n$ *monotone* if $x \in P$, $y \in [0, 1]^n$, and $y \leq x$ (coordinate-wise) implies $y \in P$. We can derive the following corollary from Theorem 5 which is the analog to Lemma 2.7 in [5].

Corollary 2. *Let* M *be admissible and let* $P \subseteq [0, 1]^n$ *be a monotone polytope with* $\max_{x \in P_I} ex = k$. *Then* $rk_{\partial\mathrm{M}}(P) \leq k + 1$.

Proof. Observe that since P is monotone, so is P_I and thus P_I possesses an inequality description $P = \{x \in [0, 1]^n \mid Ax \leq b\}$ with $A \in \mathbb{Z}_+^{m \times n}$ and $b \in \mathbb{Z}_+^n$ for some $m \in \mathbb{N}$. Therefore it suffices to consider inequalities $cx \leq d$ valid for P_I with $c, d \geq 0$. As $\max_{x \in P_I} ex = k$ and P is monotone, we claim that $P \cap F = \emptyset$ whenever F is an $n - (k + 1)$ dimensional face of $[0, 1]^n$ obtained by fixing $k + 1$ coordinates to

1. Assume by contradiction that $x \in P \cap F \neq \emptyset$. As $P \cap F$ is monotone, the point obtained by setting all fractional entries of x to 0 is contained in $P_I \cap F$ which is a contradiction to $\max_{x \in P_I} ex = k$. Therefore $cx \leq d$ is valid for all $P \cap F$ with F being an $n - (k+1)$ dimensional face of $[0,1]^n$ obtained by fixing $k+1$ coordinates to 1. The result follows now by using Theorem 5. □

5.2 Stable set polytope

Given a graph $G := (V, E)$, the fractional stable set polytope of G is given by

$$\text{FSTAB}(G) := \{x \in [0,1]^n \mid x_u + x_v \leq 1 \ \forall (u,v) \in E\}.$$

Now Lemma 17 can be used to prove the following result.

Theorem 6. *Clique Inequalities, odd hole inequalities, odd anti-hole inequalities, and odd wheel inequalities are valid for $\partial\mathrm{M}(FSTAB(G))$ with M being an admissible operator.*

Proof.

1. Let $H(V, E)$ be an induced clique. Then the clique inequality is

   $$\sum_{u \in V} x_u \leq 1.$$

   Now for every vertex v in V, fixing $x_v = 1$ in the system

   $$P^0 = \{x \in [0,1]^{|V|} \mid x_u + x_v \leq 1 \ \forall (u,v) \in E\},$$

   implies that $x_u = 0$ for $u \neq v$. Thus, the clique inequality is valid for $P^0 \cap \{x_v = 1\} \ \forall v \in V$. Now by Lemma 17 the result follows.

2. Odd hole inequalities are GC inequalities: Add all the inequalities of the form $x_u + x_v \leq 1$ along the odd hole, divide by 2, and the round down the right-hand-side. Therefore, odd hole inequalities are valid for $\partial\mathrm{M}$.

3. Let $H(V, E)$ be an induced graph which is a complement of a odd hole with $|V| \geq 5$. Then the odd anti-hole inequality is

   $$\sum_{u \in V} x_u \leq 2.$$

   Otherwise, for every vertex v in V, fixing $x_v = 1$ in the system

   $$P^0 = \{x \in [0,1]^{|V|} \mid x_u + x_v \leq 1 \ \forall (u,v) \in E\},$$

   implies that $x_u = 0$ for all u except the neighbors of vertex v in the complement graph. Moreover, the two neighbors of v in the complement graph are neighbors of each other in H (since $|V| \geq 5$). Thus, $\max \sum_{u \in V} x_u = 2$ for $x \in P^0 \cap \{x_v = 1\}$. Now by Lemma 17 the result follows.

4. Let $H(\{0, \ldots, n\}, E)$ be an induced graph which is a odd wheel, i.e. n is odd, the vertices 1 through n form a hole and the vertex 0 is a neighbor to all other vertices. Then the odd wheel inequality is

   $$\sum_{i=1}^{n} x_i + \frac{n-1}{2} x_0 \leq \frac{n-1}{2}.$$

   Now for the vertex 0, fixing $x_0 = 1$ in the system

   $$P^0 = \{x \in [0,1]^{n+1} \mid x_u + x_v \leq 1 \ \forall (u,v) \in E\},$$

   implies that $x_u = 0$ for $u \in \{1, \ldots, n\}$. Therefore, $\max \sum_{i=1}^{n} x_i + \frac{n-1}{2} x_0 = \frac{n-1}{2}$ for $x \in P^0 \cap \{x_0 = 1\}$.

On fixing $x_1 = 1$ in P^0, we obtain that $x_0 = 0$, $x_2 = 0$, $x_n = 0$ and therefore the system P^0 reduces to

$$x_k + x_{k+1} \leq 1 \; \forall k \in \{2, ..., n-2\} \tag{15}$$
$$0 \leq x_k \leq 1 \; \forall k \in \{2, ..., n-2\}. \tag{16}$$

Now observe that the constraint set (15) is totally unimodular. Therefore, $\max \sum_{i=1}^n x_i + \frac{n-1}{2} x_0 = \frac{n-1}{2}$ for $x \in P^0 \cap \{x_1 = 1\}$. Similarly, $\max \sum_{i=1}^n x_i + \frac{n-1}{2} x_0 = \frac{n-1}{2}$ for $x \in P^0 \cap \{x_v = 1\}$ for $v \in \{2, \dots, n\}$. Now by Lemma 17 the result follows.

□

5.3 The traveling salesman problem

So far we have seen that transitioning from a general cutting-plane procedure M to its $\mathbb{V}$-scheme, ∂M, can result in a significantly lower rank for valid inequalities, potentially making them accessible in a small number of rounds. However, we will now show that the rank of (the subtour elimination relaxation of) the traveling salesman polytope remains high, even when using $\mathbb{V}$-schemes of strong operators such as SC or N_+. For $n \in \mathbb{N}$, let $G = (V, E)$ be the complete graph on n vertices and $H_n \subseteq [0, 1]^n$ be the polytope given by (see [5] for more details)

$$\begin{aligned} x(\delta(\{v\})) &= 2 && \forall\, v \in V \\ x(E(W)) &\leq |W| - 1 && \forall\, \emptyset \subsetneq W \subsetneq V \\ x_e &\in [0, 1] && \forall e \in E, \end{aligned}$$

where for a given node v, $x(\delta(\{v\}))$ is the sum of the components of the vector x corresponding to edges incident to the node v and for any subset W of V, $E(W)$ is the sum of the components of the vector x corresponding to edges which are incident to nodes contained only in W. Note that the dimension of H_n is $\Theta(n^2)$. We obtain the following statement which is the analog to [5, Theorem 4.1]. A similar result for the admissible systems M in general can be found in full-length version of [19].

Theorem 7. *Let* $\mathrm{M} \in \{\mathrm{GC}, \mathrm{N}_0, \mathrm{N}, \mathrm{N}_+, \mathrm{SC}\}$*. For* $n \in \mathbb{N}$ *and* H_n *as defined above we have* $rk_{\partial \mathrm{M}}(H_n) \in \Theta(n)$*. In particular* $rk_{\partial \mathrm{M}}(H_n) \in \Theta(\sqrt{dim(P)})$*.*

Proof. We first establish the lower bound. As shown in [3] or [5, Theorem 4.1], there exists an embedding

$$f : A_{\lfloor n/8 \rfloor} \hookrightarrow H_n,$$

consisting of coordinate flips and coordinate duplications only, such that $f(\frac{1}{2}e) \in H_n \setminus (H_n)_I$. Since ∂M is almost admissible, we have that ∂M commutes with f. We obtain

$$f(\frac{1}{2}e) \in f(\partial \mathrm{M}^k(A_{\lfloor n/8 \rfloor})) = \partial \mathrm{M}^k(f(A_{\lfloor n/8 \rfloor})) \subseteq \partial \mathrm{M}^k(H_n),$$

for $k < \mathrm{rk}_{\partial \mathrm{M}}(A_{\lfloor n/8 \rfloor})$ and thus $\mathrm{rk}_{\partial \mathrm{M}}(H_n) \geq \mathrm{rk}_{\partial \mathrm{M}}(A_{\lfloor n/8 \rfloor}) \in \Omega(n)$ by Corollary 1.

For the upper bound, observe that H_n is a face of T_n given by

$$\begin{aligned} x(\delta(\{v\}) &\leq 2 && \forall\, v \in V \\ x(E(W)) &\leq |W| - 1 && \forall\, \emptyset \subsetneq W \subsetneq V \\ x_e &\in [0, 1] && \forall e \in E. \end{aligned}$$

(see [5] for details). As T_n is given by a system of inequalities of the form $Ax \leq b$ with non-negative coefficients, it follows that T_n is a monotone polytope. Furthermore, we can conclude that $\max_{x \in (T_n)_I} ex \leq n$ so that we can apply Corollary 2. We obtain that $\mathrm{rk}_{\partial \mathrm{M}}(H_n) \leq \mathrm{rk}_{\partial \mathrm{M}}(T_n) \leq n + 1$ which finishes the proof. □

The same result can be shown to hold for the asymmetric TSP problem (see [3] and [5]).

5.4 General polytopes in $\mathbb{R}^2$

The GC rank of valid inequalities for polytopes in $\mathbb{R}^2$ can be arbitrarily high; see example in [17]. The SC rank of valid inequalities for polytopes in $\mathbb{R}^2$ can be at least 2; A_2 is an example where the split rank is 2 and the instance is infeasible and see [12] for an example where the instance is feasible and the split rank is at least 2.

However, ∂GC is significantly stronger as shown next.

Theorem 8. *Let P be a rational polytope in $\mathbb{R}^2$. Then $\partial\text{GC}(P) = P_I$.*

Proof. The proof is divided into various cases based on the dimension of P_I.

1. $\dim(\text{conv}(P_I)) = 2$: We will illustrate that every facet-defining inequality can be obtained using the ∂GC operator. In this case, every facet-defining inequality $cx \leq d$ satisfies at least two integer points belonging to P_I at equality. Let $Q := P \cap \{x \in \mathbb{R}^2 \mid cx \geq d\}$. We assume that $Q \nsubseteq \{cx < d+1\}$, since otherwise $cx \leq d$ is a GC cut. Now observe that: (i) Q is a lattice-free polytope; (ii) exactly one side of Q contains multiple integer points. This is the side of Q given by the inequality $cx \geq d$. Other sides of Q contain no integer point. Let T be a maximal lattice-free convex set containing Q. By (ii), $cx \geq d$ defines a face of T that contains two or more integer points. Moreover Q is bounded. Therefore it is possible to select T as type 1 or type 2 maximal lattice-free triangle; see [13] for definition of these triangle and see [10] for construction of this triangle. Since T is a triangle of type 1 or type 2, it is contained in two sets of the form $\{\pi^1_0 \leq \pi^1 x < \pi^1_0 + 1\}$ and $\{\pi^2_0 \leq \pi^2 x \leq \pi^2_0 + 1\}$ where $\pi^1, \pi^2 \in \mathbb{Z}^2$ and $\pi^1_0, \pi^2_0 \in \mathbb{Z}$; see [9]. Moreover it is straightforward to verify that π^1 and π^2 can be selected such that (i) $\pi^1 = c$, $\pi^1_0 = d$ and (ii) there are two integer points x^1 and x^2 belonging to P that satisfy $cx = d$ and also satisfy $\pi^2 x^1 = \pi^2_0$ and $\pi^2 x^2 = \pi^2_0 + 1$. (See Figure 2.) Therefore $Q \cap \{cx \geq d+1\} \subseteq T \cap \{cx \geq d+1\} \subseteq \{\pi^2_0 \leq \pi^2 x \leq \pi^2_0 + 1\}$. Moreover, since the integer

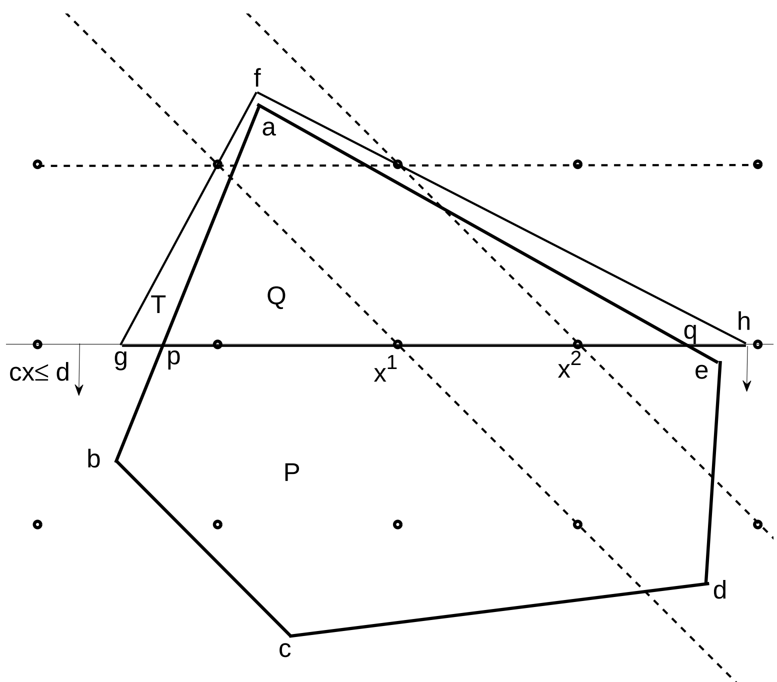

Figure 2: P is the polytope $abcde$, Q is apq, T is fgh

points belonging to the boundary of Q satisfy the condition $cx = d$, we obtain that integer points that satisfy $cx \geq d+1$ and lie on the boundary of the set $\{\pi^2_0 \leq \pi^2 x \leq \pi^2_0 + 1\}$ do not belong to Q. Now by using convexity of Q and the location of integer points x^1 and x^2, we can verify that $Q \cap \{cx \geq d+1\}$ lies in the interior of the set $\{\pi^2_0 \leq \pi^2 x \leq \pi^2_0 + 1\}$. (If there exists a point of Q on the boundary of $\{\pi^2_0 \leq \pi^2 x \leq \pi^2_0 + 1\}$, then a convex combination of this point and either x^1

and x^2 will yield a new integer point belonging to Q that does not satisfy $cx = d$, a contradiction.) Therefore $\mathrm{GC}(Q \cap \{cx \geq d+1\}) = \emptyset$. However, since $Q \cap \{cx \geq d+1\} = P \cap \{cx \geq d+1\}$, we can obtain the facet-defining inequality $cx \leq d$ using the $\partial\mathrm{GC}$ operator applied to P.

2. $\dim(\mathrm{conv}(P_I)) = 1$: Without loss of generality, we may assume that P_I is the set $\{x \in \mathbb{R}^2 \,|\, x_2 = 0, 0 \leq x_1 \leq g\}$ where $g \in \mathbb{Z}$ and $g \geq 1$. Now using arguments similar to the previous case, it is possible to obtain the inequalities $x_2 \leq 0$ and $x_2 \geq 0$ using the $\partial\mathrm{GC}$ operator. We next show that it is possible to obtain the inequality $x_1 + qx_2 \geq 0$ for some $q \in \mathbb{Z}$. There are two cases:

   (a) $\min\{x_1 \,|\, x \in P\} > -1$. In this case, the inequality $x_1 \geq 0$ is a GC inequality.

   (b) $\min\{x_1 \,|\, x \in P\} \leq -1$. Since $(-1, 0)$ does not belong to P, we obtain that all points in the set $(P \cap \{x_1 \leq -1\})$ must either satisfy $x_2 > 0$ or $x_2 < 0$. Without loss of generality, we assume that $(P \cap \{x_1 \leq -1\}) \subseteq (P \cap \{x_2 > 0\})$.

      i. $\max\{x_2 \,|\, x \in P, x_1 \leq -1\} < 1$. In this case, the set $P \cap \{x_1 \leq -1\}$ is contained in the set $\{x_2 < 1\} \cup \{x_2 > 0\}$. Thus, $\mathrm{GC}(P \cap \{x_1 \leq -1\}) = \emptyset$ and thus $x_1 \geq 0$ is a valid inequality for $\partial\mathrm{GC}(P)$.

      ii. $\max\{x_2 \,|\, x \in P, x_1 \leq -1\} \geq 1$. Let $Q := P \cap \{x \in \mathbb{R}^2 \,|\, x_2 \geq 0\}$. We first verify that Q is contained in the union of the sets of form $\{x \in \mathbb{R}^2 \,|\, 0 \leq x_2 < 1\}$ and $\{x \in \mathbb{R}^2 \,|\, 0 \leq x_1 + kx_2 \leq 1\}$ for some $k \in \mathbb{Z}_+$. Since Q contains multiple integer points on the side defined by the inequality $x_2 \geq 0$, is bounded and contains no other integer points, it is contained in a triangle of T type 1 or type 2. Since T contains multiple integer points on the side defined by the inequality $x_2 \geq 0$, it contain two integer points of the form $(t, 1)$ and $(t+1, 1)$ on its two other sides. Moreover since $T \supseteq (Q \cap \{x_1 \leq -1\} \cap \{x_2 \geq 1\}) \neq \emptyset$, these two integer points are of the form $(-k, 1)$ and $(-k+1, 1)$, where $k \in \mathbb{Z}_+$. Therefore, Q is contained in the union of the sets of form $\{x \in \mathbb{R}^2 \,|\, 0 \leq x_2 < 1\}$ and $\{x \in \mathbb{R}^2 \,|\, 0 \leq x_1 + kx_2 \leq 1\}$ for some $k \in \mathbb{Z}_+$.
      Consider the set $V := P \cap \{x \in \mathbb{R}^2 \,|\, x_1 + (k-2)x_2 \leq -1\}$. (See Figure 3.) Then we can verify the following.

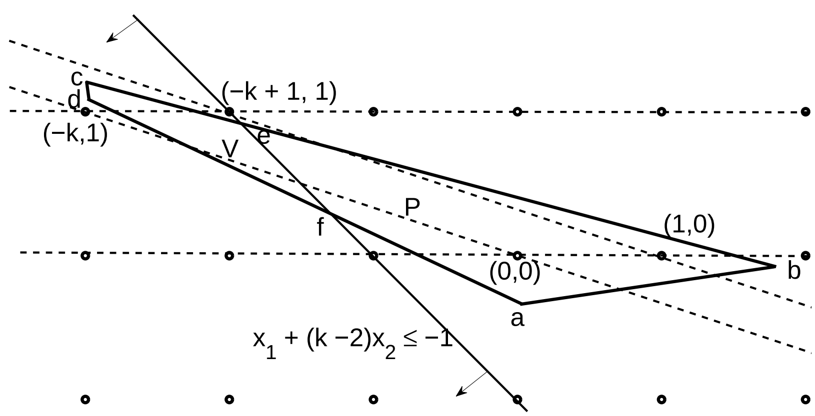

Figure 3: P is the polytope $abcd$, V is $ecdf$

      A. $\min\{x_2 \,|\, x \in V\} > 0$. Observe that since Q is contained in the union of the sets of form $\{x \in \mathbb{R}^2 \,|\, 0 \leq x_2 \leq 1\}$ and $\{x \in \mathbb{R}^2 \,|\, 0 \leq x_1 + kx_2 \leq 1\}$ and $\max_{x \in P} x_2 \geq 1$, there exists a point $\hat{x} \in P$ satisfying $\hat{x}_1 + k\hat{x}_2 \leq 1$ and $\hat{x}_2 \geq 1$. Therefore, $\hat{x}_1 + (k-2)\hat{x}_2 \leq -1$. Taking a suitable convex combination of this point and $(0, 0) \in P$, we obtain a point $x^1 \in P$ such that $x^1_1 + (k-2)x^1_2 = -1$ and $x^1_2 > 0$. Now assume by contradiction that there is a point $\tilde{x} \in P$ satisfying $\tilde{x}_1 + (k-2)\tilde{x}_2 \leq -1$ and $\tilde{x}_2 \leq 0$. Then taking a suitable convex combination of this point and $(0, 0) \in P$, we obtain a point $x^2 \in P$ such that $x^2_1 + (k-2)x^2_2 = -1$ and $x^2_2 \leq 0$. Finally note that the point $(-1, 0)$ is a convex combination of x^1 and x^2, which is the required contradiction.

B. $\max\{x_1 + kx_2 \,|\, x \in V\} < 1$. Assume by contradiction that $\max\{x_1 + kx_2 \,|\, x \in V\} \geq 1$. Then note that any point $\tilde{x}$ satisfying $\tilde{x}_1 + k\tilde{x}_2 \geq 1$ and $\tilde{x}_1 + (k-2)\tilde{x}_2 \leq -1$ satisfies $\tilde{x}_2 \geq 1$. Observe that $\exists \bar{x} \in Q \subseteq P$ satisfying $\bar{x}_1 + k\bar{x}_2 \leq 1$ and $\bar{x}_2 \geq 1$. By taking a suitable convex combination of $\tilde{x}$ and $\bar{x}$, we obtain a point $\hat{x} \in P$ satisfying $\hat{x}_1 + k\hat{x}_2 = 1$ and $\hat{x}_2 \geq 1$. However, note that the point $(-k+1, 1)$ is a convex combination of $\hat{x}$ and $(1, 0)$, which is the required contradiction.

Thus, $x_2 \geq 1$ and $x_1 + kx_2 \leq 0$ are GC cuts for V. However, now observe that $V \cap \{x_2 \geq 1\} \cap \{x_1 + kx_2 \leq 0\} \subseteq Q \cap \{x_2 \geq 1\} \cap \{x_1 + kx_2 \leq 0\} \subseteq (\{0 \leq x_2 < 1\} \cup \{0 \leq x_1 + kx_2 \leq 1\}) \cap \{x_2 \geq 1\} \cap \{x_1 + kx_2 \leq 0\} = \{x \in \mathbb{R}^2 \mid x = (-k, 1) + \lambda(-k, 1) \mid \lambda \geq 0\}$. Now note that $V \cap \{x \in \mathbb{R}^2 \mid x = (-k, 1) + \lambda(-k, 1) \mid \lambda \geq 0\} = \emptyset$, since otherwise the point $(-k, 1) \in P$. Thus, $V \cap \{x_1 \geq 1\} \cap \{x_1 + kx_2 \leq 0\} \subseteq Q \cap \{x_1 \geq 1\} \cap \{x_1 + kx_2 \leq 0\} = \emptyset$. In other words $\mathrm{GC}(V) = \emptyset$ or $x_1 + (k-2)x_2 \geq 0$ is a valid inequality for $\partial\mathrm{GC}(P)$.

A similar argument shows that an inequality $x_1 + qx_2 \leq g$ (where $q \in \mathbb{Z}$) is valid for ∂GC, completing the proof in this case.

3. $\dim(\mathrm{conv}(P_I)) = 0$: Let $P = \{x \in \mathbb{R}^2 \mid a_i x \leq b_i \; i \in \{1, \ldots, m\}\}$. We may assume that on removing any of the inequalities defining P, the resulting set contains more integer points. If not, we remove such inequalities from the description of P and call the resulting polyhedron as Q. Note that Q must be a bounded, since $|Q \cap \mathbb{Z}^2| = 1$ and Q is a rational polyhedron. It is sufficient to verify that $\partial\mathrm{GC}(Q) = Q_I$, since $Q \supseteq P$ and $Q_I = P_I = \{u\}$.

   Let $Q = \{x \in \mathbb{R}^2 \mid a_i x \leq b_i \; i \in \{1, \ldots, m\}\}$. Since $Q \neq \emptyset$ and Q is bounded, we have that $m \geq 3$. Select an integer point v such that $a_1 v > b_1$ and $a_i v \leq b_i$ for $i \in \{2, \ldots, m\}$. (This is possible due to the construction of Q.) Observe $\mathrm{conv}(\{v\} \cup Q)$ contains a nonzero, but finite number of integer points not belonging to Q. Moreover all these integer points satisfy $a_1 x > b_1$ and $a_i x \leq b_i$ for $i \in \{2, \ldots, m\}$. Now it is possible to select a integer point v^1 from this set such that $\mathrm{conv}(\{v^1\} \cup Q) \cap \mathbb{Z}^2 = \{v^1, u\}$. Let $Q^1 := \mathrm{conv}(\{v^1\} \cup Q)$. Similarly it is possible to select $v^2 \in \mathbb{Z}^2$ that satisfies $a_2 v^2 > b_2$ and $a_i v^2 \leq b_i \; \forall i \in \{1, \ldots, m\} \setminus \{2\}$, such that $Q^2 \cap \mathbb{Z}^2 = \{v^2, u\}$ where $Q^2 = \mathrm{conv}(\{v^2\} \cup Q)$.

   From the previous case, we know that $\partial\mathrm{GC}(Q^1) = P^1_I$ and $\partial\mathrm{GC}(P^2) = Q^2_I$. Moreover, we have $Q \subseteq Q^1 \cap Q^2$. Thus, $\partial\mathrm{GC}(Q) \subseteq \partial\mathrm{GC}(Q^1 \cap Q^2) \subseteq \partial\mathrm{GC}(Q^1) \cap \partial\mathrm{GC}(Q^2) = Q^1_I \cap Q^2_I = \{u\}$, where the last equality follows from the fact that $v^1 \neq v^2$ and therefore the intersection of the line segments $\mathrm{conv}\{v^1, u\}$ and $\mathrm{conv}\{v^2, u\}$ is $\{u\}$.

4. $P \cap \mathbb{Z}^2 = \emptyset$: Let $P = \{x \in \mathbb{R}^2 \mid a_i x \leq b_i \; i \in \{1, \ldots, m\}\}$. As before we may assume that on removing any of the inequalities defining P, the resulting set contains more integer points. If not, we remove such inequalities from the description of P and call the resulting polyhedron as Q. It is sufficient to verify that $\partial\mathrm{GC}(Q) = Q_I$, since $Q \supseteq P$ and $Q_I = P_I$.

   Note first that if Q is not bounded, then it must be contained in a set of the form $\{x \in \mathbb{R}^2 \,|\, a_0 \leq ax \leq a_0 + 1\}$ ([13]) where $a \in \mathbb{Z}^2$. In this case, observe that $Q \cap \{x \in \mathbb{R}^2 \,|\, ax \geq a_0 + 1\}$ is a line segment strictly contained between two integer points. In this case it is easily verified that $\mathrm{GC}(Q \cap \{x \in \mathbb{R}^2 \,|\, ax \geq a_0 + 1\}) = \emptyset$ and thus $ax \leq a_0$ is a valid inequality for $\partial\mathrm{GC}(P)$. Similarly, $ax \geq a_0 + 1$ is a valid inequality for $\partial\mathrm{GC}(P)$, completing the proof.

   Now consider the case where Q is bounded. Then following the procedure in the previous case it is possible to select integer points v^1, v^2 such that $v^1 \neq v^2$ and $Q^i := \mathrm{conv}(\{v^i\} \cup Q) \cap \mathbb{Z}^2 = \{v^i\}$, $i \in \{1, 2\}$. Therefore, $\partial\mathrm{GC}(Q) \subseteq \partial\mathrm{GC}(Q^1 \cap Q^2) \subseteq \partial\mathrm{GC}(Q^1) \cap \partial\mathrm{GC}(Q^2) = \{v^1\} \cap \{v^2\} = \emptyset$ where the second last equality follows from the previous case.

□

6 Concluding remarks

In this paper, we consider a new paradigm for generating cutting-planes. Rather than *computing* a cutting-plane we suppose that the cutting-plane is given, either by a *deliberate construction* or guessed in some other way and then we *verify* its validity using a regular cutting-plane procedure. We have shown that cutting-planes obtained via the verification scheme can be very strong, significantly exceeding the capabilities of the regular cutting-plane procedure. This superior strength is illustrated, for example, in Theorem 4, Theorem 6, Figure 1, Theorem 14, Proposition 16, Theorem 5, Theorem 6, Theorem 7 and Theorem 8. On the other hand, we also show that the verification scheme is not unrealistically strong, as illustrated by Theorem 4 and Theorem 7.

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