On the Computational Complexity of Membership Problems for the Completely Positive Cone and its Dual

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Abstract

Copositive programming has become a useful tool in dealing with all sorts of optimisation problems. It has however been shown by Murty and Kabadi [K.G. Murty and S.N. Kabadi, Some $\mathcal{NP}$-complete problems in quadratic and nonlinear programming, *Mathematical Programming*, 39, no.2:117–129, 1987] that the strong membership problem for the copositive cone, that is deciding whether or not a given matrix is in the copositive cone, is a co-$\mathcal{NP}$-complete problem. From this it has long been assumed that this implies that the question of whether or not the strong membership problem for the dual of the copositive cone, the completely positive cone, is also an $\mathcal{NP}$-hard problem. However, the technical details for this have not previously been looked at to confirm that this is true. In this paper it is proven that the strong membership problem for the completely positive cone is indeed $\mathcal{NP}$-hard. Furthermore, it is shown that even the weak membership problems for both of these cones are $\mathcal{NP}$-hard. We also present an alternative proof of the $\mathcal{NP}$-hardness of the strong membership problem for the copositive cone.

1 Introduction

Copositive programming has, in relatively recent years, been shown to be a useful tool for dealing with all kinds of combinatorial optimisation problems. For example, de Klerk and Pasechnik (2002) [7] formulated the stable set problem as a copositive program. Some other papers on copositive programming and combinatorial optimisation include the one by Gvozdenović and Laurent (2008) [12] who formulated the problem of finding the chromatic number of a graph as a copositive program and one by Burer (2009) [6] who showed that every quadratic program with both binary and linear constraints can be rewritten as a copositive program. For a more detailed overview of the field of copositive programming we refer the reader to surveys by Dür (2010) [9], Hiriart-Urruty and Seeger (2010) [13] and Berman and Shaked-Monderer (2003) [2].

A matrix, X, is defined to be copositive if for all nonnegative vectors y we have that $y^{\mathsf{T}} X y \geq 0$. We can in fact limit y such that its norm is equal to one.

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A matrix, X, is defined to be completely positive if there exists a nonnegative matrix B such that $X = BB^{\mathsf{T}}$.

We denote the sets of copositive and completely positive matrices by $\mathcal{C}$ and $\mathcal{C}^*$ respectively. These sets have been shown to be proper cones (i.e. closed, pointed, convex and full-dimensional) and to be the duals of each other, a proof of which is provided by Berman and Shaked-Monderer (2003) [2].

It has been proven by Murty and Kabadi (1987) [16] that the strong membership problem for the copositive cone is co-$\mathcal{NP}$-complete. As a result no polynomial time algorithms will exist for solving general copositive programs unless $\mathcal{P}= \mathcal{NP}$. Note that when we refer to a *polynomial (exponential) time* algorithm we mean that the maximum or worst-case computation time of the algorithm is polynomial (exponential) in the encoding lengths of the inputs. Good introductions to complexity theory are provided by Garey and Johnson (1979) [10] and Grötschel, Lovász, and Schrijver (1993) [11]. In this paper we shall assume that the reader is familiar with basic complexity theory, including an understanding of encoding lengths. Several algorithms to deal with copositive programs as well as its dual problems, the so called completely positive programs, have been introduced, for example the one by Bundfuss and Dur (2009) [5] and the one introduced by Bomze et al. (2011) [3].

Motivated by the result of Murty and Kabadi (1987) [16] we consider the as yet unanswered question of whether the strong membership problem for the completely positive cone is $\mathcal{NP}$-hard. It is commonly believed that this is true and in fact this is to such an extent that many researchers in the field believe it to have already been proven. However this is the first paper to consider the technical details in order to confirm that this is true. It was conjectured by Jarre and Schmallowsky (2009) [14] that strong membership of the completely positive cone is an $\mathcal{NP}$-complete problem. We show that the problem is indeed $\mathcal{NP}$-hard and it is left as an open question as to whether it is also a problem in $\mathcal{NP}$. In fact we show that even weak membership for the completely positive cone is $\mathcal{NP}$-hard. We prove this result in Section 5 by establishing a polynomial time Turing reduction from the stable set problem, a known $\mathcal{NP}$-complete problem (see for example Garey and Johnson (1979) [10, Section 3.1.3]) to the weak membership problem of the completely positive cone. We recall from Grötschel, Lovász, and Schrijver (1993) [11] that for two problems Π_1, Π_2, a *(polynomial time) Turing Reduction* from Π_1 to Π_2 is an algorithm A_1 which solves Π_1 using a hypothetical subroutine A_2 which solves Π_2 such that if A_2 is a polynomial time algorithm then so is A_1. A special case of Turing reductions is the many-one reduction where the algorithm A_2 is only called once. In these reductions it is vital that the encoding length of the input to the algorithm A_2 is polynomial in the input to the algorithm A_1, and providing such polynomial inputs will be the main work of this paper.

We recall that the stable set problem is the problem of whether a graph contains a stable set of a certain size, where a stable set of a graph (also referred to as an independent set or a co-clique) is a subset of vertices from the graph such that no two vertices in the subset are connected by an edge. Introductions to the stable set problem are provided by Bondy and Murty (1976) [4, Chapter 7] and Schrijver (2003) [17, Chapter 64].

In this paper we will also provide an alternative proof to the $\mathcal{NP}$-hardness of the strong membership problem for the copositive cone in Section 4 together with a proof that even the problem of weak membership of the copositive cone is an $\mathcal{NP}$-hard problem.

We will however start off this paper providing an explanation of the used notation in Section 2 followed by some definitions and known results, as well as a technical lemma, in Section 3.

2 Notation

We denote the space of real vectors by $\mathbb{R}^n$, rational vectors by $\mathbb{Q}^n$ and integer vectors by $\mathbb{Z}^n$. $\mathbb{R}^n_+$ denotes the set of nonnegative real vectors whilst $\mathbb{R}^n_{++}$ denotes that of strictly positive real vectors. For the other sets we use equivalent notation. We omit the 'n' if the dimension is equal to one.

The set of symmetric real matrices of order n we denote by $\mathcal{S}^n$ and we shall omit the 'n' if the order is apparent from the context. We denote the identity matrix and the all-ones matrix by I and E respectively.

For the inner product we use the standard dot product when dealing with vectors and the trace inner product when dealing with matrices. The accompanying norms will be the Euclidean and the Frobenius norm respectively, i.e. $\|a\| = \sqrt{\langle a, a\rangle}$. Finally, the boundary of any set K, will be denoted by $bd(K)$.

3 Problems for Convex Sets

We start off this section by recalling several definitions from Grötschel, Lovász, and Schrijver (1993) [11], which are extended for the space of symmetric matrices. In this section we let $\mathcal{X}$ be equal to either $\mathbb{R}^n$ or $\mathcal{S}^n$ and correspondingly let $\mathcal{Q}$ be equal to either $\mathbb{Q}^n$ or $(\mathbb{Q}^{n\times n} \cap \mathcal{S}^n)$ respectively.

Definition 3.1. Let $K \subseteq \mathcal{X}$ and let $\varepsilon > 0$. Then we define,

$$\mathrm{S}\,(K,\varepsilon) := \{x \in \mathcal{X} \mid ||x - y|| \leq \varepsilon \text{ for some } y \in K\},$$
$$\mathrm{S}\,(K,-\varepsilon) := \{x \in \mathcal{X} \mid \mathrm{S}\,(\{x\},\varepsilon) \subseteq K\}.$$

When $K = \{a\}$ we shall write $\mathrm{S}\,(a,\varepsilon) := \mathrm{S}\,(\{a\},\varepsilon)$.

Note that we have the following relation, $\mathrm{S}\,(K,-\varepsilon) \subseteq K \subseteq \mathrm{S}\,(K,\varepsilon)$. Hence $\mathrm{S}\,(K,-\varepsilon)$ and $\mathrm{S}\,(K,\varepsilon)$ can be seen as inner and outer approximations of K respectively.

We now consider the following problems for a set $K \subseteq \mathcal{X}$.

Definition 3.2. The Strong Membership Problem (MEM). Let $K \subseteq \mathcal{X}$ and $y \in \mathcal{Q}$. Then decide either

1. $y \in K$, or
2. $y \notin K$.

Definition 3.3. The Weak Membership Problem (WMEM). Let $K \subseteq \mathcal{X}$, $y \in \mathcal{Q}$ and $\delta \in \mathbb{Q}_{++}$. Then decide either

1. $y \in \mathrm{S}\,(K,\delta)$, or
2. $y \notin \mathrm{S}\,(K,-\delta)$.

Note that $\mathrm{S}(K,\delta) \setminus \mathrm{S}(K,-\delta) \neq \emptyset$, so for some values of y either answer would be valid.

Definition 3.4. The Weak Violation Problem (WVIOL). Let $K \subseteq \mathcal{X}$, $c \in \mathcal{Q}$, $\gamma \in \mathbb{Q}$ and $\varepsilon \in \mathbb{Q}_{++}$. Then decide either

1. $\langle c, x\rangle \leq \gamma + \varepsilon$ for all $x \in \mathrm{S}(K, -\varepsilon)$, or
2. find a vector $y \in \mathrm{S}(K,\varepsilon)$ for which $\langle c, y\rangle \geq \gamma - \varepsilon$.

Note again that for some values of the parameters either answer would be valid.

We now consider what WVIOL tells us about the values of $\langle c, x\rangle$ for $x \in K$, rather than just for x in some approximation of K. However, before we do this we first define a special type of convex body.

Definition 3.5. Consider a convex set $K \subseteq \mathcal{X}$ with the following properties,

1. $N = \dim \mathcal{X}$,
2. $\exists R \in \mathbb{Q}_{++}$ such that $K \subseteq \mathrm{S}(0, R)$, and
3. $\exists r \in \mathbb{Q}_{++}$, $a_0 \in \mathcal{Q}$ such that $\mathrm{S}(a_0, r) \subseteq K$.

Then K is an a_0-centered convex body which is denoted as the quintuple $(K; N, R, r, a_0)$.

Lemma 3.6. *Consider WVIOL with K being a convex body $(K; N, R, r, a_0)$ as defined in Definition 3.5. If we assume that $\varepsilon < r$ then we have the following,*

1. *$\langle c, x\rangle \leq \gamma + \varepsilon$ for all $x \in S(K, -\varepsilon)$ implies that*

$$\langle c, z\rangle \leq \Big(\gamma + \varepsilon - \frac{\varepsilon}{r}\langle c, a_0\rangle\Big)\Big/\Big(1 - \frac{\varepsilon}{r}\Big) \qquad \textit{for all } z \in K.$$

2. *$\exists y \in S(K,\varepsilon)$ for which $\langle c, y\rangle \geq \gamma - \varepsilon$ implies that*

$$\exists z \in K \textit{ such that } \langle c, z\rangle \geq \gamma - (1 + \|c\|)\varepsilon.$$

Proof. We shall prove both points of the theorem separately.

1. Let $z_0 \in K$, then

$$\mathrm{conv}\left(\{z_0\} \cup \mathrm{S}(a_0, r)\right) \subseteq K,$$

where conv($\bullet$) denotes the convex hull.

Therefore if we let $z_\theta = (1-\theta)z_0 + \theta a_0$, we have that $\mathrm{S}(z_\theta, \theta r) \subseteq K$ for all $0 \leq \theta \leq 1$. Hence in particular $z_{\varepsilon/r} \in \mathrm{S}(K, -\varepsilon)$. We now get that

$$\begin{aligned} \langle c, z_0\rangle &= \big(\langle c, z_\theta\rangle - \theta\langle c, a_0\rangle\big)\big/(1-\theta) \qquad \text{for all } 0 \leq \theta < 1 \\ &= \Big(\langle c, z_{\varepsilon/r}\rangle - \frac{\varepsilon}{r}\langle c, a_0\rangle\Big)\Big/\Big(1 - \frac{\varepsilon}{r}\Big) \\ &\leq \Big(\gamma + \varepsilon - \frac{\varepsilon}{r}\langle c, a_0\rangle\Big)\Big/\Big(1 - \frac{\varepsilon}{r}\Big). \end{aligned}$$

2. Let $y \in \mathrm{S}(K, \varepsilon)$ such that $\langle c, y\rangle \geq \gamma - \varepsilon$ and let $z \in \mathrm{S}(y, \varepsilon) \cap K$ $(\neq \emptyset)$, then

$$\begin{aligned}\langle c, z\rangle &\geq \min\{\langle c, u\rangle \mid u \in \mathrm{S}(y, \varepsilon)\}\\ &= \min\{\langle c, y\rangle + \varepsilon\langle c, v\rangle \mid v \in \mathrm{S}(0, 1)\}\\ &= \langle c, y\rangle - \|c\|\varepsilon\\ &\geq \gamma - (1 + \|c\|)\varepsilon.\end{aligned}$$

□

We will now consider how the problems in this section are related to each other. It is immediately apparent that an oracle for MEM would provide us with an oracle for WMEM. For $\mathcal{X} = \mathbb{R}^n$, we now connect WMEM and WVIOL using the following lemma by Yudin and Nemirovski (1976) [18].

Lemma 3.7. *For $\mathcal{X} = \mathbb{R}^n$, there exists an algorithm that solves WVIOL for every quintuple $(K; n, R, r, a_0)$ given by a weak membership oracle. This algorithm is oracle-polynomial time with respect to the input of the parameters in WVIOL and the quintuple.*

It is relatively easy to show that this theorem also holds for $\mathcal{X} = \mathcal{S}^n$. However, when doing this, care must be taken to maintain rationality. One way to do this would be to define the one-to-one mapping "svec" which maps $\mathcal{S}^n$ to $\mathbb{R}^N$, where $N = \dim(\mathcal{S}^n) = \frac{1}{2}n(n+1)$, by stacking the elements of the upper triangle, i.e.

$$\mathrm{svec}(X) = (x_{11},\ x_{12},\ x_{22}, \ldots,\ x_{1k},\ x_{2k}, \ldots,\ x_{kk}, \ldots,\ x_{nn})^{\mathsf{T}}.$$

For any two matrices $C, X \in \mathcal{S}^n$, we then have the following, where "$\circ$" denotes the Hadamard product.

$$\begin{aligned}&\langle C, X\rangle = \langle \mathrm{svec}((2E - I) \circ C), \mathrm{svec}(X)\rangle,\\ &\tfrac{2}{3}\|X\| \leq \|\mathrm{svec}(X)\| \leq \|X\|.\end{aligned}$$

The important things about this mapping are that it is linear, rationality is maintained and the norms are linearly related. (An alternative definition for the svec mapping which often appears in the literature has the off-diagonal elements multiplied by $\sqrt{2}$. Although this has the advantage of maintaining the inner product and norms, we would not maintain rationality, as required.) Using this, along with Lemma 3.7, it is now trivial to prove the following theorem.

Theorem 3.8. *For $\mathcal{X}$ equal to $\mathbb{R}^n$ or $\mathcal{S}^n$, there exists an algorithm that solves WVIOL for every quintuple $(K; N, R, r, a_0)$ given by a weak membership oracle. This algorithm is oracle-polynomial time with respect to the input of the parameters in WVIOL and the quintuple.*

4 The Copositive Cone

If a graph G contains a stable set of size λ then it is easy to see that it contains a stable set of size $t \in \mathbb{Z}_{++}$ for all $t \leq \lambda$. We denote by $\alpha \in \mathbb{Z}_{++}$ the size of the largest stable set of a graph G, referred to as the stability number of G. Note that G contains a stable set of size $t \in \mathbb{Z}_{++}$ if and only if $\alpha \geq t$.

Formulations for the stability number of a graph can be given over the cones of copositive and completely positive matrices. If we let A be the adjacency matrix of a graph G then a few of these formulations are given below,

$$\alpha = \min\left\{\lambda \mid \big((I+A)\lambda - E\big) \in \mathcal{C}\right\}, \tag{1}$$

$$= \max\left\{\langle E, X\rangle \mid \langle I+A, X\rangle = 1, X \in \mathcal{C}^*\right\}, \tag{2}$$

$$= \max\left\{\langle E, X\rangle \mid \langle I+A, X\rangle \leq 1, X \in \mathcal{C}^*\right\}, \tag{3}$$

where (1) and (2) are from a paper by de Klerk and Pasechnik (2002) [7], and (3) can easily be derived from (2). The optimal values of these problems were also shown by de Klerk and Pasechnik to be attained.

It can now be seen that the stable set problem is Turing reducible to the strong copositive membership problem.

Lemma 4.1. *The graph G contains a stable set of size $t \in \mathbb{Z}_{++}$ if and only if*

$$\big((I+A)(t-\tfrac{1}{2}) - E\big) \notin \mathcal{C}.$$

Proof. For any $\lambda \geq \alpha$ we have that $\big((I+A)\lambda - E\big) \in \mathcal{C}$. This can be seen from the fact that $\big((I+A)\alpha - E\big) \in \mathcal{C}$ and $(I+A)$ is nonnegative (so also copositive). Therefore

$$\begin{aligned}\big((I+A)(t-\tfrac{1}{2}) - E\big) \notin \mathcal{C} &\Leftrightarrow \alpha > t - \tfrac{1}{2}\\ &\Leftrightarrow \alpha \geq \left\lceil t - \tfrac{1}{2}\right\rceil = t\\ &\Leftrightarrow \text{There is a stable set of size } t.\end{aligned}$$

□

Theorem 4.2. *The stable set problem is Turing reducible to the strong copositive membership problem with a many-one reduction, and thus the strong membership problem for the copositive cone is $\mathcal{NP}$-hard.*

Proof. This comes from Lemma 4.1 and noting that the encoding length of $\big((I+A)(t-\tfrac{1}{2}) - E\big)$ is polynomial in the encoding length of the stable set problem. □

In order to extend this to weak membership of the copositive cone we provide the following two lemmas.

Lemma 4.3. *Let $G = (V, E)$ be a graph with $|V| = n \in \mathbb{Z}_{++}$, adjacency matrix A and stability number α. We* *define*

$$\begin{aligned} Y_\theta &:= (I+A)\big(\alpha + \theta(n+1-\alpha)\big) - E \\ &= (1-\theta)\big((I+A)\alpha - E\big) + \theta\big((n+1)(I+A) - E\big). \end{aligned}$$

Then we have that $Y_\theta \in S(\mathcal{C}, -\theta)$*, or equivalently* $S(Y_\theta, \theta) \subseteq \mathcal{C}$*, for all* $0 \le \theta \le 1$*.*

Proof. We have that

$$\begin{aligned} &\min\{x^\mathsf{T} Y x \mid Y \in \mathrm{S}(Y_\theta, \theta),\ x \ge 0,\ \|x\| = 1\} \\ &\quad = \min \left\{ \begin{array}{c|l} (1-\theta)x^\mathsf{T}\big((I+A)\alpha - E\big)x + \theta(n+1)x^\mathsf{T} I x & Z \in \mathrm{S}(0,1), \\ \quad + \theta(n+1)x^\mathsf{T} A x - \theta x^\mathsf{T} E x + \theta x^\mathsf{T} Z x & x \ge 0, \\ & \|x\| = 1 \end{array} \right\} \\ &\quad \ge 0 + \theta(n+1) + 0 - \theta n - \theta \\ &\quad = 0. \end{aligned}$$

□

Lemma 4.4. *Define* $Z_\lambda := (I+A)\lambda - E$ *and let* $0 < \varepsilon < 1$*. Then we have that*

$$Z_\lambda \in S(\mathcal{C}, -\varepsilon) \qquad \text{for all } \big(\alpha + \varepsilon n\big) \le \lambda \le (n+1).$$

Proof. From Lemma 4.3 we see that for all $\varepsilon \le \theta \le 1$ we have that $Y_\theta \in \mathrm{S}(\mathcal{C}, -\theta) \subseteq \mathrm{S}(\mathcal{C}, -\varepsilon)$.

Therefore $Z_\lambda \in \mathrm{S}(\mathcal{C}, -\varepsilon)$ for all $\big(\alpha + \varepsilon(n+1-\alpha)\big) \le \lambda \le (n+1)$.

Finally we note that $(n+1-\alpha) \le n$. □

We can now state the following lemma and theorem concerning the weak membership problem for the copositive cone.

Lemma 4.5. *Let* $G = (V, E)$ *be a graph with* $|V| = n \in \mathbb{Z}_{++}$ *and adjacency matrix* A*. Furthermore let*

$$\begin{aligned} Y &= (I+A)\left(t - \tfrac{1}{2}\right) - E, \\ \delta &= 1/(2n+1), \\ K &= \mathcal{C}. \end{aligned}$$

where $n \ge t \in \mathbb{Z}_{++}$*. Then considering the WMEM for these parameters we have that*

1. $Y \in S(K, \delta)$ *would imply that the graph* G *does not contain a stable set of size* t*.*
2. $Y \notin S(K, -\delta)$ *would imply that the graph* G *does contain a stable set of size* t*.*

Proof. We shall proof these results separately,

1. Suppose that $Y \in \mathrm{S}(\mathcal{C}, \delta)$. There must exist $Z \in \mathrm{S}(0,1)$ such that $(Y + \delta Z) \in \mathcal{C}$.

   From Lemma 4.3 (setting $\theta = 1$) we have that

$$((I + A)(n + 1) - E - Z) \in \mathcal{C}.$$

   Then as the copositive cone is convex the following matrix must again be copositive,

$$\begin{aligned}&\frac{\delta}{1+\delta}\Big((I + A)(n + 1) - E - Z\Big) + \left(1 - \frac{\delta}{1+\delta}\right)\Big((I + A)\big(t - \tfrac{1}{2}\big) - E + \delta Z\Big)\\&\quad= (I + A)\left(t - \frac{2t-1}{4(n+1)}\right) - E\end{aligned}$$

   From this we see that $\alpha \leq \left\lfloor t - \frac{2t-1}{4(n+1)} \right\rfloor = t - 1$.

   Therefore the graph does not contain a stable set of size t.

2. Suppose that $Y \notin \mathrm{S}(\mathcal{C}, -\delta)$. From Lemma 4.4 we have that $t - \frac{1}{2} < \alpha + \delta n$.

   Therefore $\alpha \geq \left\lceil t - \frac{1}{2} - \frac{n}{2n+1} \right\rceil = \left\lceil t - \frac{4n+1}{4n+2} \right\rceil = t$, and so the graph does contain a stable set of size t. □

Theorem 4.6. *The stable set problem is Turing reducible to the weak copositive membership problem with a many-one reduction, and thus the weak membership problem for the copositive cone is $\mathcal{NP}$-hard.*

Proof. This comes from Lemma 4.1 and noting that the encoding lengths of Y and δ from this lemma are polynomial in the encoding length of the stable set problem. □

5 The Completely Positive Cone

In this section we consider the weak membership of the completely positive cone. In order to do this, rather than reformulating to a WMEM problem as we did for the copositive case, this time we reformulate to a WVIOL problem, using the following quintuple.

Lemma 5.1. *Consider a graph G with n vertices and adjacency matrix, A. Let $K = \{X \in \mathcal{C}^* \mid \langle I + A, X\rangle \leq 1\}$ and furthermore set*

$$\begin{aligned}N &= \tfrac{1}{2}n(n+1),\\ R &= 1,\\ r &= \frac{1}{4n^2},\\ A_0 &= \tfrac{1}{2n}I + \tfrac{1}{4n^2}E,\end{aligned}$$

then the quintuple $(K; N, R, r, A_0)$ is an A_0-centered convex body as defined in Definition 3.5.

Proof. First we will show that $K \subseteq \mathrm{S}(0, R)$. Note that $(I + A)$ is in the interior of the copositive cone, therefore,

$$\begin{aligned}\max\{\|X\| \mid X \in K\} &= \max\left\{\|X\| \;\middle|\; X \in \operatorname{conv}\left(\{0\} \cup \{bb^{\mathsf{T}} \mid b \in \mathbb{R}^n_+,\ b^{\mathsf{T}}(I + A)b = 1\}\right)\right\} \\ &= \max\left\{\|bb^{\mathsf{T}}\| \;\middle|\; b \in \mathbb{R}^n_+,\ b^{\mathsf{T}}(I + A)b = 1\right\} \\ &\leq 1 = R.\end{aligned}$$

Next we will show that $S(A_0, r) \subseteq K$. Let $X \in \mathrm{S}(A_0, r)$, then we get the following entrywise inequalities for X,

$$0 \leq \frac{1}{2n} I = A_0 - Er \leq X \leq A_0 + Er = \frac{1}{2n} I + \frac{1}{2n^2} E.$$

Hence it now follows that

$$\langle I + A, X\rangle \leq \langle E, X\rangle \leq \frac{1}{2n}\langle E, I\rangle + \frac{1}{2n^2}\langle E, E\rangle = \frac{n}{2n} + \frac{n^2}{2n^2} = 1.$$

What is left to show now is that $X \in \mathcal{C}^*$. To do this we use a result from Kaykobad (1987) [15] that says that if we have an entrywise, nonnegative matrix $Y \in \mathcal{S}^n$, which is diagonally dominant, i.e. $(Y)_{ii} \geq \sum_{j\neq i}(Y)_{ij}$ for all $i = 1, \ldots, n$, then $Y \in \mathcal{C}^*$.

Note that we have already shown that $X \geq 0$. Now defining e and e_i to be respectively the all-ones vector and the unit vector such that $(e_i)_j = \delta_{ij}$, we observe the following, which implies that X must be completely positive.

$$\begin{aligned}(X)_{ii} - \textstyle\sum_{j\neq i}(X)_{ij} &= 2e_i^{\mathsf{T}} X e_i - e^{\mathsf{T}} X e_i \\ &\geq 2e_i^{\mathsf{T}}\left(\frac{1}{2n} I\right) e_i - e^{\mathsf{T}}\left(\frac{1}{2n} I + \frac{1}{2n^2} E\right) e_i \\ &= \frac{1}{n} - \frac{1}{2n} - \frac{1}{2n} = 0.\end{aligned}$$ □

We can now state the following theorem from which the $\mathcal{NP}$-hardness of the strong membership problem for the completely copositive cone will follow.

Theorem 5.2. *Consider a graph G with n vertices, let $t \in \mathbb{Z}_{++}$, let $(K; N, R, r, A_0)$ be the A_0-centered convex body as described in Lemma 5.1 and set*

$$\begin{aligned}C &= E, \\ \gamma &= t - \tfrac{1}{2}, \\ \varepsilon &= \frac{1}{16n^2 t}.\end{aligned}$$

Considering the WVIOL for these parameters we have that

1. *$\langle C, X\rangle \leq \gamma + \varepsilon$ for all $X \in S(K, -\varepsilon)$ would imply that the graph does not contain a stable set of size t.*

2. $\exists Y \in S(K,\varepsilon)$ such that $\langle C,Y\rangle \geq \gamma - \varepsilon$ would imply that the graph does contain a stable set of size t.

From this we then have that the stable set problem is Turing reducible to the problem of weak membership of K.

Proof. Using Lemma 3.6 we look at what the results of WVIOL for our choice of parameters would mean. A major point in the implications is that $\alpha \in \mathbb{Z}_{++}$ and we recall from (3) that

$$\alpha = \max\{\langle C,X\rangle \mid X \in K\}.$$

1. Let $\langle C,X\rangle \leq \gamma+\varepsilon$, for all $X \in \mathrm{S}(K,-\varepsilon)$. Then for all $Z \in K$ we have that

$$\begin{aligned}\langle C,Z\rangle &\leq \left(\gamma+\varepsilon-\frac{\varepsilon}{r}\langle C,A_0\rangle\right)\Big/\left(1-\frac{\varepsilon}{r}\right)\\ &= t-\tfrac{1}{4}-\frac{4n^2-1}{16n^2t-4n^2}\\ &\leq t-\tfrac{1}{4}.\end{aligned}$$

   Therefore $\alpha \leq \left\lfloor t-\frac{1}{4}\right\rfloor = t-1$ and so the graph does not contain a stable set of size t.

2. Assume $\exists Y \in \mathrm{S}(K,\varepsilon)$ such that $\langle C,Y\rangle \geq \gamma-\varepsilon$. Then again by Lemma 3.6, $\exists Z \in K$ such that

$$\begin{aligned}\langle C,Z\rangle &\geq \gamma-(1+\|C\|)\varepsilon\\ &= t-\frac{5}{8}+\frac{2n^2t-n-1}{16n^2t}\\ &\geq t-\frac{5}{8}.\end{aligned}$$

   Therefore $\alpha \geq \left\lceil t-\frac{5}{8}\right\rceil = t$ and so the graph does contain a stable set of size t.

It can now be seen that the stable set problem is Turing reducible to the problem of weak membership of K by noting that the encoding lengths of $(K;N,R,r,A_0)$, C, γ and ε are polynomial in the encoding length of the stable set problem □

We now get the following result for strong membership of the completely positive cone.

Theorem 5.3. *The strong membership problem for the completely positive cone is $\mathcal{NP}$-hard.*

Proof. From Theorems 3.8 and 5.2, and noting that the encoding length of $(I+A)$ is polynomial in the encoding length of the stable set problem, it follows that the stable set problem is Turing reducible to the problem of weak membership of K from Lemma 5.1, which in turn is many-one reducible to the problem of strong membership of the completely positive cone. □

In order to extend this to weak membership of the completely positive cone we need a way of solving the weak membership problem for K from Lemma 5.1 given a weak membership oracle for the completely positive cone.

Lemma 5.4. *We consider K from Lemma 5.1, let $\delta \in \mathbb{Q}_{++}$ and define $\mathcal{H} := \{X \in \mathcal{S} \mid \langle I + A, X\rangle \leq 1\}$. Then we have that*

1. *$S(K, -\delta) = S(\mathcal{H}, -\delta) \cap S(\mathcal{C}^*, -\delta) \subseteq \mathcal{H} \cap S\big(\mathcal{C}^*, -\delta/(1 + n^2)\big)$*
2. *$S(K, \delta) \supseteq \mathcal{H} \cap S\big(\mathcal{C}^*, \delta/(1 + n^2)\big)$.*

From this we then have that the problem of weak membership for K is Turing reducible to the problem of weak membership of the copositive cone.

Proof. We consider each of these parts separately.

1. We have that

$$\begin{aligned}
\mathrm{S}(K, -\delta) &= \{X \in \mathcal{S} \mid \mathrm{S}(X, \delta) \subseteq \mathcal{H} \cap \mathcal{C}^*\} \\
&= \{X \in \mathcal{S} \mid \mathrm{S}(X, \delta) \subseteq \mathcal{H}\} \cap \{X \in \mathcal{S} \mid \mathrm{S}(X, \delta) \subseteq \mathcal{C}^*\} \\
&= \mathrm{S}(\mathcal{H}, -\delta) \cap \mathrm{S}(\mathcal{C}^*, -\delta).
\end{aligned}$$

2. Consider an arbitrary $X \in \mathcal{H} \cap \mathrm{S}\big(\mathcal{C}^*, \delta/(1 + n^2)\big)$ and let $\varepsilon = \delta/(1 + n^2)$.

   Then there exists $Y \in \mathcal{C}^*$ such that $\|X - Y\| \leq \varepsilon$ and we have that

$$\begin{aligned}
\langle I + A, Y\rangle &\leq \langle I + A, X\rangle + \varepsilon\|I + A\| \leq 1 + \varepsilon n^2, \\
\|Y\| &\leq \max\left\{\|U\| \;\middle|\; U \in \mathcal{C}^*,\ \langle I + A, U\rangle \leq 1 + \varepsilon n^2\right\} \\
&= \max\left\{\|U\| \;\middle|\; U \in \operatorname{conv}\Big(\{0\} \cup \{bb^\mathsf{T} \mid b \in \mathbb{R}^n_+,\ b^\mathsf{T}(I + A)b = 1 + \varepsilon n^2\}\Big)\right\} \\
&\leq 1 + \varepsilon n^2.
\end{aligned}$$

   We now let $Z = \frac{1}{1+\varepsilon n^2} Y$. We have that $Z \in \mathcal{C}^*$ and $\langle I + A, Z\rangle \leq 1$. Therefore $Z \in \mathcal{H} \cap \mathcal{C}^*$. We finish the proof by noting that

$$\begin{aligned}
\|X - Z\| &= \left\|X - \frac{1}{1 + \varepsilon n^2} Y\right\| \\
&= \left\|X - Y + \frac{\varepsilon n^2}{1 + \varepsilon n^2} Y\right\| \\
&\leq \|X - Y\| + \frac{\varepsilon n^2}{1 + \varepsilon n^2}\|Y\| \\
&\leq \varepsilon(1 + n^2) \\
&= \delta,
\end{aligned}$$

   and hence $X \in \mathrm{S}(\mathcal{H} \cap \mathcal{C}^*, \delta) = \mathrm{S}(K, \delta)$

We then have that the problem of weak membership for K is Turing reducible to the problem of weak membership of the copositive cone by noting that $(I + A)$ and $\delta/(1 + n^2)$ are polynomial in the encoding lengths of the input δ and a symmetric matrix X. □

We now get the following result, which is the main result of this paper,

Theorem 5.5. *Both the weak and strong membership problems for the completely positive cone are $\mathcal{NP}$-hard.*

Proof. From Theorems 3.8 and 5.2, and Lemma 5.4, it follows that the stable set problem is Turing reducible to the problem of weak membership of the completely positive cone, and thus this problem is $\mathcal{NP}$-hard. To show that strong membership of the completely positive cone is also $\mathcal{NP}$-hard we can either use Theorem 5.3 or a many-one reduction from the problem of weak membership of the completely positive cone. □

Murty and Kabadi (1987) [16] showed that strong membership of the copositive cone was a co-$\mathcal{NP}$-complete problem. In this paper we have considered strong and weak membership of the copositive and completely positive cones and given a proof that they are $\mathcal{NP}$-hard. An immediate question which comes to mind is if strong membership of the completely positive cone is an $\mathcal{NP}$-complete problem. In order to show this you would need to show that for any completely positive matrix there is a polynomially bounded certificate proving that it is completely positive. For a completely positive matrix $X \in \mathbb{Q}^{n\times n}$ it would appear that such a certificate would be in the form of a number $m \in \mathbb{Z}_{++}$, a vector $\beta \in \mathbb{Q}^m_+$ and a set of vectors $\{b_1, \dots, b_m\} \in \mathbb{Q}^n_+$ such that $X = \sum_{i=1}^m (\beta)_i b_i b_i^\mathsf{T}$. As far as the authors are aware, it is still open as to whether such a certificate could be polynomially bounded, however this question has in part been considered recently for the case of sparse matrices in [8] and for weak membership in [1].

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