On semidefinite programming bounds for graph bandwidth

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Abstract

We propose two new lower bounds on graph bandwidth and cyclic bandwidth based on semidefinite programming (SDP) relaxations of the quadratic assignment problem. We compare the new bounds with two other SDP bounds in [A. Blum, G. Konjevod, R. Ravi, and S. Vempala, Semi-definite relaxations for minimum bandwidth and other vertex-ordering problems, *Theoretical Computer Science*, 235(1):25-42, 2000], and [J. Povh and F. Rendl, A copositive programming approach to graph partitioning, *SIAM Journal on Optimization*, 18(1):223-241, 2007].

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1 Introduction

Let $G = (V, E)$ be an undirected simple graph with $|V| = n$. The graph bandwidth problem is a graph labeling problem. A numbering of the vertex set V is a one-to-one mapping $\phi : V \to \{1, \ldots, n\}$. The bandwidth of a numbering ϕ of the graph G is defined as

$$\mathrm{bw}(G, \phi) = \max_{(u,v)\in E} |\phi(u) - \phi(v)|.$$

The bandwidth of G, denoted by $\mathrm{bw}(G)$ or simply bw, when the graph is clear from the context, is

$$\mathrm{bw}(G) = \min_{\phi} \{\mathrm{bw}(G, \phi) : \ \phi \text{ is a numbering of } G\}.$$

The bandwidth minimization problem appears in a wide range of applications, like sparse matrix computations, parallel computations, VLSI layout, etc; see, for example [19]. Papadimitriou [24] showed that it is NP-complete, and Garey et al. [14] proved that the bandwidth problem is NP-complete even if it is restricted to trees with maximum degree 3. The bandwidth problem can be solved in polynomial time [1] for caterpillars of strand length at most 2. Recall that caterpillar is a tree on which all vertices of degree greater than 2 lie on a single path. A strand (or hair) of the caterpillar is a path that connects a leaf to this single path. In [23] Monien showed that the bandwidth problem for caterpillars of strand length at most 3 is NP-complete.

There are graphs for which the bandwidth is known. For example, the bandwidth of:

- the path P_n is one;
- the rectangular grid graph $P_n \times P_m$ (i.e. the Cartesian product of the paths P_n and P_m) is $\min\{n, m\}$ [6];

- the complete graph K_n is $n-1$;
- the complete bipartite graph $K_{p,q}$ is $\lfloor (p-1)/2 \rfloor + q$ assuming $p \geq q \geq 1$ [5];
- the hypercube graph Q_n on 2^n vertices is $\sum_{i=0}^{n-1} \binom{i}{\lfloor i/2 \rfloor}$ [18];
- the complete k-level t-ary trees $T_{k,t}$ is $\left\lceil \frac{t(t^{k-1}-1)}{2(k-1)(t-1)} \right\rceil$ [29].

The bandwidth $\mathrm{bw}(G)$ of a graph $G = (V, E)$ may be computed exactly in

$$O\left(f(\mathrm{bw}(G))|V|^{\mathrm{bw}(G)+1}\right)$$

time, where $f(\mathrm{bw}(G))$ depends only on $\mathrm{bw}(G)$, using dynamic programming [28]. Thus $\mathrm{bw}(G)$ may be obtained in polynomial time for classes of graphs where $\mathrm{bw}(G) = O(1)$. In general it is difficult to give a constant approximation on the bandwidth problem in polynomial time. Blache et al. [3] showed that it is NP-complete to find a 3/2-approximation for general graphs and it is also NP-complete to find a 4/3-approximation for trees. Gupta [17] presented a randomized $O(\log^{2.5} n)$-approximation algorithm for general trees and chordal graphs, while for general graphs there is a $O(\log^3 n\sqrt{\log\log n})$-approximation algorithm by Dunagan and Vempala [12], based on a semidefinite programming (SDP) relaxation due to Blum et al. [4].

Main results and outline

In this paper we propose two new SDP relaxations of the minimum bandwidth problem based on the quadratic assignment problem (QAP) reformulation and compare them to SDP relaxations by Blum et al. [4] and by Povh and Rendl [26]; see Section 2. In Section 4, we test our bounds for the aforementioned special graphs and show that it is tight for paths, cliques, complete bipartite graphs, but it is not tight for hypercubes, rectangular grids and complete k-level t-ary trees.

The second part of the paper deals with the approximation of cyclic bandwidth. In this case the size of the SDP relaxation can be reduced exploiting symmetry; see Section 5. Finally, in Section 6 we present some computational results.

Notation

The space of $p \times q$ matrices is denoted by $\mathbb{R}^{p\times q}$, the space of $k \times k$ symmetric matrices is denoted by S^k. The trace inner product on S^k is denoted by $\langle \cdot, \cdot \rangle$. Vectors, scalars and indices are denoted by lowercase Latin letters, matrices by capital Latin letters. We use the notation $X \succeq 0$ for positive semidefinite (PSD) matrices. We denote the identity matrix of order n by I_n, the all-ones matrix by J_n and the zero matrix by O_n. The n-by-m all-ones matrix and the zero matrix are denoted by $J_{n\times m}$ and $O_{n\times m}$, respectively. We use u_n to denote the all-ones vector and 0_n is the zero vector in n dimensions. We omit the subscript if the order is clear from the context. We set $E_{ij} = e_i e_j^T$ where e_i is the i-th standard basis vector. The Kronecker product $A \otimes B$ of matrices $A \in \mathbb{R}^{p\times q}$ and $B \in \mathbb{R}^{r\times s}$ is defined as the $pr \times qs$ matrix composed of pq blocks of size $r \times s$, with block ij given by $a_{ij}B$ $(i = 1, \ldots, p)$, $(j = 1, \ldots, q)$. The Diag operator maps an n-vector to an $n \times n$ diagonal matrix, while $\mathrm{diag}(A)$ is the vector obtained by extracting the diagonal of A.

2 SDP relaxations for graph bandwidth

Here we describe four different lower bounds on bandwidth of graph $G = (V, E)$ ($|V| = n$) with adjacency matrix A. These bounds are all based on SDP relaxations.

2.1 Relaxation by Blum et al.

First we present the SDP problem by Blum et al. [4], which also forms the core of the approximation algorithm in [12]. In their approach the set of nodes $V = \{v_i : i = 1, \dots, n\}$ is mapped to a set of vectors on the sphere of radius n in $\mathbb{R}^n$, say $v_i \in \mathbb{R}^n$ ($i = 1, \dots, n$). These vectors are obtained from the optimal solution of a semidefinite program.

Letting $y_{ij} := v_i^T v_j$, the SDP problem is the following:

$$\begin{aligned}
\beta = \min\ & b \\
s.t\ \ & y_{ii} = n^2,\ \ 1 \le i \le n, \\
& y_{ii} - 2y_{ij} + y_{jj} \le b,\ \ (i,j) \in E, \\
& Y \succeq 0, Y \ge 0, \\
& \frac{1}{|S|} \sum_{j \in S} (y_{ii} - 2y_{ij} + y_{jj}) \ge \frac{1}{6}\left(\frac{|S|}{2} + 1\right)(|S| + 1), \forall\, S \subseteq \{1, \dots, n\},\ 1 \le i \le n,
\end{aligned} \tag{1}$$

where $Y = (y_{ij})$.

The optimal vectors $v_i \in \mathbb{R}^n$ ($i = 1, \dots, n$) are subsequently mapped to a quarter-circle of radius n in the positive orthant. This ordering of V is then used as a numbering that can be shown to yield a $\log n$-approximation of the bandwidth; see [4] and [12] for details.

There are exponentially many of the fourth type constraints, but there is a polynomial time separation oracle to decide which constraints are violated. Consequently, (1) may be solved in polynomial time by the ellipsoid method; see e.g. [16]. Finally, Povh [25] showed that a lower bound on the bandwidth is given by:

$$\mathrm{bw}(G) \ge \mathrm{bw}_{BKRV} := \left\lceil \frac{3}{\pi}\sqrt{\beta} \right\rceil. \tag{2}$$

(The constant $\frac{3}{\pi}$ is an improvement over the constant $\frac{2}{\pi}$ used in [4] and [12]; see Lemma 4.8 in [25] for details.)

2.2 Relaxation by Povh and Rendl

Another SDP relaxation was studied by Povh and Rendl [26]. Their approach is based on the graph three-partitioning problem: find a partition (S_1, S_2, S_3) of the vertex set V with $|S_i| = m_i$ for $i = 1, 2, 3$, where the total weight of edges between sets S_1 and S_2 is minimal. This problem can be reformulated equivalently as a copositive programming problem and its SDP relaxation is the following:

$$
\begin{aligned}
\alpha_{m_1,m_2,m_3} := \min\ & \langle D_{1,2} \otimes \hat{A}, Y\rangle \\
s.t.\ & \langle D_{ij} \otimes I_n, Y\rangle = m_i\delta_{ij}, \ \ 1 \le i \le j \le 3, \\
& \langle J_3 \otimes E_{ii}, Y\rangle = 1, \ \ 1 \le i \le n, \\
& \langle V_i \otimes W_j^T, Y\rangle = m_i, \ \ 1 \le i \le 3,\ 1 \le j \le n, \\
& \langle D_{ij} \otimes J_n, Y\rangle = m_i m_j, \ \ 1 \le i \le j \le 3, \\
& Y \ge 0, Y \succeq 0,
\end{aligned}
\tag{3}
$$

where $D_{ij} = (E_{ij} + E_{ji})/2 \in \mathbb{R}^{3\times 3}$, $\hat{A} = A + \frac{1}{n}(u_n^T A u_n) I_n - \mathrm{Diag}(Au_n)$, $V_i = e_i u_3^T \in \mathbb{R}^{3\times 3}$, $W_j = e_j u_n^T \in \mathbb{R}^{n\times n}$, δ_{ij} is the Kronecker delta and $m_1, m_2, m_3 \ge 0$ are given nonnegative integers such that $m_1 + m_2 + m_3 = n$.

In [26] it is proved that if the optimal value α_{m_1,m_2,m_3} of (3) is positive and $\bar{\alpha}_{m_1,m_2,m_3} = \lceil \alpha_{m_1,m_2,m_3} \rceil$, then $\mathrm{bw}(G) \ge \max\left\{m_3 + 1, m_3 + \left\lceil \sqrt{2\bar{\alpha}_{m_1,m_2,m_3}}\right\rceil - 1\right\}$. Consequently,

$$
\mathrm{bw}(G) \ge \mathrm{bw}_{PR} := \max_{\bar{\alpha}_{m_1,m_2,m_3} > 0} \left\{m_3 + 1, m_3 + \left\lceil \sqrt{2\bar{\alpha}_{m_1,m_2,m_3}}\right\rceil - 1\right\}. \tag{4}
$$

2.3 New SDP relaxations

In this paper we propose a polynomial time approximation based on the quadratic assignment problem:

$$
\gamma_{QAP} := \max_{X \in \Pi_n} \langle B, X^T A X\rangle, \tag{5}
$$

where A and B are given symmetric $n \times n$ matrices and Π_n is the set of $n \times n$ permutation matrices.

Let matrix A be the adjacency matrix of $G = (V, E)$ with $|V| = n$, and B be the symmetric Toeplitz matrix with first row $[u_k^T\ 0_{n-k}^T]$, $k > 0$, i.e:

$$
B = \begin{pmatrix}
\overbrace{1 & \dots & 1}^{k} & 0 & \dots & \dots & 0 \\
\vdots & & & \ddots & \ddots & & \vdots \\
1 & & & & \ddots & \ddots & \vdots \\
0 & \ddots & & & & \ddots & 0 \\
\vdots & \ddots & \ddots & & & & 1 \\
\vdots & & \ddots & \ddots & & & \vdots \\
0 & \dots & \dots & 0 & 1 & \dots & 1
\end{pmatrix}.
$$

Now if the optimal value of (5) is less than $2|E|$, then the bandwidth G is at least k. More precisely,

$$
\mathrm{bw}(G) = \max\{k : \gamma_{QAP} < 2|E|\}. \tag{6}
$$

Zhao, Karisch, Rendl and Wolkowicz [32] gave the following semidefinite programming relaxation for the QAP problem (see also [27] for this specific formulation of the relaxation):

$$\begin{aligned}\gamma_{ZKRW} := \max\ & \langle B \otimes A, Y\rangle \\ s.t.\ & \langle I_n \otimes E_{ii}, Y\rangle = 1,\ \langle E_{ii} \otimes I_n, Y\rangle = 1,\ \ i = 1, \ldots, n, \\ & \langle I_n \otimes (J_n - I_n) + (J_n - I_n) \otimes I_n, Y\rangle = 0, \\ & \langle J_n \otimes J_n, Y\rangle = n^2, \\ & Y \geq 0, Y \succeq 0.\end{aligned} \tag{7}$$

Therefore, a lower bound on the bandwidth that is based on the SDP problem (7) is:

$$\mathrm{bw}(G) \geq \mathrm{bw}_{ZKRW} := \max\{k : \gamma_{ZKRW} < 2|E|\}. \tag{8}$$

Finally, we introduce an SDP based bound on the bandwidth which is at least as strong as the bound (7). De Klerk and Sotirov [10] discussed a branching type approach for QAP. In the context of the bandwidth QAP problem (5), their idea is to only consider numberings ϕ such that $\phi(r) = s$, for given r, s. This corresponds to adding the following constraint to the SDP relaxation (7) (see Theorem 7 in [31]):

$$\langle E_{ss} \otimes E_{rr}, Y\rangle = 1. \tag{9}$$

Denote by $\gamma_{r,s}$ the optimal value of SDP relaxation (7) with included constraint (9), and define

$$\mathrm{bw}_{dKS}(r) := \max\{k\ :\ \max_s \gamma_{r,s} < 2|E|\}. \tag{10}$$

Then $\mathrm{bw}_{dKS}(r)$ is a valid lower bound on the bandwidth, so that

$$\mathrm{bw}(G) \geq \mathrm{bw}_{dKS} := \max_r \mathrm{bw}_{dKS}(r). \tag{11}$$

In summary, we have:

$$\mathrm{bw}(G) \geq \mathrm{bw}_{dKS} \geq \mathrm{bw}_{ZKRW}. \tag{12}$$

If the automorphism group of the graph acts transitively on V, then, for fixed s, $\gamma_{r,s}$ is the same for all r; for details see [10].

3 Exploiting algebraic symmetry

In certain cases we can exploit the algebraic symmetry of the SDP problem (7). Here we only give a short review, the interested reader may consult [7, 9, 10].

A *coherent configuration* is a set of matrices $\{A_1, \ldots, A_d\} \subseteq \{0, 1\}^{n\times n}$ with the following properties:

1. $A_iA_j \in \mathcal{A} := \mathrm{span}\{A_1, \ldots, A_d\}$ for $i, j = 1, \ldots, d$;
2. There is a set $\mathcal{I}_\mathcal{A} \subseteq \{1, \ldots, d\}$ such that $\sum_{i\in\mathcal{I}_\mathcal{A}} A_i = I$ and $A_1 + \cdots + A_d = J$;
3. $A_i^T \in \{A_1, \ldots, A_d\}$ for $i = 1, \ldots, d$.

For each $i \in \{1, \dots, d\}$ we define $i^* \in \{1, \dots, d\}$ via $A_{i^*} = A_i^T$. We also write $m_i = \langle J, A_i \rangle$ for $i = 1, \dots, d$.

We call $\mathcal{A} := \text{span}\{A_1, \dots, A_d\}$ the *associated coherent algebra*. Note that $\mathcal{A}$ is a matrix $\mathbb{C}*$-algebra. If the coherent configuration is commutative, namely $A_i A_j = A_j A_i$ for $i, j = 1, \dots, d$, we call it *association scheme*.

If $A \in \mathcal{A}$, then we can exploit the structure of the algebra $\mathcal{A}$ and the SDP relaxation (7) reduces to the following (see [7]):

$$\begin{aligned} \max\ & \langle B, \sum_{i=1}^{d} m_i^{-1} \langle A, A_i \rangle X_i \rangle \\ s.t.\ & \sum_{i \in \mathcal{I}_{\mathcal{A}}} X_i = I_n, \\ & \sum_{i=1}^{d} X_i = J_n, \\ & \sum_{i=1}^{d} m_i^{-1} A_i \otimes X_i \succeq 0, \\ & \langle J, X_i \rangle = m_i, X_{i^*} = X_i^T, X_i \geq 0,\ i = 1, \dots, d. \end{aligned} \tag{13}$$

Furthermore, if we have an algebra *-isomorphism $\phi_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}'$, then we may replace the linear matrix inequality

$$\sum_{i=1}^{d} m_i^{-1} A_i \otimes X_i \succeq 0$$

by

$$\sum_{i=1}^{d} m_i^{-1} \phi_{\mathcal{A}}(A_i) \otimes X_i \succeq 0.$$

This is useful if the size of the matrices $\phi_{\mathcal{A}}(A_i)$ are smaller than $n \times n$ or if they have block diagonal structure. Of course, we can do the reduction with a suitable algebra for matrix B as well (see Section 5.2), or even with the two algebras simultaneously.

Under some further assumptions it is also possible to reduce the size of the SDP relaxation with the fixing constraint (9). The reader may find details in [9].

4 Tightness of bounds for some graphs

In this section we inspect the bound bw_{ZKRW} in (8) on special graphs such as paths, cliques, rectangular grids, complete bipartite graphs, hypercubes and complete k-level t-ary trees.

4.1 Tightness of the bound bw_{ZKRW}

We show that the bound bw_{ZKRW} is tight for paths, cliques and complete bipartite graphs. Since $\text{bw}_{ZKRW} \leq \text{bw}_{dKS}$, the bound bw_{dKS} is also tight for these graphs.

Paths

It is easy to see, that $\mathrm{bw}_{ZKRW} \geq 1$ if the graph has at least one edge. The bandwidth of a path is 1, so for paths the relaxation (7) gives the tight bound.

Cliques

The adjacency matrix of the clique K_n is $A = J_n - I_n$. Let B be the symmetric Toeplitz matrix with first row $[u_{n-1}^T\ 0]$ (i.e., $k = n-1$). The $n^2 \times n^2$ matrix Y in relaxation (7) can be divided into n^2 blocks size n-by-n. The sum of elements in each block has to be exactly one when Y is a feasible solution, furthermore the diagonal blocks are diagonal matrices. The matrix $B \otimes A$ has a zero diagonal and the n-by-n blocks on positions $(1, n)$ and $(n, 1)$ are zero matrices. Therefore the objective value is at most $n(n-1) - 2$, so it is less than $n(n-1) = 2|E|$. Thus $\mathrm{bw}_{ZKRW}(K_n) \geq n-1$. Since the bandwidth of K_n is $n-1$, (7) gives a tight bound for cliques.

Complete bipartite graphs

Chvátal [5] proved that the bandwidth of the complete bipartite graph $K_{p,q}$ is $\lfloor (p-1)/2 \rfloor + q$, assuming $p \geq q \geq 1$. We have to prove, that the optimal value of SDP relaxation (7) with the symmetric Toeplitz matrix B, $k = \lfloor (p-1)/2 \rfloor + q$ is less than twice the number of edges, i.e. less than $2p\,q$.

The adjacency matrix of the graph $K_{p,q}$ is

$$A = \begin{pmatrix} O_{p\times p} & J_{p\times q} \\ J_{q\times p} & O_{q\times q} \end{pmatrix}.$$

There are two cases: (i) $p = q$ or (ii) $p > q$.

Case (i): The matrix A belongs to the coherent algebra spanned by matrices

$$A_1 = \begin{pmatrix} I_p & O_p \\ O_p & I_p \end{pmatrix}, \quad A_2 = \begin{pmatrix} J_p - I_p & O_p \\ O_p & J_p - I_p \end{pmatrix}, \quad A_3 = \begin{pmatrix} O_p & J_p \\ J_p & O_p \end{pmatrix},$$

and the associated *-isomorphism ϕ satisfies (see e.g. [7]):

$$\phi(A_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \phi(A_2) = \begin{pmatrix} p-1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & p-1 \end{pmatrix}, \quad \phi(A_3) = \begin{pmatrix} p & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -p \end{pmatrix}.$$

Therefore by (13) the relaxation (7) can be reduced as follows:

$$\begin{aligned} \max\ & \langle B, X_2 \rangle \\ s.t.\ & X_2 + X_3 = J - I, \\ & X_2 + X_3 \succeq -I, \\ & X_2 \preceq (p-1)I, \\ & X_2 - X_3 \succeq -I, \\ & \langle J, X_2 \rangle = 2p\,(p-1),\ \langle J, X_3 \rangle = 2p^2, \\ & X_2, X_3 \in S^{2p},\ X_2, X_3 \geq 0. \end{aligned} \tag{14}$$

The optimal value of (14) is less than $2p\,q = 2p^2$, since

$$\langle B, X_2 \rangle \leq \langle J, X_2 \rangle = 2p(p-1).$$

Case (ii): The matrix A belongs to the coherent algebra induced by the following matrices:

$$A_1 = \begin{pmatrix} I_p & O_{p\times q} \\ O_{q\times p} & O_q \end{pmatrix}, \quad A_2 = \begin{pmatrix} O_p & O_{p\times q} \\ O_{q\times p} & I_q \end{pmatrix}, \quad A_3 = \begin{pmatrix} J_p - I_p & O_{p\times q} \\ O_{q\times p} & O_q \end{pmatrix},$$

$$A_4 = \begin{pmatrix} O_p & O_{p\times q} \\ O_{q\times p} & J_q - I_q \end{pmatrix}, \quad A_5 = \begin{pmatrix} O_p & J_{p\times q} \\ O_{q\times p} & O_q \end{pmatrix}, \quad A_6 = \begin{pmatrix} O_p & O_{p\times q} \\ J_{q\times p} & O_q \end{pmatrix},$$

and the associated *-isomorphism ϕ satisfies (see e.g. [7]):

$$\phi(A_1) = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 1 & 0 \\ & & 0 & 0 \end{pmatrix}, \quad \phi(A_2) = \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 0 & 0 \\ & & 0 & 1 \end{pmatrix}, \quad \phi(A_3) = \begin{pmatrix} -1 & & & \\ & 0 & & \\ & & p-1 & 0 \\ & & 0 & 0 \end{pmatrix},$$

$$\phi(A_4) = \begin{pmatrix} 0 & & & \\ & -1 & & \\ & & 0 & 0 \\ & & 0 & q-1 \end{pmatrix}, \quad \phi(A_5) = \sqrt{p\,q}\begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & 1 \\ & & 0 & 0 \end{pmatrix}, \quad \phi(A_6) = \sqrt{p\,q}\begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & 0 \\ & & 1 & 0 \end{pmatrix}.$$

In this case the reduced SDP relaxation is the following, by (13):

$$\begin{aligned} \max\ & \langle B, (X_5 + X_5^T)\rangle \\ s.t.\ \ & X_1 + X_2 = I \\ & X_3 + X_4 + X_5 + X_5^T = J - I, \\ & \langle J, X_1\rangle = p,\ \langle J, X_2\rangle = q, \\ & \langle J, X_3\rangle = p(p-1),\ \langle J, X_4\rangle = q(q-1),\ \langle J, X_5\rangle = p\,q, \\ & (p-1)X_1 - X_3 \succeq 0, \\ & (q-1)X_2 - X_4 \succeq 0, \\ & \begin{pmatrix} X_1 + X_3 & X_5 \\ X_5^T & X_2 + X_4 \end{pmatrix} \succeq 0, \\ & X_1, X_2, X_3, X_4 \in S^n,\ X_1, X_2, X_3, X_4, X_5 \geq 0, \end{aligned} \tag{15}$$

where $n = p + q$.

Similarly as before, we can bound the optimal value of (15):

$$\langle B, (X_5 + X_5^T)\rangle \leq 2\langle J, X_5\rangle = 2p\,q,$$

and the equality holds if and only if

$$\langle B, (X_5 + X_5^T)\rangle = 2\langle J, X_5\rangle. \tag{16}$$

We will show that the equality (16) cannot hold.

First we present some technical observations which we will use later on.

Lemma 1 *Let $A \in \mathbb{R}^{n\times n}$ and $0 \preceq A \preceq J$. Then $A = \lambda J$ with $\lambda \in [0, 1]$.*

Proof. It is enough to show that the statement is true for 2 by 2 matrices. Let $A = \begin{pmatrix} a & c \\ c & b \end{pmatrix}$. Since A and $J - A$ are PSD matrices, it follows that $a, b, c \in [0, 1]$. From $z^T Az \geq 0$ and $z^T(J - A)z \geq 0$, where $z = (1, -1)^T$, it follows

$$a + b \geq 2c, \quad 2 - (a + b) \geq 2 - 2c,$$

and consequently $a + b = 2c$. On the other hand the determinant of a PSD matrix has to be nonnegative, so $ab \geq c^2$, i.e, $a = b$ and so $c = a$. ■

Lemma 2 *Let* $A = \begin{pmatrix} x & t & x \\ t & x & x \\ x & x & x \end{pmatrix}$ *or* $\begin{pmatrix} x & x & x \\ x & x & t \\ x & t & x \end{pmatrix}$ *and* $A \succeq 0$. *Then* $t = x \geq 0$.

Proof. The matrix A is PSD, so $x \geq 0$ and if $x = 0$ then $t = 0$. And if $x > 0$ then $\det(A) = x(x - t)(t - x) < 0$ unless $t = x$. ■

We can prove the next corollary by repeatedly using Lemma 2.

Corollary 3 *Let* $A \in \mathbb{R}^{2m \times 2m}$ *(*$m \geq 2$*) such that* $\operatorname{diag}(A) = c\, u_{2m}$ *and* $A_{ij} = c$ *if* $|i - j| \geq m$. *Then* $A \succeq 0$ *if and only if* $A = c\, J$ *with* $c \geq 0$.

In the sequel we show by contradiction that equality (16) cannot hold. Suppose that X_5 has zero elements on positions (i, j) where the matrix B is 0, namely if $|i - j| \geq k = \lfloor (p - 1)/2 \rfloor + q$.

From the second constraint of (15) and our assumption on matrix X_5, it follows that

$$(X_3)_{ij} + (X_4)_{ij} = 1 \quad \text{if} \ |i - j| \geq k = \lfloor (p - 1)/2 \rfloor + q. \tag{17}$$

Moreover, $X_1 + X_2 = I$, $\operatorname{diag}(X_3 + X_4) = 0$, $X_1 + X_3 \succeq 0$ and $X_2 + X_4 \succeq 0$, so using Lemma 1 on suitable 2×2 submatrices of $X_1 + X_3$, we derive the following structure:

$$\begin{array}{rcll} (X_1)_{ii} = & t, & \text{for } i \in \{1, \ldots, n - k\} \cup \{k + 1, \ldots, n\}, \\ (X_2)_{ii} = & 1 - t, & \text{for } i \in \{1, \ldots, n - k\} \cup \{k + 1, \ldots, n\}, \\ (X_3)_{ij} = & t, & \text{for } |i - j| \geq k, \\ (X_4)_{ij} = & 1 - t, & \text{for } |i - j| \geq k. \end{array} \tag{18}$$

It is easy to check that, when $k = 1$, the equality (16) cannot hold, so now we assume that $k \geq 2$. First consider the submatrix of $X_1 + X_3$ induced by the first and last $n - k$ rows and columns. It has the same structure as the matrix in Corollary 3, so $(X_1 + X_3)_{ij} = t$ for $i, j \in \{1, \ldots, n - k\} \cup \{k + 1, \ldots, n\}$. But the matrix X_1 is diagonal, so that

$$(X_3)_{ij} = t \quad \text{for } i, j \in \{1, \ldots, n - k\} \cup \{k + 1, \ldots, n\}, \ i \neq j.$$

Similarly,

$$(X_4)_{ij} = 1 - t \quad \text{for } i, j \in \{1, \ldots, n - k\} \cup \{k + 1, \ldots, n\}, \ i \neq j.$$

Therefore

$$(X_5)_{ij} = 0 \quad \text{for } i, j \in \{1, \ldots, n - k\} \cup \{k + 1, \ldots, n\}, \ i \neq j.$$

Recall that $k = \lfloor (p - 1)/2 \rfloor + q$ and $n = p + q$. Thus, the number of nonzero columns and rows in X_5 is at most $n - 2(n - k) = 2\lfloor (p - 1)/2 \rfloor - p + q \leq q - 1$. Considering that the diagonal of X_5 is zero, X_5 has at most $(n - 1)(q - 1) < 2p\,q$ nonzero elements. Since every element of $X_5 + X_5^T$ is at most 1, this contradicts the constraint $\langle J, X_5 \rangle = p\,q$.

Thus we have proved the following theorem.

Theorem 4 *The bound* bw_{ZKRW} *in (8), and – by implication – also the bound* bw_{dKS} *in (11), are tight for paths, cliques, and complete bipartite graphs.*

Remark 5 *The lower bound* bw_{PR} *in (4) on the bandwidth for complete bipartite graphs is not tight; see the computational results for* $K_{6,9}$ *in [26]. The bound* bw_{BKRW} *is also not tight for* $K_{6,9}$*, since* $\mathrm{bw}_{BKRW}(K_{6,9}) = 5 < 10 = \mathrm{bw}(K_{6,9})$.

4.2 Negative results

The bandwidth problem is NP-complete, so it is not surprising that the polynomial time bounds bw_{ZKRW} and bw_{dKS} are not tight in general. The bound bw_{ZKRW} is not tight for:

- hypercubes, the smallest example is Q_5 (see Table 6, Section 6);
- rectangular grid graphs, an example is $P_4 \times P_4$ (see Table 4, Section 6);
- complete k-level t-ary trees, small examples are $T_{5,2}$, $T_{4,3}$ and $T_{3,4}$ (see Table 2, Section 6).

The bound bw_{dKS} is not tight for

- hypercubes, the smallest example is Q_5 (see Table 6, Section 6).

The results of this section are summarized in Table 1.

	K_n	$K_{n,m}$	$P_n \times P_m$	Q_n	$T_{n,m}$
bw_{BKRV}	×	×	×	×	×
bw_{PR}	✓	×	?	×	×
bw_{ZKRW}	✓	✓	×	×	×
bw_{dKS}	✓	✓	?	×	?

Table 1: Summary of known results on the tightness of different lower bounds on $\mathrm{bw}(G)$ for different classes of G. '✓' means the bound is tight, '×' that it is not tight, and '?' that there is no proof that the bound is tight, but there are also no known counterexamples.

The result in Table 1 that the bound bw_{PR} is tight for cliques is not proven in [26], and we therefore include a short proof here for completeness.

Theorem 6 *The bound* bw_{PR} *in (4) is tight for cliques.*

Proof: Consider the clique K_n. We show that the optimal value of the SDP problem (3) is positive for $m = [1, 1, n-2]$ and so, by (4), $\mathrm{bw}_{PR}(K_n) \geq n-1$.

The adjacency matrix of K_n is $J_n - I_n$, i.e. $\hat{A} = J_n - I_n$ in (3). Considering the constraints $\langle D_{1,2} \otimes I_n, Y \rangle = m_1 \delta_{1,2} = 0$ and $\langle D_{1,2} \otimes J_n, Y \rangle = m_1 m_2 = 1$, we get that the optimal value of (3) is one. ■

5 Reduction of (7) for trees and rectangular grids

The SDP relaxation (7) is also suitable for computing the cyclic bandwidth. In this case we can take a symmetric circulant Toeplitz matrix as B, namely

$$B = \begin{array}{c} \overbrace{\hphantom{1 \quad \cdots \quad 1}}^{k} \qquad\qquad \overbrace{\hphantom{1 \quad \cdots \quad 1}}^{k-1} \\ \begin{pmatrix} 1 & \cdots & 1 & & 1 & \cdots & 1 \\ \vdots & & & \ddots & & \ddots & \vdots \\ 1 & & & & \ddots & & 1 \\ & \ddots & & & & \ddots & \\ 1 & & \ddots & & & & 1 \\ \vdots & \ddots & & \ddots & & & \vdots \\ 1 & \cdots & 1 & & 1 & \cdots & 1 \end{pmatrix} \end{array}.$$

Due to the extra symmetry that appears in the corresponding problem, we can reduce the size of the SDP relaxation (see for e.g., [7]).

5.1 Cyclic bandwidth

The cyclic bandwidth of a numbering ϕ of the graph G on n nodes is defined as

$$\mathrm{cbw}(G, \phi) = \max_{(u,v)\in E} |\phi(u) - \phi(v)|_c,$$

where $|x|_c = \min\{|x|, n - |x|\}$. The cyclic bandwidth of G, denoted by $\mathrm{cbw}(G)$ is

$$\mathrm{cbw}(G) = \min_{\phi} \{\mathrm{cbw}(G, \phi) : \ \phi \text{ is a numbering of } G\}.$$

From the definition we can see, that $\mathrm{cbw}(G) \le \mathrm{bw}(G)$. Lam et al. [20] proved that for a graph G in general $\mathrm{bw}(G) \le 2\,\mathrm{cbw}(G)$. It means that, if we give an α-approximation for the cyclic bandwidth, then we also give a 2α-approximation for the bandwidth problem. But there are graphs whose bandwidth and cyclic bandwidth are the same. Lam et al. [21] showed that trees and rectangular grids are examples of such graphs.

5.2 Reduction of (7) for cyclic bandwidth

The dimension of the symmetric circulant $n \times n$ matrix space is $d + 1$, where $d = \lfloor n/2 \rfloor$ and a $0-1$ basis is $B_0, B_1, \ldots, B_d$, where B_i has ones in the $-(n-i)^{th}$, $-i^{th}$, i^{th} and $(n-i)^{th}$ diagonal and zero everywhere else, so $B_0 = I$. These matrices form an association scheme. They share a common set of eigenvectors, given by the columns of the discrete Fourier transform matrix

$$Q_{ij} = \frac{1}{\sqrt{n}} e^{\frac{-2\pi i j\sqrt{-1}}{n}}, \quad i, j = 0, \ldots, n-1,$$

and their eigenvalues are

$$\lambda_m(B_i) = 2\cos(2\pi m i/n), \quad m = 0, \ldots, n-1, \ \ i = 1, \ldots, \lfloor (n-1)/2 \rfloor,$$

and if n is even,

$$\lambda_m(B_{n/2}) = \cos(m\pi) = (-1)^m.$$

Now the SDP problem that is equivalent to (7) in the case of the cyclic bandwidth is

$$\begin{aligned}
\max\ & \sum_{i=1}^{k-1} \langle A, X_i \rangle \\
s.t.\ & \sum_{i=1}^{d} X_i = J - I, \\
& I + \sum_{i=1}^{d} \cos\left(\frac{2\pi m i}{n}\right) X_i \succeq 0, \quad m = 1, \ldots, d, \\
& X_i \geq 0,\ X_i \in \mathcal{S}^n, \quad i = 1, \ldots, d.
\end{aligned} \tag{19}$$

This relaxation is similar to the TSP relaxation by De Klerk, Pasechnik and Sotirov [8]. In fact, it has the same constraint set.

6 Computational results

We computed the bounds bw_{BKRV}, bw_{PR}, bw_{ZKRW} and bw_{dKS} for complete k-level t-ary trees, rectangular grid graphs and hypercubes.

For the last two bounds we could exploit the algebraic symmetry of the problem as we discussed in Section 3. In Section 5.2 we saw that the bandwidth and the cyclic bandwidth are the same for trees and grid graphs, so we only needed to solve the SDP problem (19) to obtain the bound bw_{ZKRW}. Furthermore, using GAP [13] – a system for computational discrete algebra – we determined coherent configurations for adjacency matrices of trees and grid graphs, and further reduced the size of the SDP. On the other hand, since the adjacency matrix of the hypercube belongs to the Bose-Mesner algebra of the Hamming scheme [15], the size of the SDP problem (7) for hypercubes also can be reduced according to (13). Note that in each three cases either the automorphism group of matrix B or of the adjacency matrix A acts transitively, therefore each fixing gives a valid bound (see [10] for details). Moreover, when we computed the bw_{dKS} bound for complete trees we used the fact that if the label of the root node is fixed then the stabilizer group related to this fixing is isomorphic to the full automorphism group of the adjacency matrix. Unfortunately, for grid graphs we can no longer exploit symmetry after fixing. However, for hypercubes we may still exploit the structure of the Terwilliger algebra of the Hamming scheme after fixing (see [10, 15]).

For each family of graphs we present two tables: one with the different bounds and one with the computational times. The first column in the tables with numerical results contains the parameter(s) of the graphs, the second column contains the number of nodes of the graphs and the third column provides the exact value of the bandwidths. The mentioned four bounds are given in the last four columns.

We solved the SDP problems by SeDuMi [30] using the Yalmip interface [22] with Matlab 7.9., on a 2.53GHz dual-core processors with 4GB of memory, while the bound bw_{dKS} was computed on 3.33GHz dual-core processor with 32GB memory. The computational times are in seconds in Table 3, 5, 7. The second column of these tables contains the computational time of

(1), where we used a simple cutting plane scheme where one violated inequality is added at a time. In case of bw_{PR}, bw_{ZKRW} and bw_{dKS} we report the computational time of an SDP giving the best bound. The third columns of tables with computational times contain the best choice of the vector m in (3) which gives the bound bw_{PR}. Furthermore, when the bound bw_{ZKRW} was tight, we did not compute the bound bw_{dKS}, since it is also tight by (12).

(t,k)	# nodes	bw	bw_{BKRV}	bw_{PR}	bw_{ZKRW}	bw_{dKS}
(2,2)	3	1	1	1	1	1
(2,3)	7	2	2	1	2	2
(2,4)	15	3	2	1	3	3
(2,5)	31	4	3	1	3	4
(3,2)	4	2	2	1	2	2
(3,3)	13	3	3	1	3	3
(3,4)	40	6	5	1	5	6
(4,2)	5	2	2	1	2	2
(4,3)	21	5	4	2	4	5

Table 2: Bounds on the bandwidth of complete k-level t-ary trees.

(t,k)	bw_{BKRV}	m	bw_{PR}	bw_{ZKRW}	bw_{dKS}
(2,2)	0.4907	$[1,2,0]$	0.109	0.062	–
(2,3)	2.1484	$[2,5,0]$	0.937	0.359	–
(2,4)	14.7462	$[2,13,0]$	5.281	9.063	–
(2,5)	914.9767	$[2,29,0]$	311.907	22.391	87 967
(3,2)	0.7035	$[2,2,0]$	0.484	0.25	–
(3,3)	11.542	$[3,10,0]$	3.125	0.375	–
(3,4)	10 093	$[2,38,0]$	1 291.263	10.094	45 308
(4,2)	1.0165	$[2,3,0]$	0.094	0.297	–
(4,3)	94.5431	$[3,18,0]$	29.688	0.609	25.7180

Table 3: Computational times for complete k-level t-ary trees.

The numerical results show that bounds bw_{ZKRW} and bw_{dKS} are not tight for all graphs under consideration, but they dominate bw_{BKRV}, and they are also stronger than bw_{PR} for complete trees and the hypercubes. Since we could exploit the algebraic symmetry, the computational time for bound bw_{ZKRW} is the smallest in most of the cases. The time for bound bw_{dKS} is much worse, because we cannot exploit as much symmetry, and the resulting SDP problems are larger. Note that, even though the tree $T_{5,2}$ has fewer vertices than $T_{4,3}$, the computation

(n,m)	# nodes	bw	bw$_{BKRV}$	bw$_{PR}$	bw$_{ZKRW}$	bw$_{dKS}$
(2,3)	6	2	2	2	2	2
(3,3)	9	3	2	3	3	3
(3,4)	12	3	3	3	3	3
(3,5)	15	3	3	3	3	3
(4,4)	16	4	3	4	3	4

Table 4: Bounds on the bandwidth of rectangular grid graphs $P_n \times P_m$.

(n,m)	bw$_{BKRV}$	m	bw$_{PR}$	bw$_{ZKRW}$	bw$_{dKS}$
(2,3)	4.065	$[1, 4, 1]$	1.453	0.094	–
(3,3)	12.0556	$[3, 4, 2]$	3.938	2.75	–
(3,4)	17.3995	$[5, 5, 2]$	19.484	5.734	–
(3,5)	32.117	$[2, 11, 2]$	48.187	25.189	–
(4,4)	46.5472	$[5, 8, 3]$	71.281	74.534	$>$ 4 days

Table 5: Computational times for rectangular grid graphs $P_n \times P_m$.

n	# nodes	bw	bw$_{BKRV}$	bw$_{PR}$	bw$_{ZKRW}$	bw$_{dKS}$
2	4	2	2	2	2	2
3	8	4	2	4	4	4
4	16	7	4	6	7	7
5	32	13	6	10	11	12

Table 6: Bounds on the bandwidth of hypercubes Q_n.

n	bw$_{BKRV}$	m	bw$_{PR}$	bw$_{ZKRW}$	bw$_{dKS}$
2	1.4357	$[1, 2, 1]$	0.75	0.047	–
3	8.9049	$[2, 3, 3]$	0.828	0.953	–
4	64.0411	$[5, 7, 4]$	92.422	8.093	–
5	3327.9	$[10, 14, 8]$	4031.881	675.984	1.0309e+005

Table 7: Computational times for hypercubes Q_n.

of the bound bw_{dKS} on the latter graph is twice as fast as for the former, since the size of the reduced SDP is smaller in this case.

7 Conclusion and summary

In this paper we have compared SDP relaxations of graph bandwidth and cyclic bandwidth both theoretically and numerically. Two of the relaxations are new, and based on the SDP relaxations of QAP, by Zhao et al. [32], and De Klerk–Sotirov [9]. We have shown that the SDP relaxation of cyclic bandwidth based on the QAP relaxation of Zhao et al. [32] may be computed relatively efficiently. Moreover it mostly gives stronger lower bound on the bandwidth in practice than the methods of Blum et al. [4] and Povh–Rendl [26].

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