Preferences for Travel Time under Risk and Ambiguity: Implications in Path Selection and Network Equilibrium

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Abstract

In this paper, we study the preferences for uncertain travel time in which the probability distribution may not be fully characterized. In evaluating an uncertain travel time, we explicitly distinguish between *risk*, where probability distribution is precisely known, and *ambiguity*, where it is not. In particular, we propose a new criterion called *ambiguity-aware CARA travel time* (ACT) for evaluating uncertain travel time under various attitudes of risk and ambiguity, which is a preference based on blending Hurwicz criterion and Constant Absolute Risk Aversion (CARA). More importantly, we show that when the uncertain travel times of the links along the paths are independently distributed, finding the path that minimizes travel time under the ACT criterion is essentially a shortest path problem. We also study the implications on Network Equilibrium (NE) model where the travelers on the traffic network are characterized by their knowledge of the network uncertainty as well as their risk and ambiguity attitudes under ACT. We derive and analyze the existence and uniqueness of the solution under NE. Finally, we also obtain the Price of Anarchy that characterizes the inefficiency of this new equilibrium.

Keywords: risk, ambiguity, path selection, network equilibrium, Price of Anarchy.

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1 Introduction

In real world transportation networks, the travel times along paths are almost always uncertain and individuals' preferences depend on their attitudes towards uncertainty. In transportation literature, uncertain travel time is often associated with a random variable with known probability distribution. In other words, the traveler knows the exact frequency of the travel time outcomes, and her preference depends on her risk attitude, which is usually characterized by taking an expectation over a disutility function (an increase in travel time amounts to a loss). Deliberating on reliability, Fan et al. (2005) and Nie and Wu (2009) consider the probability of punctuality as the preference criterion, which could be treated as a step disutility function. Unfortunately, it is generally a computationally intractable problem to find the path with the minimum expected disutility over a transportation network, which is a severe limitation for analysis and implementation. Nevertheless, Loui (1983) and Eiger et al. (1985) consider disutility functions in the form of linear, quadratic or exponential in which the resultant static path selection problems are computationally tractable. In particular, De Palma and Picard (2005) justify empirically the relevance of the exponential disutlity function, which appeals to travelers with Constant Absolute Risk Aversion (CARA) and has the best fit on path selection behavior amongst common disutlity functions. Implications of risk in Network Equilibrium (NE) problems, which model a collective behavior of a large population of travelers, have also been studied; see for instance Mirchandani and Soroush (1987), Chen et al. (2002), Uchida and Iida (1993) and Lo et al. (2006). We refer interested readers to the review papers of Noland and Polak (2002) and Connors and Sumalee (2009).

Nevertheless, the assumption that the traveler knows the frequency of the travel time outcomes is unrealistic. In the real world, it is conceivable that the traveler is incapable of knowing the entire probability distributions of the transportation network. Major exceptional events (e.g., natural disasters) and minor regular events (e.g., minor accident, traffic signal) will incur uncertainty to travel time. Hence, complete distribution of travel time is seldom known exactly, and even the estimated distribution could be affected by the sampling procedure. If the actual travel time probability distribution is not fully known, then it would not be possible to establish the preferences for travel time based on the expected disutility criterion. In the economic literature, the distinction between risk, where outcome frequency is known, and ambiguity, where it is not, can be retrospected to Knight (1921). Ellsberg (1961) shows convincingly by means of paradoxes that ambiguity preference cannot be reconciled by classical expected utility theory. Inspired by this seminal work, numerous experimental and theoretical studies spring up to verify and accommodate this behavior issue; see for instance Camerer and Weber (1992). Notably, in

Hsu et al. (2005) ground-breaking experiment, economists and neuroscientists collaborate to establish significant physiological evidence via functional brain imaging that humans have varying and distinct attitudes towards risk and ambiguity.

From the normative perspective, ambiguity is also an active area of research within the domains of decision theory and operations research. Gilboa and Schmeidler (1989) consider ambiguity as a set of possible probability distributions, and present the Maxmin Expected Utility (MEU) model, which appeals to ambiguity averse decision makers. To accommodate the heterogeneity of ambiguity and risk attitudes found in experiments, Ghirardato et al. (2004), based on Hurwicz criterion(Hurwicz, 1951; Arrow and Hurwicz, 1972), axiomatize the $\alpha-$MEU model, which represents a compromise via a convex combination of the worst and best case expected utility. The parameter α is an index of pessimism or optimism. However, the discussion on travel time ambiguity is relatively new. Yu and Yang (1998) propose a worst-case shortest path problem over a set of discrete scenarios, which results in an NP-hard problem. Bertsimas and Sim (2003) introduce the *Budget of Uncertainty Set* in characterizing uncertain travel time and show that the worst-case shortest path problem is a tractable optimization problem. Ord*óñ*es and Stier-Moses (2010) extend the work to address an NE problem.

In contrast to the aforementioned works that consider risk and ambiguity separately, our main contribution is to explicitly distinguish between risk and ambiguity in a unified framework in articulating travelers' preferences for travel time. We present a new criterion named *ambiguity-aware CARA travel time* (ACT) for evaluating uncertain travel times for travelers with various attitudes and degrees of risk and ambiguity. Apart from the behavioral relevance of ACT, we also present a computational justification by showing that when the uncertain travel times of the links along the paths are independently distributed, finding the path that minimizes travel time under the ACT criterion is essentially a shortest path problem. We also study the implications on NE problem, in which travelers minimize their own travel time under the ACT criterion, and no traveler can improve her travel time under ACT by unilaterally changing routes. Our new NE model under the ACT criterion shares similar properties with deterministic multi-class NE model, and can be solved by the traditional Frank-Wolfe algorithm. We also examine the inefficiency of this NE model compared with System Optimum (SO), which minimizes the aggregate travel time under the ACT criterion of all travelers, by deriving its Price of Anarchy.

The remainder of this paper is organized as follows: In Section 2, we formally define the ACT criterion and its properties. In Section 3, we investigate the path selection problem under the ACT criterion. In Section 4, we turn to the study of the NE problem under the ACT criterion and discuss its computational solvability. We also analyze the corresponding NE inefficiency by calculating its Price of

Anarchy. Finally, in Section 5, we make our conclusion and some suggestions for future research.

Notations: We use boldface, e.g., $\boldsymbol{x}$ to represent a vector. We use tilde ($\tilde{}$) to denote uncertain quantities, for example $\tilde{t}$ denotes uncertain travel time. We model uncertainty $\tilde{t}$ by state-space Ω and a $\sigma-$algebra of events in Ω. To model ambiguity, $\tilde{t}$'s true distribution $\mathbb{P}$ is not necessarily specified but is known to lie in a family of distributions, $\mathbb{F}$. We denote by $\mathrm{E}_{\mathbb{P}}\left(\tilde{t}\right)$ the expectation of $\tilde{t}$ under the probability distribution $\mathbb{P}$.

2 Preferences for travel time under risk and ambiguity

In De Palma and Picard (2005) empirical study, they conclude that the exponential function, which appeals to travelers with Constant Absolute Risk Aversion (CARA), aptly characterizes traveler's preference for travel time under risk. Specifically, the exponential disutility function $u(x)$, has the following form,

$$u(x) = \begin{cases} \frac{1}{\lambda}\exp(\lambda x), & \text{when } \lambda \neq 0; \\ ax + b, & \text{when } \lambda = 0, \end{cases} \tag{1}$$

in which $a > 0$ and the parameter $\lambda \in \Re$ is known as the coefficient of absolute risk aversion. The corresponding certainty equivalent of $\tilde{t}$ is

$$CE_{\lambda}\left(\tilde{t}\right) = \begin{cases} \frac{1}{\lambda}\ln \mathrm{E}_{\mathbb{P}}\left(\exp\left(\lambda\tilde{t}\right)\right), & \text{when } \lambda \neq 0; \\ \mathrm{E}_{\mathbb{P}}\left(\tilde{t}\right), & \text{when } \lambda = 0. \end{cases} \tag{2}$$

This function can be derived through calculating moment generating function of random variable $\tilde{t}$. For example, if $\tilde{t}$ is normally distributed $N(\mu,\sigma^2)$, we have $\mathrm{E}_{\mathbb{P}}\left(\exp\left(\lambda\tilde{t}\right)\right) = \exp\left(\lambda\mu + \frac{1}{2}\sigma^2\lambda^2\right)$, thus,

$$CE_{\lambda}\left(\tilde{t}\right) = \begin{cases} \mu + \frac{1}{2}\lambda\sigma^2, & \text{when } \lambda \neq 0; \\ \mu, & \text{when } \lambda = 0. \end{cases}$$

Parameter λ specifies the risk attitudes of the traveler. If $\lambda > 0$, she is risk averse and evaluates an uncertain travel time as longer than the average travel time. In contrast, a traveler with risk seeking attitude has $\lambda < 0$ and prefers an uncertain travel time over the mean travel time. At neutrality ($\lambda = 0$), the traveler is indifferent between the uncertain travel time and its mean.

If the actual travel time probability distribution is not fully known, then it would not be possible to establish the preferences for travel time based on the expected disutility criterion. We study the preference for uncertain travel time in which the traveler is oblivious to the true probability distribution of the travel time, $\mathbb{P}$ but knows the family of probability distributions, $\mathbb{F}$ that $\mathbb{P}$ belongs to. The family of probability distributions, $\mathbb{F}$ can be characterized by the descriptive statistics of the uncertain travel time. The "size" of the set $\mathbb{F}$ indicates the level of ambiguity perceived by the traveler. For instance, the family of distributions perceived by an informed traveler is a subset of the family of distributions perceived by a clueless traveler. In the absent of ambiguity, we have $\mathbb{F} = \{\mathbb{P}\}$, where $\mathbb{P}$ is the true probability distribution of the uncertain travel time. In evaluating ambiguity preference, the Hurwicz criterion represents a compromise between the worst-case and the best case evaluation of travel time under distributional ambiguity as follows:

$$H_\alpha\left(\tilde{t}\right) = \alpha \sup_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_\mathbb{P}\left(\tilde{t}\right) + (1-\alpha) \inf_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_\mathbb{P}\left(\tilde{t}\right),$$

where the parameter $\alpha \in [0,1]$ indicates the level of optimism, with $\alpha = 0$ being the most optimistic and $\alpha = 1$ being the most pessimistic.

2.1 Ambiguity-aware CARA travel time (ACT)

Instead of considering risk and ambiguity separately, we explicitly distinguish between them in a unified framework for articulating travelers' preferences for travel time. We propose the ambiguity-aware CARA travel time (ACT) criterion for evaluating uncertain travel time under various attitudes of risk and ambiguity, which is based on blending Hurwicz and Constant Absolute Risk Aversion (CARA) criteria.

The traveler perceives $\mathbb{F}$ as the family of probability distributions that characterizes the uncertain travel time. Similar to the Hurwicz criterion, her parameter to ambiguity is given by $\alpha \in [0,1]$ and her risk attitude under CARA is given by the parameter $\lambda \in \Re$. Accordingly, we identify the traveler under ACT by $V = (\alpha, \lambda, \mathbb{F})$.

Definition 1 *The ambiguity-aware CARA travel time specified by $V = (\alpha, \lambda, \mathbb{F})$ is given by*

$$ACT_V\left(\tilde{t}\right) = \begin{cases} \alpha \sup_{\mathbb{P}\in\mathbb{F}} \frac{1}{\lambda} \ln \mathrm{E}_\mathbb{P}\left(\exp\left(\lambda\tilde{t}\right)\right) + (1-\alpha) \inf_{\mathbb{P}\in\mathbb{F}} \frac{1}{\lambda} \ln \mathrm{E}_\mathbb{P}\left(\exp\left(\lambda\tilde{t}\right)\right), & \textit{when } \lambda \neq 0; \\ \alpha \sup_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_\mathbb{P}\left(\tilde{t}\right) + (1-\alpha) \inf_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_\mathbb{P}\left(\tilde{t}\right), & \textit{when } \lambda = 0. \end{cases}$$

Observing that if probability distribution is known, i.e., $\mathbb{F} = \{\mathbb{P}\}$, we have

$$ACT_V\left(\tilde{t}\right) = ACT_{(\alpha,\lambda,\mathbb{P})}\left(\tilde{t}\right) = \alpha \frac{1}{\lambda} \ln \mathrm{E}_\mathbb{P}\left(\exp\left(\lambda\tilde{t}\right)\right) + (1-\alpha) \frac{1}{\lambda} \ln \mathrm{E}_\mathbb{P}\left(\exp\left(\lambda\tilde{t}\right)\right) = CE_\lambda\left(\tilde{t}\right).$$

Hence, the ACT criterion is therefore a generalization of certainty equivalent function under CARA. We provide some useful properties of the ACT criterion. We assume that the travel time is bounded in $\left[\underline{t}, \overline{t}\right]$,

$\infty > \overline{t} \geq \underline{t} > 0$. Hence, $\mathbb{F} \subseteq \left\{\mathbb{P} : \mathbb{P}\left(\tilde{t} \in \left[\underline{t}, \overline{t}\right]\right) = 1\right\}$. Moreover, $\left[\underline{t}, \overline{t}\right]$ is the minimum bound support, in the sense that, $\forall \epsilon > 0$, there exists $\mathbb{P} \in \mathbb{F}$ such that $\min\left\{\mathbb{P}\left(\tilde{t} \in \left[\underline{t}, \underline{t} + \epsilon\right]\right), \mathbb{P}\left(\tilde{t} \in \left[\overline{t} - \epsilon, \overline{t}\right]\right)\right\} > 0$.

Proposition 1

(a) *For $\lambda \in \Re, \alpha \in [0,1]$, $ACT_V\left(\tilde{t}\right)$ is nondecreasing in λ and α;*

(b)

$$\lim_{\lambda \to +\infty} ACT_{(1,\lambda,\mathbb{F})}\left(\tilde{t}\right) = \overline{t},$$
$$\lim_{\lambda \to -\infty} ACT_{(0,\lambda,\mathbb{F})}\left(\tilde{t}\right) = \underline{t}.$$

(c) *Suppose $\tilde{t}_1, \ldots, \tilde{t}_J$ are independent random variables, and $t_0 \in \Re$. Then*

$$ACT_V\left(t_0 + \sum_{j=1}^{J} \tilde{t}_j\right) = t_0 + \sum_{j=1}^{J} ACT_V\left(\tilde{t}_j\right).$$

Proof : (a) Note that $ACT_V\left(\tilde{t}\right)$ being nondecreasing in α follows directly from $\sup_{\mathbb{P}\in\mathbb{F}} \frac{1}{\lambda} \ln \mathrm{E}_{\mathbb{P}}\left(\exp\left(\lambda\tilde{t}\right)\right) \geq \inf_{\mathbb{P}\in\mathbb{F}} \frac{1}{\lambda} \ln \mathrm{E}_{\mathbb{P}}\left(\exp\left(\lambda\tilde{t}\right)\right)$ and $\sup_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}}\left(\tilde{t}\right) \geq \inf_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}}\left(\tilde{t}\right)$. Based on Jensen's inequality, for any $\lambda_1, \lambda_2 \in \Re \backslash \{0\}$ and $\lambda_1 \leq \lambda_2$, we can get

$$\frac{1}{\lambda_2} \ln \mathrm{E}_{\mathbb{P}}\left(\exp\left(\lambda_2\tilde{t}\right)\right) = \frac{1}{\lambda_2} \ln \mathrm{E}_{\mathbb{P}}\left(\left(\exp\left(\lambda_1\tilde{t}\right)\right)^{\lambda_2/\lambda_1}\right) \geq \frac{1}{\lambda_2}\frac{\lambda_2}{\lambda_1} \ln \mathrm{E}_{\mathbb{P}}\left(\exp\left(\lambda_1\tilde{t}\right)\right) = \frac{1}{\lambda_1} \ln \mathrm{E}_{\mathbb{P}}\left(\exp\left(\lambda_1\tilde{t}\right)\right),$$

and if $\lambda_1 < 0 < \lambda_2$,

$$\frac{1}{\lambda_2} \ln \mathrm{E}_{\mathbb{P}}\left(\exp\left(\lambda_2\tilde{t}\right)\right) \geq \frac{1}{\lambda_2} \ln \exp\left(\mathrm{E}_{\mathbb{P}}\left(\lambda_2\tilde{t}\right)\right) = \mathrm{E}_{\mathbb{P}}\left(\tilde{t}\right) \geq \frac{1}{\lambda_1} \ln \mathrm{E}_{\mathbb{P}}\left(\exp\left(\lambda_1\tilde{t}\right)\right).$$

Therefore, for any $\lambda_1 \leq \lambda_2$,

$$\begin{aligned}
ACT_{(\alpha,\lambda_2,\mathbb{F})}\left(\tilde{t}\right) &= \alpha \sup_{\mathbb{P}\in\mathbb{F}} \frac{1}{\lambda_2} \ln \mathrm{E}_{\mathbb{P}}\left(\exp\left(\lambda_2\tilde{t}\right)\right) + (1-\alpha) \inf_{\mathbb{P}\in\mathbb{F}} \frac{1}{\lambda_2} \ln \mathrm{E}_{\mathbb{P}}\left(\exp\left(\lambda_2\tilde{t}\right)\right) \\
&\geq \alpha \sup_{\mathbb{P}\in\mathbb{F}} \frac{1}{\lambda_1} \ln \mathrm{E}_{\mathbb{P}}\left(\exp\left(\lambda_1\tilde{t}\right)\right) + (1-\alpha) \inf_{\mathbb{P}\in\mathbb{F}} \frac{1}{\lambda_1} \ln \mathrm{E}_{\mathbb{P}}\left(\exp\left(\lambda_1\tilde{t}\right)\right) \\
&= ACT_{(\alpha,\lambda_1,\mathbb{F})}\left(\tilde{t}\right).
\end{aligned}$$

Equivalently, $ACT_V\left(\tilde{t}\right)$ is nondecreasing in λ.

(b) When $\alpha = 1$, the traveler is most pessimistic towards ambiguity, then

$$ACT_{(1,\lambda,\mathbb{F})}\left(\tilde{t}\right) = \begin{cases} \sup_{\mathbb{P}\in\mathbb{F}} \frac{1}{\lambda} \ln \mathrm{E}_{\mathbb{P}}\left(\exp\left(\lambda\tilde{t}\right)\right), & \text{when } \lambda \neq 0; \\ \sup_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}}\left(\tilde{t}\right), & \text{when } \lambda = 0. \end{cases}$$

We have for any $\mathbb{P} \in \mathbb{F}$ and $\lambda \in \Re \backslash \{0\}$,

$$\frac{1}{\lambda} \ln \mathrm{E}_{\mathbb{P}}\left(\exp\left(\lambda\tilde{t}\right)\right) \leq \frac{1}{\lambda} \ln\left(\exp\left(\lambda\overline{t}\right)\right) = \overline{t}.$$

Therefore,

$$\lim_{\lambda \to +\infty} ACT_{(1,\lambda,\mathbb{F})}\left(\tilde{t}\right) \leq \overline{t}.$$

Moreover, for some $\mathbb{P} \in \mathbb{F}$ such that $\min\left\{\mathbb{P}\left(\tilde{t} \in [\underline{t}, \underline{t}+\epsilon]\right), \mathbb{P}\left(\tilde{t} \in \left[\overline{t}-\epsilon, \overline{t}\right]\right)\right\} > 0$, we have

$$\begin{aligned}
& \lim_{\lambda \to +\infty} \frac{1}{\lambda} \ln \mathrm{E}_{\mathbb{P}}\left(\exp\left(\lambda \tilde{t}\right)\right) \\
= & \lim_{\lambda \to +\infty} \frac{1}{\lambda} \ln \mathrm{E}_{\mathbb{P}}\left(\exp\left(\lambda \overline{t}\right) \exp\left(\lambda\left(\tilde{t}-\overline{t}\right)\right)\right) \\
= & \ \overline{t} + \lim_{\lambda \to +\infty} \frac{1}{\lambda} \ln \mathrm{E}_{\mathbb{P}}\left(\exp\left(\lambda\left(\tilde{t}-\overline{t}\right)\right)\right) \\
\geq & \ \overline{t} + \lim_{\lambda \to +\infty} \frac{1}{\lambda} \ln\left(\exp\left(\lambda\left(\overline{t}-\epsilon-\overline{t}\right)\right) \mathbb{P}\left(\tilde{t} \in \left[\overline{t}-\epsilon, \overline{t}\right]\right)\right) && \forall \epsilon > 0 \\
= & \ \overline{t} + \lim_{\lambda \to +\infty} \frac{1}{\lambda} \ln\left(\exp(-\lambda\epsilon)\right) + \lim_{\lambda \to +\infty} \frac{1}{\lambda} \ln\left(\mathbb{P}\left(\tilde{t} \in \left[\overline{t}-\epsilon, \overline{t}\right]\right)\right) && \forall \epsilon > 0 \\
= & \ \overline{t} - \epsilon && \forall \epsilon > 0
\end{aligned}$$

which means,

$$\lim_{\lambda \to +\infty} \sup_{\mathbb{P} \in \mathbb{F}} \frac{1}{\lambda} \ln \mathrm{E}_{\mathbb{P}}\left(\exp\left(\lambda \tilde{t}\right)\right) \geq \lim_{\lambda \to +\infty} \frac{1}{\lambda} \ln \mathrm{E}_{\mathbb{P}}\left(\exp\left(\lambda \tilde{t}\right)\right) \geq \overline{t} - \epsilon \qquad \forall \epsilon > 0.$$

Combining these two inequalities together, we have

$$\lim_{\lambda \to +\infty} ACT_{(1,\lambda,\mathbb{F})}\left(\tilde{t}\right) = \overline{t}.$$

Similarly, we can modify the above proof to show that

$$\lim_{\lambda \to -\infty} ACT_{(0,\lambda,\mathbb{F})}\left(\tilde{t}\right) = \underline{t}.$$

(c) Since $\tilde{t}_1, \ldots, \tilde{t}_J$ are independently distributed, we have

$$\begin{aligned}
& ACT_V\left(t_0 + \sum_{j=1}^J \tilde{t}_j\right) \\
= & \ \alpha \sup_{\mathbb{P} \in \mathbb{F}} \frac{1}{\lambda} \ln \mathrm{E}_{\mathbb{P}}\left(\exp\left(\lambda\left(t_0 + \sum_{j=1}^J \tilde{t}_j\right)\right)\right) + (1-\alpha) \inf_{\mathbb{P} \in \mathbb{F}} \frac{1}{\lambda} \ln \mathrm{E}_{\mathbb{P}}\left(\exp\left(\lambda\left(t_0 + \sum_{j=1}^J \tilde{t}_j\right)\right)\right) \\
= & \ \alpha t_0 + \alpha \sup_{\mathbb{P} \in \mathbb{F}} \frac{1}{\lambda} \ln\left(\prod_{j=1}^J \mathrm{E}_{\mathbb{P}}\left(\exp\left(\lambda \tilde{t}_j\right)\right)\right) + (1-\alpha) t_0 + (1-\alpha) \inf_{\mathbb{P} \in \mathbb{F}} \frac{1}{\lambda} \ln\left(\prod_{j=1}^J \mathrm{E}_{\mathbb{P}}\left(\exp\left(\lambda \tilde{t}_j\right)\right)\right) \\
= & \ t_0 + \sum_{j=1}^J \left(\alpha \sup_{\mathbb{P} \in \mathbb{F}} \frac{1}{\lambda} \ln \mathrm{E}_{\mathbb{P}}\left(\exp\left(\lambda \tilde{t}_j\right)\right) + (1-\alpha) \inf_{\mathbb{P} \in \mathbb{F}} \frac{1}{\lambda} \ln \mathrm{E}_{\mathbb{P}}\left(\exp\left(\lambda \tilde{t}_j\right)\right)\right) \\
= & \ t_0 + \sum_{j=1}^J ACT_V\left(\tilde{t}_j\right).
\end{aligned}$$

■

Next, we will provide an example to illustrate travelers' preferences for travel time under the ACT criterion. Figure 1 shows three paths from the origin, O to the destination, D. The travel time on path A is deterministic, 1.5hrs; travel time on path B is stochastic and the duration is 1hr or 2hrs with equal probability; travel time on path C is uncertain and between 1hr to 2hrs. We present in Table 1 the path preferences induced by the ACT criterion under various attitudes and degrees of risk and ambiguity.

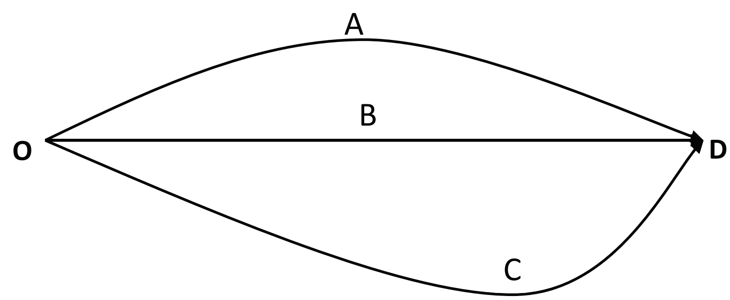

Figure 1: A simple network with uncertain travel time.

When a traveler is extremely risk averse and pessimistic towards ambiguity ($\lambda \to +\infty, \alpha = 1$), she will perceive the uncertain travel time as taking the longest duration. Hence, path A is preferred. On the other hand, when the traveler is radically risk seeking and optimistic towards ambiguity ($\lambda \to -\infty, \alpha = 0$), then path A would be least preferred. At risk neutrality, both paths A and B are equally preferred and the preference for path C depends on the traveler's attitude towards ambiguity. For instance, if she is optimistic towards ambiguity, then path C will be preferred over paths A and B.

Risk attitude λ	**Ambiguity attitude** α	$ACT_V(\tilde{t}_A)$	$ACT_V(\tilde{t}_B)$	$ACT_V(\tilde{t}_C)$	**Preferences**
$+\infty$	1	1.5	2	2	$A \succ B \sim C$
0	1	1.5	1.5	2	$A \sim B \succ C$
0	0	1.5	1.5	1	$C \succ A \sim B$
$-\infty$	0	1.5	1	1	$B \sim C \succ A$

Table 1: Preferences for travel time under the ACT criterion.

2.2 A simple uncertainty model for travel time

We propose a simple model of uncertainty for characterizing the uncertain travel time and provide the analytical form of the ACT criterion. Driven by pragmatism, we assume that the traveler has a simple description of the uncertain travel time by providing the ranges of values in which the travel time and the average travel time would fall within. Specifically, the travel time takes values in $[\underline{t}, \overline{t}]$, $0 < \underline{t} \leq \overline{t}$

and the average travel time falls within the range $[\underline{\mu}, \overline{\mu}] \subseteq [\underline{t}, \overline{t}]$. Hence, the family of distributions of the uncertain travel times, $\tilde{t}$ is given by

$$\mathbb{F} = \left\{ \mathbb{P} \ : \ \mathrm{E}_{\mathbb{P}}\left(\tilde{t}\right) \in \left[\underline{\mu}, \overline{\mu}\right], \mathbb{P}\left(\tilde{t} \in \left[\underline{t}, \overline{t}\right]\right) = 1 \right\}.$$

Proposition 2 *For the family of distributions,*

$$\mathbb{F} = \left\{ \mathbb{P} : \mathrm{E}_{\mathbb{P}}\left(\tilde{t}\right) \in \left[\underline{\mu}, \overline{\mu}\right], \mathbb{P}\left(\tilde{t} \in \left[\underline{t}, \overline{t}\right]\right) = 1 \right\},$$

the ACT criterion in which $V = (\alpha, \lambda, \mathbb{F})$ is

$$ACT_V\left(\tilde{t}\right) = \begin{cases} \frac{\alpha}{\lambda} \ln\left(\frac{\left(\overline{t}-\overline{\mu}\right)\exp(\lambda \underline{t}) + (\overline{\mu}-\underline{t})\exp\left(\lambda \overline{t}\right)}{\overline{t}-\underline{t}}\right) + (1-\alpha)\underline{\mu}, & \text{when } \lambda > 0; \\ \alpha\overline{\mu} + \frac{1-\alpha}{\lambda} \ln\left(\frac{\left(\overline{t}-\underline{\mu}\right)\exp(\lambda \underline{t}) + \left(\underline{\mu}-\underline{t}\right)\exp\left(\lambda \overline{t}\right)}{\overline{t}-\underline{t}}\right), & \text{when } \lambda < 0; \\ \alpha\overline{\mu} + (1-\alpha)\underline{\mu}, & \text{when } \lambda = 0. \end{cases}$$

Moreover,

$$\lim_{\lambda \to +\infty} ACT_V\left(\tilde{t}\right) = \alpha\overline{t} + (1-\alpha)\underline{\mu},$$
$$\lim_{\lambda \to -\infty} ACT_V\left(\tilde{t}\right) = (1-\alpha)\underline{t} + \alpha\overline{\mu}.$$

Proof : We first determine the analytical expressions for $\sup_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}}\left(\exp\left(\lambda\tilde{t}\right)\right)$ and $\inf_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}}\left(\exp\left(\lambda\tilde{t}\right)\right)$. We formulate $\sup_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}}\left(\exp\left(\lambda\tilde{t}\right)\right)$ as a convex optimization problem as follows.

$$\begin{array}{rrcl} \max\limits_{\mathbb{P}} & \mathrm{E}_{\mathbb{P}}\left(\exp\left(\lambda\tilde{t}\right)\right) & & \\ \text{s.t.} & \mathrm{E}_{\mathbb{P}}(1) & = & 1 \\ & \mathrm{E}_{\mathbb{P}}\left(\tilde{t}\right) & \leq & \overline{\mu} \\ & \mathrm{E}_{\mathbb{P}}\left(\tilde{t}\right) & \geq & \underline{\mu} \\ & \mathbb{P}\left(\left\{t \in \left[\underline{t}, \overline{t}\right]\right\}\right) & = & 1. \end{array}$$

By weak duality, we have

$$\begin{array}{rrrcl} \max\limits_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}}\left(\exp\left(\lambda\tilde{t}\right)\right) \leq & \min & y_0 + \overline{\mu}y_1 - \underline{\mu}y_2 & & \\ & \text{s.t.} & y_0 + ty_1 - ty_2 & \geq & \exp(\lambda t), \quad \forall t \in \left[\underline{t}, \overline{t}\right], \\ & & y_1, y_2 & \geq & 0. \end{array} \tag{3}$$

Since the function $\exp(\lambda t) + (y_2 - y_1)t$ is convex in t, we note that

$$y_0 \geq \max_{t\in\left[\underline{t},\overline{t}\right]} \{\exp(\lambda t) + (y_2 - y_1)t\} = \max\left\{\exp\left(\lambda\underline{t}\right) + (y_2 - y_1)\underline{t}\ , \ \exp\left(\lambda\overline{t}\right) + (y_2 - y_1)\overline{t}\right\}. \tag{4}$$

Therefore,

$$
\begin{aligned}
& \sup_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}}\left(\exp\left(\lambda\tilde{t}\right)\right) \\
\leq\ & \min_{y_1,y_2\geq 0} \max\left\{\exp\left(\lambda\underline{t}\right) + \left(\overline{\mu}-\underline{t}\right)y_1 + \left(\underline{t}-\underline{\mu}\right)y_2\ ,\ \exp\left(\lambda\overline{t}\right) + \left(\overline{\mu}-\overline{t}\right)y_1 + \left(\overline{t}-\underline{\mu}\right)y_2\right\}. \\
=\ & \begin{cases} \frac{\left(\overline{t}-\overline{\mu}\right)\exp\left(\lambda\underline{t}\right)+\left(\overline{\mu}-\underline{t}\right)\exp\left(\lambda\overline{t}\right)}{\overline{t}-\underline{t}}, & \text{when } \lambda>0; \\ \frac{\left(\overline{t}-\underline{\mu}\right)\exp\left(\lambda\underline{t}\right)+\left(\underline{\mu}-\underline{t}\right)\exp\left(\lambda\overline{t}\right)}{\overline{t}-\underline{t}}, & \text{when } \lambda<0. \end{cases}
\end{aligned}
$$

Observe that the optimal value can be achieved under the two point distribution,

$$
\begin{cases} \mathbb{P}\left(\tilde{t}=\overline{t}\right) = \frac{\overline{\mu}-\underline{t}}{\overline{t}-\underline{t}}, \quad \mathbb{P}\left(\tilde{t}=\underline{t}\right) = \frac{\overline{t}-\overline{\mu}}{\overline{t}-\underline{t}}, & \text{when } \lambda>0; \\ \mathbb{P}\left(\tilde{t}=\overline{t}\right) = \frac{\underline{\mu}-\underline{t}}{\overline{t}-\underline{t}}, \quad \mathbb{P}\left(\tilde{t}=\underline{t}\right) = \frac{\overline{t}-\underline{\mu}}{\overline{t}-\underline{t}}, & \text{when } \lambda>0, \end{cases}
$$

hence, strong duality holds.

To determine $\inf_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}}\left(\exp\left(\lambda\tilde{t}\right)\right)$, we note that by Jensen's inequality,

$$
\mathrm{E}_{\mathbb{P}}\left(\exp\left(\lambda\tilde{t}\right)\right) \geq \exp\left(\mathrm{E}_{\mathbb{P}}\left(\lambda\tilde{t}\right)\right) = \exp\left(\lambda\mathrm{E}_{\mathbb{P}}\left(\tilde{t}\right)\right),
$$

consequently,

$$
\inf_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}}\left(\exp\left(\lambda\tilde{t}\right)\right) \geq \begin{cases} \exp\left(\lambda\underline{\mu}\right), & \text{when } \lambda>0; \\ \exp\left(\lambda\overline{\mu}\right), & \text{when } \lambda<0. \end{cases}
$$

Equality holds when $\tilde{t}$ is deterministic,

$$
\begin{cases} \mathbb{P}\left(\tilde{t}=\underline{\mu}\right)=1, & \text{when } \lambda>0; \\ \mathbb{P}\left(\tilde{t}=\overline{\mu}\right)=1, & \text{when } \lambda<0. \end{cases}
$$

Accordingly, when $\lambda>0$,

$$
\begin{aligned}
ACT_V\left(\tilde{t}\right) &= \alpha \sup_{\mathbb{P}\in\mathbb{F}} \frac{1}{\lambda}\ln \mathrm{E}_{\mathbb{P}}\left(\exp\left(\lambda\tilde{t}\right)\right) + (1-\alpha)\inf_{\mathbb{P}\in\mathbb{F}} \frac{1}{\lambda}\ln \mathrm{E}_{\mathbb{P}}\left(\exp\left(\lambda\tilde{t}\right)\right) \\
&= \alpha\frac{1}{\lambda}\ln \sup_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}}\left(\exp\left(\lambda\tilde{t}\right)\right) + (1-\alpha)\frac{1}{\lambda}\ln \inf_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}}\left(\exp\left(\lambda\tilde{t}\right)\right) \\
&= \frac{\alpha}{\lambda}\ln\left(\frac{\left(\overline{t}-\overline{\mu}\right)\exp\left(\lambda\underline{t}\right)+\left(\overline{\mu}-\underline{t}\right)\exp\left(\lambda\overline{t}\right)}{\overline{t}-\underline{t}}\right) + (1-\alpha)\underline{\mu},
\end{aligned}
$$

and when $\lambda<0$,

$$
\begin{aligned}
ACT_V\left(\tilde{t}\right) &= \alpha \sup_{\mathbb{P}\in\mathbb{F}} \frac{1}{\lambda}\ln \mathrm{E}_{\mathbb{P}}\left(\exp\left(\lambda\tilde{t}\right)\right) + (1-\alpha)\inf_{\mathbb{P}\in\mathbb{F}} \frac{1}{\lambda}\ln \mathrm{E}_{\mathbb{P}}\left(\exp\left(\lambda\tilde{t}\right)\right) \\
&= \alpha\frac{1}{\lambda}\ln \inf_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}}\left(\exp\left(\lambda\tilde{t}\right)\right) + (1-\alpha)\frac{1}{\lambda}\ln \sup_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}}\left(\exp\left(\lambda\tilde{t}\right)\right) \\
&= \alpha\overline{\mu} + \frac{1-\alpha}{\lambda}\ln\left(\frac{\left(\overline{t}-\underline{\mu}\right)\exp(\lambda\underline{t})+\left(\underline{\mu}-\underline{t}\right)\exp(\lambda\overline{t})}{\overline{t}-\underline{t}}\right).
\end{aligned}
$$

Moreover, when $\lambda = 0$,

$$\begin{aligned} ACT_V\left(\tilde{t}\right) &= \alpha \sup_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}}\left(\tilde{t}\right) + (1-\alpha) \inf_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}}\left(\tilde{t}\right) \\ &= \alpha\overline{\mu} + (1-\alpha)\underline{\mu}. \end{aligned}$$

Based on L'Hôpital's rule, when $\lambda > 0$,

$$\begin{aligned} \lim_{\lambda\to+\infty} ACT_V\left(\tilde{t}\right) &= \lim_{\lambda\to+\infty}\left(\frac{\alpha}{\lambda}\ln\left(\frac{\left(\overline{t}-\overline{\mu}\right)\exp\left(\lambda\underline{t}\right)+\left(\overline{\mu}-\underline{t}\right)\exp\left(\lambda\overline{t}\right)}{\overline{t}-\underline{t}}\right)+(1-\alpha)\underline{\mu}\right) \\ &= \lim_{\lambda\to+\infty}\left(\frac{\alpha}{\lambda}\ln\left(\frac{\left(\overline{t}-\overline{\mu}\right)\exp\left(\lambda\underline{t}\right)+\left(\overline{\mu}-\underline{t}\right)\exp\left(\lambda\overline{t}\right)}{\overline{t}-\underline{t}}\right)\right)+(1-\alpha)\underline{\mu} \\ &= \alpha\lim_{\lambda\to+\infty}\left(\frac{\left(\overline{t}-\overline{\mu}\right)\exp\left(\lambda\underline{t}\right)\underline{t}+\left(\overline{\mu}-\underline{t}\right)\exp\left(\lambda\overline{t}\right)\overline{t}}{\left(\overline{t}-\overline{\mu}\right)\exp\left(\lambda\underline{t}\right)+\left(\overline{\mu}-\underline{t}\right)\exp\left(\lambda\overline{t}\right)}\right)+(1-\alpha)\underline{\mu} \\ &= \alpha\lim_{\lambda\to+\infty}\left(\frac{\left(\overline{t}-\overline{\mu}\right)\exp\left(\lambda(\underline{t}-\overline{t})\right)\underline{t}+\left(\overline{\mu}-\underline{t}\right)\overline{t}}{\left(\overline{t}-\overline{\mu}\right)\exp\left(\lambda(\underline{t}-\overline{t})\right)+\left(\overline{\mu}-\underline{t}\right)}\right)+(1-\alpha)\underline{\mu} \\ &= \alpha\overline{t}+(1-\alpha)\underline{\mu}. \end{aligned}$$

Likewise, the result extends to

$$\lim_{\lambda\to-\infty} ACT_V\left(\tilde{t}\right) = (1-\alpha)\underline{t} + \alpha\overline{\mu}.$$

■

Based on the uncertainty model, we further analyze the paths preferences of the simple network depicted in Figure 1. Travel times on path A and C remain unchanged. As for path B, we now assume that the travel time is within 1hr to 2hrs, and the mean travel time is exactly 1.5hrs.

Proposition 3 *Given the above information of three paths, traveler's preferences ranked by the ACT criterion are*

Risk attitude λ	Ambiguity attitude α	Preferences
$[0, +\infty)$	$[f(\lambda), 1]$	$A \succeq B \succeq C$
$[0, +\infty)$	$\left[\frac{1}{2}, f(\lambda)\right]$	$A \succeq C \succeq B$
$[0, +\infty)$	$\left[0, \frac{1}{2}\right]$	$C \succeq A \succeq B$
$(-\infty, 0]$	$\left[\frac{1}{2}, 1\right]$	$B \succeq A \succeq C$
$(-\infty, 0]$	$\left[g(\lambda), \frac{1}{2}\right]$	$B \succeq C \succeq A$
$(-\infty, 0]$	$[0, g(\lambda)]$	$C \succeq B \succeq A$

where $f(\lambda) = \frac{\frac{1}{2}}{\frac{3}{2}-\frac{1}{\lambda}\ln\left(\frac{1}{2}+\frac{1}{2}\exp(\lambda)\right)}$, *and* $g(\lambda) = \frac{\frac{1}{\lambda}\ln\left(\frac{1}{2}+\frac{1}{2}\exp(\lambda)\right)}{\frac{1}{2}+\frac{1}{\lambda}\ln\left(\frac{1}{2}+\frac{1}{2}\exp(\lambda)\right)}$.

Proof : From Proposition 2, we calculate the travel time under the ACT criterion for each of the three paths as follows:

$$
\begin{aligned}
ACT_V(t_A) &= \frac{3}{2}; \\
ACT_V\left(\tilde{t}_B\right) &= \begin{cases} \frac{\alpha}{\lambda}\ln\left(\frac{1}{2}\exp(\lambda)+\frac{1}{2}\exp(2\lambda)\right)+\frac{3}{2}(1-\alpha), & \text{when } \lambda>0; \\ \frac{3}{2}\alpha+\frac{1-\alpha}{\lambda}\ln\left(\frac{1}{2}\exp(\lambda)+\frac{1}{2}\exp(2\lambda)\right), & \text{when } \lambda<0; \\ \frac{3}{2}\alpha+\frac{3}{2}(1-\alpha), & \text{when } \lambda=0; \end{cases} \\
ACT_V\left(\tilde{t}_C\right) &= \begin{cases} \frac{\alpha}{\lambda}\ln\left(\exp(2\lambda)\right)+(1-\alpha), & \text{when } \lambda>0; \\ 2\alpha+\frac{1-\alpha}{\lambda}\ln\left(\exp(\lambda)\right), & \text{when } \lambda<0; \\ 2\alpha+(1-\alpha), & \text{when } \lambda=0 \end{cases} \\
&= 1+\alpha.
\end{aligned}
$$

Since the travel time under the ACT criterion is nondecreasing in both λ and α, the preference relationships between paths A and B, and between paths A and C can be readily established. When $\lambda \geq 0$, we have $A \succeq B$. Likewise, when $1 \geq \alpha \geq \frac{1}{2}$, then $A \succeq C$. Hence, we focus on the preferences between paths B and C.

$ACT_V\left(\tilde{t}_B\right) \geq ACT_V\left(\tilde{t}_C\right)$ implies

$$
\begin{cases} \frac{\alpha}{\lambda}\ln\left(\frac{1}{2}\exp(\lambda)+\frac{1}{2}\exp(2\lambda)\right)+\frac{3}{2}(1-\alpha)\geq 1+\alpha, & \text{when } \lambda>0; \\ \frac{3}{2}\alpha+\frac{1-\alpha}{\lambda}\ln\left(\frac{1}{2}\exp(\lambda)+\frac{1}{2}\exp(2\lambda)\right)\geq 1+\alpha, & \text{when } \lambda<0; \\ \frac{3}{2}\geq 1+\alpha, & \text{when } \lambda=0. \end{cases}
$$

Equivalently, path C is preferred to path B when

$$
\begin{cases} \alpha \leq f(\lambda) = \dfrac{\frac{1}{2}}{\frac{3}{2}-\frac{1}{\lambda}\ln\left(\frac{1}{2}+\frac{1}{2}\exp(\lambda)\right)}, & \text{when } \lambda\geq 0; \\ \alpha \leq g(\lambda) = \dfrac{\frac{1}{\lambda}\ln\left(\frac{1}{2}+\frac{1}{2}\exp(\lambda)\right)}{\frac{1}{2}+\frac{1}{\lambda}\ln\left(\frac{1}{2}+\frac{1}{2}\exp(\lambda)\right)} & \text{when } \lambda\leq 0. \end{cases}
$$

■

The preferences expressed by travelers with varied λ and α are depicted in Figure 2. When the traveler is risk averse ($\lambda > 0$), she prefers path A over path B, and the converse is true when the traveler is risk seeking. With α decreases from 1 to 0, the traveler's attitude towards ambiguity shifts from being pessimistic to optimistic, in which case, path C, which has complete ambiguity, will become more favorable. This example may suggest a way to empirically identify travelers' attitudes towards risk and ambiguity based on their chosen paths.

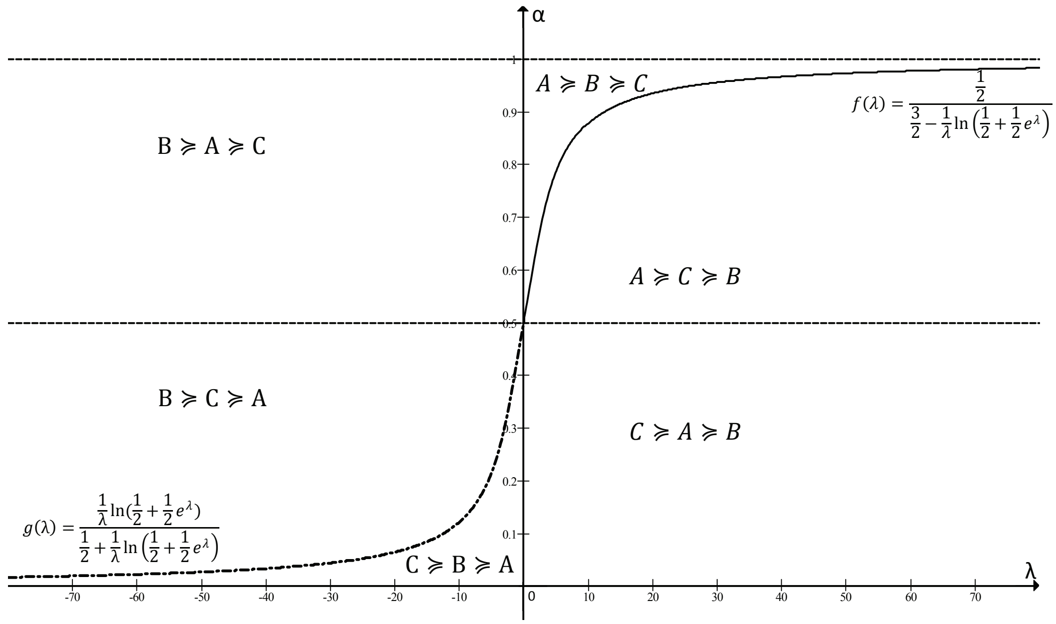

Figure 2: Path preferences under different attitudes of risk and ambiguity.

3 Path selection under the ACT criterion

In this section, we study the problem of selecting the path that minimizes the ACT criterion when the travel times along the links on the network are uncertain. We consider a directed network $G = (\mathcal{N}, \mathcal{A})$ and let $\mathcal{R}$ be the set of all admissible paths, which are sets of links connecting the origin node to the destination node. The uncertain travel time along the link, $a \in \mathcal{A}$ is denoted by $\tilde{t}_a$.

The deterministic version of path selection problem or shortest path problem is well known to be polynomial time solvable. When the travel times along the links are uncertain, the path selection problem that minimizes the travel time under the ACT criterion is given by

$$\min_{r \in \mathcal{R}} ACT_V \left(\sum_{a \in r} \tilde{t}_a \right). \tag{5}$$

In Theorem 1 below, we show that the solvability of Problem (5) depends on whether the uncertain link travel times are correlated.

Theorem 1 **(a)** *If the uncertain link travel times are independently distributed, then Problem (5) is a shortest path problem on the same network in which the link travel time, t_a on $a \in \mathcal{A}$ is given by*

$$t_a = ACT_V \left(\tilde{t}_a \right).$$

(b) *If the uncertain link travel times are correlated, then the recognition version of Problem (5) is NP-complete.*

Proof : (a) According to Proposition 1, if the travel times along the links are independently distributed, the objective function in Problem (5) can be written additively as

$$ACT_V\left(\sum_{a\in r}\tilde{t}_a\right)=\sum_{a\in r}ACT_V\left(\tilde{t}_a\right).$$

In this case, we can regard the travel time under the ACT criterion along each link as the deterministic link travel time, and polynomially solve it by the shortest path algorithm.

(b) We will prove its NP-completeness by reduction from the following problem, which is proved to be NP-completeness by Yu and Yang (1998):

$$\min_{r\in\mathcal{R}}\max\left\{\sum_{a\in r}t_a^1,\sum_{a\in r}t_a^2\right\}, \tag{6}$$

where t_a^1 and t_a^2 are two travel time scenarios on link $a\in\mathcal{A}$.

We construct an instance of Problem (5), in which the uncertain travel time on link a is

$$\tilde{t}_a=\frac{1}{2}\left(t_a^1+t_a^2\right)+\frac{1}{2}\left(t_a^1-t_a^2\right)\tilde{z},\quad \forall a\in\mathcal{A},$$

that is, the travel times of all the links are influenced by a common random variable $\tilde{z}$, which we assume is $+1$ or -1 with equal probability. Hence, for an extremely risk averse and pessimistic towards ambiguity traveler ($\lambda\to+\infty$, $\alpha=1$), finding the path with minimum travel time under the ACT criterion from origin node to destination node can be written as

$$\min_{r\in\mathcal{R}}\lim_{\lambda\to+\infty}\sup_{\mathbb{P}\in\mathbb{F}}\frac{1}{\lambda}\ln\mathrm{E}_{\mathbb{P}}\left(\exp\left(\lambda\sum_{a\in r}\left(\frac{1}{2}\left(t_a^1+t_a^2\right)+\frac{1}{2}\left(t_a^1-t_a^2\right)\tilde{z}\right)\right)\right).$$

According to Proposition 1, it can be simplified further as

$$\min_{r\in\mathcal{R}}\sum_{a\in r}\frac{1}{2}\left(t_a^1+t_a^2\right)+\max\left\{\sum_{a\in r}\frac{1}{2}\left(t_a^1-t_a^2\right),\sum_{a\in r}\frac{1}{2}\left(t_a^2-t_a^1\right)\right\},$$

which could be equivalently written as Problem (6). Thus, Problem (5) is NP-complete. ■

Theorem 1 shows that when the travel times along the links are independently distributed, we can easily find the optimal path selection under the ACT criterion, which accounts for both risk and ambiguity. The result, though simple, shows that the ACT criterion is not only descriptive relevant by being able to account for a traveler's different attitudes of risk and ambiguity over uncertain travel time, it can be used normatively to find the most preferred path using modest computational effort.

4 Analysis of network equilibrium with risk and ambiguity aware travelers

In this section, we study the network equilibrium problem when travelers are sensitive to risk and ambiguity and evaluate the travel times along paths using the ACT criterion. In Section 4.1, we characterize the network equilibrium (NE) such that no traveler could improve her travel time under the ACT criterion by unilaterally changing routes. In Section 4.2, we investigate the inefficiency of the NE by comparing with the System Optimal solution that minimizes the total travel time under the ACT criterion of all travelers. We also provide a simple network equilibrium study in Section 4.3.

4.1 Network equilibrium formulation

Given a network $(\mathcal{N}, \mathcal{A})$, we let $\mathcal{W} \subseteq \mathcal{N} \times \mathcal{N}$ be a set of Origin-Destination (OD) pairs, and $\mathcal{R}_w$ be the set of all simple paths connecting a given OD pair $w \in \mathcal{W}$. To derive a tractable model, we assume that the uncertain travel times along the links are independently distributed. We define the uncertain travel time along link $a \in \mathcal{A}$ as

$$\tilde{t}_a(v_a) = t_a(v_a)\tilde{z}_a + \tilde{\tau}_a,$$

where $t_a(v_a)$ is a differentiable, monotonically increasing function in its own link traffic flow v_a, and $\tilde{z}_a, \tilde{\tau}_a, a \in \mathcal{A}$ are independently distributed nonnegative bounded random variables. The multiplicative uncertainty $\tilde{z}_a$ can be interpreted as the flow dependent disturbance, while $\tilde{\tau}_a$, the additive uncertainty, is the flow independent disturbance.

For generality, we allow travelers to have different perceptions on uncertainty in travel times along the network links. For example, a local resident, who is very familiar with the area, would be less ambiguous, compared to a tourist, in characterizing the uncertain travel times along the network link. To characterize the heterogeneity, we classify all travelers on the network into n types. The ith type of travelers, $i \in \mathcal{I} = \{1, \ldots, n\}$ are characterized by their risk parameter, λ_i, ambiguity parameter, α_i and their perception of the family of probability distributions of the network links' travel times, $\mathbb{F}_i$. For notational convenience, we denote $V_i = (\lambda_i, \alpha_i, \mathbb{F}_i)$.

Under the ACT criterion, the uncertain travel time $\tilde{t}_a(v_a)$ perceived by the ith type of travelers is given by

$$\begin{aligned} t_{ai}(v_a) &= ACT_{V_i}\left(\tilde{t}_a(v_a)\right) \\ &= ACT_{V_i}\left(t_a(v_a)\tilde{z}_a + \tilde{\tau}_a\right) \\ &= ACT_{V_i}\left(t_a(v_a)\tilde{z}_a\right) + ACT_{V_i}\left(\tilde{\tau}_a\right). \end{aligned}$$

For a given OD pair $w \in \mathcal{W}$, let d_{wi} be the number of trips made by the ith type of travelers and f_{ri} be the flow on path $r \in \mathcal{R}_w$ contributed by the ith type of travelers, and $\boldsymbol{f} = (f_{ri})_{r\in\mathcal{R}_w, w\in\mathcal{W}, i\in\mathcal{I}}$ is the vector of flows of all travelers along all paths. The aggregate flow on link $a \in \mathcal{A}$ is

$$v_a(\boldsymbol{f}) = \sum_{i\in\mathcal{I}} \sum_{w\in\mathcal{W}} \sum_{r\in\mathcal{R}_w} f_{ri}\delta_{ar},$$

where δ_{ar} equals 1 if the link a is along the path r and 0 otherwise.

Moreover, since the travel time along any path r is given by

$$\tilde{c}_r(\boldsymbol{f}) = \sum_{a\in\mathcal{A}} \tilde{t}_a\left(v_a(\boldsymbol{f})\right)\delta_{ar},$$

the travel time along path $r \in \mathcal{R}_w$ under the ACT criterion perceived by the ith type of travelers is given by

$$\begin{aligned} c_{ri}(\boldsymbol{f}) &= ACT_{V_i}\left(\tilde{c}_r(\boldsymbol{f})\right) \\ &= ACT_{V_i}\left(\sum_{a\in\mathcal{A}} \left(t_a(v_a(\boldsymbol{f}))\tilde{z}_a + \tilde{\tau}_a\right)\delta_{ar}\right) \\ &= \sum_{a\in\mathcal{A}} \left(ACT_{V_i}\left(t_a(v_a(\boldsymbol{f}))\tilde{z}_a\delta_{ar}\right) + ACT_{V_i}\left(\tilde{\tau}_a\delta_{ar}\right)\right) \\ &= \sum_{a\in\mathcal{A}} t_{ai}(v_a(\boldsymbol{f}))\delta_{ar}. \end{aligned}$$

Let $\boldsymbol{c}(\boldsymbol{f}) = (c_{ri}(\boldsymbol{f}))_{r\in\mathcal{R}_w, w\in\mathcal{W}, i\in\mathcal{I}}$ be the vector of the travel time under the ACT criterion of all types of travelers over all paths; and $\mathcal{F}$ be the feasible set of possible flows on all paths denoted by

$$\mathcal{F} = \left\{\boldsymbol{f} \;:\; \sum_{r\in\mathcal{R}_w} f_{ri} = d_{wi}, \quad w \in \mathcal{W}, i \in \mathcal{I}, \boldsymbol{f} \geq 0\right\}.$$

We can characterize the NE as follows

Definition 2 *A path flow $\boldsymbol{f}^* \in \mathcal{F}$ is a NE if and only if*

$$\begin{aligned} &c_{ri}(\boldsymbol{f}^*) \geq \mu_{wi}, && \forall r \in \mathcal{R}_w, w \in \mathcal{W}, i \in \mathcal{I}; \\ &f^*_{ri}\left(c_{ri}(\boldsymbol{f}^*) - \mu_{wi}\right) = 0, && \forall r \in \mathcal{R}_w, w \in \mathcal{W}, i \in \mathcal{I}, \end{aligned} \tag{7}$$

where $\mu_{wi} \geq 0$.

At NE, the travel time along any path connecting the OD pair, w perceived by the ith type of travelers under the ACT criterion is at least μ_{wi}. Moreover, on the paths that have been actually traveled ($f^*_{ri} > 0$), the perceived travel times are exactly at the minimum $c_{ri}(\boldsymbol{f}^*) = \mu_{wi}$. In other words, no traveler could improve her travel time under the ACT criterion by unilaterally changing routes.

Clearly, we can also formulate the NE by means of Variational Inequalities (VI). We let $\boldsymbol{v} = (v_{ai})_{a\in\mathcal{A},i\in\mathcal{I}}$ be the vector of flows of all travelers along all links, and we have $v_a = \sum_{i\in\mathcal{I}} v_{ai}, a \in \mathcal{A}$. Let $\boldsymbol{t}(\boldsymbol{v}) = (t_{ai}(v_a))_{a\in\mathcal{A},i\in\mathcal{I}}$ be the vector of travel time under the ACT criterion of traveler types and along all links. The set of all feasible link flows is represented by

$$\mathcal{V} = \left\{ \boldsymbol{v} : v_{ai} = \sum_{w\in\mathcal{W}} \sum_{r\in\mathcal{R}_w} f_{ri}\delta_{ar}, \quad a \in \mathcal{A}, i \in \mathcal{I}, \boldsymbol{f} \in \mathcal{F} \right\}.$$

Proposition 4 *The path flow of the NE can be equivalently characterized by the following VI problem: Find $\boldsymbol{f}^* \in \mathcal{F}$, such that*

$$\langle \boldsymbol{f} - \boldsymbol{f}^*, \boldsymbol{c}(\boldsymbol{f}^*) \rangle \geq 0, \qquad \forall \boldsymbol{f} \in \mathcal{F},$$

where $\langle\cdot\rangle$ denotes the Euclidean inner product. Likewise, the link flow of NE is characterized by finding $\boldsymbol{v}^* \in \mathcal{V}$, such that

$$\langle \boldsymbol{v} - \boldsymbol{v}^*, \boldsymbol{t}(\boldsymbol{v}^*) \rangle \geq 0, \qquad \forall \boldsymbol{v} \in \mathcal{V}. \tag{8}$$

Proof : This is an extension of single class deterministic NE problem and we refer interested readers to Smith (1979) and Dafermos (1980). ■

If travelers are homogeneous, i.e., $n = 1$, the NE reduces to the single class deterministic NE model. For the general case, $n > 1$, we could adopt algorithms for solving the generic VI (see Nagurney, 1998; Facchinei and Pang, 2003). Nevertheless, for the special case in which uncertainty along links is flow independent, we will show that the corresponding NE problem can be solved via a convex optimization problem. Under this case, the uncertain travel time on link $a \in \mathcal{A}$ can be simplified as

$$\tilde{t}_a(v_a) = t_a(v_a) + \tilde{\tau}_a,$$

and travel time perceived by the ith type of travelers under the ACT criterion is as follows:

$$t_{ai}(v_a) = t_a(v_a) + ACT_{V_i}(\tilde{\tau}_a).$$

Theorem 2 *When the uncertainty is flow independent, we can compute the NE traffic flow by solving the following convex optimization problem:*

$$\min_{\boldsymbol{v}\in\mathcal{V}} \sum_{a\in\mathcal{A}} \int_0^{v_a} t_a(x)dx + \sum_{a\in\mathcal{A}} \sum_{i\in\mathcal{I}} ACT_{V_i}(\tilde{\tau}_a)\, v_{ai}. \tag{9}$$

Proof : Note that set $\mathcal{V}$ is convex and closed, and

$$\begin{aligned} &\frac{\partial t_{ai}(v_a)}{\partial v_{aj}} = \frac{\partial t_a(v_a)}{\partial v_a} = \frac{\partial t_{aj}(v_a)}{\partial v_{ai}}, \quad && \forall a \in \mathcal{A}, \forall i,j \in \mathcal{I}, i \neq j; \\ &\frac{\partial t_{ai}(v_a)}{\partial v_{bi}} = 0, && \forall a,b \in \mathcal{A}, a \neq b, \forall i \in \mathcal{I}, \end{aligned} \tag{10}$$

the Jacobian matrix of $\boldsymbol{t}(\boldsymbol{v})$ is symmetric $\forall \boldsymbol{v} \in \mathcal{V}$. Moreover, since the travel time function $t_a(v_a)$ is a differentiable, monotonically increasing function with respect to its own link flow v_a, the Jacobian matrix of $\boldsymbol{t}(\boldsymbol{v})$ is positive semidefinite over $\mathcal{V}$. Therefore, it is well known (see, e.g., Facchinei and Pang, 2003) that there exists a function $Z(\boldsymbol{v}) : \mathcal{V} \to \Re$, such that

$$\nabla Z(\boldsymbol{v}) = \boldsymbol{t}(\boldsymbol{v}), \quad \forall \boldsymbol{v} \in \mathcal{V},$$

and $\boldsymbol{v}^*$ is an optimal solution to the convex optimization problem

$$\min_{\boldsymbol{v} \in \mathcal{V}} Z(\boldsymbol{v}),$$

if and only if it solves VI Problem (8) when the uncertainty is flow independent.

Let $Z(\boldsymbol{v}) = \sum\limits_{a \in \mathcal{A}} \int_0^{v_a} t_a(x)dx + \sum\limits_{a \in \mathcal{A}} \sum\limits_{i \in \mathcal{I}} ACT_{V_i}(\tilde{\tau}_a)\, v_{ai}$, we can easily verify that

$$\frac{\partial Z(\boldsymbol{v})}{\partial v_{ai}} = t_a(v_a) + ACT_{V_i}(\tilde{\tau}_a) = t_{ai}(v_a), \quad \forall a \in \mathcal{A}, i \in \mathcal{I}.$$

■

We next derive the existence and uniqueness of the NE traffic flow under the assumption that uncertainty along links is flow independent.

Corollary 1 *If the travel time function is a strictly monotonically increasing function of its own link flow, then*

(a) *the optimal solution of flow for each type of travelers on each link exists, but may not be unique.*

(b) *the optimal solution of aggregate flow on each link exists and is unique.*

Proof : **(a)** Based on the proof of Theorem 2, the Jacobian matrix of $\boldsymbol{t}(\boldsymbol{v})$ is positive semidefinite. However, even though the travel time function is strictly increasing in its own link flow, equation (10) indicates that the Jacobian matrix is not positive definite. Therefore, the optimal solution of flow for each type of travelers on each link exists, but may not be unique.
(b) The existence conjecture is easily supported by the existence of the optimal solution of flow for each type of travelers on each link. Next, we will prove its uniqueness.

Suppose two distinct link flow solutions $\boldsymbol{v}^1$ and $\boldsymbol{v}^2$ are both optimal solutions to Problem (9). That is, $\exists a \in \mathcal{A}, v_a^1 \neq v_a^2$, and $Z(\boldsymbol{v}^1) = Z(\boldsymbol{v}^2)$. Then we will show the contradiction.

Since $t_a(x)$ is a strictly monotonic increasing function, $\int_0^{v_a} t_a(x)dx$ is a strictly convex function in v_a. For any $\eta \in (0,1)$,

$$\begin{aligned}
& Z\left(\eta\boldsymbol{v}^1 + (1-\eta)\boldsymbol{v}^2\right) - \left(\eta Z\left(\boldsymbol{v}^1\right) + (1-\eta) Z\left(\boldsymbol{v}^2\right)\right) \\
= & \sum_{a\in\mathcal{A}} \int_0^{\eta v_a^1 + (1-\eta)v_a^2} t_a(x)dx + \sum_{a\in\mathcal{A}} \sum_{i\in\mathcal{I}} ACT_{V_i}\left(\tilde{\tau}_a\right)\left(\eta v_{ai}^1 + (1-\eta)v_{ai}^2\right) \\
- & \eta \left(\sum_{a\in\mathcal{A}} \int_0^{v_a^1} t_a(x)dx + \sum_{a\in\mathcal{A}} \sum_{i\in\mathcal{I}} ACT_{V_i}\left(\tilde{\tau}_a\right) v_{ai}^1 \right) \\
- & (1-\eta) \left(\sum_{a\in\mathcal{A}} \int_0^{v_a^2} t_a(x)dx + \sum_{a\in\mathcal{A}} \sum_{i\in\mathcal{I}} ACT_{V_i}\left(\tilde{\tau}_a\right) v_{ai}^2 \right) \\
= & \sum_{a\in\mathcal{A}} \int_0^{\eta v_a^1 + (1-\eta)v_a^2} t_a(x)dx - \left(\eta \sum_{a\in\mathcal{A}} \int_0^{v_a^1} t_a(x)dx + (1-\eta) \sum_{a\in\mathcal{A}} \int_0^{v_a^2} t_a(x)dx \right) \\
< & \; 0,
\end{aligned}$$

it follows that

$$Z\left(\eta\boldsymbol{v}^1 + (1-\eta)\boldsymbol{v}^2\right) < \eta Z(\boldsymbol{v}^1) + (1-\eta) Z(\boldsymbol{v}^2) = Z(\boldsymbol{v}^1) = Z(\boldsymbol{v}^2).$$

Now we have a contradiction to the assumption that $\boldsymbol{v}^1$ and $\boldsymbol{v}^2$ are both optimal. Therefore, the optimal solution of aggregate flow on each link exists and is unique. ■

We can interpret Problem (9) as a deterministic multi-class NE problem, which is easily solved by the traditional Frank-Wolfe algorithm (Frank and Wolfe, 1956; Yang and Huang, 2004).

4.2 Inefficiency of network equilibrium

Another concept accompanied with NE is to compare with the so-called System Optimum (SO) in which the aggregate travel time of all travelers is minimized (Nash, 1951; Wardrop, 1952). As travelers choose routes without considering about possible negative impacts on the system performance, it is obvious that the NE solution usually deviates from SO and is less efficient in attaining the minimum aggregate travel time. Led by the seminal work of Koutsoupias and Papadimitriou (1999), the loss of efficiency in NE is an active area of research. The authors propose the concept of Price of Anarchy, which is formally defined as the worst-case inefficiency or the ratio between the aggregate cost of NE and that of SO. In particular, Roughgarden and Tardos (2002) and Correa et al. (2004) present a surprising, but welcoming result that NE is near optimal in the sense that the aggregate travel time of all travelers under NE is at most that under SO with double traffic in the same network. In addition, when the travel time function depends linearly on traffic flow, the aggregate travel time of all travelers under NE is at most 4/3 times that under SO. A sequence of results with respect to more general link travel time function are further developed by Roughgarden (2003), Chau and Sim (2003), Perakis (2007), and Correa et al. (2008). In this section, we will derive similar results for the NE problem for the case when travelers are sensitive to risk and ambiguity. To derive analytical results, we again assume that the uncertainty along links is flow independent.

For a given traffic flow, $\boldsymbol{v} \in \mathcal{V}$, we represent the aggregate travel time under the ACT criterion on the entire network by

$$\mathcal{C}_{\boldsymbol{v}}(\boldsymbol{v}) = \langle \boldsymbol{t}(\boldsymbol{v}), \boldsymbol{v} \rangle = \sum_{a\in\mathcal{A}} \sum_{i\in\mathcal{I}} t_{ai}(v_a) v_{ai} = \sum_{a\in\mathcal{A}} \sum_{i\in\mathcal{I}} \left(t_a\left(v_a\right) + ACT_{V_i}\left(\tilde{\tau}_a\right)\right) v_{ai}.$$

We also define

$$\mathcal{C}_{\boldsymbol{v}^*}(\boldsymbol{v}) = \langle \boldsymbol{t}(\boldsymbol{v}^*), \boldsymbol{v} \rangle = \sum_{a\in\mathcal{A}} \sum_{i\in\mathcal{I}} t_{ai}(v_a^*) v_{ai} = \sum_{a\in\mathcal{A}} \sum_{i\in\mathcal{I}} \left(t_a\left(v_a^*\right) + ACT_{V_i}\left(\tilde{\tau}_a\right)\right) v_{ai}.$$

Then variational inequalities (8) can be replaced as $\mathcal{C}_{\boldsymbol{v}^*}(\boldsymbol{v}^*) \leq \mathcal{C}_{\boldsymbol{v}^*}(\boldsymbol{v})$, where $\boldsymbol{v}^* = (v_{ai}^*)_{a\in\mathcal{A}, i\in\mathcal{I}}$ is the vector traffic flows in NE for types of travelers along all links, and $\boldsymbol{v} \in \mathcal{V}$ is the vector of any feasible flows. Let $\boldsymbol{x}^* = (x_{ai}^*)_{a\in\mathcal{A}, i\in\mathcal{I}}$ denote that at SO, which minimizes aggregate travel time under the ACT criterion. We can analyze the inefficiency of NE by comparing $\mathcal{C}_{\boldsymbol{v}^*}(\boldsymbol{v}^*)$ and $\mathcal{C}_{\boldsymbol{x}^*}(\boldsymbol{x}^*)$. In particular, we are interested in the Price of Anarchy, which is the worst-case ratio between the aggregate travel time of NE and that of SO under the ACT criterion.

Proposition 5

(a) *Consider an instance of Problem (9). The vector* $\boldsymbol{v}^* = (v_{ai}^*)_{a\in\mathcal{A}, i\in\mathcal{I}}$ *represents the link flow in NE, and* $\boldsymbol{u} = (u_{ai})_{a\in\mathcal{A}, i\in\mathcal{I}}$ *is a feasible flow for the same network but with twice as many travelers of the same type. Then*

$$\mathcal{C}_{\boldsymbol{v}^*}(\boldsymbol{v}^*) \leq \mathcal{C}_{\boldsymbol{u}}(\boldsymbol{u}).$$

(b) *If travel time function is a monomial function* $t_a(v_a) = b_a \left(\frac{v_a}{C_a}\right)^m$ $(m \geq 0)$*, then*

$$\frac{\mathcal{C}_{\boldsymbol{v}^*}(\boldsymbol{v}^*)}{\mathcal{C}_{\boldsymbol{x}^*}(\boldsymbol{x}^*)} \leq \left(1 - m(m+1)^{-(m+1)/m}\right)^{-1}.$$

(c) *If travel time function* $t(v)$ *is a general continuous, nondecreasing function of* v*, we have*

$$\frac{\mathcal{C}_{\boldsymbol{v}^*}(\boldsymbol{v}^*)}{\mathcal{C}_{\boldsymbol{x}^*}(\boldsymbol{x}^*)} \leq \frac{1}{1-\beta(t)},$$

where

$$\beta(t) = \sup_{v\in\mathcal{V}} \frac{\max_{x\in\mathcal{V}} x\left(t(v) - t(x)\right)}{t(v)v}, \quad \text{and} \quad 0 \leq \beta(t) \leq 1.$$

Proof : **(a)** Note that $t_a(v_a)$ is a differentiable, monotonically increasing function in v_a, and $\boldsymbol{u} = (u_{ai})_{a\in\mathcal{A},i\in\mathcal{I}}$ is a feasible flow for the same network but with double demands. We have

$$
\begin{aligned}
&t_a(u_a)u_a + t_a(v_a^*)v_a^* - t_a(v_a^*)u_a \geq t_a(u_a)u_a \geq 0, \quad \text{if} \quad u_a \leq v_a^*;\\
&t_a(u_a)u_a + t_a(v_a^*)v_a^* - t_a(v_a^*)u_a \geq t_a(v_a^*)v_a^* \geq 0, \quad \text{if} \quad u_a \geq v_a^*.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& \mathcal{C}_{\boldsymbol{u}}(\boldsymbol{u}) + \mathcal{C}_{\boldsymbol{v}^*}(\boldsymbol{v}^*) - \mathcal{C}_{\boldsymbol{v}^*}(\boldsymbol{u}) \\
= & \sum_{a\in\mathcal{A}}\sum_{i\in\mathcal{I}} \left(t_{ai}(u_a)u_{ai} + t_{ai}(v_a^*)v_{ai}^* - t_{ai}(v_a^*)u_{ai}\right) \\
= & \sum_{a\in\mathcal{A}}\sum_{i\in\mathcal{I}} \left(t_a(u_a)u_{ai} + t_a(v_a^*)v_{ai}^* + ACT_{V_i}\left(\tilde{\tau}_a\right) v_{ai}^* - t_a(v_a^*)u_{ai}\right) \\
= & \sum_{a\in\mathcal{A}}\sum_{i\in\mathcal{I}} \left(ACT_{V_i}\left(\tilde{\tau}_a\right)\right) v_{ai}^* + \sum_{a\in\mathcal{A}} \left(t_a(u_a)u_a + t_a(v_a^*)v_a^* - t_a(v_a^*)u_a\right) \\
\geq & \sum_{a\in\mathcal{A}}\sum_{i\in\mathcal{I}} \left(ACT_{V_i}\left(\tilde{\tau}_a\right)\right) v_{ai}^* \\
\geq & \ 0.
\end{aligned}
$$

Besides, we note that $\boldsymbol{u}/2 = (\frac{u_{ai}}{2})_{a\in\mathcal{A},i\in\mathcal{I}}$ is a feasible flow for the original instance. From the NE property,

$$
\mathcal{C}_{\boldsymbol{u}}(\boldsymbol{u}) \geq \mathcal{C}_{\boldsymbol{v}^*}(\boldsymbol{u}) - \mathcal{C}_{\boldsymbol{v}^*}(\boldsymbol{v}^*) = 2\mathcal{C}_{\boldsymbol{v}^*}\left(\frac{\boldsymbol{u}}{2}\right) - \mathcal{C}_{\boldsymbol{v}^*}(\boldsymbol{v}^*) \geq 2\mathcal{C}_{\boldsymbol{v}^*}(\boldsymbol{v}^*) - \mathcal{C}_{\boldsymbol{v}^*}(\boldsymbol{v}^*) = \mathcal{C}_{\boldsymbol{v}^*}(\boldsymbol{v}^*).
$$

(b) If travel time is a monomial function, defined as $t_a(v_a) = b_a\left(\frac{v_a}{C_a}\right)^m$ such that

$$
t_{ai}(v_a) = b_a\left(\frac{v_a}{C_a}\right)^m + ACT_{V_i}\left(\tilde{\tau}_a\right).
$$

Then, we have

$$
\begin{aligned}
\mathcal{C}_{\boldsymbol{v}^*}(\boldsymbol{x}) &= \sum_{a\in\mathcal{A}}\sum_{i\in\mathcal{I}} \left(t_a\left(v_a^*\right) + ACT_{V_i}\left(\tilde{\tau}_a\right)\right) x_{ai} \\
&= \sum_{a\in\mathcal{A}}\sum_{i\in\mathcal{I}} \left(b_a\left(\frac{v_a^*}{C_a}\right)^m + ACT_{V_i}\left(\tilde{\tau}_a\right)\right) x_{ai} \\
&= \sum_{a\in\mathcal{A}}\sum_{i\in\mathcal{I}} \left(ACT_{V_i}\left(\tilde{\tau}_a\right)\right) x_{ai} + \sum_{a\in\mathcal{A}} b_a\left(\frac{v_a^*}{C_a}\right)^m x_a \\
&\leq \sum_{a\in\mathcal{A}}\sum_{i\in\mathcal{I}} \left(ACT_{V_i}\left(\tilde{\tau}_a\right)\right) x_{ai} + \sum_{a\in\mathcal{A}} \frac{b_a}{C_a^m}\left(x_a^{m+1} + m(m+1)^{-(m+1)/m}(v_a^*)^{m+1}\right) \\
&= \sum_{a\in\mathcal{A}}\sum_{i\in\mathcal{I}} \left(b_a\left(\frac{x_a}{C_a}\right)^m + ACT_{V_i}\left(\tilde{\tau}_a\right)\right) x_{ai} + m(m+1)^{-(m+1)/m} \sum_{a\in\mathcal{A}}\sum_{i\in\mathcal{I}} b_a\left(\frac{v_a^*}{C_a}\right)^m v_{ai}^* \\
&\leq \mathcal{C}_{\boldsymbol{x}}(\boldsymbol{x}) + m(m+1)^{-(m+1)/m} \sum_{a\in\mathcal{A}}\sum_{i\in\mathcal{I}} \left(b_a\left(\frac{v_a^*}{C_a}\right)^m + ACT_{V_i}\left(\tilde{\tau}_a\right)\right) v_{ai}^* \\
&= \mathcal{C}_{\boldsymbol{x}}(\boldsymbol{x}) + m(m+1)^{-(m+1)/m}\mathcal{C}_{\boldsymbol{v}^*}(\boldsymbol{v}^*),
\end{aligned}
$$

where the first inequality is tenable because the function $f(x) = v^m x - x^{m+1}$ $(x \geq 0)$ will get its maximum $m(m+1)^{-(m+1)/m} v^{m+1}$ at $x = v(m+1)^{-1/m}$; and the second inequality holds because $\sum_{a \in \mathcal{A}} \sum_{i \in \mathcal{I}} \left(ACT_{V_i}\left(\tilde{\tau}_a\right)\right) v_{ai}^* \geq 0$. Then, since $\mathcal{C}_{\boldsymbol{v}^*}(\boldsymbol{v}^*) \leq \mathcal{C}_{\boldsymbol{v}^*}(\boldsymbol{x})$, we get

$$\left(1 - m(m+1)^{-(m+1)/m}\right) \mathcal{C}_{\boldsymbol{v}^*}(\boldsymbol{v}^*) \leq \mathcal{C}_{\boldsymbol{x}^*}(\boldsymbol{x}^*).$$

When $\boldsymbol{x}^* = (x_{ai}^*)_{a \in \mathcal{A}, i \in \mathcal{I}}$ is the system optimum, we can find the Price of Anarchy bounded at $\left(1 - m(m+1)^{-(m+1)/m}\right)^{-1}$. The fact that the bound is tight will be shown in the following example.

Moreover, when $m = 1$, travel time function is linear in link flow, the bound is exactly 4/3, the same as that in deterministic case.

(c) Similar to Correa et al. (2004), we could generalize the travel time function to continuous, nondecreasing case.

$$\begin{aligned}
\mathcal{C}_{\boldsymbol{v}^*}(\boldsymbol{x}) &= \sum_{a \in \mathcal{A}} \sum_{i \in \mathcal{I}} t_{ai}(v_a^*) x_{ai} \\
&= \sum_{a \in \mathcal{A}} t_a(v_a^*) x_a + \sum_{a \in \mathcal{A}} \sum_{i \in \mathcal{I}} \left(ACT_{V_i}\left(\tilde{\tau}_a\right)\right) x_{ai} \\
&= \sum_{a \in \mathcal{A}} \left(x_a \left(t_a(v_a^*) - t_a(x_a)\right)\right) + \mathcal{C}_{\boldsymbol{x}}(\boldsymbol{x}) \\
&= \sum_{a \in \mathcal{A}} \frac{x_a \left(t_a(v_a^*) - t_a(x_a)\right)}{t_a(v_a^*) v_a^*} t_a(v_a^*) v_a^* + \mathcal{C}_{\boldsymbol{x}}(\boldsymbol{x}) \\
&\leq \sum_{a \in \mathcal{A}} \beta(v_a^*, t_a(v_a^*)) t_a(v_a^*) v_a^* + \mathcal{C}_{\boldsymbol{x}}(\boldsymbol{x}) \\
&\leq \beta(t) \mathcal{C}_{\boldsymbol{v}^*}(\boldsymbol{v}^*) + \mathcal{C}_{\boldsymbol{x}}(\boldsymbol{x}),
\end{aligned}$$

where $\beta(v, t(v)) = \dfrac{1}{t(v)v} \max\limits_{x \geq 0} \{x\left(t(v) - t(x)\right)\}$, and $\beta(t) = \sup_{v \geq 0} \beta(v, t(v))$.

Since the travel time function $t(v)$ is a continuous nondecreasing function, the following relationship holds:

$$0 = \frac{v(t(v) - t(v))}{t(v)v} \leq \beta(v, t(v)) \leq \frac{\max_{0 \leq x \leq v} x(t(v) - t(x))}{t(v)v} \leq \frac{\max_{0 \leq x \leq v} x t(v)}{t(v)v} \leq \frac{v t(v)}{t(v)v} = 1.$$

Assuming $\boldsymbol{x}^* = (x_{ai}^*)_{a \in \mathcal{A}, i \in \mathcal{I}}$ is SO solution, we have the Price of Anarchy as

$$\frac{\mathcal{C}_{\boldsymbol{v}^*}(\boldsymbol{v}^*)}{\mathcal{C}_{\boldsymbol{x}^*}(\boldsymbol{x}^*)} \leq \frac{1}{1 - \beta(t)}.$$

■

4.3 A network equilibrium example

The following example explicitly illustrates the calculation of NE and SO under the ACT criterion, and demonstrates the inefficiency issues under various mixtures of travelers' profiles. It elucidates the importance of taking travelers' risk and ambiguity attitudes into account in analyzing traffic networks.

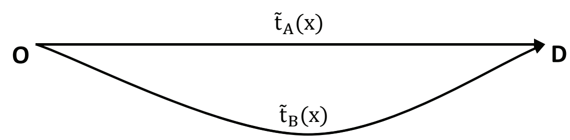

Figure 3: Two paths network with uncertain travel time.

We consider a two paths network from origin O to destination D depicted in Figure 3. The traffic rate is assumed to be 1. The paths have travel times as follows:

$$\tilde{t}_A(v_A) = (v_A)^4 + \tilde{\tau}_A; \quad \tilde{t}_B(v_B) = \frac{6}{5},$$

where the parameter $\tilde{\tau}_A$ is uncertain. We assume that all travelers have the same information on the uncertain parameter $\tilde{\tau}_A$. Specifically, $\tilde{\tau}_A$ has a mean value of $\frac{1}{5}$ and support in $[0, \Delta]$, $\Delta > \frac{1}{5}$. Hence, the corresponding family of distributions is given by

$$\mathbb{F}(\Delta) = \left\{ \mathbb{P} : \mathrm{E}_{\mathbb{P}}\left(\tilde{\tau}_A\right) = \frac{1}{5}, \mathbb{P}\left(\tilde{\tau}_A \in [0, \Delta]\right) = 1 \right\}.$$

Note that the parameter Δ represents the worst-case delay of $\tilde{\tau}_A$ and implies the level of uncertainty along the Path A. On the other hand, Path B has deterministic travel time and is unaffected by Δ. With various compositions of travelers in terms of risk and ambiguity attitudes, the NE and SO under the ACT criterion will yield different flow patterns. We intend to explore the impact of Δ on the flow patterns. We consider the following three cases:

Case 1: All travelers are risk neutral and ambiguity neutral ($\lambda = 0, \alpha = \frac{1}{2}$);

Case 2: All travelers are extremely risk averse and pessimistic towards ambiguity ($\lambda \to +\infty, \alpha = 1$).

Case 3: Travelers with composition shown in Table 2.

In Case 1, all travelers are risk and ambiguity neutral and they intuitively perceive the uncertain term as its mean value. Hence, the solutions are consistent with traditional deterministic NE and SO models. In Case 2, travelers who are radically risk averse and pessimistic towards ambiguity consider the worst-case travel time in deciding between paths. In Case 3, type 1 travelers are risk averse and

Type i	**Demand**	α_i	λ_i	$ACT_{V_i}(\tilde{\tau}_A)$
1	$\frac{2}{3}$	$\frac{4}{5}$	5	$\frac{1}{25}+\frac{4}{25}\ln\left(1+\frac{1}{5\Delta}\left(\exp(5\Delta)-1\right)\right)$
2	$\frac{1}{3}$	$\frac{1}{5}$	-5	$\frac{1}{25}-\frac{4}{25}\ln\left(1+\frac{1}{5\Delta}\left(\exp(-5\Delta)-1\right)\right)$

Table 2: Travelers' composition in Case 3.

pessimistic towards ambiguity, type 2 travelers are risk seeking and optimistic towards ambiguity, and they will exhibit different preferences towards the two paths. Here, we assume that the population is dominated by travelers who are risk averse and pessimistic towards ambiguity. We derive flow solutions of NE and SO under the ACT criterion in Table 3 and 4. Note that Δ_1 and Δ_2 are the unique solutions satisfying $\left(\frac{6-5ACT_{V_1}(\tilde{\tau}_A)}{25}\right)^{1/4}-\frac{1}{3}=0$ and $\left(\frac{6}{5}-ACT_{V_1}(\tilde{\tau}_A)\right)^{1/4}-\frac{1}{3}=0$, respectively. $\Delta_1\approx 1.8136$, and $\Delta_2\approx 1.8829$.

Case	**Condition**	**Criterion**	**Traffic flow**		**Path ACT**[1]		**Network ACT**[2]
			path A	path B	path A	path B	
1		NE	1	0	$\frac{6}{5}$	$\frac{6}{5}$	$\frac{6}{5}$
		SO	$5^{-1/4}$	$1-5^{-1/4}$	$\frac{2}{5}$	$\frac{6}{5}$	$\frac{6}{5}-4\times 5^{-5/4}$
2	$\frac{1}{5}<\Delta\leq\frac{6}{5}$	NE	$(\frac{6}{5}-\Delta)^{1/4}$	$1-(\frac{6}{5}-\Delta)^{1/4}$	$\frac{6}{5}$	$\frac{6}{5}$	$\frac{6}{5}$
		SO	$\left(\frac{6-5\Delta}{25}\right)^{1/4}$	$1-\left(\frac{6-5\Delta}{25}\right)^{1/4}$	$\frac{6+20\Delta}{25}$	$\frac{6}{5}$	$\frac{6}{5}-4\left(\frac{6-5\Delta}{25}\right)^{5/4}$
	$\frac{6}{5}\leq\Delta$	NE	0	1	Δ	$\frac{6}{5}$	$\frac{6}{5}$
		SO	0	1	Δ	$\frac{6}{5}$	$\frac{6}{5}$

[1] **Path ACT** refers to the travel time along the path under the ACT criterion;
[2] **Network ACT** refers to the aggregate travel times under the ACT criterion for all travelers on the network.

Table 3: Flow patterns of NE and SO under the ACT criterion for Case 1 and 2.

We now study the inefficiency of NE under the ACT criterion with respect to the parameter Δ. We represent the aggregate travel times under the ACT criterion in Case i under the NE and SO model by ACT_i^{NE} and ACT_i^{SO} respectively, and quantify the inefficiency of NE via the ratio $\frac{ACT_i^{NE}}{ACT_i^{SO}}$. For these

Condition	Criterion	Type	Traffic flow A	Traffic flow B	Path ACT A	Path ACT B	Network ACT
$0.2 < \Delta \leq \Delta_1$	NE	1	$\left(\frac{6}{5} - ACT_{V_1}(\tilde{\tau}_A)\right)^{1/4} - \frac{1}{3}$	$1 - \left(\frac{6}{5} - ACT_{V_1}(\tilde{\tau}_A)\right)^{1/4}$	$\frac{6}{5}$	$\frac{6}{5}$	$\frac{6}{5} - \frac{1}{3}ACT_{V_1}(\tilde{\tau}_A) + \frac{1}{3}ACT_{V_2}(\tilde{\tau}_A)$
		2	$\frac{1}{3}$	0	$\frac{6}{5} - ACT_{V_1}(\tilde{\tau}_A) + ACT_{V_2}(\tilde{\tau}_A)$	$\frac{6}{5}$	
	SO	1	$\left(\frac{6-5ACT_{V_1}(\tilde{\tau}_A)}{25}\right)^{1/4} - \frac{1}{3}$	$1 - \left(\frac{6-5ACT_{V_1}(\tilde{\tau}_A)}{25}\right)^{1/4}$	$\frac{6+20ACT_{V_1}(\tilde{\tau}_A)}{25}$	$\frac{6}{5}$	$\frac{6}{5} - \frac{1}{3}ACT_{V_1}(\tilde{\tau}_A) + \frac{1}{3}ACT_{V_2}(\tilde{\tau}_A) - 4\left(\frac{6-5ACT_{V_1}(\tilde{\tau}_A)}{25}\right)^{5/4}$
		2	$\frac{1}{3}$	0	$\frac{6-5ACT_{V_1}(\tilde{\tau}_A)}{25} + ACT_{V_2}(\tilde{\tau}_A)$	$\frac{6}{5}$	
$\Delta_1 \leq \Delta \leq \Delta_2$	NE	1	$\left(\frac{6}{5} - ACT_{V_1}(\tilde{\tau}_A)\right)^{1/4} - \frac{1}{3}$	$1 - \left(\frac{6}{5} - ACT_{V_1}(\tilde{\tau}_A)\right)^{1/4}$	$\frac{6}{5}$	$\frac{6}{5}$	$\frac{6}{5} - \frac{1}{3}ACT_{V_1}(\tilde{\tau}_A) + \frac{1}{3}ACT_{V_2}(\tilde{\tau}_A)$
		2	$\frac{1}{3}$	0	$\frac{6}{5} - ACT_{V_1}(\tilde{\tau}_A) + ACT_{V_2}(\tilde{\tau}_A)$	$\frac{6}{5}$	
	SO	1	0	$\frac{2}{3}$	$\frac{1}{81} + ACT_{V_1}(\tilde{\tau}_A)$	$\frac{6}{5}$	$\frac{4}{5} + \frac{1}{243} + \frac{1}{3}ACT_{V_2}(\tilde{\tau}_A)$
		2	$\frac{1}{3}$	0	$\frac{1}{81} + ACT_{V_2}(\tilde{\tau}_A)$	$\frac{6}{5}$	
$\Delta_2 \leq \Delta$	NE	1	0	$\frac{2}{3}$	$\frac{1}{81} + ACT_{V_1}(\tilde{\tau}_A)$	$\frac{6}{5}$	$\frac{4}{5} + \frac{1}{243} + \frac{1}{3}ACT_{V_2}(\tilde{\tau}_A)$
		2	$\frac{1}{3}$	0	$\frac{1}{81} + ACT_{V_2}(\tilde{\tau}_A)$	$\frac{6}{5}$	
	SO	1	0	$\frac{2}{3}$	$\frac{1}{81} + ACT_{V_1}(\tilde{\tau}_A)$	$\frac{6}{5}$	$\frac{4}{5} + \frac{1}{243} + \frac{1}{3}ACT_{V_2}(\tilde{\tau}_A)$
		2	$\frac{1}{3}$	0	$\frac{1}{81} + ACT_{V_2}(\tilde{\tau}_A)$	$\frac{6}{5}$	

Table 4: Flow patterns of NE and SO under the ACT criterion for Case 3.

three cases, the ratios are calculated as:

$$\frac{ACT_1^{NE}}{ACT_1^{SO}} = \frac{6}{6 - 4 \times 5^{-1/4}};$$

$$\frac{ACT_2^{NE}}{ACT_2^{SO}} = \begin{cases} \dfrac{6}{6 - 20\left(\frac{6-5\Delta}{25}\right)^{5/4}}, & \text{when } \frac{1}{5} < \Delta \leq \frac{6}{5}; \\ 1, & \text{when } \frac{6}{5} \leq \Delta; \end{cases}$$

$$\frac{ACT_3^{NE}}{ACT_3^{SO}} = \begin{cases} \dfrac{\frac{6}{5} - \frac{1}{3}ACT_{V_1}(\tilde{\tau}_A) + \frac{1}{3}ACT_{V_2}(\tilde{\tau}_A)}{\frac{6}{5} - \frac{1}{3}ACT_{V_1}(\tilde{\tau}_A) + \frac{1}{3}ACT_{V_2}(\tilde{\tau}_A) - 4\left(\frac{6-5ACT_{V_1}(\tilde{\tau}_A)}{25}\right)^{5/4}}, & \text{when } 0.2 < \Delta \leq \Delta_1; \\ \dfrac{\frac{6}{5} - \frac{1}{3}ACT_{V_1}(\tilde{\tau}_A) + \frac{1}{3}ACT_{V_2}(\tilde{\tau}_A)}{\frac{4}{5} + \frac{1}{243} + \frac{1}{3}ACT_{V_2}(\tilde{\tau}_A)}, & \text{when } \Delta_1 \leq \Delta \leq \Delta_2; \\ 1, & \text{when } \Delta_2 \leq \Delta. \end{cases}$$

Figure 4 depicts the ratios of Case 2 and 3. We observe that with the increase of the upper bound Δ, the ratios are decreasing and approach to 1. For this specific example, it suggests that if the travel time along a traffic network is highly uncertain, then there is little benefit from having the system optimal solution in which the aggregate travel time under the ACT criterion is minimized.

Next, we will highlight that it is essential to consider travelers' risk and ambiguity attitudes when determining the system optimal flow pattern. Specifically, if we ignore uncertainty and calculate the deterministic system optimal (DSO) flow pattern, the system performance may even be worse off than that of NE in terms of the aggregate travel time under the ACT criterion. We represent the DSO flow pattern by $\boldsymbol{u}^* = (u_a^*)_{a\in\mathcal{A}}$, which is the unique optimal solution of

$$\begin{aligned}
\min \quad & \sum_{a\in\mathcal{A}} (t_a(u_a) + \mathrm{E}(\tilde{\tau}_a))\, u_a \\
\text{s.t.} \quad & u_a = \sum_{w\in\mathcal{W}} \sum_{r\in\mathcal{R}_w} f_r \delta_{ar}, && \forall a \in \mathcal{A}; \\
& \sum_{r\in\mathcal{R}_w} f_r = \sum_{i\in\mathcal{I}} d_{wi}, && \forall w \in \mathcal{W}; \\
& f_r \geq 0, && \forall r \in \mathcal{R}_w, w \in \mathcal{W}.
\end{aligned}$$

Note that the flow pattern $\boldsymbol{u}^* = (u_a^*)_{a\in\mathcal{A}}$ only identifies the aggregate traffic flow on each link. Therefore, with the mixture of travelers, the traffic flow for each type of travelers on each link may not be unique. We represent its feasible set as

$$\mathcal{U} = \{\boldsymbol{v} : \boldsymbol{v} \in \mathcal{V}, \sum_{i\in\mathcal{I}} v_{ai} = u_a^*, \forall a \in \mathcal{A}\}.$$

Then, for any $\boldsymbol{v} \in \mathcal{U}$, we define $ACT^{DSO}(\boldsymbol{v})$ as the total travel time under the ACT criterion of all travelers when the traffic flow is $\boldsymbol{v}$, that is,

$$ACT^{DSO}(\boldsymbol{v}) = \sum_{a\in\mathcal{A}} \sum_{i\in\mathcal{I}} (t_a(u_a^*) + ACT_{V_i}(\tilde{\tau}_a))\, v_{ai}.$$

Since $ACT^{DSO}(\boldsymbol{v})$ is a function of $\boldsymbol{v} \in \mathcal{U}$, we define its lower and upper bound by $\underline{ACT}^{DSO}$ and $\overline{ACT}^{DSO}$ respectively, where

$$\begin{aligned}
\underline{ACT}^{DSO} &= \min_{\boldsymbol{v}\in\mathcal{U}} \sum_{a\in\mathcal{A}} \sum_{i\in\mathcal{I}} (t_a(u_a^*) + ACT_{V_i}(\tilde{\tau}_a))\, v_{ai}; \\
\overline{ACT}^{DSO} &= \max_{\boldsymbol{v}\in\mathcal{U}} \sum_{a\in\mathcal{A}} \sum_{i\in\mathcal{I}} (t_a(u_a^*) + ACT_{V_i}(\tilde{\tau}_a))\, v_{ai}.
\end{aligned}$$

Hence, for any $\boldsymbol{v} \in \mathcal{U}$, $ACT^{DSO}(\boldsymbol{v}) \in \left[\underline{ACT}^{DSO}, \overline{ACT}^{DSO}\right]$.

In our example, if the real travelers are extremely risk averse and pessimistic towards ambiguity as described in Case 2, then the traffic flow is $v_A = 5^{-1/4}$ and $v_B = 1 - 5^{-1/4}$, the aggregate travel time under the ACT criterion following DSO flow pattern is given by

$$ACT_2^{DSO}(\boldsymbol{v}) = \underline{ACT}_2^{DSO} = \overline{ACT}_2^{DSO} = \left(\left(5^{-1/4}\right)^4 + \Delta\right) \times 5^{-1/4} + \frac{6}{5} \times \left(1 - 5^{-1/4}\right) = \frac{6}{5} + 5^{-1/4}(\Delta - 1).$$

For Case 3, we solve the following two problems

$$\begin{aligned}
\underline{ACT}_3^{DSO} = \min \quad & \left(\left(5^{-1/4}\right)^4 + ACT_{V_1}\left(\tilde{\tau}_A\right)\right) v_{A1} + \left(\left(5^{-1/4}\right)^4 + ACT_{V_2}\left(\tilde{\tau}_A\right)\right) v_{A2} + \frac{6}{5} v_{B1} + \frac{6}{5} v_{B2} \\
\text{s.t.} \quad & v_{A1} + v_{A2} = 5^{-1/4}, \\
& v_{A1} + v_{B1} = \frac{2}{3}, \\
& v_{A2} + v_{B2} = \frac{1}{3}, \\
& v_{A1}, v_{B1}, v_{A2}, v_{B2} \geq 0;
\end{aligned}$$

and

$$\begin{aligned}
\overline{ACT}_3^{DSO} = \max \quad & \left(\left(5^{-1/4}\right)^4 + ACT_{V_1}\left(\tilde{\tau}_A\right)\right) v_{A1} + \left(\left(5^{-1/4}\right)^4 + ACT_{V_2}\left(\tilde{\tau}_A\right)\right) v_{A2} + \frac{6}{5} v_{B1} + \frac{6}{5} v_{B2} \\
\text{s.t.} \quad & v_{A1} + v_{A2} = 5^{-1/4}, \\
& v_{A1} + v_{B1} = \frac{2}{3}, \\
& v_{A2} + v_{B2} = \frac{1}{3}, \\
& v_{A1}, v_{B1}, v_{A2}, v_{B2} \geq 0.
\end{aligned}$$

Correspondingly, we have

$$\underline{ACT}_3^{DSO} = \frac{6}{5} - 5^{-1/4} + \left(5^{-1/4} - \frac{1}{3}\right) ACT_{V_1}\left(\tilde{\tau}_A\right) + \frac{1}{3} ACT_{V_2}\left(\tilde{\tau}_A\right),$$

the minimum value is achieved when $v_{A1} = 5^{-1/4} - \frac{1}{3}, v_{A2} = \frac{1}{3}, v_{B1} = 1 - 5^{-1/4}, v_{B2} = 0$. And

$$\overline{ACT}_3^{DSO} = \frac{6}{5} - 5^{-1/4} + \frac{2}{3} ACT_{V_1}\left(\tilde{\tau}_A\right) + \left(5^{-1/4} - \frac{2}{3}\right) ACT_{V_2}\left(\tilde{\tau}_A\right),$$

the maximum value is achieved when $v_{A1} = \frac{2}{3}, v_{A2} = 5^{-1/4} - \frac{2}{3}, v_{B1} = 0, v_{B2} = 1 - 5^{-1/4}$.

Likewise, we quantify the inefficiency of DSO under the ACT criterion via the ratios $\frac{\underline{ACT}_i^{DSO}}{ACT_i^{SO}}$ and $\frac{\overline{ACT}_i^{DSO}}{ACT_i^{SO}}$ as following:

$$\frac{\underline{ACT}_2^{DSO}}{ACT_2^{SO}} = \frac{\overline{ACT}_2^{DSO}}{ACT_2^{SO}} = \begin{cases} \dfrac{6 + 5^{3/4}\left(\Delta - 1\right)}{6 - 20\left(\frac{6-5\Delta}{25}\right)^{5/4}}, & \text{when } \frac{1}{5} < \Delta \leq \frac{6}{5}; \\ \dfrac{6 + 5^{3/4}\left(\Delta - 1\right)}{6}, & \text{when } \frac{6}{5} \leq \Delta; \end{cases}$$

$$\frac{\underline{ACT}_3^{DSO}}{ACT_3^{SO}} = \begin{cases} \dfrac{\frac{6}{5} - 5^{-1/4} + \left(5^{-1/4} - \frac{1}{3}\right) ACT_{V_1}\left(\tilde{\tau}_A\right) + \frac{1}{3} ACT_{V_2}\left(\tilde{\tau}_A\right)}{\frac{6}{5} - \frac{1}{3} ACT_{V_1}\left(\tilde{\tau}_A\right) + \frac{1}{3} ACT_{V_2}\left(\tilde{\tau}_A\right) - 4\left(\frac{6-5ACT_{V_1}\left(\tilde{\tau}_A\right)}{25}\right)^{5/4}}, & \text{when } 0.2 < \Delta \leq \Delta_1; \\ \dfrac{\frac{6}{5} - 5^{-1/4} + \left(5^{-1/4} - \frac{1}{3}\right) ACT_{V_1}\left(\tilde{\tau}_A\right) + \frac{1}{3} ACT_{V_2}\left(\tilde{\tau}_A\right)}{\frac{4}{5} + \frac{1}{243} + \frac{1}{3} ACT_{V_2}\left(\tilde{\tau}_A\right)}, & \text{when } \Delta_1 \leq \Delta; \end{cases}$$

$$\frac{\overline{ACT}_3^{DSO}}{ACT_3^{SO}} = \begin{cases} \dfrac{\frac{6}{5} - 5^{-1/4} + \frac{2}{3} ACT_{V_1}\left(\tilde{\tau}_A\right) + \left(5^{-1/4} - \frac{2}{3}\right) ACT_{V_2}\left(\tilde{\tau}_A\right)}{\frac{6}{5} - \frac{1}{3} ACT_{V_1}\left(\tilde{\tau}_A\right) + \frac{1}{3} ACT_{V_2}\left(\tilde{\tau}_A\right) - 4\left(\frac{6-5ACT_{V_1}\left(\tilde{\tau}_A\right)}{25}\right)^{5/4}}, & \text{when } 0.2 < \Delta \leq \Delta_1; \\ \dfrac{\frac{6}{5} - 5^{-1/4} + \frac{2}{3} ACT_{V_1}\left(\tilde{\tau}_A\right) + \left(5^{-1/4} - \frac{2}{3}\right) ACT_{V_2}\left(\tilde{\tau}_A\right)}{\frac{4}{5} + \frac{1}{243} + \frac{1}{3} ACT_{V_2}\left(\tilde{\tau}_A\right)}, & \text{when } \Delta_1 \leq \Delta. \end{cases}$$

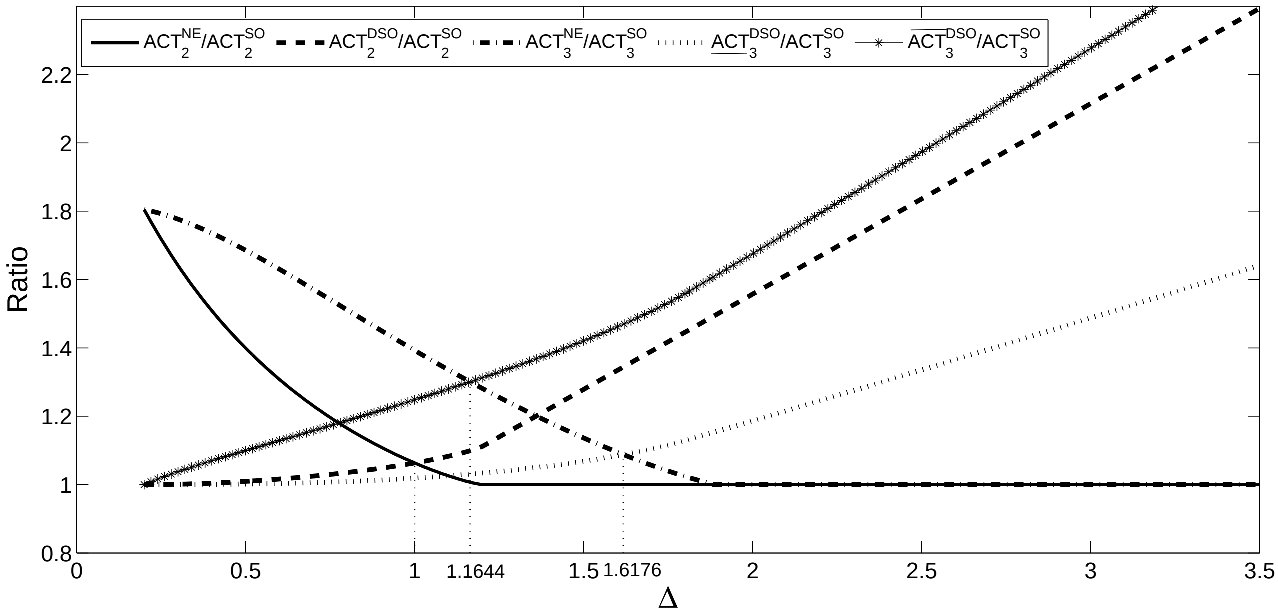

Figure 4: Inefficiency of NE and DSO under the ACT criterion in Case 2 and 3.

Figure 4 demonstrates the inefficiency of NE and DSO under the ACT criterion of Cases 2 and 3. In Case 2, for the network where travelers are extremely risk averse and pessimistic towards ambiguity, with the increase of Δ, the NE flow pattern under the ACT criterion becomes less inefficient, while the inefficiency of DSO grows increasingly severe. When $\Delta > 1$, $ACT_2^{DSO} > ACT_2^{NE}$ suggests if we instruct the traffic flow following DSO criterion, which does not account for travelers' attitudes towards risk and ambiguity, the performance will turn worse than its original anarchy state. Similarly, in Case 3, with two types of travelers, the ratio $\frac{ACT_3^{DSO}(\boldsymbol{v})}{ACT_3^{SO}}$ lies between the two lines $\frac{\underline{ACT}_3^{DSO}}{ACT_3^{SO}}$ and $\frac{\overline{ACT}_3^{DSO}}{ACT_3^{SO}}$. The increase of upper bound Δ will cut down the inefficiency of NE, but result in the deterioration of DSO in terms of system performance. Moreover, when the level of travel time uncertainty increases to some specific value, the DSO performance will be no better than the NE performance, which suggests this guidance effort is in vain.

5 Conclusion

This paper studies the preferences for uncertain travel time in which the probability distribution may not be fully characterized. By explicitly distinguishing risk and ambiguity concepts, we propose a new criterion called ambiguity-aware CARA travel time for ranking uncertain travel time, which systematically integrates the travelers' inability to capture the exact information of uncertain travel time, and their attitudes towards risk and ambiguity. This setting is based on Hurwicz criterion and constant absolute risk aversion, which is empirically supported and helps us derive tractable formulation.

With this criterion, we explore computational solvability of path selection problem on a network where the travel time is uncertain. We show that finding the path with the minimum travel time under the ACT criterion is polynomially solvable when the links' travel times are independently distributed. We also prove that the problem becomes intractable when the links' travel times are correlated. Focusing on independently distributed links' travel times, we present the general VI formulation of NE under the ACT criterion. We analyze the case when the uncertainty along links is flow independent and show that it can be addressed as a convex optimization problem. We also determine the inefficiency of the NE by deriving the Price of Anarchy, which is similar to the deterministic NE case.

The ACT criterion could potentially enhance the predictive capability of path selection and also traffic equilibrium. First, it does not require the traveler to know the probability distributions of the network. Second, it has the potential to incorporate risk and ambiguity in the travelers' decision making. Third, the path selection problem and network equilibrium established retain the computational tractability of the deterministic counterpart.

Several opportunities exist for future research. First, it will be valuable to establish empirically the risk and ambiguity profiles of a population of travelers residing in different cities and possibly having different cultures. Second, since travel times along the links are often correlated, albeit the intractability, there is a computational need to address the path selection problem for this situation. Finally, we could go one step further and extend the network equilibrium model under the ACT criterion from various aspects. For example, we could consider the elastic demand and link capacity constraints, or assume the travel time on a given link depends on the traffic flows of the entire network. We hope that our work could encourage future research in these directions.

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