Competitive location in networks with threshold-sensitive customer behaviour *

D.R. Santos-Peñate †, R. Súarez-Vega

and P. Dorta-González

Departamento de Métodos Cuantitativos en Economía y Gestión

Universidad de Las Palmas de Gran Canaria (Spain)

e-mail: {drsantos, rsuarez, pdorta}@dmc.ulpgc.es

February 28, 2012

Abstract

We consider the $(r|X_p)-$medianoid problem in networks. Goods are assumed to be essential and the only decision criterion is the travel distance. The portion of demand captured by the competitors is modelled by a general capture function which includes the binary, partially binary and proportional customer choice rules as specific cases. We prove that, under certain conditions of the capture function, a finite dominating set

*Partially financed by Gobierno de España and FEDER grant ECO2008-05589

†Corresponding author

exists, extending some known results given for maximal covering location problems.

Keywords: competitive location, networks, medianoid, customer behaviour, covering decay function, finite dominating set.

1 Introduction

Competitive location problems refer to situations where two or more competing firms offer services or goods to clients. Firms make decisions on facility locations and attributes, as well as on prices and quantities of the offered goods, in order to achieve certain goals. On the other side, customers choose the facilities to visit taking into account the travel distance or time, and other attributes of the facilities and goods offered by the firms. Normally, firms want to maximize profits and consumers desire to minimize the total cost of getting the product. Competitive location models represent the interactions between the decision makers involved in the transference of goods and services in a competitive environment, considering that both firms and consumers make decisions (see [8, 9, 10, 11, 17, 18] and references therein for a review of competitive location models).

If we consider that the only decision criterion for consumers is the distance to the facilities, the simpler customer choice rule is the binary one which represents the "all or nothing" behaviour. A customer uses all her/his buying power at the closest facility, disregarding all other facilities which are more distant than the closest one, even those whose difference in travel distance is very small. Other customer choice rules replace this hyper-sensitive consumer conduct, implicit in the binary rule, by a threshold sensitive behaviour; a customer uses exclusively firm A if the distance from this customer to the competitors exceeds the distance to firm A in an amount greater than or equal to a sensitivity threshold, the demand captured by each firm in the *shared* zone, is given by a non-increasing function of the travel distance [6, 7]. In the case of the partially binary choice rule, the demand is distributed among the closest facilities of the firms operating in the market, the proportional choice rule shares out the demand among all the facilities [13].

A threshold sensitivity behaviour can be modelled by a *coverage decay function* [4] or, using a more competitive term, a *capture decay function*. A capture decay function represents the demand captured by a firm. In this paper, we consider the problem of the $(r|X_p)-$medianoid [13], which is the follower problem in the Stackelberg location model. The Stackelberg or leader-follower location problem in networks was formalised by Hakimi who introduced the terms $(r|p)-$centroid and $(r|X_p)-$medianoid to refer to the solutions of the leader and follower problems respectively. An $(r|X_p)-$medianoid is an optimal solution to the follower problem when he/she opens r facilities and the leader has p facilities located at the points of set X_p. The $(r|p)-$centroid is an optimal solution to the leader problem when the leader opens p facilities and the follower opens r facilities [12, 13].

We assume that goods are essential. Essential goods must be consumed and customers visit one or more facilities to get them. This means that consumers satisfy all their demand which is totally captured by the firms operating in the market. The only criterion taken into account by the customers in the decision making is the distance to the facilities. In the more general case, the demand captured by a firm is a function of the distance from the customers to the $r+p$ facilities operating in the market.

Initially, we consider the $(r|X_p)-$medianoid problem in networks with a customer choice rule modelled by a capture decay function. Later, assuming that this function satisfies certain properties, we identify a finite dominating set. A finite dominating set is a finite set which contains an optimal solution. If such a set exists, we can solve the location problem in networks using tools designed to solve problems in discrete spaces (for a review of finite dominating sets for network location problems see [14], some optimality results for competitive location problems in networks can be found in [13, 16, 19, 20]). Since the follower problem is essentially a maximal covering location problem, this paper is directly connected with the gradual covering decay location problem and extends some results presented in [4] (see [1] for a review of generalized covering location models).

The paper is organised as follows. In Section 2 we describe the competitive location problem and introduce the capture decay function used to define the customer choice rule, some known choice rules are presented as specific cases. In Section 3 we study the existence of a finite dominating set. First, in Section 3.1, we consider the case where customers visit only the closest facility from the firms they patronize. This means that a consumer may visit only one facility (the closest facility of one of the firms), or he/she may patronize both firms and visit the closest facility of each. Later, we consider the case where customers may visit several facilities for both firms. Finally, Section 4 includes some concluding remarks.

2 Problem description and particular choice rules

Let $N = N(V, E)$ be an undirected network with node set $V = \{v_i\}_{i=1}^m$ and edge set E. An edge connecting $u, v \in V$ is denoted by $[u, v]$. The distance on networks between any two points $x_1,\ x_2 \in N$ is denoted by $d(x_1, x_2)$. The distance from point $z \in N$ to set $X \subset N$ is $d(z, X) = \min\{d(z, x) : x \in X\}$. Network N represents the market where firms offer an essential good to customers concentrated in the set of nodes, being w_v (≥ 0) the demand at node v. The total demand is $W = \sum\limits_{v \in V} w_v$.

There are p facilities located at points $X = \{x_1, x_2, ..., x_p\}$ belonging to firm A. A competitor, firm B, plans to enter the market with r facilities and wants to determine the locations that maximize the captured demand. Any point on the network is a candidate facility location.

Assuming that firm B opens facilities at $Y = \{y_1, \ldots, y_r\}$, customers at v are served by only one of the competing firms if distances $d(v, X)$ and $d(v, Y)$ are *sufficiently different*; otherwise demand at v is shared among the competing firms in some way. The distance values which lead to the *all or nothing* choice rule are given by the *sensitivity thresholds* denoted by $\underline{\alpha}_v$ and $\overline{\alpha}_v$. If $d(v,Y) < d(v,X) - \underline{\alpha}_v$ or $d(v,Y) > d(v,X) + \overline{\alpha}_v$ then customers at v patronize the closest firm. Given X, the demand at v captured by firm B is $w_v H_v(Y)$ where $H_v(Y)$ is a non-negative real function, non-increasing with respect to $d(v,Y)$, such that $H_v(Y) = 1$ if $d(v,Y) < d(v,X) - \underline{\alpha}_v$, and $H_v(Y) = 0$ if $d(v,Y) > d(v,X) + \overline{\alpha}_v$. The total demand captured by firm B is $W_B(X,Y) = \sum_{v \in V} w_v H_v(Y)$. As demand is assumed to be totally satisfied (essential goods), the demand captured by firm A is $W_A(X,Y) = W - W_B(X,Y)$.

In its more general form, function H_v is defined as follows:

$$H_v(Y) = \begin{cases} 1 & if \quad d(v,Y) < L_v \\ h_v(Y) & if \quad L_v \leq d(v,Y) \leq U_v \\ 0 & if \quad d(v,Y) > U_v \end{cases} \tag{1}$$

if $L_v > 0$, and

$$H_v(Y) = \begin{cases} h_v(Y) & if \quad 0 \leq d(v,Y) \leq U_v \\ 0 & if \quad d(v,Y) > U_v \end{cases} \tag{2}$$

if $L_v \leq 0$,

where $L_v = d(v,X) - \underline{\alpha}_v$, $U_v = d(v,X) + \overline{\alpha}_v$, and $h_v(Y) = h_v(d(v,y_1),...,d(v,y_r))$, with h_v a non-negative, non-increasing real function, that is, if $z_i \leq u_i$ for $1 \leq i \leq r$, then $0 \leq h_v(z_1,...,z_r) \leq h_v(u_1,...,u_r)$. Function H_v is the capture function for firm B.

Given X, the problem of firm B is to find the set $Y^* = Y(X)$ that maximizes its market share, that is, the set $Y^* = \{y_1^*, \ldots, y_r^*\}$ such that

$$W_B(X, Y^*) = \max_{Y \subset N, |Y|=r} W_B(X, Y). \tag{3}$$

Using the denomination introduced by Hakimi [13], a solution to Problem (3) is an $(r|X)$-medianoid. Observe that the demand captured by firm B is a non-increasing function of $\underline{\alpha}$ and a non-decreasing function of $\overline{\alpha}$. Observe also that, if $h_v(Y) = h_v(d(v, Y))$ with $h_v(L_v) = 1$ and $h_v(U_v) = 0$, we have a decay function equal to those introduced in [4].

Particular sensitivity threshold values and functions h_v lead to different customer choice rules:

- **The binary choice rule**

  When $\underline{\alpha}_v = \overline{\alpha}_v = 0$, and $h_v(Y) = \mu$, where $0 \leq \mu \leq 1$, for all $v \in V$, we obtain a binary choice rule. If $\mu = 0$ we have the binary rule oriented to firm A, if $\mu = 1$ we have the binary rule oriented to firm B, if $\mu = \frac{1}{2}$ each firm takes half of demand at v when $d(v, X) = d(v, Y)$.

- **The partially binary choice rule**

  If for all $v \in V$, $\underline{\alpha}_v = d(v, X)$, $\overline{\alpha}_v = \max\{d(x, y) : x, y \in N\}$ and $h_v(Y) = \frac{f_v(d(v,X))}{f_v(d(v,Y))+f_v(d(v,X))}$, where f_v is non-decreasing and $f_v(0) \geq 0$, with $h_v(Y) = \frac{1}{2}$ if $d(v, X) = d(v, Y) = 0$, we have the partially binary choice rule.

- **The proportional choice rule**

If for all $v \in V$, $\underline{\alpha}_v = d(v,X)$, $\overline{\alpha}_v = \max\{d(x,y) : x,y \in N\}$ and $h_v(Y) = \frac{\sum_{1\le i\le r} \frac{1}{f_v(d(v,y_i))}}{\sum_{1\le i\le r} \frac{1}{f_v(d(v,y_i))} + \sum_{1\le i\le p} \frac{1}{f_v(d(v,x_i))}}$, where f_v is non-decreasing and $f_v(0) > 0$, we obtain the proportional choice rule.

- **Piecewise linear choice rule**

  Let $h_v(Y) = h_v(d(v,Y))$ for all $v \in V$, with h_v a piecewise linear function, then we have a piecewise linear choice rule. Two specific functions are the following:

  – Step-function

$$h_v(z) = \begin{cases} a_{v1} & if \quad L_v \le z < r_1 \\ \vdots & \vdots \qquad \vdots \\ a_{vq_v} & if \quad r_{q_v-1} \le z \le U_v \end{cases} \tag{4}$$

  where $1 \ge a_{v1} > \cdots > a_{vq_v} \ge 0$.

  – Continuous function

$$h_v(z) = \begin{cases} \frac{(1-\mu_v)(L_v - z) + \underline{\alpha}_v}{\underline{\alpha}_v} & if \quad L_v \le z \le d(v,X) \\ \mu_v \frac{U_v - z}{\overline{\alpha}_v} & if \quad d(v,X) < z \le U_v \end{cases} \qquad \text{if } d(v,X) \ne 0, \tag{5}$$

$$H_v(z) = \begin{cases} \mu_v \frac{\overline{\alpha}_v - z}{\overline{\alpha}_v} & if \quad 0 \le z \le U_v \\ 0 & if \quad z > U_v \end{cases} \qquad \text{if } d(v,X) = 0, \tag{6}$$

where

$L_v = d(v, X) - \underline{\alpha}_v$ with $\underline{\alpha}_v = \underline{k}_v d(v, X)$ and $0 < \underline{k}_v \leq 1$,

$U_v = d(v, X) + \overline{\alpha}_v$ with $\overline{\alpha}_v > 0$, and $0 < \mu_v < 1$, for all $v \in V$.

Observe that, in our competitive location problem, value μ_v in expressions (5) and (6) is the portion of demand at v captured by firm B when $d(v, Y) = d(v, X)$.

Example 1 *Consider the network represented in Figure 1 where the demand at node v_i is $w_i = 1$, for $1 \leq i \leq 4$, and the lengths are shown on the edges. Consider $p = r = 1$ and $X_1 = \{x_1\}$ with $x_1 = v_1$. Figure 2 represents the demand captured by firm B if its facility is located at $x \in [v_2, v_3]$ for the following customer choice rules: binary (1), partially binary (2), and those defined by the piecewise linear functions (5) and (6) with $L_v = 0, 0, 0, 2$, $U_v = 1, 4, 4, 6$ (3), $L_v = 0, 1, 2, 4$, $U_v = 3, 5, 6, 8$ (4), and $L_v = 0, 2, 3, 5$, $U_v = 3, 5, 6, 8$ (5), where $\mu_v = \frac{1}{2}$ for $v \in V$.*

3 Finite dominating sets

A finite dominating set (FDS) for a network location problem is *a finite set of points to which some optimal solution must belong* [14]. If a FDS is available, the candidate facility locations can be constrained to this set and the network problem becomes a discrete problem. It is known that for the binary choice rule (oriented to the leader) there is an $(r|X)-$medianoid consisting of points

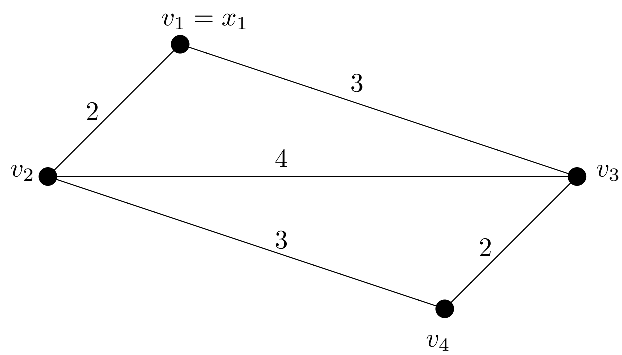

Figure 1: Network used in Example 1

which are non-*isodistant* nodes or non-isodistant points between two isodistant points on an edge [15, 16, 20]. For the partially binary and proportional choice rules described in Section 2, if f_v is concave for all $v \in V$, we have an $(r|X)-$medianoid included in V [13, 20]. For functions (1) and (2), with $h_v(Y) = h_v(d(v,Y))$, if $h_v(z)$ is convex for all v, $h_v(L_v) = 1$, if $L_v > 0$, and $h_v(U_v) = 0$, a finite dominating set is $\mathcal{F} = V \cup \{x \in N : d(v,x) = L_v$ or $d(v,x) = U_v, v \in V\}$ [4]. As we will see in Section 3.1, taking into account the convexity of H_v on the open interval $(L_v, +\infty)$, this finite dominating set can be reduced to $\mathcal{F} = V \cup \{x \in N : d(v,x) = L_v, v \in V\}$. A FDS result for the step-functions defined as (4) is included in [3].

In this section we determine a FDS for the $(r|X)-$medianoid problem when the customer choice rule is defined by a capture decay function that verifies certain conditions. First, we consider the case where customers visit only the closest facility of the firms they patronize. Later we study the situation where

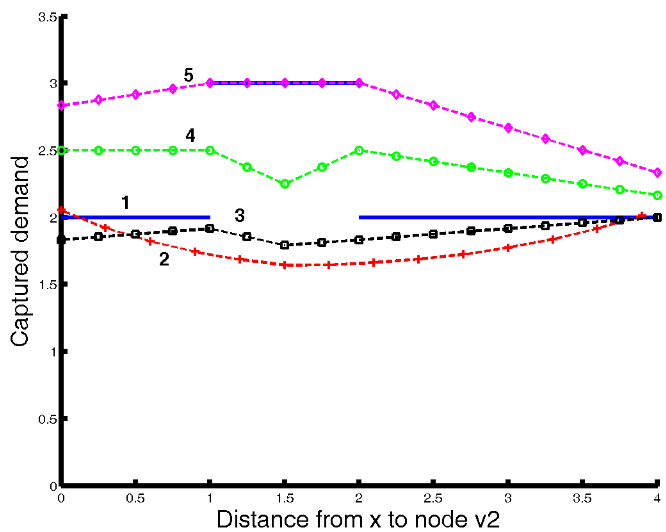

Figure 2: Captured demand on edge $[v_2, v_3]$ for the customer choice rules of Example 1

customers may visit several facilities.

3.1 Customers visit the closest facility

Suppose that customers may patronize both firms but they visit only the closest facility of each of them. Consider $H_v(Y) = H_v(d(v,Y))$ where

$$H_v(z) = \begin{cases} 1 & if \quad z < L_v \\ h_v(z) & if \quad L_v \leq z \leq U_v \\ 0 & if \quad z > U_v \end{cases} \tag{7}$$

if $L_v > 0$, and

$$H_v(z) = \begin{cases} h_v(z) & if \quad 0 \leq z \leq U_v \\ 0 & if \quad z > U_v \end{cases} \tag{8}$$

if $L_v \leq 0$,

where h_v is a univariate, continuous, and non-increasing function, such that $h_v(L_v) = 1$, if $L_v > 0$, and $h_v(U_v) = 0$. It implies the continuity of H_v. Without any loss of generality, we can assume $L_v \geq 0$, for all v.

Definition 1 *Let $h : D \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ where D is a convex set. Then, h is piecewise convex if there exists a set of points $S = \{b_k\}_{k=1}^{q} \subset \mathbb{R}$ and h is convex on $D \cap (a,b)$, for any open interval (a,b) such that $(a,b) \cap S = \emptyset$. Points b_k, $1 \leq k \leq q$, are the breakpoints.*

Proposition 1 *Let h_v be a piecewise convex, non-negative, continuous, and non-increasing real function defined on $\mathbb{R}_+ = \{z \in \mathbb{R} : z \geq 0\}$. Let $\{b_{vk}\}_{k=1}^{q_v}$ be the set of breakpoints of h_v in $[L_v, U_v]$, $v \in V$. Then $\mathcal{F} = V \cup \{x \in N :$*

$d(v,x) = L_v$ or $d(v,x) = b_{vk},\ k = 1,...,q_v,\ v \in V\}$ is a finite dominating set for the $(r|X)-$medianoid problem.

Proof. Under the conditions assumed for h_v, function H_v is continuous and piecewise convex in $\mathbb{R}_+$, with breakpoints $b_{v0} = L_v,\ b_{vk},\ k = 1,...,q_v$, for $v \in V$. Let $Y^* = \{y_i^*\}_{i=1}^r$ be an optimal solution to Problem (3) and suppose that there exists y_i^* which does not belong to set $\mathcal{F}$. Without loss of generality we can suppose $i = 1$ and $y_1^* \in (u,u')$, with $[u,u'] \in E$. Consider that $y = y_1^*$ varies along the edge $[u,u']$ and the rest of locations remain fixed. Then, if $Y = \{y, y_i^*\}_{2\leq i\leq r}$, $d(v,Y) = \min\{d(v,y), K\}$ where $K = \min\limits_{2\leq i\leq r}\{d(v,y_i^*)\}$.

Let $\mathcal{F} \cap (u,u') = \{c_1,...,c_q\}$ where points c_k are enumerated by increasing value of the distance to node u. Let $u = c_0$ and $u' = c_{q+1}$. Then, there exists k, $0 \leq k \leq q$ such that $y \in (c_k, c_{k+1})$. Given v, the distance $d(v,y)$ is a continuous and concave function of y on any edge and, consequently, function $g_v(y) = \min\{d(v,y), K\}$ has these same properties. In particular, $d(v,y)$ and $\min\{d(v,y), K\}$ are concave functions of y when y varies on the segment $[c_k, c_{k+1}]$. Let $a = \min\{d(v,x), x \in [c_k, c_{k+1}]\}$, $b = \max\{d(v,x), x \in [c_k, c_{k+1}]\}$, and $J = \{z \in \mathbb{R} : g_v(y) = z,\ y \in [c_k, c_{k+1}]\}$, then only three cases can occur: 1) if $K \leq a$ then $J = \{K\}$, 2) if $a < K < b$ then $J = [a, K]$, and 3) if $K \geq b$ then $J = [a,b]$. If $J = \{K\}$, $H_v(d(v,Y)) = H_v \circ g_v(y) = H_v(K)$, leading to a constant function on $[c_k, c_{k+1}]$. For all case, since $(c_k, c_{k+1}) \cap \mathcal{F} = \emptyset$ and $H_v(z)$ is continuous, it follows that $H_v(z)$ constrained to J is convex. Then, as $g_v(y)$ is a concave function of y on $[c_k, c_{k+1}]$, $H_v(z)$ is non-increasing and $H_v(z)$ constrained to J is convex, we deduce that $H_v \circ g_v(y)$ constrained to $[c_k, c_{k+1}]$ is convex for all $v \in V$. Then, $W(y) = \sum_{v \in V} w(v) H_v(g_v(y))$ constrained to $[c_k, c_{k+1}]$ is convex, and we conclude that $W(y)$ reaches the maximum at c_k or at c_{k+1}. Therefore, if $c^* = \arg\max\{W(c_k), W(c_{k+1})\}$ then $W(c^*) \geq W(y_1^*)$, and we can replace y_1^* by c^* without decreasing of W. ■

Corollary 1 *Let h_v be a convex, non-negative, continuous, and non-increasing real function defined on $\mathbb{R}_+$. Then $\mathcal{F} = V \cup \{x \in N : d(v, x) = L_v,\ v \in V\}$ is a finite dominating set for the $(r|X)-$medianoid problem.*

Proof. It follows directly from proposition 1. ■

Example 2 *Consider the market share function used in Example 1 and numbered 4 in Figure 2. The breakpoints are $L_{v_2} = 1$, $b_{v_2 1} = d(v_2, v_1) = 2$, $L_{v_3} = 2$, $b_{v_3 1} = d(v_3, v_1) = 3$, $L_{v_4} = 4$, $b_{v_4 1} = d(v_4, v_1) = 5$. Then, $\mathcal{F} \cap [v_2, v_3] = \{v_2, v_3, c_1, c_2\}$ where $d(c_1, v_2) = 1$ and $d(c_2, v_2) = 2$.*

3.2 Customers may visit several facilities

Now, we consider the case where $r \geq 2$ and the customers may visit several facilities of both firms and where facilities different to the closest one may capture demand. For $Y = \{y_1, ..., y_r\} \subset N$, consider function $H_v(Y) = H_v(d(v, y_1), ..., d(v, y_r))$, $v \in V$. For $v \in V$ and $\vec{z} = (z_1, ..., z_r) \in \mathbb{R}_+^r$, $H_v(\vec{z})$ is defined as follows:

$$H_v(\vec{z}) = \begin{cases} 1 & if \quad \min\limits_{1\le i\le r}\{z_i\} < L_v \\ h_v(\vec{z}) & if \quad L_v \le \min\limits_{1\le i\le r}\{z_i\} \le U_v \\ 0 & if \quad \min\limits_{1\le i\le r}\{z_i\} > U_v \end{cases} \tag{9}$$

if $L_v > 0$, and

$$H_v(\vec{z}) = \begin{cases} h_v(\vec{z}) & if \quad 0 \le \min\limits_{1\le i\le r}\{z_i\} \le U_v \\ 0 & if \quad \min\limits_{1\le i\le r}\{z_i\} > U_v \end{cases} \tag{10}$$

if $L_v = 0$,

where h_v is a non-negative r-variate continuous and non-increasing real function such that, if $\vec{z} = (z_1, ..., z_r) \in \mathbb{R}^r_+$ and $\min\limits_i\{z_i\} = L_v > 0$ then $h_v(\vec{z}) = 1$, and if $\vec{z} = (z_1, ..., z_r) \in \mathbb{R}_+$ and $\min\limits_i\{z_i\} = U_v$ then $h_v(\vec{z}) = 0$. Under these conditions, H_v is continuous.

Definition 2 *Let $h : \mathbb{R}^r_+ \rightarrow \mathbb{R}$. Then, h is partially piecewise convex if there exists a set of real values $S = \{b_k\}_{k=1}^q$ and h is convex with respect to any coordinate i on (a, b), for any open interval (a, b) such that $(a, b) \cap S = \emptyset$. Points b_k, $1 \le k \le q$, are the breakpoints.*

That is to say, for any $i \in \{1, \ldots, r\}$ and $\vec{z} = (z_1, \ldots, z_r) \in \mathbb{R}^r_+$, function $h_{vi}(z) = h_v(z_1, \ldots, z_i, z, z_{i+1}, \ldots, z_r)$ is piecewise convex with breakpoints belonging to $\{b_k,\ 1 \le k \le q\}$.

Proposition 2 *For any $v \in V$, let h_v be a partially piecewise convex, non-negative, continuous, and non-increasing real function defined on $\mathbb{R}^r_+$. For each $v \in V$, let $\{b_{vk}\}_{k=1}^{q_v}$ be the set of breakpoints of h_v. Then $\mathcal{F} = V \cup \{x \in N :$*

$d(v,x) = L_v$ or $d(v,x) = b_{vk},\ k = 1,...,q_v,\ v \in V\}$ is a finite dominating set for the $(r|X)-$medianoid problem.

Proof.

For all v, under the conditions assumed for h_v, function H_v is continuous and partially piecewise convex in $\mathbb{R}^r_+$, with breakpoints $b_{v0} = L_v$, b_{vk}, $k = 1,...,q_v$, $v \in V$. Let $Y^* = \{y_i^*\}_{i=1}^r$ be an optimal solution to Problem (3) and suppose that there exists y_i^* which does not belong to set $\mathcal{F}$. Without loss of generality we can suppose $i = 1$ and $y_1^* \in (u,u')$, with $[u,u'] \in E$. Consider that $y = y_1^*$ varies along the edge $[u,u']$ and the rest of locations remain fixed. Let $\mathcal{F} \cap (u,u') = \{c_1,...,c_q\}$ where points c_k are enumerated by increasing value of the distance to node u. Let $u = c_0$ and $u' = c_{q+1}$. Then, there exists k, $0 \le k \le q$ such that $y \in (c_k, c_{k+1})$.

Given v, the distance $d(v,y)$ is a continuous and concave function of y on any edge, in particular $d(v,y)$ is continuous and concave on $[c_k,c_{k+1}]$. Let $g_v : \mathbb{R}_+ \to \mathbb{R}$ be the function defined as $g_v(z) = H_v(z, d(v,y_2^*),...,d(v,y_r^*))$ and let $J = \{z \in \mathbb{R} : d(v,y) = z, y \in [c_k,c_{k+1}]\}$. Then $J = [a,b]$ where $a = \min\{d(v,x), x \in [c_k,c_{k+1}]\}$, $b = \max\{d(v,x), x \in [c_k,c_{k+1}]\}$. From the properties of H_v it follows that g_v is continuous and non-increasing. Moreover, since $(c_k,c_{k+1}) \cap \mathcal{F} = \emptyset$ and $g_v(z)$ is continuous, we conclude that $g_v(z)$ constrained to J is convex. As $d(v,y)$ is a concave function of y in $[c_k,c_{k+1}]$ and g_v constrained to J is convex and non-increasing, the composite function $G_v(y) = g_v \circ d(v,y) = H_v(d(v,y), d(v,y_2^*)..., d(v,y_r^*))$ constrained to $[c_k,c_{k+1}]$ is convex for all $v \in V$. Then, $W(y) = \sum_{v\in V} w(v)G_v(y)$ constrained to $[c_k,c_{k+1}]$

is convex, and we conclude that $W(y)$ reaches the maximum at c_k or at c_{k+1}. Therefore, if $c^* = \arg\max\{W(c_k), W(c_{k+1})\}$ then $W(c^*) \geq W(y_1^*)$, and we can replace y_1^* by c^* without decreasing of W. ■

Corollary 2 *Let h_v be a convex, non-negative, continuous, and non-increasing real function defined on $\mathbb{R}_+^r$. Then $\mathcal{F} = V \cup \{x \in N : d(v,x) = L_v,\ v \in V\}$ is a finite dominating set for the $(r|X)-$medianoid problem.*

Proof. It follows directly from Proposition 2. ■

Example 3 *Consider $r = 2$. For $v \in V$, let $I_1 = [0, L_v) \times [0, +\infty)$, $I_2 = [L_v, +\infty) \times [0, L_v)$, $I_3 = [L_v, U_v] \times [L_v, U_v]$, $I_4 = (U_v, +\infty) \times [L_v, U_v]$, $I_5 = [L_v, U_v] \times (U_v, +\infty)$, $I_6 = (U_v, +\infty) \times (U_v, +\infty)$. Observe that $\{I_j\}_{j=1}^6$ is a partition of $\mathbb{R}_+^2$. Let H_v be the function defined as follows:*

$$H_v(z_1,z_2) = \begin{cases} 1 & if \quad (z_1,z_2) \in I_1 \cup I_2 \\ \max\limits_{i=1,2} \left\{ \frac{U_v - z_i}{U_v - L_v} exp[-(z_1 - L_v)^{\frac{1}{2}} (z_2 - L_v)^{\frac{1}{2}}] \right\} & if \quad (z_1,z_2) \in I_3 \\ \max\limits_{i=1,2} \left\{ \frac{U_v - z_i}{U_v - L_v} exp[-(z_i - L_v)^{\frac{1}{2}} (U_v - L_v)^{\frac{1}{2}}] \right\} & if \quad (z_1,z_2) \in I_4 \cup I_5 \\ 0 & if \quad (z_1,z_2) \in I_6 \end{cases}$$

if $L_v > 0$, and

$$H_v(z_1,z_2) = \begin{cases} \mu \cdot \max\limits_{i=1,2} \left\{ \frac{U_v - z_i}{U_v} exp[-z_1^{\frac{1}{2}} z_2^{\frac{1}{2}}] \right\} & if \quad (z_1,z_2) \in I_3 \\ \mu \cdot \max\limits_{i=1,2} \left\{ \frac{U_v - z_i}{U_v} exp[-z_i^{\frac{1}{2}} U_v^{\frac{1}{2}}] \right\} & if \quad (z_1,z_2) \in I_4 \cup I_5 \\ 0 & if \quad (z_1,z_2) \in I_6 \end{cases}$$

if $L_v = 0$, where $0 < \mu \leq 1$, for all $v \in V$.

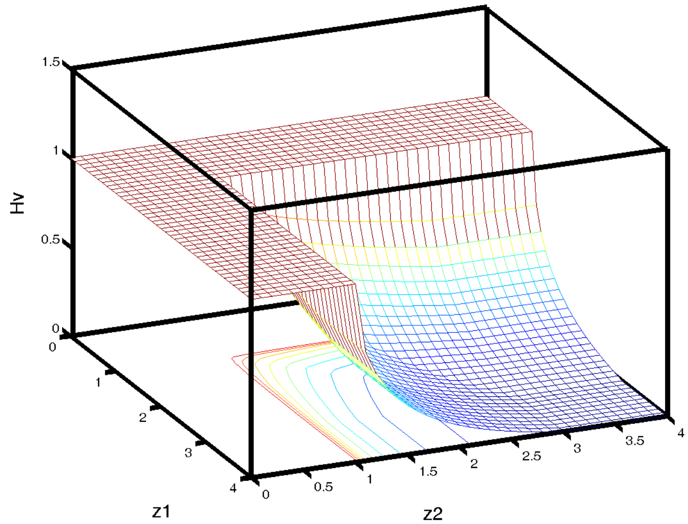

Figure 3: Function H_v of Example 3

Function H_v, represented in Figure 3, is continuous, non-increasing, and partially piecewise convex in $\mathbb{R}^2_+$ with breakpoints L_v, $v \in V$. Then $\mathcal{F} = V \cup \{x \in N : d(v,x) = L_v, v \in V\}$ is a finite dominating set for the $(r|X_p)-$medianoid. In particular, for the network of Figure 1, with $L_v = 0, 1, 2, 4$, $U_v = 3, 5, 6, 8$, and $\mu = 0.5$, the candidate locations are $V \cup \{c_1, c_2, c_3, c_4, c_5\}$, where $c_1 \in [v_1, v_2]$ with $d(v_1, c_1) = 1$, $c_2, c_5 \in [v_2, v_3]$ with $d(v_2, c_2) = 1$, $d(v_2, c_5) = 2$, $c_3 \in [v_2, v_4]$ with $d(v_2, c_3) = 1$, and $c_4 \in [v_1, v_3]$ and $d(v_1, c_4) = 1$. The optimal pair of locations is $\{c_1, c_4\}$ which captures 3.12 units of demand.

4 Concluding remarks

In this paper we considered the $(r|X_p)-$medianoid problem in networks with a general customer choice rule that includes several specific rules studied in the literature. We considered two scenarios, in the first scenario customers visit at most the closest facility from each firm, in the second scenario customers may visit several facilities for both competing firms. A customer visits exclusively one of the firms if the difference in distances between this customer and the competitors exceeds certain threshold, otherwise the demand is shared among the competing firms in some way. In its more general form, the capture (coverage) decay function is multivariate and the demand captured by a firm depends on the distance from the demand point to the locations of all the facilities. For both scenarios, assuming some convexity conditions for the capture function, we proved the existence of a finite dominating set. If a finite dominating set is found, the location problem on networks can be solved as a discrete problem. In this case, for the univariate capture function (Section 3.1), the Linear Programming formulations and results shown in [2, 3, 4, 5] can be applied to our competitive location problem.

The translation of our discretization results to maximal covering location problems leads to an extension of those presented in [4]. On one hand, we replace the convexity of the univariate coverage function by the piecewise convexity, and on the other hand we give a result for multivariate coverage functions.

Finally, our results can be useful to solve location problems where the capture function is approximated by a piecewise convex function and/or where several facilities and not only the closest one attract demand. Moreover, the model can be adapted to unessential goods; if we consider that customers visit only facilities which are close enough, we can use capture functions similar to (1)-(2) with $U_v = \min\{\beta_v, d(v, X) + \overline{\alpha}_v\}$, where β_v is the maximum distance that a customer at v is willing to travel to get the product, in this case $W_A(X, Y) + W_B(X, Y) \leq W$.

References

[1] O. Berman, Z. Drezner, D. Krass, Generalized coverage: New developments in covering location models, Computers & Operations Research 37 (2010) 1675-1687.

[2] O. Berman, D. Krass, Constraint expansion and alternate formulations of the maximum cover and uncapacitated facility location problems, Working paper (2001).

[3] O. Berman, D. Krass, The generalized maximal covering location problem, Computers & Operations Research 29 (2002) 563-581.

[4] O. Berman, D. Krass, Z. Drezner, The gradual covering decay location problem on a network, European Journal of Operational Research 151 (2003) 474-480.

[5] G. Cornuejols, G.L. Nemhauser, L. A. Wolsey, The Uncapacitated facility location problem. In Discrete Location Theory, P.B. Mirchandani and R.L.

Francis Edit. 1990, pp. 119-171.

[6] N.E. Devletoglou, A dissenting view of duopoly and spatial competition, Economica May (1965) 141-160

[7] N.E. Devletoglou, P.A. Demetriou, Choice and threshold: a further experiment in spatial duopoly, Economica November (1967) 351-371.

[8] H.A. Eiselt, G. Laporte, Competitive spatial models, European Journal of Operational Research 39 (1989) 231-242.

[9] H.A. Eiselt, G. Laporte, Sequential location problems, European Journal of Operational Research 96 (1996) 217-231.

[10] H.A. Eiselt, G. Laporte, J.F. Thisse, Competitive location models: A framework and bibliography, Transportation Science 27(1) (1993) 44-54

[11] T.L. Friesz, T. Miller, R.L. Tobin, Competitive network facility location models: a survey, Papers of the Regional Science Association 65 (1988) 47-57.

[12] S.L. Hakimi, On locating new facilities in a competitive environment. European Journal of Operational Research, 12 (1983) 29-35.

[13] S.L. Hakimi, Location with spatial interactions: competitive locations and games, in: P.B. Mirchandani, R.L. Francis (ed), Discrete Location Theory. Wiley, New York, 1990 pp. 439-478.

[14] j.n. Hooker, R.S. Garfinkel, C.K. Chen, Finite dominating sets foe network location problems, Operations Research 39(1) (1991) 110-118.

[15] N. Meggido, E. Zemel, S.L. Hakimi, The maximum coverage location problem, SIAM Journal on Algebraic and Discrete Methods 4(2)(1983) 253-261.

[16] P.H. Peeters, F. Plastria, Discretization results for the Huff and Pareto-Huff competitive location models on networks, TOP 6(2) (1998) 247-260.

[17] F. Plastria, Static competitive facility location: an overview of optimization approaches, European Journal of Operational Research 129 (2001) 461-470

[18] D. Serra, C. ReVelle, Competitive location in discrete space, in: Drezner Z (ed) Facility location: A survey of applications and methods, Springer, Berlin, 1995, pp 367-386.

[19] J. Spoerhase,H.C. Wirth, (r, p)-centroid problems on paths and trees, Technical Report 441, Inst. Comp. Science, University of Würzburg. http://www.informatik.uni-wuerzburg.de/ (2007)

[20] R. Suárez-Vega, D.R. Santos-Peñate, P. Dorta-González, Competitive multifacility location on networks: the $(r|X_p)$-medianoid problem, Journal of Regional Science 44(3) (2004) 569-588.