Learning how to play Nash, potential games and alternating minimization method for structured nonconvex problems on Riemannian manifolds

J.X. da Cruz Neto* P. R. Oliveira† P.A. Soares Jr ‡ A. Soubeyran §

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Abstract

In this paper we consider minimization problems with constraints. We show that if the set of constraints is a Riemannian manifold of non positive curvature and the objective function is lower semicontinuous and satisfies the Kurdyka-Lojasiewicz property, then the alternating proximal algorithm in Euclidean space is naturally extended to solve that class of problems. We prove that the sequence generated by our algorithm is well defined and converges to an inertial Nash equilibrium under mild assumptions about the objective function. As an application, we give a welcome result on the difficult problem of "learning how to play Nash" (convergence, convergence in finite time, speed of convergence, constraints in action spaces in the context of " alternating potential games" with inertia).

Key words: Nash equilibrium; convergence; finite time; proximal algorithm; alternation; learning in games; inertia; Riemannian manifold; Kurdyka-Lojasiewicz property.

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1 Introduction.

The "learning to play Nash" problem: four central questions In non cooperative game theory, one of the most important topic is how do players learn to play Nash equilibria (Chen and Gazzale [17]). Learning dynamics include Bayesian learning, fictitious play, Cournot best reply, adaptive learning, evolutionary dynamics, reinforcement learning and others ... Players can play sequentially (as in Cournot best reply), moving in alternation, one player at each period. In this case they follow a backward dynamic if the deviating

*Partially supported by CNPq Grant 301625/2008-5. Department of Mathematics, Federal University of Piauí, Teresina, Brazil. e-mail: jxavier@ufpi.br

†PESC/COPPE - Programa de Engenharia de Sistemas e Computação, Rio de Janeiro, Brazil
e-mail: poliveir@cos.ufrj.br

‡Partially supported by CAPES. PESC/COPPE - Programa de Engenharia de Sistemas e Computação, Rio de Janeiro, Brazil. e-mail:pedrosoares@cos.ufrj.br

§GREQAM, Université d'Aix-Marseille II, France
e-mail: antoine.soubeyran@univmed.fr

player have observed (know) what all other players have done in the past. Each period the deviating player chooses a best reply, taking as given the actions played just before by the non deviating players. Players can play simultaneously each period (as in fictitious play, Monderer and Shapley [39]). They first form beliefs over what each other players will do this period. These beliefs are usually the Cesaro mean of the actions played in the past by each other players. Then, they give a best reply to these beliefs. In this case they follow a forward dynamic. Beliefs are updated each period, as a mean of the previous actions of each other players and their previous beliefs. In both cases games are given in normal form. They are defined by their payoffs functions over their strategies (actions) spaces. For two players with action spaces M and N their payoffs functions are $F(x, y) \in \mathbb{R}$ and $G(x, y) \in \mathbb{R}$, $x \in M$ and $y \in N$.

In this dynamic context the "learning to play Nash" problem poses four central questions: i) how these learning dynamics converge to the set of Nash equilibria (positive convergence) or converge to a Nash equilibrium? ii) does the process converges in finite time, iii) what is the speed of convergence, does plays converge gradually or abruptly? iv) how, for each player, constraints on the spaces of actions can be included?

An alternating proximal algorithm model of exploration-exploitation This paper gives an answer to these "four difficult questions", using the worthwhile to change exploration-exploitation model of (Soubeyran [44, 43]) in the context of alternating proximal algorithms (Attouch et al. [3]), including constraints on the spaces of actions. We examine proximal algorithms when the feasible subsets of actions are Riemannian manifolds, allowing to modelize the striking case of players engaged in multi-tasks activities where resource constraints, like time constraints, play a role, which is the usual case. Moreover Riemannian manifolds help us to modelize "vital dynamic constraints", almost always neglected in the economic litterature. To be able to play again and again agents must produce, conserve and regenerate their energy as time evolves. Some activities produce daily vital energy for each agent (like eating, resting, holidays, sports, healthy activities, arts, ...), giving further motivations to act. Other activities consume energy (working, thinking, stress, ...). Each period, agents must maintain a balance between regeneration and consumption of vital energy. This define, each period, a constraint (conservation law).

Moreover, to give a more realistic description of "alternating games with learning and inertia", our paper modelizes costs to change as quasi distances (see Soubeyran [44, 43]). This means that the cost to move from "being able to do an action" to "become able to do an other action" can be different from the reverse.

Aiming at answering the four central questions posed, we propose and analysis the alternating proximal algorithm in the setting of Riemannian manifolds. Let us describe: consider the following minimization problem

$$\begin{aligned} &\min H(x, y) \\ &s.t.\ (x, y) \in M \times N, \end{aligned} \tag{1.1}$$

where M and N are complete Riemannian manifolds and $H : M \times N \to \mathbb{R} \cup \{+\infty\}$ is a proper lower semicontinuous function bounded from below. The alternating proximal algorithm to solve optimization problems of the form (1.1) generates, for a starting point $z_0 = (x_0, y_0) \in M \times N$, a sequence (z_k), with $z_k = (x_k, y_k) \in M \times N$, as it follows:

$$(x_k, y_k) \to (x_{k+1}, y_k) \to (x_{k+1}, y_{k+1})$$

$$\begin{cases} x_{k+1} \in \arg\min\{H(x, y_k) + \frac{1}{2\lambda_k} C_M^2(x_k, x), x \in M\} \\ \\ y_{k+1} \in \arg\min\{H(x_{k+1}, y) + \frac{1}{2\mu_k} C_N^2(y_k, y), y \in N\} \end{cases}$$

where C_M and C_N are quasi distances associated with the manifolds M and N respectively, (λ_k) and (μ_k) are sequences of positive numbers and the function H consists of a separable term $(x, y) \longmapsto f(x)+g(y)$ and a coupling term Ψ. Previous related works can be found in Attouch et al. [5, 3, 7], but there, the setting is $M = \mathbb{R}^m$, $N = \mathbb{R}^n$ and a quadratic coupling Ψ. Lewis and Malick [36] studied the method of alternating projections in the context in which $M \subset \mathbb{R}^n$ and $N \subset \mathbb{R}^n$ are two smooth manifolds intersect transversally.

In Attouch et al. [3] the Kurdyka-Lojasiewicz property is used to derive the convergence of the alternating proximal algorithm for nonconvex problems where regularization functions are symmetrical. In the context of Hadamard manifolds we work with non symmetrical regularization functions . Theses are the most appropriate for applications. Then, from a mathematical point - of - view, our paper generalizes the work of Attouch et al. [3], using quasi distances instead of distances and adding resources constraints as manifolds.

In each iteration we must solve the following subproblems:

$$\begin{array}{c} \min H(x, y_k) + \frac{1}{2\lambda_k} C_M^2(x_k, x), \\ s.t. \;\; x \in M, \end{array} \tag{1.2}$$

and

$$\begin{array}{c} \min H(x_{k+1}, y) + \frac{1}{2\mu_k} C_N^2(y_k, y), \\ s.t. \;\; y \in N. \end{array} \tag{1.3}$$

To solve the subproblem of the form (1.2) or (1.3), we use the exact proximal point algorithm that generates, for a starting point $x_0 \in M$, a sequence (x_k), with $x_k \in M$ as it follows:

$$x_{k+1} \in \arg\min_{x\in M} \left\{ f(x) + \frac{1}{2\lambda_k} C_M^2(x_k, x) \right\},$$

with $f(x) = H(x, y_k)$, $\alpha d_M(x_k, x) \leq C_M(x_k, x) \leq \beta d_M(x, x_k)$, where d_M is the Riemannian distance (see Section 3.1) and α, β are positive numbers.

The exact proximal point algorithm was first considered, in this context, by Ferreira and Oliveira [20], in the particular case that M is a Hadamard manifold (see Section 3.1), $C_M(x_k, \cdot) = d_M(x_k, \cdot)$, $\mathrm{dom} f = M$ and f is convex. They proved that, for each $k \in \mathbb{N}$ the function $f(\cdot) + \frac{1}{2} d_M^2(x_k, \cdot) : M \to \mathbb{R}$ is 1-coercive and, consequently, that the sequence $\{x_k\}$ is well defined, with x_{k+1} being uniquely determined. Moreover, supposing $\sum_{k=0}^{+\infty} 1/\lambda_k = +\infty$ and that f has a minimizer, the authors proved convergence of the sequence $(f(x_k))$ to the minimum value and convergence of the sequence (x_k) to a minimizer point. With the result of convergence of the sequence generated by the alternating proximal algorithm (1.1) (see Theorem 4.1), from a mathematical point - of - view, our paper generalizes the work of Ferreira and Oliveira [20], using quasi distances instead of distances and when we consider $H(x, y) = f(x)$.

Several authors have proposed in the last three decades the generalized proximal point algorithm for certain nonconvex minimization problems. As far as we know the first direct generalization, in the case where M is a Hilbert space, was performed by Fukushima and Mine [22]. In the Riemannian context, Papa Quiroz and Oliveira [42] considered the proximal point algorithm for quasiconvex function (not necessarily convex) and proved full convergence of the sequence $\{x^k\}$ to a minimizer point with M being a Hadamard manifold. Bento et al. [12] considered the proximal point algorithm for C^1-lower type functions and obtained local convergence of the generated sequence to a minimizer, also in the case that M is a Hadamard manifold. With the result of convergence of the sequence generated by our algorithm (1.1) (see Theorems 4.1 and 4.3), our paper generalizes on the work of Papa Quiroz and Oliveira [42] and Bento et al. [12], using quasi distances instead of distances and when we consider $H(x, y) = f(x)$.

So far, in the convergence analysis of the exact proximal point algorithm for solving convex or quasiconvex minimization problems, it was necessary to consider Hadamard type manifolds. This is because the convergence analysis is based on the Fejér convergence to the minimizers set of f and these manifolds, apart from having the same topology and differentiable structures of Euclidean spaces, it also has geometric properties satisfactory to the characterization of Fejér convergence of the sequence. In case f is not convex, but M is a Hadamard manifold the convergences of the sequence is derived from the objective function and satisfies a property well-known as Kurdyka-Lojasiewicz inequality. This inequality has been introduced by Kurdyka [31], for differentiable functions definables in an o-minimal structure defined in $\mathbb{R}^n$ (see section 3.3), through the following result:

Given $U \subset \mathbb{R}^n$ a bounded open set and $g : U \to \mathbb{R}_+$ a differentiable function definable on an o-minimal structure, there exists $c, \eta > 0$ and a strictly increasing positive function definable $\varphi : \mathbb{R}_+ \to \mathbb{R}$ of class C^1, such that

$$\|\nabla (\varphi \circ g)(x)\| \geq c, \qquad x \in U \cap g^{-1}(0, \eta). \tag{1.4}$$

Note that taking $\varphi(t) = t^{1-\alpha}$, $\alpha \in [0, 1)$, the inequality (1.4) yields

$$\|\nabla g(x)\| \geq c|g(x)|^{\alpha}, \tag{1.5}$$

where $c = 1/(1-\alpha)$, which is know as Lojasiewicz inequality (see Lojasiewicz [37]). For extensions of Kurdyka-Lojasiewicz inequality to subanalytic nonsmooth functions (defined in Euclidean spaces) see, for example, Bolte et al. [15], Attouch and Bolte [4]. A more general extension, yet in the context of Euclidean spaces, was developed by Bolte et al. [16] mainly for Clarke's subdifferentiable of a lower semicontinuous function definable in an o-minimal structure. Lageman [33] extended the Kurdyka-Lojasiewicz inequality (1.4) for analytic manifolds and differentiable $\mathcal{C}$-functions in an analytic-geometric category (see Section 3.3), aiming at establishing an abstract result of convergence of gradient-like method, see of (Lageman [33, Theorem 2.1.22]). It is important to note that Kurdyka et al. [32] had already established an extension of the inequality (1.5) for analytic manifolds and analytic functions to solve René Thom's conjecture.

Literature on Supermodular and Potential Games In the literature, answers to the three first questions have been given in three main cases: supermodular games, potential games and very recently the new class of inertial games of equilibration (see Attouch and Soubeyran [8, 9]) and Attouch et al. [7, 5].

i) Supermodular games (games with strategic complementarities). In this case, when one player takes a higher action, the others want to do the same (payoffs functions have increased differences). Then, best replies are increasing. For references see (Chen and Gazzale [17], Levin [35], Milgrom and Roberts [38], Vives [49] and Topkis [45]). Convergence towards a Nash equilibrium have been proved under the following hypothesis (Levin [35]): i) strategy spaces are subsets of $\mathbb{R}^n$, each of them being a complete sublattice. ii) the payoff function $u_i(x_i, x_{-i})$ of each player $i \in I$ is supermodular in x_i with increasing differences in (x_i, x_{-i}). A "convergence in finite time" result is given only for supermodular games with a finite number of actions for each player (finite games). To our knowledge there exists no result about convergence in finite time and about the speed of convergence for infinite supermodular games.

ii) Potential games and their variants (games with common interests, like exact potential, weighted potential, ordinal potential, best reply potential, and pseudo potential games). In the most general class of pseudo potential games the best reply correspondence of each player is included in the best reply correspondence of a unique function, a pseudo potential. It is "as if" each player maximizes a common objective. For references (see Voorneveld [50] and Jensen-Oyama [28]). For approximate solutions of potential games (see Awerbuch et al. [10]). With two players, a canonical form (our case) decomposes the payoff of each player into a common goal $\Psi(x, y)$ and an individual goal $f(x)$ or $g(y)$.

Then the normal form of the game is $F(x,y) = f(x) + \Psi(x,y)$ and $G(x,y) = g(y) + \Psi(x,y)$. The potential function is $H(x,y) = f(x) + g(y) + \Psi(x,y)$. This is a best reply potential game because it is "as if" each player maximizes the same objective: $\sup\{F(p,y), p \in X\} = \sup\{H(p,y), p \in X\}$ and $\sup\{G(x,q), q \in Y\} = \sup\{H(x,q), q \in Y\}$. This nice property comes from the separability assumption of each payoff between a joint and an individual payoff. Interactions (coupling) between players interact weakly with their individual incentives which depend only on their own action.

The most general known results are the following (Jensen and Oyama [28]). Consider a best reply potential game with single valued continuous best reply functions and compact strategy sets. Then, any admissible sequential best reply path converges to a set of pure strategy Nash equilibria. This result concerns the so called "positive convergence to a set", i.e. sequences for which every convergent subsequence converges to a point of this set, in this setting the set of Nash equilibria. Admissible sequences require that whenever $cardI$ successive periods have passed, all $i \in I$ players have moved. Results concerning finite time and speed of convergence are very rare. One recent exception is the interesting but restrictive case of fast convergence to nearly optimal solutions in potential games with ≥ 2 (Awerbuch et al. [10]). These authors consider approximate solutions to a Nash equilibrium (with rate of deviations lower than one for the payoff of each player, instead of a negative rate of deviation). But their assumptions are quite specific: a γ bounded jump condition, a β nice hypothesis. Convergence occurs in exponential time. For zero sum games see Hofbauer-Sorin [25].

Inertia Games Attouch and Soubeyran [8, 9] have considered quite realistic behaviours where it is costly to change (inertia matters). Following this line of research Attouch et al. [7, 5] have defined Inertia games as repeated normal form games with costs to change. For two players the usual normal form of a game is $F(x,y) \in \mathbb{R}$ and $G(x,y) \in \mathbb{R}$, with $x \in M$ and $y \in N$. They define costs to change as distances or squares of distances, $C_M(x,p) = d_M(x,p) \geq 0$ or $C_M(x,p) = d_M^2(x,p)$ and $C_N(y,q) = d_N(y,q) \geq 0$, or $C_N(y,q) = d_N^2(y,q)$. Then, the inertial normal form of the game becomes $J_F(p,y)/x) = \lambda F(p,y) + C_M(x,p)$ and $J_G(x,q)/y) = \mu G(x,q) + C_N(y,q)$, where $x,p \in X$ and $y,q \in Y$ and $\lambda > 0, \mu > 0$ are weights over the normal form payoffs F and G , the weighs over costs to change being normalized to one. In the case of potential games and convexity assumptions, these authors give a result about convergence, without giving any result on convergence in finite time and speed of convergence. Using the theory of tame problems (Bolte et al. [14, 15, 16]) we are able to give an answer to the four main questions on "how to learn to play Nash equilibria ?". As the previous revue of literature can show our paper is one of the very few (if not the first) to be able to give an answer to these four questions with very weak hypothesis, using our alternating proximal minimization algorithm for non convex functions (tame problems). However our analysis is restricted to two players.

This paper is organized as follows. In Section 2, we propose a synchronized model of "exploration and exploitation" in alternation to motivate the use of alternating proximal algorithms with quasi distances. In Section 3, we recall several tools and properties of Riemannian manifolds, present elements of nonsmooth analysis on manifolds, Kurdyka-Lojasiewicz property in the Riemannian context and some basic notions on o-minimal structures on $(\mathbb{R}, +, \cdot)$ and analytic-geometric categories. In section 4, the alternating proximal algorithm with quasi distances is presented and its convergence under mild conditions is established. In Section 5, we gives an application: Learning to Play Nash. In Section 6, show when a critical point is an Inertial Nash equilibrium. In Section 7, a concrete example from Psychology: "the course of motivation" is shown. Finally, in Section 8 we present our conclusions.

2 Exploration and exploitation in alternation: the model.

Convergence in finite time and speed of convergence to learn how to play Nash equilibria can be examined in a very simple "exploration-exploitation" model which is a special, but important, case of the general "Variational rationality theory" Soubeyran [44, 43].

"Exploration-exploitation" models can be either synchronic or diachronic. When agents cannot do at the same period exploration and exploitation, the model is diachronic (this is implicit in alternating games, see Attouch et al. [7, 5]). In our present paper with two agents, we propose a synchronized "exploration-exploitation" model where each agent can do, each period, both exploitation and exploration.

Exploration (Knowledge) Costs We use the following benchmark model of "exploration-exploitation" with costs to change (Soubeyran [44, 43]). In this model costs to change are "capability costs" (costs "to be able to do") or "reactivity costs" of learning along a path of change, i.e. costs which depend on distance and speed of moving. We start with a simple but very important stylized fact: learning takes time and it can be very costly to learn quickly. Learning requires delays before knowing how to do a new action, departing from doing an old action. This represents a major form of inertia (assimilation costs, accumulation costs, application costs, i.e. absorptive capacities aspects,...). Let $t_F(x,p) \geq 0$ and $t_G(y,q) \geq 0$ be the times spent to be " able to know" how to do the new actions $p \in M$ and $q \in N$ for players F and G , starting from being able to do their current actions $x \in M$ and $y \in N$. Let $d_M(x,p) \geq 0$ and $d_N(y,q) \geq 0$ be the distances between actions x and p in the manifold M and actions y and q in the manifold N. These distances represent dissimilarity indexes. The more dissimilar actions are, the more costly it is and the more time it takes to know "how to do" a new action, starting from being able to do an old action. Let $C_M(x,p) \geq 0$ and $C_N(y,q) \geq 0$ be the costs to change from x to p and from y to q on the manifolds M and N. Agents F and G must spend these costs to "know how" to do and "to be able to do" actions $p \in M$, $q \in N$ starting from doing actions $x \in M, y \in N$. Costs to change are asymmetric. For example the cost to change from x to p can be different from the cost to change from p to x, $C_M(p,x) \neq C_M(x,p)$. Let $D_F(C_M) = \delta_F C_M^2 \geq 0$ and $D_G(C_N) = \delta_G C_N^2 \geq 0$ be the desutilities of costs to change, with desutility weights $\delta_F > 0$ and $\delta_G > 0$. Then, these desutilities are $D_F(x,p) = \delta_F C_M^2(x,p)$ and $D_G(y,q) = \delta_G C_N^2(y,q)$.

"Per unit of distance" costs to change are exactly $e_M(x,p) \geq 0$, $e_N(y,q) \geq 0$ for agents F and G, with $C_M(x,p) = e_M(x,p)d_M(x,p)$ for F and $C_N(y,q) = e_N(y,q)d_N(y,q)$ for G.

Notice that "per unit of distance" costs to change $e_M(x,p), e_N(y,q)$ are not symmetric, while distances $d_M(x,p)$, $d_N(y,q)$ on the manifolds M and N are. This means that "per unit of distance" costs to change from x to p (from y to q) can be different from "per unit of distance" costs to change from p to x (from q to y).

"Per unit of time" costs to change (i.e. "reactivity costs") are $c_M(x,p) \geq 0$, $c_N(y,q) \geq 0$ for agents F and G, with $C_M(x,p) = c_M(x,p)t_F(x,p)$ for F and $C_N(y,q) = c_N(y,q)t_G(y,q)$ for G.

To escape to repetition, consider only costs to change $C_M(x,p)$ of agent F. We will make two hypothesis (the same for G):

A_1: constant speed to move:
$\omega_M(x,p) = d_M(x,p)/t_F(x,p) = \omega_M > 0 \Longleftrightarrow t_F(x,p) = (1/\omega_M)d_M(x,p)$ for all $x,p \in M$.

A_2: "per unit of distance" costs to change $e_M(x,p)$" are bounded above and below:
$0 < \alpha \leq \ e_M(x,p) \leq \beta < +\infty$.

Remark 2.1. *: In section 4 , we will consider the convergence of the alternating "exploration-exploitation" process (modelized by an alternating proximal algorithm), an important hypothesis (see hypothesis A) which* *drive the convergence result:* $\alpha d_M(x,p) \leq C_M(x,p) \leq \beta d_M(x,p)$, $0 < \alpha < \beta < +\infty$, *It means that the "per unit of distance" cost to change is bounded above and below (hypothesis* A_2*).*

Usually, "per unit of distance" costs to change increase with the speed of moving and increase (from resource depletion and fatigue) or decrease (learning) with the overall distance, $e_M = \widetilde{e}_M(\omega_M, d_M)$. In our case, the speed of moving ω_M being constant, "per unit of distance" costs to change vary only with the distance d_M. Hypothesis A_2 supposes that this variation is bounded whatever the distance: $0 < \alpha \leq \widetilde{e}_M(\omega_M, d_M) \leq \beta < +\infty$.

A Simple "Exploration-Exploitation" Model Consider $F(x,y) = f(x) + \Psi(x,y)$ and $G(x,v) = g(y) + \Psi(x,y)$ the daily normal form game payoffs of agents F and G when, this day, F performs $x \in M$ and G carries out y. We will suppose that $F(x,y)$ and $G(x,v)$ represent losses they want to minimize. Usually, in game theory, agents maximize gains $\overline{F} - F(x,y) \geq 0$, where $\overline{F} = \sup\{F(x,y), (x \in M, y \in N\} < +\infty$. and $\overline{G} - G(x,v) \geq 0$, where $\overline{G} = \sup\{G(x,y), (x \in M, y \in N\} < +\infty$ (see Soubeyran [44, 43]). In our main theorem we will suppose $\underline{H} = \inf_{M \times N} H > -\infty$ where the function $H(x,y) = f(x) + g(y) + \Psi(x,y)$.

In our present paper with two agents, we propose a synchronized "exploration-exploitation" model where each agent can do, each period, both exploitation and exploration. We suppose that players play in alternation because they have only a partial knowledge of the joint payoff $\Psi(x,y) \in \mathbb{R}$. More precisely player F knows the marginal function $p \in M \longmapsto \Psi(p,y) \in \mathbb{R}$ as soon as he can observe the action $y \in N$ done by player G, and player G knows the marginal function $q \in N \longmapsto \Psi(x,q) \in \mathbb{R}$ as soon as he can observe the action $x \in M$ done by player F. Initially players F and G know "how to do" actions $x_0 \in M$ and $y_0 \in N$ and effectively do these actions.

First period. Player F starts the process. Starting from $x_0 \in M$, F wants to move from doing action x_0 to do a new improving action $x_1 \in M$. Suppose that F thinks that, i) G will not have the intention to move as long as he can see that F has not moved and, ii) G starts thinking to move (before being able to move) as soon as he observed that F has effectively changed, doing x_1 instead of x_0 (this is a kind of "Cournot-Nash belief"). Then, each day t, F spends time, both i) to exploit (doing, each day, the old action x_0) and ii) to explore, trying first to find a new valuable action x_1 and then learning and building capabilities to "become able to do it". Suppose that this occurs $t_F(x_0,x_1) \geq 0$ days after he starts exploration. This defines the endogenous length of the first period. This first period, cumulated exploitation payoffs of each player are $t_F(x_0,x_1)F(x_0,y_0)$ and $t_F(x_0,x_1)G(x_0,y_0)$ where $F(x_0,y_0)$ and $G(x_0,y_0)$ are their per day payoff. Suppose that agent F can determine optimally the choice of a new action x_1. This means that, starting from doing x_0, he can estimate correctly both costs to change $C_M(x_0,p)$ and their desutility $\delta_{0,F} C_M^2(x_0,p)$, for all new feasible actions $p \in M$, where $\delta_{0,F} > 0$ is his first period desutility weigth. This requires that he must explore his whole action space M to evaluate $C_M(x_0,p)$ for all $p \in M$. Let $\varphi_{0,F} > 0$ be his expected length of the second period where he will only exploit x_1, given that player G performs again his old action y_0. This determines his expected second period exploitation gains $\varphi_{0,F} F(p,y_0)$. In this context, player F can choose in the first period $x_1 \in \arg\min\left\{\varphi_{0,F} F(p,y_0) + \delta_{0,F} C_M^2(x_0,p)\right\}$.

Second period. Suppose that, as soon as player F stops doing action x_0 and starts performing the new action x_1, player G can see it. Suppose that this observation will push him to react to try to be able to do a new improving action y_1, given that he has repeated, each day, action y_0 during the first period (without doing any exploration). Then, he starts exploration as well as exploits his old action y_0 each day of this second period. Exploration means that he tries first to find a new valuable action y_1 and then learns to "become able to do it". Suppose that this occurs $t_G(y_0,y_1) \geq 0$ days after he starts exploration. This defines the endogenous length of the second period. This second period, cumulated exploitation payoffs of each player are $t_G(y_0,y_1)F(x_1,y_0)$ and $t_G(y_0,y_1)G(x_1,y_0)$ where $F(x_1,y_0)$ and $G(x_1,y_0)$ are their per day second period payoffs. Suppose that agent G also can determine optimally the choice of a new action y_1. This means that, starting from doing y_0, he can estimate correctly both costs to change $C_N(y_0, q)$ and their desutility $\delta_{1,G} C_N^2(y_0, q)$, for all new feasible actions $q \in N$, where $\delta_{1,G} > 0$ is his second period desutility weigth. This requires that he must explore his whole action space N to evaluate $C_N(y_0, q)$ for all $q \in N$. Let $\varphi_{1,G} > 0$ be his expected length of the third period where he will only exploit y_1, given that player F performs again his action x_1. Then, given his expected second period exploitation gains $\varphi_{1,G} G(x_1, q)$, player G can choose in the second period $y_1 \in \arg\min \left\{\varphi_{1,G} G(x_1, q) + \delta_{1,G} C_N^2(y_0, q)\right\}$.

Desutility weights $\delta_{k,F} > 0$ and $\delta_{k,G} > 0$ vary from period to period. Each period, the length of exploitation is determined by the length of exploration activity done by each player, in alternation. At periods $k, k+1$, the alternating game is

$$x_{k+1} \in \arg\min \left\{\varphi_{k,F} F(p, y_k) + \delta_{k,F} C_M^2(x_k, p)\right\}$$

and

$$y_{k+1} \in \arg\min \left\{\varphi_{k+1,G} G(x_{k+1}, q) + \delta_{k+1,G} C_N^2(y_k, q)\right\}.$$

This game can be written

$$x_{k+1} \in \arg\min \left\{F(p, y_k) + (1/2\lambda_k) C_M^2(x_k, p)\right\}$$

and

$$y_{k+1} \in \arg\min \left\{G(x_{k+1}, q) + (1/2\mu_k) C_N^2(y_k, q)\right\}$$

where $\delta_{k,F}/\varphi_{k,F} = 1/2\lambda_k$ and $\delta_{k+1,G}/\varphi_{k+1,G} = 1/2\mu_{k+1}$.

This is our alternating minimization proximal algorithm ! Notice that, each period, anticipations by one player about the length of exploitation he can benefit does not coincide with the length of the exploration period determined by the other player!

Alternating or Simultaneous Moves. Delays The merit of our process is that it fully justifies endogenous delays and alternating moves. Each player, giving a best reply to the action done by the other player, must take some time to "know how to give" his best reply. This exploration time varies from step to step. It depends, each step, of the shape of his best reply at the current profile of actions. The other player, "who has" given his best reply to the previous action done by the first player rationally waits before engaging in a new search to "know how to give" a best reply. And so on...This process with endogenous delays seems to be entirely new in the dynamic game literature. It shows that playing sequentially can be rational. This economizes on systematic errors of moving, when players move simultaneously. In the simultaneous case, each step, each player must anticipate the best reply of the other player which can be different from what he will do. Then, waiting to change (to give his best reply), until the other player have changed, can be an economizing behaviour because, with costs to change, it is costly to change again and again, and it is not efficient to give repeatedly a best reply to a wrong anticipation of the rival best reply. When costs to change are present, playing simultaneously can be very costly. Then, sequential moves seem more realistic than simultaneous moves when it is advantageous to wait to see what have done the other player. Simultaneous moves are more realistic when it is costly to wait to change. Simultaneous moves pose a lot of coordination problems, to anticipate what other players will do. This anticipation process can generate a lot of systematic errors, which are more and more costly, the higher will be costs to change.

Action spaces as Riemannian manifolds For two players F and G the usual normal form of a game is $F(x, y) \in \mathbb{R}$ and $G(x, y) \in \mathbb{R}$, $x \in M$ and $y \in N$. Let us show on an example how the sets of feasible actions

M and N of the two players F and G can be modelled as Riemannian manifolds, allowing to consider the important case of players doing multi-tasks activities where resources and time constraints matter. Suppose that players F and G can perform the list of activities $i \in I_F$ and $j \in I_G$. Let $l_F^i(x^i) \in \mathbb{R}_+$ and $l_G^j(y^j) \in \mathbb{R}_+$ be the times or resources spent to produce the quantities x^i and x^j of intermediate inputs $i \in I_F$ and $j \in I_G$. Let, each period, $L_F \in \mathbb{R}_+$ and $L_G \in \mathbb{R}_+$ be the amounts of resources available for players F and G . Then, each period, the resource constraints of the players are $\Sigma_{i\in I_F} l_F^i(x^i) - L_F = 0$ and $\Sigma_{j\in I_G} l_G^j(y^j) - L_G = 0$.

If $x = (x^i,\ i \in I_F) \in \mathbb{R}_+^{cardI_F} = X$ and $y = (y^j,\ \ j \in I_G) \in \mathbb{R}_+^{cardI_G} = Y$ are the intermediate quantities of inputs produced by each player (their actions), the outputs (performances) produced by each player are $\varphi_F(x) \in \Phi$ and $\varphi_G(y) \in \Phi$, where Φ is the space of outputs. The revenues of each player are $R_F\left[\varphi_F(x), \varphi_G(y)\right]$, $R_G\left[\varphi_F(x), \varphi_G(y)\right]$ and the costs to do these actions are $K_F(x, y)$, $K_G(x, y)$. Then, the explicit payoff functions of each player are

$$F(x, y) = R_F\left[\varphi_F(x), \varphi_G(y)\right] - K_F(x, y)$$

and

$$G(x, y) = R_G\left[\varphi_F(x), \varphi_G(y)\right] - K_G(x, y)$$

The subsets of feasible actions of the two players are the manifolds

$$M = \{x \in X : L_F(x) - L_F = 0\} \text{ and } N = \{y \in Y : L_G(y) - L_G = 0\},$$

where $L_F(x) = \Sigma_{i\in I_F} l_F^i(x^i)$ and $L_G(y) = \Sigma_{j\in I_G} l_G^j(y^j)$. For each player, they constraint the utilization of resources for each activity.

A first example related to manifolds with positive curvature is the following: $L_F(x) = \Sigma_{i\in I_F}(x^i)^2$ and $L_G(y) = \Sigma_{j\in I_G} l_G^j(y^j)^2$ or $L_F(x) = \Sigma_{i\in I_F}(x^i)^{1/2}$ and $L_G(y) = \Sigma_{j\in I_G} l_G^j(y^j)^{1/2}$ for decreasing or increasing returns to scale, where producing more of each input use, marginally, more (or less) resources. In these cases manifolds are portions of the usual several dimensions spheres or ellipsoids.

A second example related to manifolds with negative curvature (as Hadamard manifolds, our case) is the following: consider one player doing the bundle of activities described by $x = ((x^i, i \in I), (x^j, j \in J))$. All activities $i \in I$ produce daily vital energy for the agent (like eating, resting, holidays, sports, healthy activities, arts, ...), giving further motivations to act. All activities $j \in J$ consume energy (working, thinking, ...). An important constraint, almost always neglected in the economic literature is that the agent can conserve (regenerate) his energy as time evolves. Let $e_+^i(x^i) \geq 0$ be the energy produced by doing action $x^i \in \mathbb{R}$ and $e_-^j(x^j) \geq 0$ be the energy consumed by doing action $x^j \in \mathbb{R}$. Then, the regeneration of vital energy imposes the constraint $\Sigma_{i\in I} e_+^i(x^i) - \Sigma_{j\in J} e_-^j(x^j) = E > 0$. Production and consumption functions of energy can be quadratic (the more an agent does an activity, the more he produces and consumes energy, at an increasing rate). In this case the expression $\Sigma_{i\in I}(x^i)^2 - \Sigma_{j\in J}(x^j)^2 = E > 0$ defines an hyperboloid. A more realistic example can be given where production functions of energy are increasing, concave, and consumption functions of energy are increasing convex. More generally each activity can both consume and produce some energy.

3 Constraints and action's spaces: elements of Riemannian geometry.

The classical optimization theory considers optimization problems only in Euclidean spaces. However, in some applications an optimization problem appears naturally on smooth manifolds. To deal with such a situation in a classical setting, the manifold has to embedded into an Euclidean space. Then a standard algorithm for constrained optimization can be applied to the optimization problem. However, this approach has several disadvantages. The dimension of the Euclidean space, in which the manifold is embedded, can be very large, leading to inefficient algorithms. Further, standard constrained optimization algorithms will not, in general, produce iterates on the manifold itself, thus requiring complicated projections onto the manifold. Optimization algorithms on manifolds try to avoid these problems by using the structure of the manifold itself and not relying on any embeddings. There has been a signification interest in such optimization algorithms on manifolds in the recent years, see e.g., Absil et al. [1].

In the next subsection, we introduce some fundamental properties and notations of Riemannian manifold. These basics facts can be found in any introductory book of Riemannian geometry, for example in Do Carmo [19].

3.1 Riemannian manifolds.

Let M be a connected m-dimensional C^∞ manifold and let

$$TM = \{(x, v) : x \in M, v \in T_x M\}$$

be its tangent bundle. T_xM is a linear space and has same dimension of M , moreover, as we restrict ourselves to real manifolds, it is isomorphic to R^m. If M is endowed with a Riemannian metric g, then M is a Riemannian manifold and we denoted it by (M, g). The inner product of two vectors u and v in T_xM is written $\langle u, v\rangle := g_x(u, v)$ where g_x is the metric at the point x. The norm of a vector $u \in T_xM$ is defined by $\|u\| := \langle u, u\rangle_x^{1/2}$. Recall that the metric can be used to define the length of piecewise smooth curve $c : [a, b] \to M$ joining x' to x, i.e., such that $c(a) = x'$ and $c(b) = x$, by $l(c) = \int_a^b \|c'(t)\|dt$. Minimizing this length functional over the set of all such curves we obtain a Riemannian distance $d(x, x')$ which induces the original topology on M. Let ∇ be the Levi-Civita connection associated to (M, g). A vector field V along c is said to be parallel if $\nabla_{c'} V = 0$. If c' itself is parallel we say that c is a geodesic. The geodesic equation $\nabla_{\gamma'}\gamma' = 0$ is a second order nonlinear ordinary differential equation, then $\gamma = \gamma_v(\cdot, x)$ is determined by its position x and velocity v at x. It is easy to verify that $\|\gamma'\|$ is constant. We say that γ is normalized if $\|\gamma'\| = 1$. The restriction of a geodesic to a closed bounded interval is called a geodesic segment. A geodesic segment joining x' to x in M is said to be minimal if its length equals $d(x, x')$ and this geodesic is called a minimizing geodesic.

A Riemannian manifold is *complete* if the geodesics are defined for any values of t. Hopf-Rinow's theorem (see, for example, Theorem 2.8, page 146 of Do Carmo [19]) asserts that if this is the case then any pair of points, say p and q, in M can be joined by a (not necessarily unique) minimal geodesic segment. Moreover, (M, d) is a complete metric space so that bounded and closed subsets are compact. From the completeness of the Riemannian manifold M, the *exponential map* $\exp_p : T_pM \to M$ is defined by $\exp_p v = \gamma_v(1, p)$, for each $p \in M$.

We denote by R the curvature tensor defined by

$$R(X, Y) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[Y,X]} Z,$$

where X, Y and Z are vector fields of M and $[X, Y] = YX - XY$. Then the sectional curvature with respect to X and Y is given by

$$K(X, Y) = (\langle R(X, Y)Y, X\rangle)/(\|X\|^2\|Y\|^2 - \langle X, Y\rangle^2),$$

where $\|X\|^2 = \langle X, X\rangle$. If $K(X, Y) \leq 0$ for all X and Y, then M is called a Riemannian manifold of nonpositive curvature and we use the short notation $K \leq 0$. Some interesting results are obtained when the curvature is constant.

A complete simply connected Riemannian manifold of nonpositive sectional curvature is called a Hadamard manifold. Now we proceed to define useful notions.

The Riemannian distance plays a fundamental role in the next section. We proceed now to stating a result which we go to use. Let M be a Hadamard manifold. For any $x \in M$ we can define the exponential inverse map

$$exp_x^{-1} : M \to T_x M,$$

which is C^{∞}. In this case $d(x, x')= \|exp_{x'}^{-1}(x)\|$, then the map $C_{x'} : M \to \mathbb{R}$ defined by $C_{x'}(x) = \frac{1}{2}d^2(x, x')$ is C^{∞} too. Recall the well-known result that map $C_{x'}$ is strictly convex, 1-coercive and its gradient at x is given by $gradC_{x'}(x) = -exp_x^{-1}(x')$, (see Ferreira and Oliveira [20]).

3.2 Nonsmooth analysis on manifolds.

In this subsection, we present elements of non smooth analysis on a manifold, which can be found in Ledyaev and Zhu [34]. Let $f : M \to \mathbb{R} \cup \{+\infty\}$ be a real extended-valued function and denote by

$$\text{dom} f := \{x \in M : f(x) < +\infty\}$$

its domain. We recall that f is said to be proper when $\text{dom} f \neq \emptyset$. The graph of a real-extended-valued function $f : M \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$Graphf := \{(x, y) \in M \times \mathbb{R} : y = f(x)\}.$$

Similarly if $F : M \rightrightarrows N$ is a point-to-set mapping , its graph is defined by

$$GraphF := \{(x, y) \in M \times N : y \in F(x)\}.$$

Definition 3.1. *Let f be a lower semicontinuous function. The Fréchet-subdifferential of f at $x \in M$ is defined by*

$$\hat{\partial} f(x) = \begin{cases} \{dh_x : h \in C^1(M) \text{ and } f - h \text{ attains a local minimum at } x\}, & \text{if } \ x \in domf \\ \emptyset, & \text{if } \ x \notin domf, \end{cases}$$

where $dh_x \in (T_x M)^*$ is given by $dh_x(v) = \langle grad\, h(x), v\rangle$, $v \in T_x M$.

Note that if f is differentiable at x, then $\hat{\partial} f(x) = \{gradf(x)\}$.

Definition 3.2. *Let f be a lower semicontinuous function. The (limiting) subdifferential of f at $x \in M$ is defined by*

$$\partial f(x) := \{v \in T_x M : \exists (x_n, v_n) \in Graph(\hat{\partial} f) \text{ with } (x_n, v_n) \to (x, v),\ f(x_n) \to f(x)\},$$

where $Graph(\hat{\partial} f) := \{(y, u) \in TM : u \in \hat{\partial} f(y)\}$.

When $M = \mathbb{R}^m$, then the function h in the definition of Fréchet- subgradient can be chosen to be a quadratic one. In this case the definition becomes a definition of a proximal subgradient which has a natural geometric interpretation in terms of normal vectors to the epigraph of function f (it is useful to recall that in the case of a smooth function f , the vector $(f'(x), -1)$ is a normal vector to its epigraph).

Returning to the manifold case, remember that x is a critical point of a function f if $0 \in \partial f(x)$. Note that it follows directly from the definition that $\hat{\partial} f(x) \subset \partial f(x)$ and that $\hat{\partial} f(x)$ may be empty. However, if f attains a local minimum at x, then $0 \in \hat{\partial} f(x)$. These are the usual properties to be expected for a subdifferential.

Now, if S is a subset of M, the distance between a point $x \in M$ and the set S is defined by

$$dist(x, S) := \inf\{d(x, p) : p \in S\},$$

If S is empty defines $dist(x, S) = +\infty$ for all $x \in M$, and for any real-extended valued function f on M and any $x \in M$,

$$dist(0, \partial f(x)) = \inf\{\|v\| : v \in \partial f(x)\}.$$

Proposition 3.1. *Let $f : M \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function. Suppose that (U, ϕ) is a local coordinate neighborhood and $x \in U$. Then,*

$$\partial f(x) = (\phi_x^*)\partial(f \circ \phi^{-1})(\phi(x)),$$

where ϕ_x^* denote the adjunct of the Fréchet derivative of the function ϕ.

Proof. See Ledyaev and Zhu [34, Corollary 4.2]. ∎

Let $H : M \times N \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function. Given y in N, the subdifferential of the function $H(\cdot, y)$ at p is denoted by $\partial_x H(p, y)$. Similarly, given x in M, the subdifferential of the function $H(x, \cdot)$ at q is denoted by $\partial_y H(x, q)$.

The following result, though elementary, is central to the paper. The demonstration is similar to that in Attouch et al. [3].

Proposition 3.2. *Let H be as in Definition 4.1. Then for all (x, y) in*

$$domH = domf \times domg$$

we have

$$\begin{aligned} \partial H(x, y) &= \{\partial f(x) + grad_x \Psi(x, y)\} \times \{\partial g(y) + grad_y \Psi(x, y)\} \\ &= \partial_x H(x, y) \times \partial_y H(x, y). \end{aligned}$$

3.3 Kurdyka-Lojasiewicz inequality on Riemannian manifolds.

In this subsection we present Kurdyka-Lojasiewicz inequality in the Riemannian context and we recall some basic notions on o-minimal structures on $(\mathbb{R}, +, \cdot)$ and analytic-geometric categories. Our main interest here is to observe that Kurdyka-Lojasiewicz inequality, in a Riemannian context, holds for lower semicontinuous functions, not necessarily differentiable. The differentiable case was presented by Lageman [33, Corollary 1.1.25]. It is important to note that Kurdyka et al. [32] had already established such inequality for analytic manifolds and analytic functions. For a detailed discussion on o-minimal structures and analytic geometric categories see, for example, van den Dries and Miller [48], and references therein.

Let $f : M \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function. Consider the following set:

$$[\eta_1 < f < \eta_2] := \{x \in M : \eta_1 < f(x) < \eta_2\}, \quad -\infty < \eta_1 < \eta_2 < +\infty.$$

Definition 3.3. *The function f is said to have the Kurdyka-Lojasiewicz property at $\bar{x} \in dom\, \partial f$ if there exist $\eta \in (0, +\infty]$, a neighbourhood U of $\bar{x}$ and a continuous concave function $\varphi : [0, \eta) \to \mathbb{R}_+$ such that:*

(i) $\varphi(0) = 0$, $\varphi \in C^1(0, \eta)$ *and, for all* $s \in (0, \eta)$, $\varphi'(s) > 0$;

(ii) for all $x \in U \cap [f(\bar{x}) < f < f(\bar{x}) + \eta]$, *the Kurdyka-Lojasiewicz inequality holds*

$$\varphi'(f(x) - f(\bar{x}))dist(0, \partial f(x)) \geq 1. \tag{3.1}$$

We call f a KL function if it satisfies the Kurdyka - Lojasiewicz inequality at each point of $dom\partial f$.

Next we show that if $\bar{x}$ is a noncritical point of a lower semicontinuous function then a Kurdyka-Lojasiewicz inequality holds in $\bar{x}$.

Lemma 3.1. *Let $f : M \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function and $\bar{x} \in dom\, \partial f$ such that $0 \notin \partial f(\bar{x})$. Then the Kurdyka-Lojasiewicz inequality holds in $\bar{x}$.*

Proof. Since $\bar{x}$ is a noncritical point of f and $\partial f(\bar{x})$ is a closed set, we have

$$\delta := dist(0, \partial f(\bar{x})) > 0.$$

Take $\varphi(t) := t/\delta$, $U := B(\bar{x}, \delta/2)$, $\eta := \delta/2$ and note that, for each $x \in \mathrm{dom}\partial f$,

$$\varphi'(f(x) - f(\bar{x}))dist(0, \partial f(x)) = dist(0, \partial f(x))/\delta. \tag{3.2}$$

Now, for each arbitrary $x \in U \cap [f(\bar{x}) - \eta < f < f(\bar{x}) + \eta]$, notice

$$d(x, \bar{x}) + |f(x) - f(\bar{x})| < \delta.$$

claim that, for each x satisfying the last inequality, it holds

$$dist(0, \partial f(x)) \geq \delta. \tag{3.3}$$

Let us suppose, by contradiction, that this does not hold. Then, there exist sequences $\{(y_k, v_k)\} \subset \mathrm{Graph}\partial f$ and $\{\delta_k\} \subset \mathbb{R}_{++}$ such that

$$d(y_k, \bar{x}) + |f(y_k) - f(\bar{x})| < \delta_k, \quad \text{and} \quad \|v_k\| \leq \delta_k$$

with $\{\delta_k\}$ converging to zero. Thus, using that $\{(y_k, v_k)\}$ and $\{f(y_k)\}$ converge to $(\bar{x}, 0)$ and $f(\bar{x})$ respectively, and ∂f is a closed mapping, it follows that $\bar{x}$ is a critical point of f, which proves the statement. Therefore, the result of the lemma follows by combining (3.2) with (3.3). ∎

It is known that a C^2 function $f : M \to \mathbb{R}$ is a Morse function if each critical point $\bar{x}$ of f is nondegenerate, i.e., if $Hess\ f(\bar{x})$ has all its eigenvalues different from zero. From the inverse function theorem, it follows that the critical points are isolated. It is known , see Hirsh [24, Theorem 1.2, page 147], that Morse functions form a dense and open set in the space of C^2 function, more precisely

Theorem 3.1. *Let M be a manifold and denote by $C^r(M, \mathbb{R})$, the set of all C^r functions $g : M \to \mathbb{R}$. The collection of all the Morse functions $f : M \to \mathbb{R}$ form a dense and open set in $C^r(M, \mathbb{R})$, $2 \leq r \leq +\infty$.*

Proof. Let $\bar{x} \in M$ be , a critical point of f and let $U = B(\bar{x}, \delta) \subset \mathcal{U}_{\bar{x}}$ be such that it does not contain another critical point. Using the Taylor formula for f and *grad* f, we obtain, for $x \in U$

$$f(x) - f(\bar{x}) = \frac{1}{2}\langle \operatorname{Hess} f(\bar{x}) \exp_{\bar{x}}^{-1} x, \exp_{\bar{x}}^{-1} x\rangle + o(d^2(x, \bar{x})),$$

$$\operatorname{grad} f(x) = \operatorname{Hess} f(\bar{x}) \exp_{\bar{x}}^{-1} x + o(d(x, \bar{x})).$$

Reducing, if necessary, the size of the radius δ, we can ensure the existence of positive constants δ_1, δ_2 such that

$$|f(x) - f(\bar{x})| \leq \delta_1 d^2(x, \bar{x}) \quad \text{and} \quad \delta_2 d(x, \bar{x}) \leq \|grad\, f(x)\|.$$

From the last two inequalities, it is easy to verify that (3.1) holds for $\varphi(s) = 2\delta_1\sqrt{s}/\delta_2$, $U = B(\bar{x}, \delta)$ and $\eta = \delta$. Therefore, it follows from Lemma 3.1 that the Morse functions are KL functions. ∎

Remark 3.1. *It is worth to point that last examples among others also have appeared in Attouch et al.[3] in the Euclidean context.*

Next, we recall some definitions which refer to o-minimal structures on $(\mathbb{R}, +, \cdot)$, following the notations of Bolte et al. [16].

Definition 3.4. *Let $\mathcal{O} = \{\mathcal{O}_n\}_{n\in\mathbb{N}}$ be a sequence such that each $\mathcal{O}_n$ is a collection of subsets of $\mathbb{R}^n$. $\mathcal{O}$ is said to be an o-minimal structure on the real field $(\mathbb{R}, +, \cdot)$ if, for each $n \in \mathbb{N}$:*

(i) $\mathcal{O}_n$ is a boolean algebra;

(ii) If $A \in \mathcal{O}_n$, then $A \times \mathbb{R} \in \mathcal{O}_{n+1}$ and $\mathbb{R} \times A \in \mathcal{O}_{n+1}$;

(iii) If $A \in \mathcal{O}_{n+1}$, then $\pi_n(A) \in \mathcal{O}_n$, where $\pi_n : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the projection on the first n coordinates;

(iv) $\mathcal{O}_n$ contains the family of algebraic subsets of $\mathbb{R}^n$;

(v) $\mathcal{O}_1$ consists of all finite unions of points and open intervals.

The elements of $\mathcal{O}$ are said to be *definable* in $\mathcal{O}$. A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be *definable* in $\mathcal{O}$ if its graph belongs to $\mathcal{O}_{n+1}$. Moreover, according to Coste [18] a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is said to be definable in $\mathcal{O}$ if the inverse image of $f^{-1}(+\infty)$ is a definable subset of $\mathbb{R}^n$ and the restriction of f to $f^{-1}(\mathbb{R})$ is a definable function with values in $\mathbb{R}$. It is worth noting that an o-minimal structure on the real field $(\mathbb{R}, +, \cdot)$ is a generalization of a semi-algebraic set on $\mathbb{R}^n$, i.e., a set which can be written as a finite union of sets of the form

$$\{x \in \mathbb{R}^n : p_i(x) = 0,\ q_i(x) < 0,\ i = 1, \ldots, r\},$$

with p_i, q_i, $i = 1, \ldots, r$, being real polynomial functions. Bolte et al. [16], presented a nonsmooth extension of the Kurdyka-Lojasiewicz inequality for definable functions, but in the case that the function φ, which appears in Definition 3.3, is not necessarily concave. Attouch et al. [3], reconsidered the mentioned extension by noting that φ may be taken concave. For an extensive list of examples of definable sets and functions on an o-minimal structure and properties see, for example, van den Dries and Miller [48] and Attouch et al. [3]), and references therein. We limit ourselves to present the material required for our purposes.

The first elementary class of examples of definable sets is given by the semi-algebraic sets, which we denote by $\mathbb{R}_{alg}$. An other class of examples, which we denoted by $\mathbb{R}_{an}$, is given by restricted analytic functions, i.e., the smallest structure containing the graphs of all $f\mid_{[0,1]^n}$ analytic functions , where $f:\mathbb{R}^n\to\mathbb{R}$ is an arbitrary function.

Fulfilling the same role as the semi-algebraic sets on X on analytic manifolds we have the semi-analytic and sub-analytic sets which we define below, see Bierstone and Milman [13] and van den Dries [47]:

A subset of an analytic manifold is said to be semi-analytic if it is locally described by a finite number of analytic equations and inequalities, while the sub-analytic ones are locally projections of relatively compact semi-analytic sets.

A generalization of semi-analytic and sub-analytic sets, analogous to what was given to semi-algebraic sets in terms of the o-minimal structure, leads us to the analytic-geometric categories which we define below:

Definition 3.5. *An analytic-geometric category $\mathcal{C}$ assigns to each real analytic manifold M a collection of sets $\mathcal{C}(M)$ such that for all real analytic manifolds M, N the following conditions hold:*

(i) *$\mathcal{C}(M)$ is a boolean algebra of subsets of M, with $M\in\mathcal{C}(M)$;*

(ii) *If $A\in\mathcal{C}(M)$, then $A\times\mathbb{R}\in\mathcal{C}(A\times\mathbb{R})$;*

(iii) *If $f:M\to N$ is a proper analytic map and $A\in\mathcal{C}(M)$, then $f(A)\in\mathcal{C}(N)$;*

(iv) *If $A\subset M$ and $\{U_i\mid i\in\Lambda\}$ is an open covering of M, then $A\in\mathcal{C}(M)$ if and only if $A\cap U_i\in\mathcal{C}(U_i)$, for all $i\in\Lambda$;*

(v) *Every bounded set $A\in\mathcal{C}(\mathbb{R})$ has finite boundary, i.e. the topological boundary, ∂A, consists of a finite number of points.*

The elements of $\mathcal{C}(M)$ are called $\mathcal{C}$-sets. When the graph of a continuous function $f:A\to B$ with $A\in\mathcal{C}(M), B\in\mathcal{C}(N)$ is contained in $\mathcal{C}(M\times N)$, then f is called a $\mathcal{C}$-function. All subanalytic subsets and continuous subanalytic maps of a manifold are $\mathcal{C}$-sets and $\mathcal{C}$-functions respectively, in that manifold. We denote this collection by $\mathcal{C}_{an}$ which represents the "smallest" analytic-geometric category.

The next theorem provides us a biunivocal correspondence between o-minimal structures containing $\mathbb{R}_{an}$ and analytic-geometric category.

Theorem 3.2. *For any analytic-geometric category $\mathcal{C}$ there is an o-minimal structure $\mathcal{O}(\mathcal{C})$ and for any o-minimal structure $\mathcal{O}$ on $\mathbb{R}_{an}$ there is an analytic geometric category $\mathcal{C}(\mathcal{O})$, such that*

(i) *$A\in\mathcal{C}(\mathcal{O})$ if for all $x\in M$ it exists an analytic chart $\phi:U\to\mathbb{R}^n$, $x\in U$, which maps $A\cap U$ onto a set definable in $\mathcal{O}$.*

(ii) *$A\in\mathcal{O}(\mathcal{C})$ if it is mapped onto a bounded $\mathcal{C}$-set in an Euclidean space by a semialgebraic bijection.*

Furthermore, for $\mathcal{C}=\mathcal{C}(\mathcal{O})$ we get back to an o-minimal structure $\mathcal{O}$ by this correspondence, and for $\mathcal{O}=\mathcal{O}(\mathcal{C})$ we get again $\mathcal{C}$.

Proof. See van den Dries and Miller [48] and Lageman [33, Theorem 1.1.3]. ∎

As a consequence of the correspondence between o-minimal structures containing $\mathbb{R}_{an}$ and analytic-geometric categories, the definable sets associated allows us to provide examples of $\mathcal{C}$-sets in $\mathcal{C}(\mathcal{O})$. Furthermore, $\mathcal{C}$-functions are locally mapped to definable functions by analytic charts.

Proposition 3.3. *Let $f : M \to \mathbb{R}$ be a $\mathcal{C}$-function and $\phi : U \to \mathbb{R}^n$, $U \subset M$ be an analytic local chart. Assume that $U \subset domf$ and $V \subset M$ is a bounded open set such that $\overline{V} \subset U$. If f restricted to U is a bounded $\mathcal{C}$-function, then*

$$f \circ \phi^{-1} : \phi(V) \to \mathbb{R}, \tag{3.4}$$

is definable in $\mathcal{O}(\mathcal{C})$.

Proof. See Lageman [33, Proposition 1.1.5.]. ∎

Next result provides us with the nonsmooth extension of the Kurdyka-Lojasiewicz properties for $\mathcal{C}$-functions defined on analytic manifolds.

Theorem 3.3. *Let M be a analytic Riemannian manifold and $f : M \to \mathbb{R}$ be a continuous $\mathcal{C}$-function. Then, f is a KL function. Moreover, the function φ which appears in (3.1) is definable in $\mathcal{O}$.*

Proof. Let $\bar{x} \in M$ be a critical point of f and let $\phi : V \to \mathbb{R}^n$ be an analytic local chart with $V \subset M$ a neighbourhood of $\bar{x}$ chosen such that V and $f(V)$ are bounded. Thus, from Proposition 3.3, we have $f \circ \phi^{-1} : \phi(V) \to \mathbb{R}$ is a definable function in $\mathcal{O}(\mathcal{C})$. Thus, as $\phi(V)$ is a bounded open definable set containing $\bar{y} = \phi(\bar{x})$ and ϕ is definable, applying Theorem 11 of (Bolte et al. [16]) with $U = \phi(V)$ and taking into account the proof of Theorem 4.1 of (Attouch et al. [3]), Kurdyka-Lojasiewicz properties holds at $\bar{y} = \phi(\bar{x})$, i.e., there exists $\eta \in (0, +\infty]$ and a continuous concave function $\Phi : [0, \eta) \to \mathbb{R}_+$ such that:

(i) $\Phi(0) = 0$, $\Phi \in C^1(0, \eta)$ and, for all $s \in (0, \eta)$, $\Phi'(s) > 0$;

(ii) for all $y \in U \cap [h(\bar{y}) < h < h(\bar{y}) + \eta]$, it holds

$$\Phi'(h(y) - h(\bar{y}))dist(0, \partial h(y)) \geq 1.$$

Since ϕ is a diffeomorphism and using that $y = \phi(x)$, $\bar{y} = \phi(\bar{x})$ and $h = f \circ \phi^{-1}$, from Proposition 3.1 last inequality yields

$$\Phi'(f(x) - f(\bar{x}))dist(0, (\phi_x^*)^{-1}\partial f(x)) \geq 1, \quad x \in V \cap [0 < f < f(\bar{x}) + \eta],$$

where ϕ_x^* denote the adjunct of the Fréchet derivative of the function ϕ.

Take $V' \subset V$ be an open set such that $K = \overline{V'}$ is contained in the interior of the set V and $\bar{x} \in V'$. Thus, K is compact set and for each $x \in K$ there exists $C_x > 0$ with

$$\|(\phi_x^*)^{-1}w\| \leq C_x\|w\|, \qquad w \in T_xM.$$

Since K is a compact set and $(\phi_x^*)^{-1}$ is a diffeomorphism, there exists $C > 0$, $C := \sup\{C_x : x \in K\}$ such that

$$\|(\phi_x^*)^{-1}w\| \leq C\|w\|, \qquad w \in T_xM, \ x \in K.$$

Hence, for $x \in V' \cap [0 < f < f(\bar{x}) + \eta]$, we have

$$1 \leq \Phi'(f(x) - f(\bar{x}))dist(0, (\phi_x^*)^{-1}\partial f(x)) \leq C\,\Phi'(f(x) - f(\bar{x}))dist(0, \partial f(x)),$$

and the Kurdyka-Lojasiewicz properties holds at $\bar{x}$ with $\varphi = C\,\Phi$. Therefore, from Lemma 3.1 we conclude that f is a KL function. The second part also follows from Theorem 11 of (Bolte et al. [16]) and the proof is done. ∎

The following result gives us the nonsmooth extension of the Kurdyka-Lojasiewicz properties for definable functions defined on Euclidean space submanifolds. Coste [18] devotes chapter 6 to establish properties of such submanifolds. For that we need of the following:

Theorem 3.4. *Let $f : M \subset \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous definable function in an o-minimal structure $\mathcal{O}$. If M is endowed with the induced metric of Euclidean spaces , then f is a KL function. Moreover, the function φ which appears in* (3.1) *is definable in $\mathcal{O}$.*

Proof. Let $\bar{x} \in M$ be a critical point of f and W be a bounded definable subset of $\mathbb{R}^n$ such that $\bar{x} \in W$. Since $\mathrm{dom}f$ and W are definable sets in $\mathbb{R}^n$ and W is bounded, it follows that $\mathrm{dom}f \cap W$ is a bounded definable set in $\mathbb{R}^n$. Thus, applying Theorem 11 of (Bolte et al.[16]) with $U = \mathrm{dom}f \cap W$ and, taking into account the proof of Theorem 4.1 of (Attouch et al. [3]), the Kurdyka-Lojasiewicz properties hold at $\bar{x}$. Therefore, from Lemma 3.1, we conclude the first part of the theorem. The second part also follows from Theorem 11 of (Bolte et al.[16]) and the proof is done. ∎

Remark 3.2. *A large class of examples of definable submanifolds of Euclidean spaces are given by manifolds which are obtained as inverse images of regular values of a definable function, more specifically, if $F : \mathbb{R}^{n+k} \to \mathbb{R}^k$ is a C^p definable function and "0" is a regular value of F, then $M = F^{-1}(0)$ is a definable submanifold of $\mathbb{R}^n$. Moreover, by Nash Theorem ([41]), we can isometrically embed in some $\mathbb{R}^n$ a small piece $\mathcal{Y}$ of M, which is a regular submanifold of $\mathbb{R}^n$. Indeed, if $\epsilon > 0$ is small enough, then the set of normal segments of radius ϵ centered at points of $\mathcal{Y}$ determines a tubular neighbourhood $\mathcal{V}$ of $\mathcal{Y}$. Clearly, $\mathcal{V}$ has a natural coordinate system given by $y = (x,t) \in \mathcal{Y} \times B_\epsilon(0)$, where $B_\epsilon(0) \subset \mathbb{R}^m$ is an ϵ-ball (here, $n-m$, $m < n$, is the dimension of M). We identify $(x,0)$ with x. Define $h(x,t) = t$. It is obvious that h is a definable function and $\mathcal{Y} = \{y \in V; h(y) = 0\}$ is a definable submanifold of $\mathbb{R}^n$.*

In the next section we show that the algorithm is well defined and that the generated sequence converges.

4 Alternating proximal algorithms.

From now on fix our areas of activity. We consider M and N finite dimension complete Riemannian manifolds, $m = dim(M)$, $n = dim(N)$ and a function $H : M \times N \to \mathbb{R} \cup \{+\infty\}$.

Definition 4.1. *Let H be a function consisting of the following set of hypotheses:*

(i) $H(x,y) = f(x) + g(y) + \Psi(x,y)$;

(ii) $f : M \to \mathbb{R} \cup \{+\infty\}$, $g : N \to \mathbb{R} \cup \{+\infty\}$ *are proper lower semicontinuous;*

(iii) $grad\Psi$ *is Lipschitz continuous on bounded subsets of $M \times N$.*

Definition 4.2. *A mapping $C_M : M \times M \to \mathbb{R}_+$ is called a quasi distance if:*

(i) for all $x_1, x_2 \in M$, $C_M(x_1,x_2) = C_M(x_2,x_1) = 0 \Leftrightarrow x_1 = x_2$;

(ii) for all $x_1, x_2, x_3 \in M$, $C_M(x_1,x_3) \leq C_M(x_1,x_2) + C_M(x_2,x_3)$.

Note that if C_M is symmetric, that is, for all $x_1, x_2 \in M$, $C_M(x_1,x_2) := C_M(x_2,x_1)$, then C_M is a distance.

Definition 4.3. *Let*

$$d : (M \times N) \times (M \times N) \to \mathbb{R}_+$$

be given by

$$d(z_1, z_2) = [d_M^2(x_1, x_2) + d_N^2(y_1, y_2)]^{1/2}$$

for all $z_1 = (x_1, y_1),\ z_2 = (x_2, y_2)$ *in* $M \times N$*, where* $d_M,\ d_N$ *are distances in* M, N *respectively. It is easy to see that* d *is a distance in* $M \times N$*.*

Being given $(x_0, y_0) \in M \times N$, the alternating discrete dynamical system we are to study is of the form:

$$(x_k, y_k) \to (x_{k+1}, y_k) \to (x_{k+1}, y_{k+1})$$

$$\begin{cases} x_{k+1} \in \arg\min\{H(p, y_k) + \frac{1}{2\lambda_k} C_M^2(x_k, p), p \in M\} \\ \\ y_{k+1} \in \arg\min\{H(x_{k+1}, q) + \frac{1}{2\mu_k} C_N^2(y_k, q), q \in N\} \end{cases} \tag{4.1}$$

where $C_M,\ C_N$ are quasi distances associated with the manifolds $M,\ N$ respectively, and $(\lambda_k),\ (\mu_k)$ are sequences of positive numbers.

4.1 Convergence.

Now we establish the conditions to ensure proper definition and convergence to our algorithm. Consider the following assumptions concerning (4.1),

$$(\mathcal{A}) : \begin{cases} M \text{ and } N \text{ are Hadamard manifolds;} \\ C_M,\ C_N \text{ are } C^1 \text{and there exist } 0 < \alpha < \beta < +\infty \text{ such that} \\ \alpha d_M(x, p) \leq C_M(x, p) \leq \beta d_M(x, p),\ \alpha d_N(y, q) \leq C_N(y, q) \leq \beta d_N(y, q); \\ \inf_{M \times N} H > -\infty; \\ \text{the function } H(\cdot, y_0) \text{ is proper;} \\ \text{for some positive } t_1 < t_2, \text{ the sequences } (\lambda_k),\ (\mu_k) \text{ are subsets of } (t_1, t_2)\ . \end{cases}$$

The following proposition is fundamental to this end and will be widely used. We use H as in definition 4.1 and consider the vectors

$$u_{k+1} := grad_x \Psi(x_{k+1}, y_{k+1}) - grad_x \Psi(x_{k+1}, y_k) - \lambda_k^{-1} C_M(x_k, x_{k+1}) grad C_M(x_k, x_{k+1})$$

and

$$v_{k+1} := -\mu_k^{-1} C_N(y_k, y_{k+1}) grad C_N(y_k, y_{k+1}).$$

Notice that $grad\Psi(x, y) = (grad_x \Psi(x, y), grad_y \Psi(x, y)) \in T_x M \times T_y N$ and consequently (u_{k+1}, v_{k+1}) belongs to $T_{x_{k+1}} M \times T_{y_{k+1}} N$.

Proposition 4.1. *Assume that* H *verifies hypothesis* $(\mathcal{A})$*. Then the sequences* $(x_k),\ (y_k)$ *are well defined. Moreover*

(i) the following estimate holds

$$H(x_{k+1}, y_{k+1}) + \frac{1}{2\lambda_k} C_M^2(x_k, x_{k+1}) + \frac{1}{2\mu_k} C_N^2(y_k, y_{k+1}) \leq H(x_k, y_k); \tag{4.2}$$

(ii)

$$\sum_{k=0}^{+\infty}[C_M^2(x_k,x_{k+1})+C_N^2(y_k,y_{k+1})]<+\infty \ .$$

As a consequence we have $lim_{k\to+\infty}[d_M(x_k,x_{k+1})+d_N(y_k,y_{k+1})]=0$*;*

(iii) For all $k\geq 0$

$$(u_{k+1},v_{k+1})\in\partial H(x_{k+1},y_{k+1}). \tag{4.3}$$

For all bounded subsequences $(x_{k'},y_{k'})$ *of* (x_k,y_k) *we have* $(u_{k'},v_{k'})\to 0$ *and* $dist(0,\partial H(x_{k'},y_{k'}))\to 0$.

Proof. In view of hypothesis $(\mathcal{A})$ we have that for any positive real number t, $(\bar{x},\bar{y})\in M\times N$ the functions

$$x\to H(x,\bar{y})+\frac{1}{2t}C_M^2(\bar{x},x)$$

and

$$y\to H(\bar{x},y)+\frac{1}{2t}C_N^2(\bar{y},y)$$

are coercive. A simple induction ensures that the sequences are well defined and that (i) and (ii) hold for all integer $k\geq 1$.

Now we prove the item (iii). Since 0 must lie in the subdifferential at point x_{k+1} of the function

$$p\to H(p,y_k)+\frac{1}{2\lambda_k}C_M^2(x_k,p)$$

we have

$$0\in(\frac{1}{\lambda_k}C_M(x_k,x_{k+1})gradC_M(x_k,x_{k+1})+\partial_x H(x_{k+1},y_k)),\forall k\geq 0.$$

Similarly, 0 must lie in the subdifferential at point y_{k+1} of the function

$$q\to H(x_{k+1},q)+\frac{1}{2\lambda_k}C_N^2(y_k,q),$$

which implies

$$0\in(\frac{1}{\mu_k}C_N(y_k,y_{k+1})gradC_N(y_k,y_{k+1})+\partial_y H(x_{k+1},y_k)),\forall k\geq 0.$$

Due to the structure of H, it follows that

$$\begin{aligned}\partial_x H(x_{k+1},y_k) &= \partial f(x_{k+1})+grad_x\Psi(x_{k+1},y_k)+grad_x\Psi(x_{k+1},y_{k+1})-grad_x\Psi(x_{k+1},y_{k+1})\\ &= \partial_x H(x_{k+1},y_{k+1})+grad_x\Psi(x_{k+1},y_k)-grad_x\Psi(x_{k+1},y_{k+1}).\end{aligned}$$

Thus,

$$grad_x\Psi(x_{k+1},y_{k+1})-grad_x\Psi(x_{k+1},y_k)-\frac{1}{\lambda_k}C_M(x_k,x_{k+1})gradC_M(x_k,x_{k+1})$$

belongs

$$\partial_x H(x_{k+1},y_{k+1}),$$

and

$$-\frac{1}{\mu_k}C_N(y_k, y_{k+1})gradC_N(y_k, y_{k+1}) \in \partial_y H(x_{k+1}, y_{k+1}).$$

By Proposition 3.2 this yields (4.3).

Let (x', y') be accumulation point. Without any loss of generality we can assume that $(x_{k'}, y_{k'}) \to (x', y')$. By (ii), triangle inequality, with the uniform continuity of $grad_x\Psi$ and the fact that $C_M,\ C_N$ is C^1 and from expressions of u_k and v_k it follows that $(u'_k, v'_k) \to (0,0)$ and

$$dist(0, \partial H(x_{k'+1}, y_{k'+1})) \to 0.$$

■

The next lemma, especially points (ii),(iii), gives the first convergence results about sequences generated by (4.1). Theorems 4.1 and 4.3 below make the convergence properties much more precise. We denote $\Gamma(x_0, y_0)$ the set of accumulation points of the sequence (x_k, y_k).

Lemma 4.1. *Assuming the hypotheses of Proposition 4.1. Let (x_k, y_k) be a sequence complying with (4.1). Then ,*

(i) $((x_k, y_k))$ is bounded implies that $\Gamma(x_0, y_0)$ is a nonempty compact connected set and

$$dist((x_k, y_k), \Gamma(x_0, y_0)) \to 0, k \to +\infty;$$

(ii) $\Gamma(x_0, y_0) \subset critH$;

(iii) H is finite and constant on $\Gamma(x_0, y_0)$, equal to

$$\inf_{k\in N} H(x_k, y_k) = \lim_{k\to+\infty} H(x_k, y_k).$$

Proof.
(i) follows by using the sequence $(d_M(x_k, x_{k+1}) + d_N(y_k, y_{k+1}))$ converges to 0, together with Hopf-Rinow's theorem;
(ii) By the very definition of (x_{k+1}, y_{k+1}), $k \geq 0$ we have

$$H(x_{k+1}, y_{k+1}) + \frac{1}{2\lambda_k}C_M^2(x_k, x_{k+1}) \leq H(p, y_{k+1}) + \frac{1}{2\lambda_k}C_M^2(x_k, p), \forall p \in M$$

and

$$H(x_{k+1}, y_{k+1}) + \frac{1}{2\mu_k}C_N^2(y_k, y_{k+1}) \leq H(x_{k+1}, q) + \frac{1}{2\mu_k}C_N^2(y_k, q), \forall q \in N.$$

Due to the special form of H to $0 < t_1 \leq \lambda_k \leq t_2$ and $0 < t_1 \leq \mu_k \leq t_2$, we have

$$f(x_{k+1}) + \Psi(x_{k+1}, y_k) + \frac{1}{2t_2}C_M^2(x_k, x_{k+1}) \leq f(p) + \Psi(p, y_k) + \frac{1}{2t_1}C_M^2(x_k, p), \forall p \in M \tag{4.4}$$

and

$$g(y_{k+1}) + \Psi(x_{k+1}, y_{k+1}) + \frac{1}{2t_2}C_N^2(y_k, y_{k+1})$$

$$\leq g(q) + \Psi(x_{k+1}, q) + \frac{1}{2t_1} C_N^2(y_k, q), \forall q \in N. \tag{4.5}$$

Let $(\bar{x}, \bar{y})$ be a point in $\Gamma(x_0, y_0)$. There exists a subsequence $((x_{k'}, y_{k'}))$ of $((x_k, y_k))$ converging to $(\bar{x}, \bar{y})$. Since $C_M^2(x_k, x_{k+1}) \to 0$, we deduce from (4.4) that

$$\liminf f(x_{k'}) + \Psi(\bar{x}, \bar{y}) \leq f(p) + \Psi(p, \bar{y}) + \frac{1}{2t_1} C_M^2(\bar{x}, p), \forall p \in M.$$

For $p = \bar{x}$ we obtain

$$\liminf f(x_{k'}) \leq f(\bar{x}).$$

Since f is lower semicontinuous we get further

$$\liminf f(x_k) = f(\bar{x}).$$

Without any loss of generality we can assume that the whole sequence $f(x_{k'})$ converges to $f(\bar{x})$, i.e.,

$$\lim f(x'_k) = f(\bar{x}).$$

Similarly, using (4.5), we may assume that $\lim g(y_{k'}) = g(\bar{y})$. Since Ψ is continuous, we have

$$\lim \Psi(x_{k'}, y_{k'}) = \Psi(\bar{x}, \bar{y})$$

and hence

$$\lim H(x_{k'}, y_{k'}) = H(\bar{x}, \bar{y}).$$

From Proposition 4.1 (iii), and using the same notation, we have

$$(u_{k'}, v_{k'}) \in \partial H(x_{k'}, y_{k'})$$

and

$$(u_{k'}, v_{k'}) \to 0.$$

Due to the closedness properties of ∂H, we finally obtain

$$0 \in \partial H(\bar{x}, \bar{y}),$$

which concludes the proof of (ii);
(iii) since the sequence $(H(x_k, y_k))$ is nonincreasing and $\inf H > -\infty$ the sequence $(H(x_k, y_k))$ converges to $\inf H$. Now let $(\bar{x}, \bar{y})$ be an accumulation point of sequence $((x_k, y_k))$ and $((x_{k_j}, y_{k_j}))$ a subsequence such that $((x_{k_j}, y_{k_j})) \to (\bar{x}, \bar{y})$. From similar argument of proof item (ii) $H(x_{k_j}, y_{k_j}) \to H(\bar{x}, \bar{y})$. Thus $H(\bar{x}, \bar{y}) = \inf H$ and therefore H is finite and constant on $\Gamma(x_0, y_0)$

■

4.2 Convergence to a critical point.

This section is devoted to the convergence analysis of the alternating algorithm (4.1). It provides the main mathematical results of this paper. Previous related work may be found in Attouch et al. [5, 7], but there, the setting is convex with a quadratic coupling Ψ, and the mathematical analysis relies on the monotonicity of the convex subdifferential operators.

Let us prove the main theorem of this work. For this purpose we introduce the notations: $z_k = (x_k, y_k)$, $h_k = H(z_k)$, $\bar{z} = (\bar{x}, \bar{y})$, $\bar{h} = H(\bar{z})$. We denote by U, η and φ as in definition 3.3, relative to H at $\bar{z}$ and let $\rho > 0$ be such that $B(\bar{z}, \rho) = \{z \in M \times N, d(\bar{z}, z) < \rho\} \subset U$. Also assume that

$$\bar{h} < h_k < \bar{h} + \eta, k \geq 0, \tag{4.6}$$

and

$$\varphi(h_0 - \bar{h})E + \alpha^{-1}\sqrt{2t_2(h_0 - \bar{h})} + d(z_0, \bar{z}) < \rho \tag{4.7}$$

with $E = 2t_2(\delta + 2\tau\beta t_1{}^{-1})\alpha^{-2}$, where δ is a Lipschitz constant for $grad\Psi$, τ is an upper bound for the norms of $gradC_M$, $gradC_N$ on $B(\bar{z}, \sqrt{2}\rho)$, i.e., $\|gradC_M\| < \tau$ and $\|gradC_N\| < \tau$, and α , β are given by assumption $(\mathcal{A})$. Denote by $D(z_k) = \sum_{i=k+1}^{+\infty} d(z_i, z_{i+1})$.

Remark 4.1. *When* (z_k) *is a bounded sequence we can consider* $\bar{z}$ *as an accumulation point of* (z_k). *Therefore,* $\varphi(h_k - \bar{h}) \to 0$. *Then*

$$b_k := \varphi(h_k - \bar{h})E + \alpha^{-1}\sqrt{2t_2(h_k - \bar{h})} + d(\bar{z}, z_k)$$

admits 0 *as a cluster point. Thus, given* $\rho > 0$, *we obtain the existence of* k_0 *such that* $b_{k_0} < \rho$. *Then, taking* $z_0 = z_{k_0}$,, *we get the inequality (4.7).*

Theorem 4.1. *Assume that* H *satisfies the hypothesis* $(\mathcal{A})$ *and is a KL function at* $\bar{z}$. *Let* (z_k) *be the sequence generated by (4.1), with* z_0 *as an initial point. Let us assume that* (4.6) *and* (4.7) *hold . Then, the sequence* (z_k) *converges to the critical point* $\bar{z}$ *of* H *and the following estimates hold* ($\forall k \geq 0$)*:*

(i) $z_k \in B(\bar{z}, \rho)$;

(ii) $D(z_k) \leq \varphi(h_k - \bar{h})E + \alpha^{-1}\sqrt{2t_2(h_{k-1} - \bar{h})}$.

Proof. Assume that $\bar{h} = 0$ (replace if necessary H by $(H - \bar{h})$). By Remark 4.1 we assume that z_0 is such that (4.6) holds. From (4.2)
we have, for $i \geq 0$:

$$\frac{1}{2t_2}\alpha^2 d^2(z_i, z_{i+1}) \leq h_i - h_{i+1}. \tag{4.8}$$

Since $\varphi'(h_i)$ makes sense in view of (4.6) and $\varphi'(h_i) > 0$, we have

$$\frac{\varphi'(h_i)}{2t_2}\alpha^2 d^2(z_i, z_{i+1}) \leq \varphi'(h_i)(h_i - h_{i+1}).$$

Owing to φ being concave, we get further:

$$\frac{\varphi'(h_i)}{2t_2}\alpha^2 d^2(z_i, z_{i+1}) \leq \varphi(h_i) - \varphi(h_{i+1}).$$

Let us first check (i) for $k = 0$ and $k = 1$. In view of (4.7), z_0 lies in $B(\bar{z}, \rho)$. From (4.8),

$$\frac{\alpha^2}{2t_2} d^2(z_0, z_1) \leq h_0 - h_1 \leq h_0 \tag{4.9}$$

using triangle inequality and (4.7).

Let us now prove by induction that $z_k \in B(\bar{z}, \rho)$ for all $k \geq 0$. Assume that it holds up to some $k \geq 2$. Now, for $0 \leq i \leq k$, since $z_i \in B(\bar{z}, \rho)$ and $0 < h_i < \eta$, we can write the Kurdyka-Lojasiewicz inequality at z_i:

$$1 \leq \varphi'(h_i) dist(0, \partial H(z_i)),$$

taking (u_i, v_i) as given in Proposition 4.1 (iii) and recalling that (u_i, v_i) is an element of $\partial H(z_i)$. Hence, for $1 \leq i \leq k$:

$$1 \leq \varphi'(h_i) \|(u_i, v_i)\|. \tag{4.10}$$

Let us examine $\|(u_i, v_i)\|$, for $1 \leq i \leq k$. Being

$$gradC_M^2(x_{i-1}, x_i) = 2C_M(x_{i-1}, x_i) gradC_M(x_{i-1}, x_i)$$

and

$$gradC_N^2(y_{i-1}, y_i) = 2C_N(y_{i-1}, y_i) gradC_N(y_{i-1}, y_i),$$

we have

$$\begin{aligned} &\|(\frac{1}{2\lambda_{i-1}} gradC_M^2(x_{i-1}, x_i), \frac{1}{2\mu_{i-1}} gradC_N^2(y_{i-1}, y_i))\| \\ &\leq \frac{\tau\beta}{t_1}(d_M(x_{i-1}, x_i) + d_N(y_{i-1}, y_i)) \leq \frac{2\tau\beta}{t_1} d(z_{i-1}, z_i), \end{aligned}$$

where we use the fact that $d(z_{i-1}, z_i) = [d_M^2(x_{i-1}, x_i) + d_N^2(y_{i-1}, y_i)]^{1/2}$. Moreover,

$$d^2(\bar{z}, (x_i, y_{i-1})) \leq d^2(\bar{z}, z_i) + d^2(\bar{z}, z_{i-1}) \leq 2\rho^2.$$

Thus (x_i, y_{i-1}) and $z_i = (x_i, y_i)$ lies in $B(\bar{z}, \sqrt{2}\rho)$, which allows us to apply the Lipschitz inequality between these points

$$\|(grad_x \Psi(x_i, y_i) - grad_x \Psi(x_i, y_{i-1})\| \leq \delta d(z_{i-1}, z_i).$$

Therefore, for $1 \leq i \leq k$

$$\|(u_i, v_i)\| \leq (\delta + 2\tau\beta {t_1}^{-1}) d(z_{i-1}, z_i), \tag{4.11}$$

using (4.10) yields

$$[(\delta + 2\tau\beta t_1^{-1}) d(z_{i-1}, z_i)]^{-1} \leq \varphi'(h_i)$$

and using (4.9) yields

$$\begin{aligned} \varphi(h_i) - \varphi(h_{i+1}) \quad &\geq \quad \frac{\varphi'(h_i)\alpha^2}{2t_2} d^2(z_i, z_{i+1}) \\ &\geq \quad \frac{\alpha^2}{2t_2(\delta + 2\tau\beta {t_1}^{-1})} \frac{d^2(z_i, z_{i+1})}{d(z_{i-1}, z_i)}, \end{aligned}$$

which can be rewritten as

$$d(z_i, z_{i+1}) \leq ((\varphi(h_i) - \varphi(h_{i+1})) E d(z_{i-1}, z_i))^{1/2},$$

where $E = 2t_2(\delta + 2\tau\beta {t_1}^{-1})\alpha^{-2}$. Now given that $2ab \leq a^2 + b^2$ we obtain

$$2d(z_i, z_{i+1}) \leq (\varphi(h_i) - \varphi(h_{i+1}))E + d(z_{i-1}, z_i).$$

This inequality holds for $1 \leq i \leq k$. Then, summing (4.2) over i gives

$$\sum_{i=1}^{k} d(z_i, z_{i+1}) + d(z_k, z_{k+1}) \leq (\varphi(h_1) - \varphi(h_{k+1}))E + d(z_0, z_1).$$

Hence, in view of the monotonicity properties of φ and h_k

$$\sum_{i=1}^{k} d(z_i, z_{i+1}) \leq \varphi(h_0)E + d(z_0, z_1). \tag{4.12}$$

By (4.9), we have

$$\begin{aligned} d(z_{k+1}, \bar{z}) &\leq \sum_{i=1}^{k} d(z_i, z_{i+1}) + d(z_1, \bar{z}) \\ &\leq \varphi(h_0)E + d(z_0, z_1) + d(z_0, \bar{z}) \\ &\leq \varphi(h_0)E + \alpha^{-1}\sqrt{2t_2 h_0} + d(\bar{z}, z_0). \end{aligned}$$

Thus $z_{k+1} \in B(\bar{z}, \rho)$ in view of (4.7). That completes the proof of (i). Since inequality (4.2) holds for $i \geq 1$, let us sum it for i running from some k to some $j > k$

$$\sum_{i=k}^{j} d(z_i, z_{i+1}) + d(z_j, z_{j+1}) \leq (\varphi(h_k) - \varphi(h_{j+1}))E + d(z_{k-1}, z_k).$$

Thus

$$\sum_{i=k}^{j} d(z_i, z_{i+1}) \leq \varphi(h_k)E + d(z_{k-1}, z_k).$$

Letting $j \to +\infty$ yields

$$\sum_{i=k}^{+\infty} d(z_i, z_{i+1}) \leq \varphi(h_k)E + d(z_{k-1}, z_k). \tag{4.13}$$

To complete the proof, (4.8) gives us,

$$\sum_{i=k+1}^{+\infty} d(z_i, z_{i+1}) \leq \varphi(h_k)E + \alpha^{-1}\sqrt{2t_2(h_{k-1} - h_k)} \leq \varphi(h_k - \bar{h})E + \alpha^{-1}\sqrt{2t_2(h_{k-1} - \bar{h})},$$

thus

$$\sum_{i=k+1}^{+\infty} d(z_i, z_{i+1}) < +\infty$$

which implies that (z_k) is a Cauchy sequence and therefore convergent. By Lemma 4.1, we obtain that its limit is a critical point of H. ■

The next theorem has two important consequences:

Theorem 4.2. *Assume that H satisfies ($\mathcal{A}$) and is a KL function. Then, either the sequence $(d(z_0, z_k))$ is unbounded or*

$$\sum_{i=1}^{+\infty} d(z_{k-1}, z_k) < +\infty,$$

which implies that (z_k) converges to a critical point of H.

Proof. Assume that $(d(z_0, z_k))$ does not tend to infinity and let $\bar{z}$ be a limit-point of (z_k) for which we denote by ρ, η, φ, the associated objects given in definition 3.3. Lemma 4.1 shows that $\bar{z}$ is a critical point and that (h_k) converges to 0.

If there exists an integer k_0 for which $H(z_{k_0}) = 0$, it is straightforward to check (recall Proposition 4.1) that $z_k = z_{k_0}$ for all $k \geq k_0$, so $z_{k_0} = \bar{z}$. Let assume that $h_k > 0$. Since $\max\{\varphi(h_k), d(\bar{z}, z_k)\}$ admits 0 as a cluster point, we obtain $k_0 \geq 0$ such that (4.7) is fulfilled with z_{k_0} as a new initial point. The conclusion is then a consequence of Theorem 4.1. ■

In the following result we do not assume here any standard nondegeneracy conditions such as, for instance, uniqueness of the minimizers or second-order conditions.

Theorem 4.3. *(local convergence to global minima)*
Let $\bar{z}$ be a global minimum point of H and (z_k) the sequence generated by (4.1), with z_0 as an initial point. Assume that H satisfies ($\mathcal{A}$) and is a KL function at $\bar{z}$. Then there exist ϵ, η such that

$$d(z_0, \bar{z}) < \epsilon, \quad \min H < H(z_0) < \min H + \eta$$

implies that the sequence (z_k) has the finite length property and converges to z^* with $H(z^*) = \min H$.

Proof. A straightforward application of Theorem 4.1 yields the convergence of (z_k) to some z^*, a critical point of H with $H(z^*) \in [\min H, \min H + \eta)$. Now, if $H(z^*)$ were not equal to $H(\bar{z})$ then the Kurdyka-Lojasiewicz inequality would entail

$$\varphi'(H(z^*) - H(\bar{z})) dist(0, \partial H(z^*) \geq 1,$$

a clear contradiction since $0 \in \partial H(z^*)$. ■

Theorem 4.4. *Assume that $(z_k = (x_k, y_k))$ converges to $z^* = (x^*, y^*)$. If H satisfies ($\mathcal{A}$) and is a KL function at z^* with $\varphi(s) = cs^{1-\theta}$, $\theta \in [0, 1)$, $c > 0$, then the following estimations hold:*

(i) If $\theta = 0$ then the sequence (z_k) converges in a finite number of steps;

(ii) If $\theta \in (0, \frac{1}{2}]$ then there exist $c_0 > 0$ and $\varsigma \in [0, 1)$ such that

$$d(z_k, z^*) \leq c_0 \varsigma^k;$$

(iii) If $\theta \in (\frac{1}{2}, 1)$ then there exist $\xi > 0$ such that

$$d(z_k, z^*) \leq \xi k^{-\frac{1-\theta}{2\theta-1}}.$$

Proof. The notations are those of Theorem 4.3. Then by Proposition 4.1 $H(z^*) = 0$.

(i) If (h_k) is stationary, so is (z_k) in view of Proposition 4.1. If (h_k) is not stationary, the Kurdyka-Lojasiewicz inequality yields for any k sufficiently large, i.e, $c\,dist(0, \partial H(z_k)) \geq 1$. On the other hand, since the sequence $z_k \to z^*$ and $\partial H(\cdot)$ is closed then, $c\,dist(0, \partial H(z_k)) \to 0$ which is a contradiction.

Now to complete the proof of the theorem, set for any $k \geq 0$

$$D(z_k) := \sum_{i=k+1}^{+\infty} [d_M^2(x_i, x_{i+1}) + d_N^2(y_i, y_{i+1})]^{1/2},$$

which is finite by Theorem 4.2. Now, given $k \in \mathbb{N}$, let $j \in \mathbb{N}$, $j > k$. Thus, using the triangle inequality,

$$d(z_k, z^*) \leq d(z_k, z_j) + d(z_j, z^*) \leq \sum_{i=k+1}^{j} d(z_i, z_{i+1}) + d(z_j, z^*),$$

making $j \to +\infty$ in the last inequality, we have

$$d(z_k, z^*) = [d_M^2(x_k, x^*) + d_N^2(y_k, y^*)]^{1/2} \leq D(z_k),$$

so, to estimate $d(z_k, z^*)$ it is sufficient to estimate $D(z_k)$. The following is a rewriting of (4.13) with these notations

$$D(z_k) \leq \varphi(h_k)E + D(z_{k-1}) - D(z_k). \tag{4.14}$$

The Kurdyka-Lojasiewicz inequality successively yields

$$1 \leq \varphi'(h_k)dist(0, \partial H(z_k)) = c(1-\theta)h_k^{-\theta} dist(0, \partial H(z_k)),$$

thus

$$h_k^{\theta} \leq c(1-\theta)dist(0, \partial H(z_k)).$$

But with (4.11) we have

$$dist(0, \partial H(z_k)) \leq \|(u_k, v_k)\| \leq (\delta + 2\tau\beta t_1^{-1})(D(z_{k-1}) - D(z_k)).$$

Thus,

$$h_k^{\theta} \leq c(1-\theta)(\delta + 2\tau\beta t_1^{-1})(D(z_{k-1}) - D(z_k)).$$

$$h_k^{1-\theta} \leq [c(1-\theta)(\delta + 2\tau\beta t_1^{-1})]^{\frac{1-\theta}{\theta}} (D(z_{k-1}) - D(z_k))^{\frac{1-\theta}{\theta}}.$$

Using the previous two inequalities and taking $\Theta = c[c(1-\theta)(\delta + 2\tau\beta t_1^{-1})]^{\frac{1-\theta}{\theta}}$, we obtain

$$\varphi(h_k) = c h_k^{1-\theta} \leq \Theta(D(z_{k-1}) - D(z_k))^{\frac{1-\theta}{\theta}}.$$

Therefore (4.14) gives

$$D(z_k) \leq E\Theta(D(z_{k-1}) - D(z_k))^{\frac{1-\theta}{\theta}} + (D(z_{k-1}) - D(z_k)). \tag{4.15}$$

(ii) By (4.15) and $\theta \in (0, 1/2]$ there exists a positive constant c_1 such that

$$D(z_k) \leq c_1(D(z_{k-1}) - D(z_k)),$$

for k sufficiently large. This yields

$$D(z_k) \leq \frac{c_1}{1+c_1} D(z_{k-1}).$$

Therefore, using recurrence in k, we obtain

$$d(z_k, z^*) \leq D(z_k) \leq c_0 \varsigma^k,$$

where $\varsigma = c_1/(1+c_1)$ and c_0 is a positive constant.

(iii) We use identical arguments of Attouch and Bolte, (see [4, Theorem 2]). First, by (4.15) there exists an integer $n_1 > n_0$ and $c_2 > 0$ such that

$$D(z_k)^{\frac{\theta}{1-\theta}} \leq c_2(D(z_{k-1}) - D(z_k)),$$

for all $k \geq n_1$, where $c_2 = (E\Theta + 1)^{\frac{\theta}{1-\theta}}$. Second, from of the fact that $\theta \in (1/2, 1)$ we can find a constant $r > 0$ such that

$$0 < r \leq D(z_k)^{\frac{1-2\theta}{1-\theta}} - D(z_{k-1})^{\frac{1-2\theta}{1-\theta}}.$$

Finally for n large enough,

$$r(n - n_1) \leq D(z_n)^{\frac{1-2\theta}{1-\theta}} - D(z_{n_1})^{\frac{1-2\theta}{1-\theta}},$$

then

$$d(z_n, z^*) \leq D(z_n) \leq [r(n-n_1) + D(z_{n_1})^{\frac{1-2\theta}{1-\theta}}]^{\frac{1-\theta}{1-2\theta}} \leq \xi n^{-\frac{1-\theta}{2\theta-1}},$$

for some $\xi > 0$. ■

5 Application: "Learning to Play Nash".

Using our main result (Theorem 4.1) we are able to give an answer to the first three main questions on "how to learn to play Nash equilibria ?": i) how these learning dynamics converge to the set of Nash equilibria (positive convergence) or converges to a Nash equilibrium ? ii) does the process converges in finite time, iii) what is the speed of convergence, does play converge gradually or abruptly ? Remember that we have yet shown how using Riemannian manifolds as constraints help to modelize repeated multi-tasks activities and the necessary conservation of the minimal energy "to be able to play Nash again and again" (the fourth main question iv)).

Take M and N be the spaces of actions $x \in M$ and $y \in N$ of the two players. The distances between actions x and x' for the first player and actions y and y' for the second player are $d_M(x, x')$ and $d_N(y, y')$. Take also $d(z, z') = \sqrt{d_M^2(x, x') + d_N^2(y, y')}$ be the distance between the couple of actions $z = (x, y)$ and $z' = (x', y')$. From $a + b \leq 2\sqrt{a^2 + b^2}$, with $a \geq 0, b \geq 0$, we have

$$d_M(x, x') + d_N(y, y') \leq 2\sqrt{d_M^2(x, x') + d_N^2(y, y')} = 2d(z, z').$$

We have supposed a constant speed of moving, $\omega_F > 0$ and $\omega_G > 0$ for each player. Then, for both players, the time spent to give his best reply is proportional to the distance of move $(1/\omega_F)d_M(x_i, x_{i+1})$ and $(1/\omega_G)d_N(y_i, y_{i+1})$. Take, to simplify, a unit speed of moving $\omega_F = \omega_G = \omega > 0$. Then, starting from z_k, the time spent to converge is

$$T(z_k) = (1/\omega)\Sigma_{i=k}^{+\infty} [d_M(x_i, x_{i+1}) + d_N(y_i, y_{i+1})]$$

and $T(z_k) \leq (2/\omega) D(z_k)$. This implies

$$T(z_k) \leq (2/\omega)\left[E\varphi(h_k - \bar{h}) + \alpha^{-1}\sqrt{2t_2 h_{k-1} - \bar{h}}\right],$$

where $h_k = H(z_k) = f(x_k) + \Psi(x_k, y_k) + g(y_k)$, $\bar{h} = H(\bar{z})$, and $E = 2t_2(\delta + 2\tau\beta t_1{}^{-1})\alpha^{-2}$ with $0 < t_1 \leq \lambda_k \leq t_2$ and $0 < t_1 \leq \mu_k \leq t_2$. From this majoration we can draw several nice consequences.

First consequence: the length of the convergence time is shorter, the higher is the speed of learning $\omega = \omega_F = \omega_G$.

To give a second consequence, start from the normalized proximal form game,

$$x_{k+1} \in \arg\min\left\{F(p, y_k) + (1/2\lambda_k)C_M^2(x_k, p)\right\}$$

and

$$y_{k+1} \in \arg\min\left\{G(x_{k+1}, q) + (1/2\mu_k)C_N^2(y_k, q)\right\},$$

where $2\lambda_k = \varphi_{k,F}/\delta_{k,F} > 0$ and $2\mu_k = \varphi_{k,G}/\delta_{k,G}$. Then, the lower is the upper bound t_2, i.e. the lower must be the coefficients λ_k and μ_k, where $0 < t_1 \leq \lambda_k \leq t_2$ and $0 < t_1 \leq \mu_k \leq t_2$.

Second consequence: shorter expected lengths of exploitation $\varphi_{k,F}$, $\varphi_{k,G}$ and higher desutility of costs to change $\delta_{k,F}$ and $\delta_{k,G}$ imply a shorter length of the convergence time.

i) shorter anticipated lengths $\varphi_{k,F}$ and $\varphi_{k,G}$ of the exploration periods represent a self confirming anticipation process because along the path distances between states converge to zero. Then, the lengths of the exploration periods (which are also the lengths of the exploitation periods) converge to zero. Hence players anticipate shorter and shorter exploitation periods. This makes our game not only convergent in the action's space but also convergent in the belief's space (see Mondeer-Shapley (1996) for fictitious play with convergence in actions and beliefs). Then, our game has a very nice property in term of stability.

ii) observed that larger desutilities of costs to change $\delta_{k,F}$, $\delta_{k,G}$ represent larger desutilities to change $D_M(p, x_k) = (1/2\lambda_k)C_M^2(x_k, p)$ and $D_N(q, y_k) = (1/2\mu_k)C_N^2(y_k, q)$ for given costs to change $C_M(x_k, p)$ and $C_N(y_k, q)$. This means more inertia, i.e., more resistance to change.

Third consequence: the length of the convergence time is also lower

i) the lower is the distance $\sqrt{d_M^2(x_k, \bar{x}) + d_N^2(y_k, \bar{y})}$ from the initial point (x_k, y_k) to $(\bar{x}, \bar{y})$.

ii) the lower is the joint payoff $h_k = H(x_k, y_k)$, i.e., the less potential gains can be realized by moving from the initial state to the Nash equilibrium

iii) the lower is the coefficient $E = 2t_2(\delta + 2\tau\beta t_1{}^{-1})\alpha^{-2}$, i.e., for a given t_1, the lower is t_2 (hence the lower is λ_k, and the lower is the Lipschitz constant C of the gradient $grad\Psi$, i.e., the flatter is the joint payoff $\Psi(x, y)$, i.e., the less there is to gain jointly.

The interpretation of Theorem 4.4 is the following: the Kurdyka-Lojasiewicz inequality with $\varphi(s) = cs^{1-\theta}$ and exponent $\theta \in [0, 1)$, $c > 0$, means that the discrepancy $|H(z) - H(z^*)|$ that the players want to fill between the present value of their common interest payoff $H(z)$ and its minimum value $H(z^*)$ is not too high, lower than some power of the subgradient of $H(z)$ at point z: $|H(z) - H(z^*)| \leq c\,|g|^{1/\theta}$ for all $g \in \partial H(z)$ and all $z \in B(z^*, \varepsilon)$,$\varepsilon > 0$. This power $1/\theta \geq 1$ is lower the larger is θ. Consider the two first cases (similar comments can be given for the third case):

i) if $\theta = 0$, then, there is convergence to a Nash equilibrium in a finite number of steps. A welcome result for game theory!

ii) if $\theta \in (0, \frac{1}{2}]$ the power $1/\theta$ is high enough (≥ 2). Theorem 4.4 shows that there exist $c_0 > 0$ and $\varsigma \in [0, 1)$ such that $d(z_k, z^*) \leq D(z_0)\varsigma^k$ where $D(z_0) = \Sigma_{i=1}^{+\infty} d(z_i, z_{i+1}) < +\infty$ represents the length of the path starting from z_k which converges to z^*, see the remark below. This means that, starting from any point z_k of this path, the distance from z_k to its limit z^* is lower than a fraction ς^k of the total length $D(z_0)$ of this path. Then, the remaining distance $d(z_k, z^*)$ between any z_k of the sequence and its limit z^*, hence the remaining time spent to converge (if we suppose a constant speed of moving ω) is lower, the lower is the constant $0 < \varsigma = c_1/(1+c_1) < 1$, i.e., the lower is the constant $c_1 = E\Theta + 1 > 0$, i.e., the lower are the constants E and Θ, where $E = 2t_2(\delta + 2\tau\beta t_1^{-1})\alpha^{-2}$ and $\Theta = c\left[c(1-\theta)(\delta + 2\tau\beta t_1^{-1})\right]^{(1-\theta)/\theta}$. Then, for given reference values t_1, α, the remaining time spent to converge is lower, the lower is the coefficient t_2 which majors the weights $0 < t_1 \leq \lambda_k, \mu_k \leq t_2$ on costs to change, the lower is the Lipschitz constant δ for the gradient variations, the lower is the upper bound τ on $gradC_M, gradC_N$, the lower is the cost per unit of distance majoration constant β, $(0 < \alpha \leq c_M, c_N \leq \beta)$, and the lower is the majoration ratio c for the discrepancy $|H(z) - H(z^*)|$ that the players want to fill.

Remark 5.1. *when $\theta \in (0, \frac{1}{2}]$, the proof of part ii) of Theorem 4.4 shows that there exists c_1 such that $D(z_k) \leq c_1(D(z_{k-1}) - D(z_k))$ and for k sufficiently large $D(z_k) \leq \varsigma D(z_{k-1})$, where $\varsigma = c_1/(1+c_1)$. Then, $d(z_k, z^*) \leq D(z_k) \leq c_0\varsigma^k$, where $c_0 = D(z_0) = \Sigma_{i=1}^{+\infty} d(z_i, z_{i+1}) < +\infty$. Similar precisions can be given for the case $\theta \in]1/2, 1[$.*

6 When a critical point is an Inertial Nash equilibrium

Critical points as Inertial Nash equilibria The alternating proximal algorithm generates, for a starting point $z_0 = (x_0, y_0) \in M \times N$, a sequence (z_k), with $z_k = (x_k, y_k) \in M \times N$, as it follows:

$$(x_k, y_k) \to (x_{k+1}, y_k) \to (x_{k+1}, y_{k+1})$$

$$\begin{cases} x_{k+1} \in \arg\min\{H(x, y_k) + \frac{1}{2\lambda_k} C_M^2(x_k, x), x \in M\} \\ \\ y_{k+1} \in \arg\min\{H(x_{k+1}, y) + \frac{1}{2\mu_k} C_N^2(y_k, y), y \in N\} \end{cases}$$

Let $\Gamma_{M,k}(x_k, x/y_k) = H(x, y_k) + \frac{1}{2\lambda_k} C_M^2(x_k, x)$ and $\Gamma_{N,k+1}(y_k, y/x_{k+1}) = H(x_{k+1}, y) + \frac{1}{2\mu_k} C_N^2(y_k, y)$ be the perturbed normal form game, including the desutility of costs to change (see Attouch et al. [7], 2007, for this concept).

Then, each two successive steps, $k, k+1$, x_{k+1} is a minimum of the perturbed unsatisfied needs function $\Gamma_{M,k}(x_k, x/y_k)$ and y_{k+1} is a minimum of the perturbed unsatisfied needs function $\Gamma_{N,k+1}(y_k, y/x_{k+1})$, i.e.,

$$\begin{aligned} \Gamma_{M,k}(x_{k+1}, x_{k+1}/y_k) = H(x_{k+1}, y_k) &\leq \Gamma_{M,k}(x_k, x_{k+1}/y_k) = H(x_{k+1}, y_k) + \frac{1}{2\lambda_k} C_M^2(x_k, x_{k+1}) \\ &\leq \Gamma_{M,k}(x_k, x/y_k) = H(x, y_k) + \frac{1}{2\lambda_k} C_M^2(x_k, x) \end{aligned}$$

for all $x \in M$, and

$$\begin{aligned} \Gamma_{N,k+1}(y_{k+1}, y_{k+1}/x_{k+1}) = H(x_{k+1}, y_{k+1}) &\leq \Gamma_{N,k+1}(y_k, y_{k+1}/x_{k+1}) = H(x_{k+1}, y_{k+1}) + \frac{1}{2\mu_k} C_N^2(y_k, y_{k+1}) \\ &\leq \Gamma_{N,k+1}(y_k, y/x_{k+1}) = H(x_{k+1}, y) + \frac{1}{2\mu_k} C_N^2(y_k, y) \end{aligned}$$

for all $y \in N$.

This implies the two inequalities

$$H(x_{k+1}, y_k) \leq H(x, y_k) + \frac{1}{2\lambda_k} C_M^2(x_k, x) \text{ for all } x \in M \tag{6.1}$$

and

$$H(x_{k+1}, y_{k+1}) \leq H(x_{k+1}, y) + \frac{1}{2\mu_k} C_N^2(y_k, y) \text{ for all } y \in N. \tag{6.2}$$

Suppose that $\lim_{k\to+\infty} \lambda_k = \lambda^* > 0$ and $\lim_{k\to+\infty} \mu_k = \mu^* > 0$ with $\lambda_k \geq \lambda_{k+1}\ \forall\, k$ and $\mu_k \geq \mu_{k+1}\ \forall\, k$.

If the process converges to a critical point, $\lim_{k\to+\infty} x_k = x^*$, and $\lim_{k\to+\infty} y_k = y^*$, the limiting perturbed unsatisfied need functions are, starting from (x^*, y^*), $\Gamma_{M,+\infty}(x^*, x/y^*) = H(x, y^*) + \frac{1}{2\lambda^*} C_M^2(x^*, x)$ and $\Gamma_{N,+\infty}(y^*, y/x^*) = H(x^*, y) + \frac{1}{2\mu^*} C_N^2(y^*, y)$. Then, if H is continuous with respect to each variable x, y, the inequalities (6.1) and (6.2) imply

$$H(x_{k+1}, y^*) = \lim_{y_k \to y^*} H(x_{k+1}, y_k) \leq \lim_{y_k \to y^*} \left\{ H(x, y_k) + \frac{1}{2\lambda_k} C_M^2(x_k, x) \right\} = H(x, y^*) + \frac{1}{2\lambda_k} C_M^2(x_k, x)$$

for all $x \in M$, and

$$H(x^*, y_{k+1}) = \lim_{x_{k+1} \to x^*} H(x_{k+1}, y_{k+1}) \leq \lim_{x_{k+1} \to x^*} \left\{ H(x_{k+1}, y) + \frac{1}{2\mu_k} C_N^2(y_k, y) \right\} = H(x^*, y) + \frac{1}{2\mu_k} C_N^2(y_k, y)$$

for all $y \in N$. Which give, dropping the intermediate terms, the two inequalities

$$H(x_{k+1}, y^*) \leq H(x, y^*) + \frac{1}{2\lambda_k} C_M^2(x_k, x) \text{ for all } x \in M$$

and

$$H(x^*, y_{k+1}) \leq H(x^*, y) + \frac{1}{2\mu_k} C_N^2(y_k, y) \text{for all} y \in N.$$

Then, from $\lambda_k \geq \lambda_{k+1} \geq \lambda^* > 0$, $\forall\, k$ and $\mu_k \geq \mu_{k+1} \geq \mu^* > 0$, $\forall\, k$, we get

$$H(x_{k+1}, y^*) \leq H(x, y^*) + \frac{1}{2\lambda^*} C_M^2(x_k, x) \text{ for all } x \in M$$

and

$$H(x^*, y_{k+1}) \leq H(x^*, y) + \frac{1}{2\mu^*} C_N^2(y_k, y) \text{ for all} y \in N,$$

which imply, if H is continuous with respect to each variable and $C_M(\cdot, x), C_N(\cdot, y)$ are continuous for all $x \in M, y \in N$

$$H(x^*, y^*) = \lim_{x_{k+1} \longrightarrow x^*} H(x_{k+1}, y^*) \leq H(x, y^*) + \frac{1}{2\lambda^*} C_M^2(x^*, x) \text{ for all } x \in M$$

and

$$H(x^*, y^*) = \lim_{y_{k+1} \longrightarrow y^*} H(x^*, y_{k+1}) \leq H(x^*, y) + \frac{1}{2\mu^*} C_N^2(y^*, y) \text{ for all } y \in N,$$

i.e.,

$$H(x^*,y^*) = \Gamma_{M,+\infty}(x^*,x^*/y^*) \leq \Gamma_{M,+\infty}(x^*,x/y^*) = H(x,y^*) + \frac{1}{2\lambda^*}C_M^2(x^*,x) \text{ for all } x \in M$$

and

$$H(x^*,y^*) = \Gamma_{N,+\infty}(y^*,y^*/x^*) \leq \Gamma_{N,+\infty}(y^*,y/x^*) = H(x^*,y) + \frac{1}{2\mu^*}C_N^2(y^*,y) \text{ for all } y \in N.$$

This shows that the critical point (x^*,y^*) is an Inertial Nash equilibrium defined as a Nash equilibrium of the perturbed normal form game $\Gamma_{M,k}(x_k,x/y_k)$ and $\Gamma_{N,k+1}(y_k,y/x_{k+1})$.

When a critical point is an epsilon Nash equilibrium Start from the two inequalities (6.1) and (6.2). Let $\lim_{k\to+\infty} x_k = x^*$ and $\lim_{k\to+\infty} y_k = y^*$. Suppose that costs to change from (x^*,y^*) are bounded above: $C_M(x,x') \leq \overline{C}_M < +\infty, C_N(y,y') \leq \overline{C}_N < +\infty$ for all $x,x' \in M$ and all $y,y' \in N$. This means that costs to change can be very high, up to some bound.

Suppose also that $\lim_{k\to+\infty} \lambda_k = \lambda^* < +\infty$ and that $\lim_{k\to+\infty} \mu_k = \mu^* < +\infty$ with $0 < \lambda^*\overline{C}_M < \varepsilon$ and $0 < \mu^*\overline{C}_N < \varepsilon$. This is equivalent to $0 <\lambda^* < \varepsilon/\overline{C}_M = \overline{\lambda}$ and $0 < \mu^* < \varepsilon/\overline{C}_N = \overline{\mu}$.

Then, $\lim_{k\to+\infty} \frac{1}{2\lambda_k}C_M^2(x_*,x) \leq \varepsilon$ and $\lim_{k\to+\infty} \frac{1}{2\mu_k}C_N^2(y^*,y) \leq \varepsilon$ for all $x \in M, y \in N$. The continuity of the functions $H$and $C_M(\cdot,x), C_N(\cdot,y)$ for all $x \in M, y \in N$ which appear in the two sides of each inequality (6.1) and (6.2) gives

$$H(x^*,y^*) \leq H(x,y^*) + \varepsilon \text{ for all } x \in M$$

and

$$H(x^*,y^*) \leq H(x^*,y) + \varepsilon \text{ for all } y \in N.$$

This shows that (x^*,y^*) is an epsilon Nash equilibrium of the normal form game.

7 An example from Psychology: "the course of motivation "

How to decrease the (finite) time of convergence ? 1) In this paper we have not only shown finite time convergence of the alternating process, a very nice result in the context of game theory (as we have shown in the introduction, very few finite time convergence results exist in the literature on "how to play Nash"), but, even better, we have shown how agents can decrease this finite time of convergence. This is important because transitions matter ! Hence, even if it is true that when trajectories have finite length, it is always possible to parametrize them so as to obtain finite time convergence, this is not the main point, in our context where we can give minorations and majorations of the convergence time, as a function of several parameters. The central point has been to exhibit mechanisms which endogenously decrease the time of convergence, using speed as an unknown variable and not a control variable. To simplify we have supposed a constant speed of moving. However this hypothesis must be reconsidered because, for example, a non vanishing speed of moving can cause a final shock problem and burn out effects (depletion of resources).

2) Then, in this final section we will consider the general case of non constant speed of moving, giving a detailed formalization of the minorization-majorization of the speed of moving, using the standard definition of the length of a curve on a manifold. For the special case of constant speed of moving we will show the existence of geodesics (constant speed trajectories) where convergence in finite time can be proved. We will examine a very important and concrete example in Psychology ("the course of motivation"), using a potential game with "vital" or "regeneration of resource constraints" and inertia (costs to change).

A leading example: "the course of motivation" Come back to our hypothesis A_1 of a constant speed of moving along a trajectory. We see it as a benchmark case and consider this hypothesis and other variants (increasing, decreasing or cycling speed of moving) in the context of the " course of motivation" (Touré-Tillery, Fishbach, 2011 [46]) which examines how "motivation variables" like instantaneous effort, speed of moving and persistence in effort evolve along a trajectory as an agent approaches his final goal (desired end state). Our context of alternating games where two agents approach their own final goal is more general. For even more general and diverse aspects see Soubeyran [44, 43]) who showed how human behavior can be modelized as a course between motivation and resistance to change (aspirations, capabilities and beliefs).

The "course of motivation" presents two polar cases of increasing and decreasing motivation.

Increasing motivation. The "goal gradient hypothesis" or the "goal looms larger effect" find that people (and other animals) spend more effort and persistence as they get closer to a goal's end state (Hull [26],1932). Hull [27],(1934), has shown that "rats in a straight alley progressively increased their running speed as they proceeded from the beginning of the alley toward the food at the end of the alley". More recently, Kivetz et al. [30], (2006), demonstrated goal-gradient effects for a variety of human behaviors. "They found, for example, that participants who rated songs online to obtain reward certificates increased their efforts as they approached the reward goal. Specifically, as they got closer to receiving the reward, participants increased the frequency of their visits to the rating site, rated more songs per visit, and were less likely to abandon uncompleted rating efforts" (Touré-Tillery and Fishbach [46], (2011)). There are at least three explanations for the pattern of increasing motivation as the distance to the goal decreases.

i) "closure" or the need to finish what one starts, particularly when agents are close to finishing it (see Hull [26, 27]).

ii) an increasing relative marginal reduction of the remaining distance to the final goal. As shown by Touré-Tillery and Fishbach [46],(2011), "if the completion of a goal requires a given number of identical steps, each new step would reduce a larger proportion of the remaining distance to the goal and therefore would appear more impactful". For example, suppose that, to reach the goal, the agent has to make 10 similar operations (the initial distance to the goal). Doing the first operation reduce the distance to goal by 10% (1 out of 10 remaining operations). The next steps reduce the distance by 1/9, 1/8,... and so on....

iii) loss aversion.Prospect theory shows that the value of outcomes follows an S-shaped function (Kahneman and Tversky [29], 1979). Following Touré-Tillery, Fishbach [46], (2011), "outcomes of goal pursuit that fall short of the goal (i.e., losses) have a greater marginal impact when they are closer to the goal's end state (i.e. reference point)—at points where the loss function is steep—than when they are distant from the end state. This diminishing sensitivity principle suggests falling short of a goal when one is close to (vs. far from) the end state would be more painful (i.e., perceived as a greater loss)".

Decreasing motivation Touré-Tillery and Fishbach [46], (2011), have identified several explanations for the pattern of decreasing motivation as the distance to the goal decreases.

i) **Diminishing goal accessibility.** A decrease in motivation can occur over the course of pursuing endless goals (never fully completed), because a vague end state does not satisfy the need for closure, hidds the perception of an increasing relative marginal reduction of the remaining distance to the goal and the loss aversion effect. Moreover, in the absence of a clear end state, motivation to follow a goal will also decrease as goal accessibility decreases over time, because "motivation is at its peak soon after the goal is primed by contextual cues, including images, words, and sounds" (see the literature on "goal priming", and among many others, Fishbach-Ferguson [21], 2007, for review).

ii) **Depletion of resources.** The decline in physiological and psychological (self control) resources (physical and mental fatigue) following previous engagement can lead to a decreasing course of motivation called "depletion". Efforts made to achieve goals can reduce or exhaust physiological and psychological resources (Wright, Martin and Bland [51], 2003). However to resist to energy-depletion the agent can mobilize " motivational resources" (Muraven and Slessareva [40], 2003).

In Adly-Attouch-Cabot [2], (2011), finite time convergence is obtained, while at the same time the speed tends to zero ! A nice result ! But this paper (basically the proposition 24.4) uses strong assumptions of convexity and uniqueness of solution. Our structure is far weaker, and may require a particularization that would not fit in this article. Moreover, in the case of renewable resources "ego depletion" is less important. Then, in our paper where resources are renewable, this effect which can lead to decreasing motivation can disappear.

Cyclical motivation As Touré-Tillery and Fishbach [46],(2011), say "When goal pursuit taps into limited but renewable resources, motivation can also take a cyclical down-and-up pattern depending on the length of the pursuit and the type of tasks involved. Indeed, if depletion due to prior efforts leads to lower efforts in a subsequent task then this decreased effort would not interfere with the replenishment of depleted resources, which could potentially result in increased effort in the next task. This pattern of down-and-up motivation could repeat itself until the goal is reached, depending on the number of steps required for goal attainment. Moreover, the duration of the low-motivation (low-resource) stage following depletion would depend on the amount of time required to replenish depleted resources. Highly depleting initial activities might require more recovery time, hence prolonging the low-motivation (low-resource) phase, whereas marginally depleting tasks might require less recovery time, leading to shorter low-motivation (low-resource) stages".

Our paper can be easily generalized to this case, with a variable (but not too much) speed of moving. Let $C_M(x,p) = e_M\left[\omega_M(x,p)\right] d_M(x,p)$ be the cost to change from x to p on M. Suppose that the "per unit of distance" cost to change e_M depends only of the speed of moving from x to p, $\omega_M(x,p) = d_M(x,p)/t_F(x,p)$. The "reactivity cost" e_M usually increases with ω_M. If the speed of moving is bounded below and above, i.e.,

$$0 < \underline{\omega} \leq \omega_M(x,p) \leq \overline{\omega} < \infty \text{ for all } x,p \in M,$$

then, $e_M\left[\underline{\omega}\right] \leq e_M\left[\omega_M(x,p)\right] \leq e_M\left[\overline{\omega}\right]$. Thus, both

$$\underline{\omega} d_M(x,p) \leq t_F(x,p) \leq \overline{\omega} d_M(x,p)$$

and

$$e_M\left[\underline{\omega}\right] d_M(x,p) \leq C_M(x,p) = e_M\left[\omega_M(x,p)\right] d_M(x,p) \leq e_M\left[\overline{\omega}\right] d_M(x,p)$$

where $\alpha = e_M\left[\underline{\omega}\right]$ and $\beta = e_M\left[\overline{\omega}\right]$. These inequalities are sufficient to give the finite distance and finite time convergence result.

Minorization-majorization of the speed of moving Let us give some more details than in section 3 (Riemannian manifolds) to modelize precisely the minorization-majorization of the speed of moving hypothesis seen just above.

Length spaces and speed of moving. Let (M,d) be a metric space and $I = [t,t'] \subset \mathbb{R}$ be a non empty interval. A move from x to p on M can be modelized by a piecewise continuous curve σ, given by $\sigma(\cdot) : \tau \in [t,t'] = I \subset R \longmapsto \sigma(\tau) \in M$ where $\sigma(t) = x$ and $\sigma(t') = p$. The length of this curve is $L\left[\sigma(\cdot)\right] = \sup\left\{\Sigma_{i=1}^{k} d\left[\sigma(t_{i-1}), \sigma(t_i)\right]\right\}$ where the supremum is taken over all $k \in N$ and all sequences $t_0 \leq t_1 \leq .... \leq t_k$ in I. The piecewise continuous curve $\sigma(\cdot)$ is rectifiable if $L\left[\sigma(\cdot)\right] < +\infty$. Let $\inf L\left[\sigma(\cdot)\right]$ be the infimum of all $L[\sigma(\cdot)]$, where this infimum is taken over all rectifiable piecewise continuous curves from x to $p, \sigma(\cdot) : \tau \in [t, t'] = I \subset R \longmapsto \sigma(\tau) \in M$ where $\sigma(t) = x$ and $\sigma(t') = p$. Then, $d_M(x,p) = \inf L[\sigma(\cdot)]$ defines the inner distance or length metric on the metric space (M, d). If $d_M(x,p) = d(x,p)$ for all $x, p \in M$, then, (M, d) is called a length space. Example: let (X, g) be a connected Riemannian manifold with the metric $d(x,p) = \inf \left\{ L[\sigma(\cdot)], \sigma(\cdot) : I = [0,1] \longmapsto M \text{ a piecewise } C^1 \text{ curve from } x \text{ to } p \right\}$. Then, (M, d) is a length space.

Minorization-majorization of the speed of moving.

This is the case, if each move $(x_k = x, x_{k+1} = p)$, $k \in \mathbb{N}$, we can on each manifold, minorize-majorize the speed of moving of any given finite length trajectory by some minimal and maximal speeds.

Existence of constant speed trajectories: geodesics Let $(X = M, d)$ be a metric space. A curve $\sigma(\cdot) : I = [0,1] \longmapsto X$ is a geodesic if $\sigma(\cdot)$ has constant speed and if $L[\sigma[\tau, \tau']] = d(\sigma(\tau), \sigma(\tau'))$ for all $\tau, \tau' \in I, \tau < \tau'$. (X, d) is called a geodesic space if for every pair of points $x, p \in X$, there exists a geodesic $\sigma(\cdot) : I = [0,1] \longmapsto X$ joining x to p.

Theorem of Hopf-Rinow

Let $X = M$ be a length space. If X is complete and locally compact, then, X is proper (i.e. every closed bounded subset of X is compact), and X is a geodesic space.

This shows that, for each agent, say agent F, each step $(x_k = x \in M, x_{k+1} = p \in M)$ of a finite length trajectory which converges (which exists, as shown by theorems 4.2, 4.3, 4.4), it is always possible to find a geodesic joining x to p with a constant (but eventually different) speed of moving (the same for the other player). Then finite time convergence follows if these step by step speeds of moving are minorized and majorized by the same constants.

Example of potential games In the Public Economics literature a very important example of potential game is the following: within an organization (a family, a team, a firm, a group, the society) two agents F and G have both individual and joint interests. Their individual payoffs to do actions $x \in M$ and $y \in N$ are $I_F(x) \in \mathbb{R}$ and $I_G(y) \in \mathbb{R}$. Their joint payoffs are $\lambda_F J(x,y)$ and $\lambda_G J(x,y)$ where $\lambda_F, \lambda_G > 0$ are the weights they put on the joint payoff $J(x,y) \in \mathbb{R}_+$. For example the organization can be a family, F and G being the father and the mother. The manifolds M and F represent the constraints on the resources (time, energy, money) they can spend for their hobbies (individual interests) and their joint activities. $I_F(x)$ and $I_G(y)$ are their utilities to spend resources for hobbies and $\lambda_F J(x,y)$ and $\lambda_G J(x,y)$ are their relative utilities to give a good education. The weights they put on joint activities represent how much a good education is important for them. A good education requires that both parents will be involved in, each period. Then, the dilemma is that the more resources parents spend for hobbies, the less they have for education !

Inexact proximal alternated algorithms and "Learning to play Nash" In our paper we have used exact proximal point algorithms to modelize agents who optimize in alternation. This is not very realistic because most of the time they will not have the time and the resources to optimize. This requires to explore again and again the whole state space of actions M and N, because the payoff of a player changes as soon as the other player carries out a new action (preferences change along the path of best responses). Then agents will prefer to give in alternation an approximate best response, say a satisficing (i.e. improving enough) response, to economize time and resources. Would an agent optimizes some given period, and the second period the other agent does the same, then, the third period the preference of the first player will have changed and his previous best responses will be useless. Inexact proximal algorithms played in alternation would modelize very well such a succession of satisficing responses. This will be left for future research. For inexact proximal methods which are not played in alternation see Attouch et al. [6] and Hare and Sagatizabal [23], in the Euclidean context and Bento et al. [11] in Riemannian setting where they solve approximatively the subproblem of the form (1.2) or (1.3).

At a mathematical level, an exact descent method on a manifold generalizes an exact proximal algorithm on such a manifold and an inexact descent method (or proximal algorithm) on a manifold generalizes an exact descent method (proximal algorithm) on such a manifold (Bento et al. [11], 2011).

Let us give an indication of how to proceed to generalize the analysis from an exact to an inexact alternating proximal algorithm on manifolds. Let M and N be complete Riemannian manifolds and let $H : M \times N \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function bounded from below. The alternating proximal algorithm to solve optimization problems of the form (1.1) generates, for a starting point z_0, with $z_0 = (x_0, y_0) \in M \times N$, a sequence (z_k), with $z_k = (x_k, y_k) \in M \times N$, as follows:

$$(x_k, y_k) \to (x_{k+1}, y_k) \to (x_{k+1}, y_{k+1})$$

$$\begin{cases} x_{k+1} \in \arg\min\{H(x, y_k) + \frac{1}{2\lambda_k} C_M^2(x_k, x), x \in M\} \\ \\ y_{k+1} \in \arg\min\{H(x_{k+1}, y) + \frac{1}{2\mu_k} C_N^2(y_k, y), y \in N\} \end{cases}$$

where C_M and C_N are quasi distances associated with the manifolds M and N respectively, (λ_k) and (μ_k) are sequences of positive numbers and the function H consists of a separable term $(x, y) \longmapsto f(x) + g(y)$ and a coupling term Ψ.

An inexact alternating proximal algorithm on such manifolds will consider a sequence of actions (x_k, y_k), when the first player changes and the second player stays (the reverse one period later). Instead of doing an optimizing change $(x_k, y_k) \curvearrowright (x_{k+1}, y_k)$, the first player will carry out

i) some "worthwhile change" $(x_k, y_k) \curvearrowright (x_{k+1}, y_k)$ such that $H(x_{k+1}, y_k) + \frac{1}{2\lambda_k} C_M^2(x_k, x_{k+1}) \leq H(x_k, y_k)$, for some $x_k \in M$.

ii) which is, "marginally, worthwhile enough" to do not have the motivation to choose an other change $(x_k, y_k) \curvearrowright (x', y_k)$. This means that it exists $m_{k+1} \in \partial H(x_{k+1}, y_k)$ such that $\|m_{k+1}\| \leq b C_M(x_k, x_{k+1})$, $b > 0$, i.e. , such that marginal gains are lower than some proportion of the desutility of costs to change.

This is an alternating descent algorithm. Bento et al. [11],(2011), have examined the specific case of descent methods on a Riemannian manifold, but not played in alternation. The extension to alternating descent methods on manifolds could consider two hypothesis made by Bento et al. [11],

iii) H restricted to its domain is continuous and $C_M(x, .), C_N(x, .)$ are continuous.

iv) $\Sigma_{k=0}^{+\infty} C_M(x_k, x_{k+1}) < +\infty$ implies that $\{x_k\}$ is convergent on M and $\Sigma_{k=0}^{+\infty} C_N(y_k, y_{k+1}) < +\infty$ implies that $\{y_k\}$ is convergent on N.

8 Conclusion.

We present and analyze the alternating proximal algorithm on Riemannian manifold for minimizing nondifferentiable KL functions. We use intrinsic relations between the o-minimal structure and analytic-geometric categories for ensure the existence of KL functions. We derive important theoretical results of convergence (Theorem 4.1) and convergence rate (Theorem 4.4) in the Riemannian context. These results are applied in our alternating "exploration-exploitation" model and have a very interesting interpretation. Starting from $z_k = (x_k, y_k)$ the length of the alternating learning path is $D(z_k) = \Sigma_{i=k+1}^{+\infty} d(z_i, z_{i+1})$. Our main Theorem (Theorem 4.1) shows that $D(z_k) \leq E\varphi(h_k - \bar{h})\alpha^{-1}\sqrt{2t_2(h_k - \bar{h})}$. As a consequence of the convergence results, we show that our application model is naturally connected with proximal algorithm and quasi distances. One other merit of our model is that it can incorporate non convex constraints by considering individual goals such as characteristic functions of non convex sets $C \subset X$ and $D \subset Y$, $\delta_C = f$ and $\delta_D = g$.

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