Forbidding extreme points from the 0 − 1 hypercube

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May 10, 2012

Abstract

Let $B$ be the set of $n$-dimensional binary vectors and let $V$ be a subset of $m$ of its elements. We give an extended formulation of the convex hull of $B \setminus V$ which is polynomial in $n$ and $m$. In developing this result, we give a two-sided extension of the result in [3] for knapsack sets with superincreasing coefficients.

1 Introduction

We consider the following problem. Let $B := \{0, 1\}^n$ and let $V \subseteq B$ contain $m$ elements which are given as a list. Is it possible to give a description of $\text{conv}(B \setminus V)$ which is polynomial in $n$ and $m$? We give a positive answer to this question by presenting an extended formulation involving $O(mn)$ variables and constraints. Our approach relies on being able to give the convex hull of binary knapsack sets having a superincreasing weight. This generalizes a result due to [3].

2 Binary expansion

Let $N := \{1, \ldots, n\}$ and $N := \{0, \ldots, 2^n - 1\}$. There exists a bijection between $B$ and $N$ given by the mapping $\sigma(v) := \sum_{i \in N} 2^{i-1}v_i$ for all $v \in B$. Therefore, we can write $B = \{v^0, \ldots, v^{2^n-1}\}$, where $v^k$ gives the binary expansion of $k$ for each $k \in N$, that is, $v^k = \sigma^{-1}(k)$. Let $V = \{v^{k_1}, \ldots, v^{k_m}\}$, where without loss of generality we assume $k_l < k_{l+1}$ for all $l = 1, \ldots, m-1$. Also, let $N_V := \{k \in N : v^k \in V\}$. Then we have

$$B \setminus V = \left\{ x \in B : \sum_{i \in N} 2^{i-1}x_i \notin N_V \right\}.$$ 

Now, for integers $a$ and $b$, let

$$K(a, b) = \left\{ x \in B : a \leq \sum_{i \in N} 2^{i-1}x_i \leq b \right\}.$$ 

\(^1\text{After completion of this note, it was brought to our attention that recently Muldoon et al. [4] independently provided a convex hull description of two-sided binary knapsack sets with superincreasing weights.}\)
If \( b < a \), then \( K(a, b) \) is empty. Set \( k_0 = -1 \) and \( k_{m+1} = 2^n \). Then we can write

\[
B \setminus V = \bigcup_{l=0}^{m} K(k_l + 1, k_{l+1} - 1).
\]

Thus

\[
\text{conv}(B \setminus V) = \text{conv} \left( \bigcup_{l=0}^{m} K(k_l + 1, k_{l+1} - 1) \right) = \text{conv} \left( \bigcup_{l=0}^{m} \text{conv}(K(k_l + 1, k_{l+1} - 1)) \right). \tag{1}
\]

Therefore, if we have a polynomial description of \( \text{conv}(K(a, b)) \) for any \( a, b \in \mathcal{N} \), then we have a compact extended formulation for \( \text{conv}(B \setminus V) \) as the convex hull of the union of \( m + 1 \) polytopes as given in [1].

### 3 Superincreasing two-sided knapsack sets

**Definition 1.** A vector \( w \in \mathbb{Z}_+^n \) is a superincreasing sequence if \( \sum_{j<i} w_j \leq w_i \) for all \( i \in \mathcal{N} \).

A prime example of a superincreasing sequence is given by \( w = w^* \), where \( w_i^* := 2^{i-1} \) for all \( i \in \mathcal{N} \).

Let \( w \in \mathbb{Z}_+^n \). For integers \( 0 \leq a \leq b \leq \sum_{i \in \mathcal{N}} w_i \), let

\[
K^\geq_w(a) := \left\{ x \in \{0, 1\}^n : a \leq \sum_{i \in \mathcal{N}} w_i x_i \right\},
\]

\[
K^\leq_w(b) := \left\{ x \in \{0, 1\}^n : \sum_{i \in \mathcal{N}} w_i x_i \leq b \right\},
\]

\[
K_w(a, b) := \left\{ x \in \{0, 1\}^n : a \leq \sum_{i \in \mathcal{N}} w_i x_i \leq b \right\} = K^\geq_w(a) \cap K^\leq(b).
\]

We say that \( C \subseteq \mathcal{N} \) is a minimal cover of \( b \) with respect to \( w \) if \( \sum_{i \in C} w_i > b \) and \( \sum_{i \in C'} w_i \leq b \) for all \( C' \subseteq C \). In [3] it is shown that if \( w \in \mathbb{Z}_+^n \) is a superincreasing sequence, then the non-trivial facets of the convex hull of \( K^\leq_w(b) \) are all minimal cover inequalities, that is

\[
\text{conv}(K^\leq_w(b)) = \left\{ x \in [0, 1]^n : \sum_{i \in C} x_i \leq |C| - 1 \forall C \in \mathcal{C}_w(b) \right\},
\]

where \( \mathcal{C}_w(b) \subseteq 2^\mathcal{N} \) is the set of all minimal covers of \( b \) with respect to \( w \). It is not hard to verify that an analogous result holds for the convex hull of \( K^\geq_w(a) \). We say that \( R \subseteq \mathcal{N} \) is a minimal reverse cover of \( a \) with respect to \( w \) if \( \sum_{i \in \mathcal{N} \setminus R} w_i < a \) and \( \sum_{i \in \mathcal{N} \setminus R'} w_i \geq a \) for all \( R' \subseteq R \). Then we have

\[
\text{conv}(K^\geq_w(a)) = \left\{ x \in [0, 1]^n : \sum_{i \in R} x_i \geq 1 \forall R \in \mathcal{R}_w(a) \right\},
\]

where \( \mathcal{R}_w(a) \subseteq 2^\mathcal{N} \) is the set of all minimal reverse covers of \( a \) with respect to \( w \). We show that

\[
\text{conv}(K_w(a, b)) = \text{conv} \left( K^\geq_w(a) \right) \cap \text{conv} \left( K^\leq_w(b) \right).
\]
Theorem 2. If \( w \in \mathbb{Z}_+^n \) is a superincreasing sequence, then for all integers \( 0 \leq a \leq b \leq \sum_{i \in N} w_i \), we have

\[
\text{conv}(K_w(a, b)) = \left\{ x \in [0, 1]^n : \sum_{i \in \bar{R}} x_i \geq 1 \quad \forall \bar{R} \in \mathcal{R}_w(a) \right\}
\]

Proof. We prove the result by induction on \( n \). When \( n = 1 \), the result is trivial. Fix \( n > 1 \) and assume the result is true for \( n - 1 \). Let \( w \in \mathbb{Z}_+^n \) be a superincreasing sequence, and let \( 0 \leq a \leq b \leq \sum_{i \in N} w_i \). Consider the set

\[
K_w(a, b) = \left\{ x \in \{0, 1\}^n : a \leq \sum_{i < n} w_i x_i \leq b, \quad x_n = 0 \right\} = K_w(a, b) \times \{0\}.
\]

Let \( \bar{w} \in \mathbb{Z}_+^{n-1} \) be the vector obtained by removing the last component from \( w \). Clearly, \( \bar{w} \) is a superincreasing sequence.

Case 1: \( w_n > b \).

If \( w_n > b \), then \( x_n = 0 \) for all \( x \in K_w(a, b) \). Thus

\[
K_w(a, b) = \left\{ x \in \{0, 1\}^n : a \leq \sum_{i < n} w_i x_i \leq b, \quad x_n = 0 \right\} = K_{\bar{w}}(a, b) \times \{0\}.
\]

Notice that \( \bar{C} := \{n\} \) is a minimal cover of \( b \) with respect to \( w \), and therefore \( n \) cannot be contained in any other cover in \( \mathcal{C}_{\bar{w}}(b) \). Then we have \( \mathcal{C}_w(b) = \mathcal{C}_{\bar{w}}(b) \cup \{\bar{C}\} \). We also have \( \bar{R} \cup \{n\} \in \mathcal{R}_{\bar{w}}(a) \) for all \( \bar{R} \in \mathcal{R}_{\bar{w}}(a) \). On the other hand, if \( R \in \mathcal{R}_{\bar{w}}(a) \) and \( n \notin R \), then \( a > \sum_{N \setminus R} w_i \geq w_n \geq b \), a contradiction. Furthermore, if \( \bar{R} \subseteq N \setminus \{n\} \) and \( \bar{R} \notin \mathcal{R}_{\bar{w}}(a) \), then \( \bar{R} \cup \{n\} \notin \mathcal{R}_{\bar{w}}(a) \). Thus \( \mathcal{R}_w(a) = \{\bar{R} \cup \{n\} : \bar{R} \in \mathcal{R}_{\bar{w}}(a)\} \). Hence, by the inductive hypothesis, we have

\[
\text{conv}(K_w(a, b)) = \text{conv}(K_{\bar{w}}(a, b) \times \{0\})
\]

\[
= \left\{ x \in [0, 1]^n : \sum_{i \in \bar{R}} x_i \geq 1 \quad \forall \bar{R} \in \mathcal{R}_{\bar{w}}(a) \right\}
\]

\[
= \left\{ x \in [0, 1]^n : \sum_{i \in \bar{R}} x_i \leq |\bar{C}| - 1 \quad \forall \bar{C} \in \mathcal{C}_{\bar{w}}(b) \right\}
\]

\[
= \left\{ x \in [0, 1]^n : \sum_{i \in \bar{R}} x_i \geq 1 \quad \forall \bar{R} \in \mathcal{R}_{\bar{w}}(a) \right\}
\]

\[
= \left\{ x \in [0, 1]^n : \sum_{i \in \bar{R}} x_i \leq |\bar{C}| - 1 \quad \forall \bar{C} \in \mathcal{C}_{\bar{w}}(b) \right\}
\]

Since the result holds for \( w_n > b \), we may assume \( w_n \leq b \). Notice that this implies \( n \in C \) for all \( C \in \mathcal{C}_{\bar{w}}(b) \).

Case 2: \( a > \sum_{i < n} w_i \), \( w_n \leq b \).
If $a > \sum_{i<n} w_i$, then $x_n = 1$ for all $x \in K_w(a, b)$. Thus

$$K_w(a, b) = \left\{ x \in \{0, 1\}^n : a - w_n \leq \sum_{i<n} w_i x_i \leq b - w_n, \ x_n = 1 \right\} = K_w(a - w_n, b - w_n) \times \{1\}.$$ 

It is not hard to verify that $C_w(b) = \{\bar{C} \cup \{n\} : \bar{C} \in C_w(b - w_n)\}$. Also, since $\sum_{i<n} w_i < a$ and $\sum_{i\in N} w_i \geq a$, we have that $\bar{R} := \{n\}$ is a minimal reverse cover of $a$ with respect to $w$. Therefore $n$ cannot be contained in any other reverse cover in $R_w(a)$. Then it is straightforward to verify that $R_w(a) = R_w(a - w_n) \cup \{\bar{R}\}$. Hence, by the inductive hypothesis, we have

$$\text{conv}(K_w(a, b)) = \text{conv}(K_w(a - w_n, b - w_n) \times \{1\})$$

$$= \left\{ x \in [0, 1]^n : \sum_{i \in \bar{R}} x_i \geq 1 \ \forall \bar{R} \in R_w(a - w_n) \right\}$$

$$= \left\{ x \in [0, 1]^n : \sum_{i \in \bar{R}} x_i + x_n \leq |\bar{C} \cup \{n\}| - 1 \ \forall \bar{C} \in C_w(b - w_n) \right\}$$

$$= \left\{ x \in [0, 1]^n : \sum_{i \in \bar{R}} x_i \geq 1 \ \forall \bar{R} \in R_w(a) \right\}$$

$$= \left\{ x \in [0, 1]^n : \sum_{i \in \bar{C}} x_i \leq |C| - 1 \ \forall C \in C_w(b) \right\}.$$ 

Since the result holds for $a > \sum_{i<n} w_i$, we may assume $a \leq \sum_{i<n} w_i$. Notice that this implies $n \in R$ for all $R \in R_w(a)$.

Let $w_0 := \sum_{i<n} w_i$. We have $w_0 \leq w_n$.

Case 3: $a \leq w_0 \leq w_n \leq b$.

Consider the sets

$$K_w^0(a, b) := K_w(a, b) \cap \{ x \in \mathbb{R}^n : x_n = 0 \} = K_w(a, w_0) \cap \{ x \in \mathbb{R}^n : x_n = 0 \} = K_w(a, w_0) \times \{0\},$$

$$K_w^1(a, b) := K_w(a, b) \cap \{ x \in \mathbb{R}^n : x_n = 1 \} = K_w(w_n, b) \cap \{ x \in \mathbb{R}^n : x_n = 1 \} = K_w(0, b - w_n) \times \{1\},$$

which are nonempty. Notice that $C_w(w_0) = \emptyset$ and $R_w(0) = \emptyset$. By the inductive hypothesis, we have

$$\text{conv}(K_w^0(a, b)) = \left\{ x \in [0, 1]^n : \sum_{i \in \bar{R}} x_i \geq 1 \ \forall \bar{R} \in R_w(a) \right\},$$

$$\text{conv}(K_w^1(a, b)) = \left\{ x \in [0, 1]^n : \sum_{i \in \bar{C}} x_i \leq |\bar{C}| - 1 \ \forall \bar{C} \in C_w(b - w_n) \right\}.$$ 

Furthermore, by arguments similar to that of Cases 1 and 2, we obtain

$$\text{conv}(K_w^0(a, b)) = \left\{ x \in [0, 1]^n : \sum_{i \in \bar{R}} x_i \geq 1 \ \forall \bar{R} \in R_w(a) \right\},$$

$$\text{conv}(K_w^1(a, b)) = \left\{ x \in [0, 1]^n : \sum_{i \in \bar{C}} x_i \leq |\bar{C}| - 1 \ \forall \bar{C} \in C_w(b - w_n) \right\}.$$
\[
\text{conv}(K_w^1(a, b)) = \left\{ x \in [0,1]^n : \sum_{i \in C} x_i \leq |C| - 1 \quad \forall C \in \mathcal{C}_w(b) \right\}.
\]

Clearly,
\[
\text{conv}(K_w(a, b)) = \text{conv} \left( \text{conv}(K_w^0(a, b)) \cup \text{conv}(K_w^1(a, b)) \right).
\]

Let \( Q \subseteq \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \) be the polyhedron given by
\[
\begin{align*}
x_i &= y_i + z_i \quad \forall i \\
\sum_{i \in R} y_i &\geq \lambda \quad \forall R \in \mathcal{R}_w(a) \\
y_i &\geq 0 \quad \forall i \leq n - 1 \\
y_i &\leq \lambda \quad \forall i \leq n - 1 \\
y_n &= 0 \\
\sum_{i \in C} z_i &\leq (|C| - 1)(1 - \lambda) \quad \forall C \in \mathcal{C}_w(b) \\
z_i &\geq 0 \quad \forall i \leq n - 1 \\
z_i &\leq 1 - \lambda \quad \forall i \leq n - 1 \\
z_n &= 1 - \lambda \\
\lambda &\geq 0 \\
\lambda &\leq 1.
\end{align*}
\]

From [1], we have that \( \text{conv}(K_w(a, b)) = \text{proj}_x(Q) \). Eliminating \( z \) from \( Q \) using \( z_i = x_i - y_i \) for all \( i \in N \), we obtain \( \text{proj}_{x,y,\lambda}(Q) \) as
\[
\begin{align*}
\sum_{i \in R} y_i &\geq \lambda \quad \forall R \in \mathcal{R}_w(a) \\
y_i &\geq 0 \quad \forall i \leq n - 1 \\
y_i &\leq \lambda \quad \forall i \leq n - 1 \\
y_n &= 0 \\
\sum_{i \in C} (x_i - y_i) &\leq (|C| - 1)(1 - \lambda) \quad \forall C \in \mathcal{C}_w(b) \\
x_i - y_i &\geq 0 \quad \forall i \leq n - 1 \\
x_i - y_i &\leq 1 - \lambda \quad \forall i \leq n - 1 \\
x_n - y_n &= 1 - \lambda \\
\lambda &\geq 0 \\
\lambda &\leq 1.
\end{align*}
\]
Eliminating \( \lambda \) from \( \text{proj}_{x,y,\lambda}(Q) \) using \( \lambda = 1 - x_n + y_n \), we obtain \( \text{proj}_{x,y}(Q) \) as
\[
\sum_{i \in R} y_i \geq 1 - x_n + y_n \quad \forall R \in R_w(a)
\]
\[
y_i \geq 0 \quad \forall i \leq n - 1
\]
\[
y_i \leq 1 - x_n + y_n \quad \forall i \leq n - 1
\]
\[
y_n = 0
\]
\[
\sum_{i \in C} (x_i - y_i) \leq (|C| - 1)(x_n - y_n) \quad \forall C \in C_w(b)
\]
\[
x_i - y_i \geq 0 \quad \forall i \leq n - 1
\]
\[
x_i - y_i \leq x_n - y_n \quad \forall i \leq n - 1
\]
\[
1 - x_n + y_n \geq 0
\]
\[
1 - x_n + y_n \leq 1.
\]
Let \( P^n := \text{proj}_{x,y}(Q) \). For each \( k \in \{n - 1, \ldots, 0\} \), let \( P^k \) be the polyhedron obtained after projecting out \( y_{k+1} \) from \( P^{k+1} \). Also, for each \( C \in C_w(b) \) and \( k \in \{n - 1, \ldots, 0\} \), let \( C^k := \{i \in C : i > k\} \).

Claim: For each \( k \in \{n - 1, \ldots, 0\} \), \( P^k \) is given by
\[
\sum_{i \in R: i \leq k} y_i + \sum_{i \in R: i > k} x_i \geq 1 \quad \forall R \in R_w(a) \quad (2)
\]
\[
y_i \geq 0 \quad \forall i \leq k \quad (3)
\]
\[
y_i \leq 1 - x_n \quad \forall i \leq k \quad (4)
\]
\[
\sum_{i \in C: i \leq k} (x_i - y_i) + \sum_{i \in C: i > k} x_i \leq |C^k| - 1 + (|C| - |C^k|)x_n \quad \forall C \in C_w(b) \quad (5)
\]
\[
x_i - y_i \geq 0 \quad \forall i \leq k \quad (6)
\]
\[
x_i - y_i \leq x_n \quad \forall i \leq k \quad (7)
\]
\[
x_i \geq 0 \quad \forall i > k
\]
\[
x_i \leq 1 \quad \forall i > k.
\]
We verify the claim for \( k = n - 1 \). Following Assumptions 1 and 2, \( n \in C \) for all \( C \in C_w(b) \) and \( n \in R \) for all \( R \in R_w(a) \). Given that \( y_n = 0 \) and \( C^{n-1} = \{n\} \) for all \( C \in C_w(b) \), we obtain \( P^{n-1} \) as
\[
\sum_{i \in R: i \leq n-1} y_i + \sum_{i \in R: i > n-1} x_i \geq 1 \quad \forall R \in R_w(a)
\]
\[
y_i \geq 0 \quad \forall i \leq n - 1
\]
\[
y_i \leq 1 - x_n \quad \forall i \leq n - 1
\]
\[
\sum_{i \in C: i \leq n-1} (x_i - y_i) + \sum_{i \in C: i > n-1} x_i \leq (|C| - 1)x_n \quad \forall C \in C_w(b)
\]
\[
x_i - y_i \geq 0 \quad \forall i \leq n - 1
\]
\[
x_i - y_i \leq x_n \quad \forall i \leq n - 1
\]
\[
x_n \geq 0
\]
\[
x_n \leq 1.
\]
Now, fix \( 1 \leq k < n \) and assume the claim holds for \( k \). We want to show that after projecting out \( y_k \) from \( P^k \) we obtain \( P^{k-1} \) having the desired form. We check the inequalities obtained by appropriately combining \([2] - [7]\). We begin combining \([3], [4], [5], \) and \([7]\) for \( i = k \).
(3), (4) gives \( x_n \leq 1 \), which is redundant.

(6), (7) gives \( x_n \geq 0 \), which is redundant.

(3), (6) gives \( x_k \geq 0 \).

(4), (7) gives \( x_k \leq 1 \).

Inequalities (3), (4), (6), and (7) remain for \( i < k \). Inequalities (2) and (5) remain if \( R \not\in \mathcal{R}_w(a) \) and \( C \not\in \mathcal{C}_w(b) \), respectively. Notice that (2) and (5) cannot be combined. The remaining cases are the following.

(2), (4) for \( R \in \mathcal{R}_w(a) \) such that \( k \in R \) gives
\[
\sum_{i \in R: i \leq k-1} y_i + \sum_{i \in R: i > k} x_i \geq 0,
\]
which is redundant since \( y_i \geq 0 \) for all \( i \leq k-1 \) and \( x_i \geq 0 \) for all \( i > k-1 \).

(2), (6) for \( R \in \mathcal{R}_w(a) \) such that \( k \in R \) gives
\[
\sum_{i \in R: i \leq k-1} y_i + \sum_{i \in R: i > k-1} x_i \geq 1,
\]

(5), (4) for \( C \in \mathcal{C}_w(b) \) such that \( k \in C \) gives
\[
\sum_{i \in C: i \leq k-1} (x_i - y_i) + \sum_{i \in C: i > k-1} x_i \leq |C^k| + (|C| - |C^k| - 1)x_n = |C^{k-1}| - 1 + (|C| - |C^{k-1}|)x_n,
\]
where we have used that \( k \in C \) implies \( k \notin C^k \) and \( C^{k-1} = C^k \cup \{k\} \).

(5), (6) for \( C \in \mathcal{C}_w(b) \) such that \( k \in C \) gives
\[
\sum_{i \in C: i \leq k-1} (x_i - y_i) + \sum_{i \in C: i > k} x_i \leq |C^k| - 1 + (|C| - |C^k|)x_n,
\]
which is redundant since \( x_i - y_i \leq x_n \) for all \( i \leq k-1 \) and \( x_i \leq 1 \) for all \( i > k-1 \).

Thus, \( P^{k-1} \) has the desired form, which completes the proof of the claim. \( \diamond \)

For \( k = 0 \), we have that \( P^0 = \text{proj}_e(Q) \) has the required form, which completes the proof of the proposition. \( \square \)

4 Description of \( \text{conv}(B \setminus V) \)

Recall the definition \( w^*_i := 2^{i-1} \) for all \( i \in \mathcal{N} \). For each \( k \in \mathcal{N} \), let \( N^k := \{i \in \mathcal{N} : v^k_i = 1\} \). From (2) and (3), we have
\[
\mathcal{C}_{w^*}(k) = \{\{i\} \cup \{j \in N^k : j > i\} : i \notin N^k\}.
\]

Similarly, we have
\[
\mathcal{R}_{w^*}(k) = \{\{i\} \cup \{j \notin N^k : j > i\} : i \in N^k\}.
\]
Thus, for $0 \leq a \leq b \leq 2^n - 1$, applying Theorem 2 yields

$$
\text{conv}(K(a, b)) = \text{conv}(K_w(a, b)) = \left\{ x \in [0, 1]^n : \begin{align*}
    x_i + \sum_{j \notin N_i, j > i} x_j &\geq 1 & \forall i \in N^a \\
    x_i + \sum_{j \in N_i, j > i} x_j &\leq c_i^b & \forall i \notin N^b
\end{align*} \right\},
$$

where $c_i^k := |\{j \in N^k : j > i\}|$ for each $k \in \mathcal{N}$ and $i \notin N^k$. Finally, from the above result and [1], we have that $\text{conv}(B \setminus V)$ can be described by an extended formulation having $O(m \cdot n)$ variables and constraints.

References


