Forbidding extreme points from the $0-1$ hypercube

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Abstract

Let B be the set of n-dimensional binary vectors and let V be a subset of m of its elements. We give an extended formulation of the convex hull of $B \setminus V$ which is polynomial in n and m. In developing this result, we give a two-sided extension of the result in [3] for knapsack sets with superincreasing coefficients.

1 Introduction

We consider the following problem. Let $B := \{0,1\}^n$ and let $V \subseteq B$ contain m elements which are given as a list. Is it possible to give a description of $conv(B \setminus V)$ which is polynomial in n and m? We give a positive answer to this question by presenting an extended formulation involving $\mathcal{O}(mn)$ variables and constraints. Our approach relies on being able to give the convex hull of binary knapsack sets having a superincreasing weight. This generalizes a result due to [3][1]

2 Binary expansion

Let $N := \{1, \dots, n\}$ and $\mathcal{N} := \{0, \dots, 2^n - 1\}$. There exists a bijection between B and $\mathcal{N}$ given by the mapping $\sigma(v) := \sum_{i \in N} 2^{i-1} v_i$ for all $v \in B$. Therefore, we can write $B = \{v^0, \dots, v^{2^n-1}\}$, where v^k gives the binary expansion of k for each $k \in \mathcal{N}$, that is, $v^k = \sigma^{-1}(k)$. Let $V = \{v^{k_1}, \dots, v^{k_m}\}$, where without loss of generality we assume $k_l < k_{l+1}$ for all $l = 1, \dots, m-1$. Also, let $\mathcal{N}_V := \{k \in \mathcal{N} : \ v^k \in V\}$. Then we have

$$B \setminus V = \left\{ x \in B : \ \sum_{i \in N} 2^{i-1} x_i \notin \mathcal{N}_V \right\}.$$

Now, for integers a and b, let

$$K(a,b) = \left\{ x \in B : \ a \le \sum_{i \in N} 2^{i-1} x_i \le b \right\}.$$

[1]After completion of this note, it was brought to our attention that recently Muldoon et al.[4] independently provided a convex hull description of two-sided binary knapsack sets with superincreasing weights.

If $b < a$, then $K(a, b)$ is empty. Set $k_0 = -1$ and $k_{m+1} = 2^n$. Then we can write

$$B \setminus V = \bigcup_{l=0}^{m} K(k_l + 1, k_{l+1} - 1).$$

Thus

$$conv\,(B \setminus V) = conv\left(\bigcup_{l=0}^{m} K(k_l + 1, k_{l+1} - 1)\right) = conv\left(\bigcup_{l=0}^{m} conv(K(k_l + 1, k_{l+1} - 1))\right). \quad (1)$$

Therefore, if we have a polynomial description of $conv(K(a, b))$ for any $a, b \in \mathcal{N}$, then we have a compact extended formulation for $conv(B \setminus V)$ as the convex hull of the union of $m+1$ polytopes as given in [1].

3 Superincreasing two-sided knapsack sets

Definition 1. *A vector $w \in \mathbb{Z}^n_+$ is a superincreasing sequence if $\sum_{j<i} w_j \leq w_i$ for all $i \in N$.*

A prime example of a superincreasing sequence is given by $w = w^*$, where $w^*_i := 2^{i-1}$ for all $i \in N$.

Let $w \in \mathbb{Z}^n_+$. For integers $0 \leq a \leq b \leq \sum_{i \in N} w_i$, let

$$K^{\geq}_w(a) := \left\{ x \in \{0,1\}^n : a \leq \sum_{i \in N} w_i x_i \right\},$$

$$K^{\leq}_w(b) := \left\{ x \in \{0,1\}^n : \sum_{i \in N} w_i x_i \leq b \right\},$$

$$K_w(a, b) := \left\{ x \in \{0,1\}^n : a \leq \sum_{i \in N} w_i x_i \leq b \right\} = K^{\geq}_w(a) \cap K^{\leq}_w(b).$$

We say that $C \subseteq N$ is a minimal cover of b with respect to w if $\sum_{i \in C} w_i > b$ and $\sum_{i \in C'} w_i \leq b$ for all $C' \subsetneq C$. In [3] it is shown that if $w \in \mathbb{Z}^n_+$ is a superincreasing sequence, then the non-trivial facets of the convex hull of $K^{\leq}_w(b)$ are all minimal cover inequalities, that is

$$conv(K^{\leq}_w(b)) = \left\{ x \in [0,1]^n : \sum_{i \in C} x_i \leq |C| - 1 \ \forall C \in \mathcal{C}_w(b) \right\},$$

where $\mathcal{C}_w(b) \subseteq 2^N$ is the set of all minimal covers of b with respect to w. It is not hard to verify that an analogous result holds for the convex hull of $K^{\geq}_w(a)$. We say that $R \subseteq N$ is a minimal reverse cover of a with respect to w if $\sum_{i \in N \setminus R} w_i < a$ and $\sum_{i \in N \setminus R'} w_i \geq a$ for all $R' \subsetneq R$. Then we have

$$conv(K^{\geq}_w(a)) = \left\{ x \in [0,1]^n : \sum_{i \in R} x_i \geq 1 \ \forall R \in \mathcal{R}_w(a) \right\},$$

where $\mathcal{R}_w(a) \subseteq 2^N$ is the set of all minimal reverse covers of a with respect to w. We show that

$$conv(K_w(a, b)) = conv\left(K^{\geq}_w(a)\right) \cap conv\left(K^{\leq}_w(b)\right).$$

Theorem 2. *If $w \in \mathbb{Z}_+^n$ is a superincreasing sequence, then for all integers $0 \le a \le b \le \sum_{i \in N} w_i$, we have*

$$conv(K_w(a,b)) = \left\{ x \in [0,1]^n : \begin{array}{rl} \sum_{i \in R} x_i \ge 1 & \forall R \in \mathcal{R}_w(a) \\ \sum_{i \in C} x_i \le |C| - 1 & \forall C \in \mathcal{C}_w(b) \end{array} \right\}.$$

Proof. We prove the result by induction on n. When $n = 1$, the result is trivial. Fix $n > 1$ and assume the result is true for $n - 1$. Let $w \in \mathbb{Z}_+^n$ be a superincreasing sequence, and let $0 \le a \le b \le \sum_{i \in N} w_i$. Consider the set

$$K_w(a,b) = \left\{ x \in \{0,1\}^n : \ a \le \sum_{i \in N} w_i x_i \le b \right\}.$$

Let $\bar{w} \in \mathbb{Z}_+^{n-1}$ be the vector obtained by removing the last component from w. Clearly, $\bar{w}$ is a superincreasing sequence.

Case 1: $w_n > b$.

If $w_n > b$, then $x_n = 0$ for all $x \in K_w(a,b)$. Thus

$$K_w(a,b) = \left\{ x \in \{0,1\}^n : \ a \le \sum_{i<n} w_i x_i \le b, \ x_n = 0 \right\} = K_{\bar{w}}(a,b) \times \{0\}.$$

Notice that $\widehat{C} := \{n\}$ is a minimal cover of b with respect to w, and therefore n cannot be contained in any other cover in $\mathcal{C}_w(b)$. Then we have $\mathcal{C}_w(b) = \mathcal{C}_{\bar{w}}(b) \cup \{\widehat{C}\}$. We also have $\bar{R} \cup \{n\} \in \mathcal{R}_w(a)$ for all $\bar{R} \in \mathcal{R}_{\bar{w}}(a)$. On the other hand, if $R \in \mathcal{R}_w(a)$ and $n \notin R$, then $a > \sum_{N \setminus R} w_i \ge w_n \ge b$, a contradiction. Furthermore, if $\widetilde{R} \subseteq N \setminus \{n\}$ and $\widetilde{R} \notin \mathcal{R}_{\bar{w}}(a)$, then $\widetilde{R} \cup \{n\} \notin \mathcal{R}_w(a)$. Thus $\mathcal{R}_w(a) = \{\bar{R} \cup \{n\} : \ \bar{R} \in \mathcal{R}_{\bar{w}}(a)\}$. Hence, by the inductive hypothesis, we have

$$\begin{aligned} conv(K_w(a,b)) &= conv(K_{\bar{w}}(a,b) \times \{0\}) \\ &= \left\{ x \in [0,1]^n : \begin{array}{rl} \sum_{i \in \bar{R}} x_i \ge 1 & \forall \bar{R} \in \mathcal{R}_{\bar{w}}(a) \\ \sum_{i \in \bar{C}} x_i \le |\bar{C}| - 1 & \forall \bar{C} \in \mathcal{C}_{\bar{w}}(b) \\ x_n = 0 & \end{array} \right\} \\ &= \left\{ x \in [0,1]^n : \begin{array}{rl} \sum_{i \in \bar{R}} x_i + x_n \ge 1 & \forall \bar{R} \in \mathcal{R}_{\bar{w}}(a) \\ \sum_{i \in \bar{C}} x_i \le |\bar{C}| - 1 & \forall \bar{C} \in \mathcal{C}_{\bar{w}}(b) \\ x_n \le 0 & \end{array} \right\} \\ &= \left\{ x \in [0,1]^n : \begin{array}{rl} \sum_{i \in R} x_i \ge 1 & \forall R \in \mathcal{R}_w(a) \\ \sum_{i \in C} x_i \le |C| - 1 & \forall C \in \mathcal{C}_w(b) \end{array} \right\}. \end{aligned}$$

Since the result holds for $w_n > b$, we may assume $w_n \le b$. Notice that this implies $n \in C$ for all $C \in \mathcal{C}_w(b)$.

Case 2: $a > \sum_{i<n} w_i, \ w_n \le b$.

If $a > \sum_{i<n} w_i$, then $x_n = 1$ for all $x \in K_w(a,b)$. Thus

$$K_w(a,b) = \left\{ x \in \{0,1\}^n : \; a - w_n \le \sum_{i<n} w_i x_i \le b - w_n, \; x_n = 1 \right\} = K_{\bar{w}}(a - w_n, b - w_n) \times \{1\}.$$

It is not hard to verify that $\mathcal{C}_w(b) = \{\bar{C} \cup \{n\} : \; \bar{C} \in \mathcal{C}_{\bar{w}}(b - w_n)\}$. Also, since $\sum_{i<n} w_i < a$ and $\sum_{i \in N} w_i \ge a$, we have that $\widehat{R} := \{n\}$ is a minimal reverse cover of a with respect to w. Therefore n cannot be contained in any other reverse cover in $\mathcal{R}_w(a)$. Then it is straightforward to verify that $\mathcal{R}_w(a) = \mathcal{R}_{\bar{w}}(a - w_n) \cup \{\widehat{R}\}$. Hence, by the inductive hypothesis, we have

$$\begin{aligned}
conv(K_w(a,b)) &= conv(K_{\bar{w}}(a - w_n, b - w_n) \times \{1\}) \\
&= \left\{ x \in [0,1]^n : \begin{array}{rl} \sum_{i \in \bar{R}} x_i \ge 1 & \forall \bar{R} \in \mathcal{R}_{\bar{w}}(a - w_n) \\ \sum_{i \in \bar{C}} x_i \le |\bar{C}| - 1 & \forall \bar{C} \in \mathcal{C}_{\bar{w}}(b - w_n) \\ x_n = 1 & \end{array} \right\} \\
&= \left\{ x \in [0,1]^n : \begin{array}{rl} \sum_{i \in \bar{R}} x_i \ge 1 & \forall \bar{R} \in \mathcal{R}_{\bar{w}}(a - w_n) \\ \sum_{i \in \bar{C}} x_i + x_n \le |\bar{C} \cup \{n\}| - 1 & \forall \bar{C} \in \mathcal{C}_{\bar{w}}(b - w_n) \\ x_n \ge 1 & \end{array} \right\} \\
&= \left\{ x \in [0,1]^n : \begin{array}{rl} \sum_{i \in R} x_i \ge 1 & \forall R \in \mathcal{R}_w(a) \\ \sum_{i \in C} x_i \le |C| - 1 & \forall C \in \mathcal{C}_w(b) \end{array} \right\}.
\end{aligned}$$

Since the result holds for $a > \sum_{i<n} w_i$, we may assume $a \le \sum_{i<n} w_i$. Notice that this implies $n \in R$ for all $R \in \mathcal{R}_w(a)$.

Let $w_0 := \sum_{i<n} w_i$. We have $w_0 \le w_n$.

Case 3: $a \le w_0 \le w_n \le b$.

Consider the sets

$$\begin{aligned}
K_w^0(a,b) &:= K_w(a,b) \cap \{x \in \mathbb{R}^n : \; x_n = 0\} = K_w(a, w_0) \cap \{x \in \mathbb{R}^n : \; x_n = 0\} = K_{\bar{w}}(a, w_0) \times \{0\}, \\
K_w^1(a,b) &:= K_w(a,b) \cap \{x \in \mathbb{R}^n : \; x_n = 1\} = K_w(w_n, b) \cap \{x \in \mathbb{R}^n : \; x_n = 1\} = K_{\bar{w}}(0, b - w_n) \times \{1\},
\end{aligned}$$

which are nonempty. Notice that $\mathcal{C}_{\bar{w}}(w_0) = \emptyset$ and $\mathcal{R}_{\bar{w}}(0) = \emptyset$. By the inductive hypothesis, we have

$$conv(K_w^0(a,b)) = \left\{ x \in [0,1]^n : \begin{array}{rl} \sum_{i \in \bar{R}} x_i \ge 1 & \forall \bar{R} \in \mathcal{R}_{\bar{w}}(a) \\ x_n = 0 & \end{array} \right\},$$

$$conv(K_w^1(a,b)) = \left\{ x \in [0,1]^n : \begin{array}{rl} \sum_{i \in \bar{C}} x_i \le |\bar{C}| - 1 & \forall \bar{C} \in \mathcal{C}_{\bar{w}}(b - w_n) \\ x_n = 1 & \end{array} \right\}.$$

Furthermore, by arguments similar to that of Cases 1 and 2, we obtain

$$conv(K_w^0(a,b)) = \left\{ x \in [0,1]^n : \begin{array}{rl} \sum_{i \in R} x_i \ge 1 & \forall R \in \mathcal{R}_w(a) \\ x_n = 0 & \end{array} \right\},$$

$$conv(K^1_w(a,b)) = \left\{ x \in [0,1]^n : \begin{array}{ll} \sum_{i \in C} x_i \leq |C| - 1 & \forall C \in \mathcal{C}_w(b) \\ x_n = 1 & \end{array} \right\}.$$

Clearly,

$$conv(K_w(a,b)) = conv\left(conv(K^0_w(a,b)) \cup conv(K^1_w(a,b))\right).$$

Let $Q \subseteq \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ be the polyhedron given by

$$\begin{aligned}
x_i = y_i + z_i & \quad \forall i \\
\sum_{i \in R} y_i \geq \lambda & \quad \forall R \in \mathcal{R}_w(a) \\
y_i \geq 0 & \quad \forall i \leq n-1 \\
y_i \leq \lambda & \quad \forall i \leq n-1 \\
y_n = 0 & \\
\sum_{i \in C} z_i \leq (|C|-1)(1-\lambda) & \quad \forall C \in \mathcal{C}_w(b) \\
z_i \geq 0 & \quad \forall i \leq n-1 \\
z_i \leq 1-\lambda & \quad \forall i \leq n-1 \\
z_n = 1-\lambda & \\
\lambda \geq 0 & \\
\lambda \leq 1. &
\end{aligned}$$

From [1], we have that $conv(K_w(a,b)) = proj_x(Q)$. Eliminating z from Q using $z_i = x_i - y_i$ for all $i \in N$, we obtain $proj_{x,y,\lambda}(Q)$ as

$$\begin{aligned}
\sum_{i \in R} y_i \geq \lambda & \quad \forall R \in \mathcal{R}_w(a) \\
y_i \geq 0 & \quad \forall i \leq n-1 \\
y_i \leq \lambda & \quad \forall i \leq n-1 \\
y_n = 0 & \\
\sum_{i \in C} (x_i - y_i) \leq (|C|-1)(1-\lambda) & \quad \forall C \in \mathcal{C}_w(b) \\
x_i - y_i \geq 0 & \quad \forall i \leq n-1 \\
x_i - y_i \leq 1-\lambda & \quad \forall i \leq n-1 \\
x_n - y_n = 1-\lambda & \\
\lambda \geq 0 & \\
\lambda \leq 1. &
\end{aligned}$$

Eliminating λ from $proj_{x,y,\lambda}(Q)$ using $\lambda = 1 - x_n + y_n$, we obtain $proj_{x,y}(Q)$ as

$$\begin{aligned}
\sum_{i \in R} y_i &\geq 1 - x_n + y_n \quad && \forall R \in \mathcal{R}_w(a) \\
y_i &\geq 0 && \forall i \leq n-1 \\
y_i &\leq 1 - x_n + y_n && \forall i \leq n-1 \\
y_n &= 0 && \\
\sum_{i \in C}(x_i - y_i) &\leq (|C| - 1)(x_n - y_n) && \forall C \in \mathcal{C}_w(b) \\
x_i - y_i &\geq 0 && \forall i \leq n-1 \\
x_i - y_i &\leq x_n - y_n && \forall i \leq n-1 \\
1 - x_n + y_n &\geq 0 && \\
1 - x_n + y_n &\leq 1. &&
\end{aligned}$$

Let $P^n := proj_{x,y}(Q)$. For each $k \in \{n-1, \dots, 0\}$, let P^k be the polyhedron obtained after projecting out y_{k+1} from P^{k+1}. Also, for each $C \in \mathcal{C}_w(b)$ and $k \in \{n-1, \dots, 0\}$, let $C^k := \{i \in C : \ i > k\}$.

Claim: For each $k \in \{n-1, \dots, 0\}$, P^k is given by

$$\sum_{i \in R:\ i \leq k} y_i + \sum_{i \in R:\ i > k} x_i \geq 1 \quad \forall R \in \mathcal{R}_w(a) \tag{2}$$

$$y_i \geq 0 \quad \forall i \leq k \tag{3}$$

$$y_i \leq 1 - x_n \quad \forall i \leq k \tag{4}$$

$$\sum_{i \in C:\ i \leq k} (x_i - y_i) + \sum_{i \in C:\ i > k} x_i \leq |C^k| - 1 + (|C| - |C^k|)x_n \quad \forall C \in \mathcal{C}_w(b) \tag{5}$$

$$x_i - y_i \geq 0 \quad \forall i \leq k \tag{6}$$

$$x_i - y_i \leq x_n \quad \forall i \leq k \tag{7}$$

$$x_i \geq 0 \quad \forall i > k$$

$$x_i \leq 1 \quad \forall i > k.$$

We verify the claim for $k = n-1$. Following Assumptions 1 and 2, $n \in C$ for all $C \in \mathcal{C}_w(b)$ and $n \in R$ for all $R \in \mathcal{R}_w(a)$. Given that $y_n = 0$ and $C^{n-1} = \{n\}$ for all $C \in \mathcal{C}_w(b)$, we obtain P^{n-1} as

$$\begin{aligned}
\sum_{i \in R:\ i \leq n-1} y_i + \sum_{i \in R:\ i > n-1} x_i &\geq 1 \quad && \forall R \in \mathcal{R}_w(a) \\
y_i &\geq 0 && \forall i \leq n-1 \\
y_i &\leq 1 - x_n && \forall i \leq n-1 \\
\sum_{i \in C:\ i \leq n-1} (x_i - y_i) + \sum_{i \in C:\ i > n-1} x_i &\leq (|C| - 1)x_n && \forall C \in \mathcal{C}_w(b) \\
x_i - y_i &\geq 0 && \forall i \leq n-1 \\
x_i - y_i &\leq x_n && \forall i \leq n-1 \\
x_n &\geq 0 && \\
x_n &\leq 1. &&
\end{aligned}$$

Now, fix $1 \leq k < n$ and assume the claim holds for k. We want to show that after projecting out y_k from P^k we obtain P^{k-1} having the desired form. We check the inequalities obtained by appropriately combining (2)-(7). We begin combining (3), (4), (6), and (7) for $i = k$.

(3),(4) gives $x_n \le 1$, which is redundant.

(6),(7) gives $x_n \ge 0$, which is redundant.

(3),(6) gives $x_k \ge 0$.

(4),(7) gives $x_k \le 1$.

Inequalities (3), (4), (6), and (7) remain for $i < k$. Inequalities (2) and (5) remain if $R \notin \mathcal{R}_w(a)$ and $C \notin \mathcal{C}_w(b)$, respectively. Notice that (2) and (5) cannot be combined. The remaining cases are the following.

(2),(4) for $R \in \mathcal{R}_w(a)$ such that $k \in R$ gives

$$\sum_{i \in R:\ i \le k-1} y_i + \sum_{i \in R:\ n > i > k} x_i \ge 0,$$

which is redundant since $y_i \ge 0$ for all $i \le k-1$ and $x_i \ge 0$ for all $i > k-1$.

(2),(6) for $R \in \mathcal{R}_w(a)$ such that $k \in R$ gives

$$\sum_{i \in R:\ i \le k-1} y_i + \sum_{i \in R:\ i > k-1} x_i \ge 1.$$

(5),(4) for $C \in \mathcal{C}_w(b)$ such that $k \in C$ gives

$$\sum_{i \in C:\ i \le k-1} (x_i - y_i) + \sum_{i \in C:\ i > k-1} x_i \le |C^k| + (|C| - |C^k| - 1)x_n = |C^{k-1}| - 1 + (|C| - |C^{k-1}|)x_n,$$

where we have used that $k \in C$ implies $k \notin C^k$ and $C^{k-1} = C^k \cup \{k\}$.

(5),(6) for $C \in \mathcal{C}_w(b)$ such that $k \in C$ gives

$$\sum_{i \in C:\ i \le k-1} (x_i - y_i) + \sum_{i \in C:\ i > k} x_i \le |C^k| - 1 + (|C| - |C^k|)x_n,$$

which is redundant since $x_i - y_i \le x_n$ for all $i \le k-1$ and $x_i \le 1$ for all $i > k-1$.

Thus, P^{k-1} has the desired form, which completes the proof of the claim. $\Diamond$

For $k = 0$, we have that $P^0 = proj_x(Q)$ has the required form, which completes the proof of the proposition. $\square$

4 Description of $conv(B \setminus V)$

Recall the definition $w_i^* := 2^{i-1}$ for all $i \in N$. For each $k \in \mathcal{N}$, let $N^k := \{i \in N :\ v_i^k = 1\}$. From [2] and [3], we have

$$\mathcal{C}_{w^*}(k) = \left\{ \{i\} \cup \{j \in N^k :\ j > i\} :\ i \notin N^k \right\}.$$

Similarly, we have

$$\mathcal{R}_{w^*}(k) = \left\{ \{i\} \cup \{j \notin N^k :\ j > i\} :\ i \in N^k \right\}.$$

Thus, for $0 \leq a \leq b \leq 2^n - 1$, applying Theorem 2 yields

$$conv(K(a,b)) = conv(K_{w^*}(a,b)) = \left\{ x \in [0,1]^n : \begin{array}{ll} x_i + \sum_{j \notin N^a:\ j>i} x_j \geq 1 & \forall i \in N^a \\ x_i + \sum_{j \in N^b:\ j>i} x_j \leq c_i^b & \forall i \notin N^b \end{array} \right\},$$

where $c_i^k := |\{j \in N^k : \ j > i\}|$ for each $k \in \mathcal{N}$ and $i \notin N^k$. Finally, from the above result and (1), we have that $conv(B \setminus V)$ can be described by an extended formulation having $\mathcal{O}(mn)$ variables and constraints.

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