About the Convergence of Parametric Sum of Squares Relaxations

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Abstract

In this paper we look at a parametric polynomial optimization problem. We assume, that the parameter space, and, for each parameter, the feasible set is compact. We show, that under these assumptions, the optimal solution of the sum of squares relaxations, using Schmüdgen’s Positivstellensatz, converge uniformly over the parameter space to the optimal value functions. The proof’s key idea is to use the bound obtained in Schweighofer (2004), and generalize it to the parametric case. By adding some additional constraints, we obtain as a consequence explicit bounds on the degree of the relaxation, i.e. there are no unknown constants contained in it.

Keywords: parametric optimization, Schmüdgen’s positivstellensatz, convergence rate

1 Introduction

In this paper we study the convergence properties of the relaxation methods first introduced by Lasserre (2001) and Parrilo (2000). They showed, if we have a polynomial optimization problem over a compact semi-algebraic set, then we can relax this to a semi-definite optimization problem. The optimal value thus obtained is a lower bound on the real objective. The key algebraic tools are the Positivstellensätze, which give sum of squares representations of positive polynomials over certain sets.

We are looking at relaxations obtained by using Schmüdgen’s Positivstellensatz, see Schmüdgen (1991). We are interested in the convergence properties of the relaxation to the following parametric problem.

\[ f^*(p) = \min_{x} f(x, p) \]
\[ \text{s.t. } g_i(x, p) \geq 0 \quad i = 1, \ldots, m \]

Given that the feasible set is nonempty and compact, we can relax this problem, using

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the Positivstellensatz, to the following semi-definite optimization problem.

\[
 f^*_k(p) = \sup_{\rho, \sigma} \rho \\
s.t. \ f(X, p) - \rho = \sum_{\delta \in \{0, 1\}^m} \sigma_\delta g_1^\delta(X, p) \cdots g_m^\delta(X, p) \\
\sigma_\delta \in \Sigma[X] \\
\deg(\sigma_\delta g_1^\delta(X, p) \cdots g_m^\delta(X, p)) \leq k \text{ for all } \delta \in \{0, 1\}^m.
\]

The dual version of this problem has been studied by Lasserre (2010). There Lasserre proves that a piecewise polynomial function converges almost uniformly to the optimal value function, over a compact parameter set. We show that the optimal value function \( f^*_k(p) \) converges uniformly to \( f^*(p) \) over a compact semi-algebraic set \( P \). Furthermore given that the constraint set contains polynomials \( 1 - \varepsilon + X_i, 1 - \varepsilon - X_i \) for all \( i \) and a constraint \( 2n\varepsilon - \sum_i g_i \), then we obtain explicit bounds on the relaxation order \( k \). The fundamental idea to prove this is to generalize the bounds obtained by Schweighofer (2004) to the parametric case.

This result is of interest to us, since we studied generalized Nash equilibria in Couzoudis and Renner (forthcoming). We found a way to compute them in the non-convex case, by using parametrized sum of squares relaxations. The main idea was to use such a relaxation for each agent, instead of the usual first order conditions. With the uniform convergence, we now can conclude that our approximation scheme, for a sufficiently high relaxation order, captures all equilibria.

2 Main result

We are using the following notation and conventions:

- We denote the ring of real polynomials as \( \mathbb{R}[X_1, \ldots, X_n] = \mathbb{R}[X] \),
- a monomial \( X_1^{\alpha_1} \cdots X_n^{\alpha_n} \) as \( X^\alpha \),
- the set of sum of squares in \( X \) as \( \Sigma[X] \),
- \( |\alpha| = |\alpha_1| + \ldots + |\alpha_n| \),
- for a polynomial \( f = \sum_\alpha a_\alpha x^\alpha \) with coefficients \( a_\alpha \in \mathbb{R} \), set
  \[
  \|f\| = \max_\alpha \frac{|a_\alpha|}{|\alpha|!^{\alpha_1 \cdots \alpha_n}},
  \]
- for \( f \in \mathbb{R}[X, P] \) set \( \deg_X f \) the degree of \( f \) in the polynomial ring \( R[X] \), where \( R = \mathbb{R}[P] \), and
- set \( \deg(0) = -1 \).

To prove the convergence rate in a parametric setting, we look at the following theorem from Schweighofer (2004), and generalize it.
Theorem 1. (Schweighofer, 2004, Th.3) For all polynomials \( g_1, \ldots, g_m \in \mathbb{R}[X_1, \ldots, X_n] \) such that the set

\[
K = \{ x \in \mathbb{R}^n \mid g_1(x) \geq 0, \ldots, g_m(x) \geq 0 \} \subset (-1, 1)^n
\]
is non-empty, there is some \( c \in \mathbb{N} \) with the following property:

Any \( f \in \mathbb{R}[X] \) of \( \deg f = d \geq 1 \) with \( f^* = \min_{x \in K} f(x) > 0 \) can be written as

\[
\sum_{\delta \in \{0,1\}^m} \sigma_\delta g_1^{\delta_1} \cdots g_m^{\delta_m} \quad \text{where} \quad \sigma_\delta \in \Sigma[X],
\]

such that

\[
\deg(\sigma_\delta g_1^{\delta_1} \cdots g_m^{\delta_m}) \leq cd^2 \left( 1 + \left( \frac{d^2 n^d \|f^*\|}{f^*} \right)^c \right) \quad \text{for all} \quad \delta \in \{0,1\}^m.
\]

In the parametric setting, we obtain a different bound on the complexity. This is mainly the case, since we have to use a different version of Pólya’s theorem. So theorem 1 in the parametric case looks as follows.

Theorem 2. Let \( P \subset \mathbb{R}^n \) be a compact semi-algebraic set. For polynomials \( g_1, \ldots, g_m \in \mathbb{R}[X_1, \ldots, X_n, P_1, \ldots, P_s] \) such that the set

\[
K(P) = \{ x \in \mathbb{R}^n \mid g_1(x, p) \geq 0, \ldots, g_m(x, p) \geq 0 \} \subset (-1, 1)^n
\]
is non-empty for all \( p \in P \). Furthermore assume that \( \deg g_1 \geq \ldots \geq \deg g_m \). Then there is some \( c \in \mathbb{N} \) with the following property:

Let \( \tilde{p} \in P \). Any \( f \in \mathbb{R}[X] \) of \( \deg f = d \geq 1 \) with \( f^*(\tilde{p}) = \min_{x \in K(\tilde{p})} f(x) > 0 \) can be written as

\[
\sum_{\delta \in \{0,1\}^m} \sigma_\delta g_1^{\delta_1}(X, \tilde{p}) \cdots g_m^{\delta_m}(X, \tilde{p}) \quad \text{where} \quad \sigma_\delta \in \Sigma[X],
\]

such that

\[
\deg(\sigma_\delta g_1^{\delta_1}(X, \tilde{p}) \cdots g_m^{\delta_m}(X, \tilde{p})) \leq \frac{8(2n)^D \ln D}{f^*} + D(2n + m + 1) + 1 + c \quad \forall \delta \in \{0,1\}^m,
\]

where

\[
D = \max \left\{ d, 2 \left( \frac{1}{2} \prod_j \deg g_j + \deg g_1 \right)^{\frac{2n+m+1}{2}} \right\}.
\]

The large \( D \) stems from the degree of an elimination ideal’s generators. Since, in general, there are no known bounds for the minimal degree of the generating polynomials, we have to use bounds on Gröbner bases.

We can give the bound more explicitly, if we add some additional constraints. More precisely we can simply set \( c \) to 0.

Corollary 3. Let the assumptions of theorem 2 hold. Furthermore let \( \varepsilon > 0 \) such that \( K(p) \subset [-1 + \varepsilon, 1 + \varepsilon]^n \) for all \( p \in P \). Now scale \( g_j(X, p) \), such that \( 2n\varepsilon - \sum_j g_j(X, p) \geq 0 \) for all feasible \((X, p)\). Then, we add the following constraints

\[
q_1 = 1 - \varepsilon + X_1, \ldots, q_n = 1 - \varepsilon + X_n,
\]

\[
q_{n+1} = 1 - \varepsilon - X_1, \ldots, q_{2n} = 1 - \varepsilon - X_n,
\]

\[
q_{2n+1} = 2n\varepsilon - \sum_i g_i.
\]
Thus the degree bound from theorem 2 is satisfied and so $F$ equals an expression of the form

$$F = \max \left\{ d, 2 \left( \frac{1}{2} \deg g_1 \right)^2 \prod_{j \geq 2} \deg g_j + \deg g_1 \right\}^{2^{4n+m+1}}. $$

Now the bound on the degree is

$$\frac{8(2n)^D \ln D}{f^*} + D(4n + m + 2) + 1$$

Proof. Follows directly from the way $c$ is chosen in the proof of theorem 2.

Corollary 4. Let $P \subset \mathbb{R}^s$ be a compact semi-algebraic set. Furthermore let $g_1, \ldots, g_m \in \mathbb{R}[X_1, \ldots, X_n, P_1, \ldots, P_s]$ define the set valued mapping

$$K(p) = \{ x \in \mathbb{R}^n \mid g_1(x, p) \geq 0, \ldots, g_m(x, p) \geq 0 \} \subset (-1, 1)^n,$$

with $K(p)$ a nonempty compact set. Assume that $\deg g_1 \geq \ldots \geq \deg g_m$. Then there is some $c \in \mathbb{N}$ with the following property:

Every $f \in \mathbb{R}[X, P]$ with $\deg_X f(X, \tilde{p}) \geq 1$ for all $\tilde{p} \in P$, for all $k \in \mathbb{N}$ with $k \geq 2D(2n + m + 1) + 2 + 2c$ and for all $\tilde{p} \in P$, the polynomial

$$(f(X, \tilde{p}) - f^*(\tilde{p})) + 8(2n)^D \ln D \frac{2}{k}$$

equals an expression of the form

$$\sum_{\delta \in \{0, 1\}^m} \sigma_\delta g_1^{\delta_1}(X, \tilde{p}) \cdots g_m^{\delta_m}(X, \tilde{p}) \quad \text{with} \quad \sigma_\delta \in \Sigma[X].$$

Where $D$ is defined as in theorem 2, $f^*(\tilde{p}) = \min_{x \in K(\tilde{p})} f(x, \tilde{p})$ and

$$\deg(\sigma_\delta g_1^{\delta_1}(X, \tilde{p}) \cdots g_m^{\delta_m}(X, \tilde{p})) \leq k \quad \text{for all} \ \delta \in \{0, 1\}^m.$$

Proof. Denote the polynomial (5) by $F(X, \tilde{p})$ and let $c$ be the constant, that exists due to theorem 2. Note that we have $\min_{x \in K(\tilde{p})} F(x, \tilde{p}) = 8(2n)^D \ln D \frac{2}{k}$. So for

$$k = 2 \geq D(2n + m + 1) + 1 + c \quad \Leftrightarrow \quad k - D(2n + m + 1) - 1 - c \geq \frac{8(2n)^D \ln D}{\min_{x \in K(\tilde{p})} F(x, \tilde{p})}$$

Thus the degree bound from theorem 2 is satisfied and so $F(X, \tilde{p})$ has a representation (6) with degree bound $k$.

As a consequence of this we get that the relaxation method, using Schmüdgen’s Positivstellensatz, parametrized over a compact semi-algebraic set, converges uniformly to the global optimal solution.
Theorem 5. Let $P \subset \mathbb{R}^s$ be a compact semi-algebraic set. Furthermore let $g_1, \ldots, g_m \in \mathbb{R}[X_1, \ldots, X_n, P_1, \ldots, P_s]$ define the set valued mapping

$$K(p) = \{x \in \mathbb{R}^n \mid g_1(x, p) \geq 0, \ldots, g_m(x, p) \geq 0\},$$

with $K(p)$ a nonempty compact set and let $k \in \mathbb{N}$ sufficiently large. Look at the following parametrized optimization problem

$$f_k^*(p) = \sup_{\rho, \sigma_\delta} \rho$$

s.t. $f(X, p) - \rho = \sum_{\delta \in \{0, 1\}^m} \sigma_\delta g_1^{\delta_1}(X, p) \cdots g_m^{\delta_m}(X, p)$

$$\sigma_\delta \in \Sigma[X]$$

$$\deg(\sigma_\delta g_1^{\delta_1}(X, p) \cdots g_m^{\delta_m}(X, p)) \leq k \quad \text{for all } \delta \in \{0, 1\}^m.$$  \hfill (7)

Then $f_k^*(p)$ converges uniformly to $f^*(p) = \min_{x \in K(p)} f(x, p)$ on $P$ for $k \to \infty$.

Proof. Since $K(P)$ is a compact set, we can scale it such that $K(P) \subset (-1, 1)^n \times P$. Thus, defining $D$ just as in (3), by theorem 4 we know that there is a $c$ such that for any $k \geq 2(2n+1)^2 + 2+2c$ and any $p \in P$ the polynomial $(f(X, \tilde{p}) - f^*(\tilde{p}) + 8(2n)^D \ln D^2_k$ has a representation as in (6). In particular for all $p \in P$ the gap between $f^*(p)$ and $f_k^*(p)$ is bounded by $8(2n)^D \ln D^2_k$, which goes to 0 for $k \to \infty$. \hfill $\square$

3 Proof of main theorem

3.1 Preliminaries

Before we can start proving our main result, we need some additional theorems.

The next theorem, Lojasiewicz’s inequality, is, as in Schweighofer (2004), necessary for our proof.

Theorem 6. (Bochnak et al., 1998, Prop.2.6.7) Let $A$ be a locally closed semi-algebraic set and $f, g : A \to \mathbb{R}$ continuous semi-algebraic functions, such that $f^{-1}(0) \subseteq g^{-1}(0)$. Then there exists an integer $N > 0$ and a constant $c$, such that $|g|^N \leq c|f|$ on $A$.

The important theorem, where we deviate from the original proof, is this different bound for Pólya’s theorem.

Theorem 7. (de Loera and Santos, 1996, 2001, Th 1.2) Let $F(X_1, \ldots, X_n)$ be a real homogeneous polynomial of degree $d$. Suppose that $F$ is strictly positive on the non-negative orthant (minus the origin). Denote $F^*$ the minimum of the function $F$ on the unit simplex $\Delta_1 = \{x \in \mathbb{R}^n \mid \sum_{i=1}^n x_i = 1, x_i \geq 0 \forall i\}$. Then for any $N > \frac{2\ln d^2 + dn}{F^*}$, the product $F(X_1, \ldots, X_n)(X_1, \ldots, X_n)^N$ has all its coefficients strictly positive.

Note that this bound does not depend on the coefficients of the polynomial $F$. This is going to be crucial later on.

Lastly, to make our complexity estimate exact, we use the following bound on the degree of the generators for a Gröbner basis with respect to a global monomial ordering.

Theorem 8. (Mayr and Ritscher, 2010, Cor.3.21) Let $I = \langle f_1, \ldots, f_s \rangle$ be an ideal in the ring $\mathbb{R}[X_1, \ldots, X_n]$, generated by arbitrary polynomials of degree $d_1 \geq \ldots \geq d_s$. Then for any global monomial ordering $\prec$, the degree required in a Gröbner basis for $I$ with respect to $\prec$ is bounded by $2(\frac{1}{2}d_1 \cdots d_{n-r} + d_1)^2$, where $r$ is the dimension of $I$. 

5
The main tool in the proof of theorem 1 in Schweighofer (2004) is the next lemma. Thus our main objective is to generalize it to the parametric case.

**Lemma 9.** *(Schweighofer, 2004, Lem.9)* Assume \( g_1, \ldots, g_m \in \mathbb{R}[X], \varepsilon > 0 \) and

\[
K = \left\{ x \in [-1 + \varepsilon, 1 - \varepsilon]^n \mid g_1(x) \geq 0, \ldots g_m(x) \geq 0, \sum_{i=1}^m g_i(x) \leq 2n\varepsilon \right\}
\]

is nonempty. Setting

\[
q_1 = 1 - \varepsilon + X_1, \ldots, q_n = 1 - \varepsilon + X_n, \\
q_{n+1} = 1 - \varepsilon - X_1, \ldots, q_{2n} = 1 - \varepsilon - X_n, \\
q_{2n+1} = g_1, \ldots, q_{2n+m} = g_m, \\
q_{2n+m+1} = 2n\varepsilon - \sum_{i=1}^m g_i.
\]

Then there is some \( c \in \mathbb{N} \) such that every \( f \in \mathbb{R}[X] \) of degree \( d \) with \( f^* = \min_{x \in K} f(x) > 0 \) and \( \|f\| = 1 \) can be written as

\[
\sum_{\alpha_1 + \ldots + \alpha_{2n+m+1} = M} a_\alpha q_1^{\alpha_1} \cdots q_{2n+m+1}^{\alpha_{2n+m+1}}
\]

where \( 0 < a_\alpha \in \mathbb{R} \) for all \( \alpha \in \mathbb{N}^{2n+1} \) with \( |\alpha| = M \) and

\[
M \leq cd^2 \left( 1 + \left( \frac{d^2n^d}{f^*} \right)^c \right)
\]

### 3.2 Main part

Now we can state and prove the parametric version of lemma 9.

**Lemma 10.** Let \( P \subset \mathbb{R}^s \) be a compact semi-algebraic set. Assume that \( g_1, \ldots, g_m \in \mathbb{R}[X_1, \ldots, X_n, P_1, \ldots, P_s], \varepsilon > 0 \) and

\[
K(p) = \left\{ x \in [-1 + \varepsilon, 1 - \varepsilon]^n \mid g_1(x, p) \geq 0, \ldots g_m(x, p) \geq 0, \sum_{i=1}^m g_i(x, p) \leq 2n\varepsilon \right\}
\]

is nonempty. Furthermore let \( \deg g_1 \geq \ldots \geq \deg g_m \). Setting

\[
q_1 = 1 - \varepsilon + X_1, \ldots, q_n = 1 - \varepsilon + X_n, \\
q_{n+1} = 1 - \varepsilon - X_1, \ldots, q_{2n} = 1 - \varepsilon - X_n, \\
q_{2n+1} = g_1, \ldots, q_{2n+m} = g_m, \\
q_{2n+m+1} = 2n\varepsilon - \sum_{i=1}^m g_i.
\]

Then there is some \( c \in \mathbb{N} \) such that every \( f \in \mathbb{R}[X] \) of degree \( d \) with \( \|f\| = 1 \) and \( f^* = \min_{x \in K(p)} f(x) > 0 \), for some \( p \in P \), can be written as

\[
\sum_{\alpha_1 + \ldots + \alpha_{2n+m+1} = M} a_\alpha q_1^{\alpha_1}(X, p) \cdots q_{2n+m+1}^{\alpha_{2n+m+1}}(X, p)
\]
where \(0 < a_\alpha \in \mathbb{R}\) for all \(\alpha \in \mathbb{N}^{2n+m+1}\) with \(|\alpha| = M\) and

\[
M \leq \frac{8(2n)^D \ln D}{f^*} + D(2n + m + 1) + 1.
\]

Where

\[
D = \max \left\{ d, 2 \left(\frac{1}{2} \prod_j \deg g_j + \deg g_1\right)^{2n+m+s} \right\}.
\]

The main idea of the proof is to show that the constant \(c\) in lemma 9 can be chosen independently of the parameters. To accomplish this we examine the details of the proof of lemma 9 and modify it where necessary. The reason for the different bound is the fact that we had to apply an older bound for Pólya’s theorem.

Proof. Look at the \(\mathbb{R}\)-algebra homomorphism \(\varphi : \mathbb{R}[Y_1, \ldots, Y_{2n+m+1}, P_1, \ldots, P_s] \rightarrow \mathbb{R}[X, P]\) with \(Y_i \mapsto q_i\) and \(P_j \mapsto P_j\). It’s kernel is an ideal of the form

\[
\ker \varphi = \langle Y_1 + \ldots + Y_{2n+m+1} - 2n, r_1, \ldots, r_l \rangle \subset \mathbb{R}[Y, P],
\]

where \(r_i\) is non constant in \(Y\) for all \(i\). Analogously to the proof of the lemma set

\[
\Delta = \{ y \in [0, \infty)^{2n+m+1} \mid y_1 + \ldots + y_{2n+m+1} - 2n = 0 \}
\]

and

\[
Z(p) = \{ y \in \Delta \mid r_1(y, p) = 0, \ldots, r_l(y, p) = 0 \}.
\]

Define \(R_0(Y, P)\) a \(d_0\) form (in the variables \(Y\)) such that \(R_0 \geq 0\) on \(\Delta \times P\) and \(Z(p) = \{ y \in \Delta \mid R_0(y, p) = 0 \}\). Note that the degree \(d_0\) depends on the parameters \(P\), but it is bounded from above by \(\max_i \deg_Y r_i\). Thus we may set \(d_0 = \max_i \deg_Y r_i\).

Now fix some \(\tilde{p} \in P\). For a fixed parameter we can now proceed as in Schweighofer (2004).

The next point, where we deviate from the original proof, is that we do not directly define \(c\). Instead we examine the different bound for the Pólya-form found in theorem 7. Define \(P\) as in Schweighofer (2004) and set \(d_1 = \max\{d, d_0\}\). Thus the \((d_1 + N)\)-form

\[
Q(Y, \tilde{p}) := (P(Y) + \lambda(\tilde{p})R(Y, \tilde{p})) \left(\frac{Y_1 + \ldots + Y_{2n+m+1}}{2n}\right)^N
\]

has positive coefficients for all

\[
N > \frac{4 \ln d_1}{\min_{y \in \Delta_1} Q(y, p)} + d_1(2n + m + 1) \tag{11}
\]

By the same methods as in Schweighofer (2004), we can get the following estimate for the minimum

\[
Q(y, \tilde{p}) \geq \frac{f^*}{2(2n)^{d_1}} \text{ for all } y \in \Delta_1.
\]

Note that it does not depend on the parameters. In particular our choice of \(\tilde{p}\) did not matter. Also since \(\sum_j Y_j - 2n \in \ker \varphi\) we know that \(\dim \mathbb{R}[Y, P]/\ker \varphi \leq 2n + m + s\).
Furthermore (11) grows monotonically in $d_1$. Thus by applying theorem 8 the following inequality holds

$$M := \deg Q \leq \frac{8(2n)d_1 \ln d_1}{f^*} + d_1(2n + m + 1) + 1 \leq \frac{8(2n)^D \ln D}{f^*} + D(2n + m + 1) + 1,$$

where

$$D = \max \left\{ d, 2 \left( \frac{1}{2} \prod_j \deg g_j + \deg g_1 \right)^{2n+m+1} \right\}.$$  

Proof of theorem 2. Since $K = \{(x, p) \in \mathbb{R}^{n+1} | x \in K(p) \text{ and } p \in P\} \subset (-1,1)^n \times P$ is compact its projection $S$ to the first $n$ coordinates is again a compact semi-algebraic set contained in $(-1,1)^n$. Thus we can choose $\varepsilon > 0$ such that $K(\tilde{p}) \subseteq [-1 + 2\varepsilon, 1 - 2\varepsilon]^n$ for all $\tilde{p} \in P$. We define $q_i$ as in (8).

Since $g_i$ are continuous functions, we scale them with a small positive number in such a way that $q_{2n+m+1} = 2n \varepsilon - \sum_i g_i$ is positive on $K$. Also note that since $P$ is a closed semi-algebraic set, we can assume w.l.o.g. that it is given by inequality constraints on the polynomials $h_j \in \mathbb{R}[P]$. Adding these to our set of constraints, we know, due to Schmüdgen’s Positivstellensatz, that they have a representation of the form (1), where all the polynomials are in $\mathbb{R}[X,P]$. Therefore for each $\tilde{p} \in P$ we get a representation (2) over $\mathbb{R}[X]$ by plugging in $\tilde{p}$. And since $h_j$ only depend on the variables $P$, they become positive constants, when plugging in $\tilde{p}$. Let $c$ be the maximal degree in any of those representations.

Let $f$ be any polynomial of degree $d$ such that $f^* = \min_{x \in K(\tilde{p})} f(x) > 0$ for some $\tilde{p} \in P$. Then applying lemma 10 we obtain a representation for $f$ of the form (10). By plugging in the Schmüdgen representation for $q_j$ we get that $f$ equals an expression of the form (2), where the degree of the summands does not exceed $\frac{8(2n)^D \ln D}{f^*} + D(2n + m + 1) + 1 + c.1$  

4 Conclusion

We have shown that the optimal value of the convex relaxations, to a parametric polynomial optimization problem over a compact set, using Schmüdgen’s Positivstellensatz, converge uniformly to the optimal value function. This extends the convergence of the result in Lasserre (2001) to the parametric setting. Furthermore, as a byproduct, we found that in a special case we do have explicit bounds on the complexity. In Nie and Schweighofer (2007) similar methods are used to prove a convergence result for the relaxations using Putinar’s Positivstellensatz. Thus it should be possible to extend the results obtained here to this framework as well.

References


If we have exponents bigger than 1 for $g_j$, then we can add the even part of them to the sum of squares coefficient.


