SECOND-ORDER VARIATIONAL ANALYSIS AND CHARACTERIZATIONS OF TILT-STABLE OPTIMAL SOLUTIONS IN FINITE AND INFINITE DIMENSIONS[1]

B. S. MORDUKHOVICH[2] and T. T. A. NGHIA[3]

Abstract. The paper is devoted to developing second-order tools of variational analysis and their applications to characterizing tilt-stable local minimizers of constrained optimization problems in finite-dimensional and infinite-dimensional spaces. The importance of tilt stability has been well recognized from both theoretical and numerical aspects of optimization. Based on second-order generalized differentiation, we obtain qualitative and quantitative characterizations of tilt stability in general frameworks of constrained optimization and establish its relationships with strong metric regularity of subgradient mappings and uniform second-order growth. The results obtained are applied to deriving new necessary and sufficient conditions for tilt-stable minimizers in problems of nonlinear programming with twice continuously differentiable data in Hilbert and Euclidean spaces.

1 Introduction

The concept of *tilt stability* of local minimizers introduced by Poliquin and Rockafellar [26] has recently attracted much attention in the literature; see [9, 12, 18, 22, 23, 24] along with the earlier studies in [6, 17, 26] and other publications. Tilt stability postulates single-valued Lipschitzian behavior of local minimizers with respect to a special class of "tilt" perturbations. This property of local minimizers is important not only for theoretical aspects of optimization but also plays a fundamental role in the justification of numerical algorithms; see [26] for more discussions. Except the book [6], where tilt stability is connected with certain quadratic growth conditions in the framework of conic programming with twice continuously differentiable data, all the known results in this direction are available for optimization problems in finite-dimensional settings, and finite dimensionality seems to be essential for the methods and proofs developed in the aforementioned publications.

The seminal paper [26] not only introduces the notion of tilt-stable local minimizers but provides their complete characterization, in the unconstrained format of optimization described by extended-real-valued functions on $I\!R^n$ satisfying natural requirements, via the *second-order subdifferential/generalized Hessian* in the sense of Mordukhovich [20]. Based on the extensive *second-order calculus* for this construction and its partial counterpart (see [21, 23] and the references therein), the results of [26] were extended in [23] to tilt-stable minimizers in various problems of constrained optimization and further in [24] to the more general notion of *full stability* of locally optimal solutions introduced in [17].

The main goal of this paper is to develop new approaches to tilt stability that allow us, in particular, to extend the major results of [23, 26] to both unconstrained and constrained frameworks of *infinite-dimensional optimization* by using appropriate counterparts of the

[1]This research was supported by the National Science Foundation under grant DMS-1007132.
[2]Department of Mathematics, Wayne State University, Detroit, MI 48202 (boris@math.wayne.edu). Research of this author was also partly supported by the Australian Research Council under grant DP-12092508 and by the Portuguese Foundation of Science and Technologies under grant MAT/11109.
[3]Department of Mathematics, Wayne State University, Detroit, MI 48202 (nghia@math.wayne.edu).

second-order subdifferential from [20]. Many of the results obtained provide essentially new *qualitative* and *quantitative* information even in *finite-dimensional spaces.* In this way we derive, e.g., a complete *second-order characterization* of tilt-stable local minimizers in classical nonlinear programming *without* imposing the linear independence constraint qualification (LICQ) and with *non-unique* Lagrange multipliers; this improves the corresponding sufficient condition from [22]. Besides the derivative-like conditions for tilt-stable local minimizers, our results provide infinite-dimensional characterizations of such minimizers via *strong metric regularity* of the (first-order) subgradient mappings, which extend the corresponding finite-dimensional results of [9, 18, 22]. Furthermore, we study strong metric regularity for its own sake, independently of tilt stability, and derive its characterizations by certain *second-order growth* conditions for convex and nonconvex functions in Banach spaces. The results discussed above for tilt stability in optimizing general extended-real-valued functions are applied to *nonlinear programs* with $\mathcal{C}^2$ (twice continuously differentiable) data in Hilbert spaces, where they are specified and developed via first-order and second-order qualification and optimality conditions, both well known and new ones, and also relate to Robinson's *strong regularity* of the associate KKT variational inequality.

The rest of the paper is organized as follows. Section 2 recalls some definitions and facts from variational analysis largely used in the paper. Among known constructions we introduce here a new notion of the *combined second-order subdifferential* important for subsequent results in both finite and infinite dimensions.

Section 3 is devoted to characterizing *strong metric regularity* of subgradient mappings for convex and nonconvex extended-real-valued functions in Banach and Asplund spaces, with providing their specifications in the Hilbert space setting. Second-order growth conditions for the functions under consideration play a crucial role in these characterizations.

In Section 4 we derive complete *qualitative* and *quantitative* characterizations of *tilt stability* for local minimizers of extended-real-valued functions in various infinite-dimensional settings (general Banach, Asplund, and Hilbert spaces) with new consequences in finite-dimensions. Besides obtaining necessary and sufficient conditions for tilt stability, *precise formulas* for tilt stability *moduli* are derived and relate to the corresponding moduli of strong metric regularity of the subdifferentials. The results obtained are expressed in terms of the second-order subdifferential and growth conditions.

Section 5 concerns tilt stability and strong metric regularity in *nonlinear programming* (NLP) with equality and inequality constraints described by $\mathcal{C}^2$ functions in Hilbert spaces while the major results are new even in finite dimensions. We introduce the new *uniform second-order sufficient condition* (USOSC), which is strictly weaker than the more conventional *strong second-order sufficient condition* (SSOSC), and use it for characterizing tilt-stable local minimizers in NLP under additional qualification conditions. In particular, we show that tilt stability of local minimizers for finite-dimensional NLP is *equivalent* to USOSC under the validity of both Mangasarian-Fromovitz and constant rank constraint qualifications (MFCQ and CRCQ). We also establish the equivalence, in the general Hilbert space setting, between tilt stability of local minimizers for NLP and two strong regularity notions: strong metric regularity of the auxiliary Lagrangian mapping under MFCQ and also Robinson's strong regularity of the associate KKT variational system under LICQ.

The last Section 6 is an appendix, which contains some technical lemmas of their own interest while needed in the proofs of the main results developed in the paper.

Our notation is basically standard in variational analysis and generalized differentiation; cf. [21, 29]. Unless otherwise stated, the space X in question is Banach with the norm $\|\cdot\|$

and the topological dual X^*, where $\langle\cdot,\cdot\rangle$ indicates the canonical pairing between X and X^*, and where w^* signifies the weak* convergence in X^*. We denote by $I\!B$ and $I\!B^*$ the closed unit ball in the space in question and its dual space, respectively, with $I\!B_\eta(x) := x + \eta I\!B$ standing for the closed ball centered at x with radius $\eta > 0$. Given a set-valued mapping $F\colon X \rightrightarrows X^*$ between X and X^*, the symbol

$$\mathop{\rm Lim\,sup}_{x\to\bar{x}} F(x) := \Big\{x^* \in X^* \Big|\ \exists \text{ sequences } x_k \to \bar{x},\ x_k^* \stackrel{w^*}{\to} x^* \text{ such that } x_k^* \in F(x_k) \text{ for all } k \in I\!N := \{1,2,\ldots\}\Big\} \tag{1.1}$$

signifies the *sequential Painlevé-Kuratowski outer/upper limit* of $F(x)$ as $x \to \bar{x}$.

2 Basic Definitions and Preliminaries

In what follows $f : X \to \overline{I\!R} := (-\infty,\infty]$ is an extended-real-valued function, which is always assumed to be *proper*, i.e., $\mathrm{dom}\, f := \{x \in X|\ f(x) < \infty\} \neq \emptyset$. Recall first some constructions and facts from *convex analysis* needed in the paper; see, e.g., [29, 31]. If f is convex, its (Fenchel) *conjugate* $f^* : X^* \to \overline{I\!R}$ is defined by

$$f^*(x^*) := \sup_{x\in X}\big\{\langle x^*, x\rangle - f(x)\big\} \quad \text{for all} \quad x^* \in X^*, \tag{2.1}$$

and its *convex subdifferential* (collection of subgradients) at $\bar{x} \in \mathrm{dom}\, f$ is given by

$$\partial f(\bar{x}) := \big\{x^*|\ f(x) - f(\bar{x}) \ge \langle x^*, x - \bar{x}\rangle, x \in X\big\}, \tag{2.2}$$

which can be equivalently represented via the conjugate function by $\{x^* \in X^*|\ \langle x^*, \bar{x}\rangle - f(\bar{x}) \ge f^*(x^*)\}$. The *biconjugate* f^{**} of f is $(f^*)^*$, which is known to agree with the convex function f if and only if f is lower semicontinuous (l.s.c.) on X. The following well-known result is useful for our subsequent considerations.

Lemma 2.1 (subdifferential duality). *Let $f : X \to \overline{I\!R}$ be a convex and l.s.c. function, and let f^* be its conjugate* (2.1). *Then we have the relationship*

$$x^* \in \partial f(x) \ \textit{ if and only if } \ x \in \partial f^*(x^*),$$

which implies that $\partial f^*(x^*) = \mathrm{argmin}_X\{f(x) - \langle x^*, x\rangle\}$ for any $x^* \in X^*$.

Next we proceed with subdifferential constructions for *arbitrary* extended-real-valued functions following mainly [21]; see also [7, 29, 30] for related and additional material as well as for the alternative terminology. Given $f\colon X \to \overline{I\!R}$ and $\bar{x} \in \mathrm{dom}\, f$, the *regular/Fréchet subdifferential* (known also as the presubdifferential or viscosity subdifferential) of f at $\bar{x}$ is

$$\widehat{\partial} f(\bar{x}) := \Big\{x^* \in X^* \Big|\ \liminf_{x\to\bar{x}} \frac{f(x) - f(\bar{x}) - \langle x^*, x - \bar{x}\rangle}{\|x - \bar{x}\|} \ge 0\Big\}. \tag{2.3}$$

Then the *limiting/Mordukhovich subdifferential* (known also as the basic or general subdifferential) of f at $\bar{x} \in \mathrm{dom}\, f$ is defined via the sequential outer limit (1.1) by

$$\partial f(\bar{x}) := \mathop{\rm Lim\,sup}_{x \xrightarrow{f} \bar{x}} \widehat{\partial} f(x), \tag{2.4}$$

where the symbol $x \xrightarrow{f} \bar{x}$ means that $x \to \bar{x}$ with $f(x) \to f(\bar{x})$. Note that definition (2.4) is more appropriate for l.s.c. functions in *Asplund spaces* (i.e., Banach spaces where every separable subspace has a separable dual, which is automatic, e.g., under reflexivity). This is actually the setting of using the limiting subdifferential and related normal and coderivative constructions in what follows. In the general Banach space framework (see [21]) we need to *enlarge* (2.3) in the limiting procedure of (2.4), which surely keeps the inclusion

$$\widehat{\partial} f(\bar{x}) \subset \partial f(\bar{x}) \ \text{ for any } \ \bar{x} \in \operatorname{dom} f. \tag{2.5}$$

Observe also that both regular and limiting subdifferentials reduce to the subdifferential (2.2) of convex analysis if the function f is convex.

Given further a set $\Omega \subset X$ with its indicator function $\delta(x;\Omega)$ equal to 0 for $x \in \Omega$ and to ∞ otherwise, the regular and limiting *normal cones* to Ω at $\bar{x} \in \Omega$ are defined, respectively, via the corresponding subdifferentials (2.3) and (2.4) by

$$\widehat{N}(\bar{x};\Omega) := \widehat{\partial}\delta(\bar{x};\Omega) \ \text{ and } \ N(\bar{x};\Omega) := \partial\delta(\bar{x};\Omega). \tag{2.6}$$

Now we consider a set-valued mapping $F : X \rightrightarrows Y$ and associate with it the *domain* $\operatorname{dom} F$ and the *graph* $\operatorname{gph} F$ given by

$$\operatorname{dom} F := \big\{x \in X \big|\ F(x) \neq \emptyset\big\} \ \text{ and } \ \operatorname{gph} F := \big\{(x,y) \in X \times Y \big|\ y \in F(x)\big\}.$$

Then the *regular coderivative* of F at $(\bar{x},\bar{y}) \in \operatorname{gph} F$ is defined by

$$\widehat{D}^*F(\bar{x},\bar{y})(y^*) := \big\{x^* \in X^* \big|\ (x^*,-y^*) \in \widehat{N}((\bar{x},\bar{y});\operatorname{gph} F)\big\} \ \text{ for all } \ y^* \in Y^* \tag{2.7}$$

and the (limiting) *mixed coderivative* of F at $(\bar{x},\bar{y})$ is

$$D^*_M F(\bar{x},\bar{y})(y^*) := \mathop{\operatorname{Lim\,sup}}_{\substack{(x,y)\to(\bar{x},\bar{y})\\ z^*\to y^*}} \widehat{D}^*F(x,y)(z^*) \ \text{ for all } \ y^* \in Y^*, \tag{2.8}$$

where the convergence $z^* \to y^*$ is strong in Y^* while the outer limit in (2.8) is taken by (1.1) in the weak* topology of X^*; cf. [21] for other coderivative constructions not used in this paper. We omit the subscript "M" in (2.8) when X is finite-dimensional and $\bar{y} = F(\bar{x})$ when F is single-valued. Note that

$$\widehat{D}^*F(\bar{x})(y^*) = D^*_M F(\bar{x})(y^*) = \big\{\nabla F(\bar{x})^* y^*\big\} \ \text{ for all } \ y^* \in Y^*$$

if F is single-valued and continuously differentiable around $\bar{x}$ (or merely strictly differentiable at this point) with the adjoint derivative operator $\nabla F(\bar{x})^* \colon Y^* \to X^*$.

It has been well recognized that the coderivative constructions in (2.7) and (2.8) are appropriate tools for the study and characterization of well-posedness and sensitivity in variational analysis; see [21, Chapter 4] for more details and references. The following property of this type is used in what follows: $F \colon X \rightrightarrows Y$ is *metrically regular* with modulus $\kappa > 0$ around $(\bar{x},\bar{y}) \in \operatorname{gph} F$ if there are neighborhoods U of $\bar{x}$ and V of $\bar{y}$ with

$$\operatorname{dist}\big(x;F^{-1}(y)\big) \le \kappa \operatorname{dist}\big(y;F(x)\big) \ \text{ for all } \ x \in U \ \text{ and } \ y \in V, \tag{2.9}$$

where $\operatorname{dist}(x;\Omega)$ stands for the distance from x to Ω. An important specification of (2.9) is studied in [11] under the name of "strong metric regularity." Recall first that $\widehat{F}$ is a *localization* of $F \colon X \rightrightarrows Y$ around $(\bar{x},\bar{y}) \in \operatorname{gph} F$ if there are neighborhoods U of $\bar{x}$ and V of

$\bar{y}$ such that $\operatorname{gph}\widehat{F} = \operatorname{gph} F \cap (U \times V)$. Then F is *strongly metrically regular* around $(\bar{x}, \bar{y})$ with modulus $\kappa > 0$ if the inverse mapping F^{-1} admits a single-valued localization around $(\bar{y}, \bar{x})$ that is Lipschitz continuous with modulus κ around $\bar{y}$. It is easy to check that F is strongly metrically regular at $(\bar{x}, \bar{y})$ if and only if F is metrically regular around $(\bar{x}, \bar{y})$ and F^{-1} has a single-valued Lipschitzian localization around $(\bar{y}, \bar{x})$. Moreover, the domain of such a single-valued localization must be a neighborhood of $\bar{y}$. Recall also that for any single-valued mapping $g : X \to Y$, which is Lipschitz continuous around $\bar{x} \in \operatorname{dom} g$, the *exact Lipschitzian bound* of f at $\bar{x}$ is defined by

$$\operatorname{lip} g(\bar{x}) := \limsup_{\substack{x,u \to \bar{x} \\ x \neq u}} \frac{\|g(x) - g(u)\|}{\|x - u\|}. \tag{2.10}$$

Next we formulate two significant concepts of prox-regularity and subdifferential continuity taken from [25, 29], where they are comprehensively studied in finite dimensions. A l.s.c. function $f : X \to \overline{I\!R}$ is *prox-regular* at $\bar{x} \in \operatorname{dom} f$ for $\bar{x}^* \in \partial f(\bar{x})$ if there are constants $r > 0$ and $\varepsilon > 0$ such that for all $x, u \in I\!B_\varepsilon(\bar{x})$ with $|f(u) - f(\bar{x})| \le \varepsilon$ we have

$$f(x) \ge f(u) + \langle u^*, x - u\rangle - \frac{r}{2}\|x - u\|^2 \;\text{ whenever }\; u^* \in \partial f(u) \cap I\!B_\varepsilon(\bar{x}^*). \tag{2.11}$$

Further, f is *subdifferentially continuous* at $\bar{x} \in \operatorname{dom} f$ for $\bar{x}^* \in \partial f(\bar{x})$ if the function $(x, x^*) \mapsto f(x)$ is continuous relative to the subdifferential graph $\operatorname{gph}\partial f$ at $(\bar{x}, \bar{v})$.

These notions have been also studied in the frameworks of Hilbert and more general Banach spaces; see, e.g., [4, 5]. When f is both prox-regular and subdifferentially continuous at $\bar{x}$ for $\bar{x}^* \in \partial f(\bar{x})$, it is easy to observe that the condition "$|f(u) - f(\bar{x})| \le \varepsilon$" can be omitted in the definition of prox-regularity. The class of prox-regular and subdifferentially continuous functions is rather broad including, in particular, *strongly amenable* functions in finite dimensions, l.s.c. convex functions in Banach spaces, etc.; see [5, 25, 29] for further details. Moreover, in the general Banach space X the *graph* of ∂f is *closed* near $(\bar{x}, \bar{x}^*)$ in the norm$\times$norm topology of $X \times X^*$when f is prox-regular and subdifferentially continuous at $\bar{x}$ for $\bar{x}^*$. Indeed, for any sequence $(x_k, x_k^*) \in \operatorname{gph}\partial f \cap I\!B_{\frac{\varepsilon}{2}}(\bar{x}, \bar{x}^*)$ converging to some $(x_0, x_0^*) \in I\!B_{\frac{\varepsilon}{2}}(\bar{x}, \bar{x}^*)$ in the norm$\times$norm topology of $X \times X^*$ we get from (2.11) that

$$f(x) \ge f(x_0) + \langle x_0^*, x - x_0\rangle - \frac{r}{2}\|x - x_0\|^2 \;\text{ for all }\; x \in I\!B_\varepsilon(\bar{x}),$$

which readily implies that $x_0^* \in \widehat{\partial} f(x_0) \subset \partial f(x_0)$ by (2.5) and hence justifies the inclusion $(x_0, x_0^*) \in \operatorname{gph}\partial f \cap I\!B_{\frac{\varepsilon}{2}}(\bar{x}, \bar{x}^*)$. This proves the aforementioned closedness of $\operatorname{gph}\partial f$.

As shown in [25], the limiting subdifferential of prox-regular functions is strongly connected to *monotonicity*. Recall that a set-valued mapping $T : X \rightrightarrows X^*$ (or sometimes $T : X^* \rightrightarrows X$) is *monotone* if it satisfies the relationship

$$\langle x^* - u^*, x - u\rangle \ge 0 \;\text{ whenever }\; (x, x^*),\ (u, u^*) \in \operatorname{gph} T.$$

In addition $T\colon X \rightrightarrows X^*$ is said to be *maximal monotone* if $T = S$ for any monotone mapping $S : X \rightrightarrows X^*$ with $\operatorname{gph} T \subset \operatorname{gph} S$. Further, we say that T is *locally monotone* around $(\bar{x}, \bar{x}^*) \in \operatorname{gph} T$ if it admits a monotone localization around this point. Moreover, T is *locally maximal monotone* around $(\bar{x}, \bar{x}^*) \in \operatorname{gph} T$ if there are neighborhoods U of $\bar{x}$ and U^* of $\bar{x}^*$ such that for any mapping $S : X \rightrightarrows X^*$ with $\operatorname{gph} T \cap (U \times U^*) \subset \operatorname{gph} S$ we have the equality $\operatorname{gph} T \cap (U \times U^*) = \operatorname{gph} S \cap (U \times U^*)$.

Now we formulate the main optimization-related property studied in this paper.

Definition 2.2 (tilt stability, [26]). *Given $f\colon X \to \overline{I\!R}$, a point $\bar{x} \in \mathrm{dom}\, f$ is a* TILT-STABLE LOCAL MINIMIZER *of f if there is $\gamma > 0$ such that the mapping*

$$M_\gamma : x^* \mapsto \mathrm{argmin}\big\{ f(x) - \langle x^*, x\rangle \big| \, x \in I\!B_\gamma(\bar{x}) \big\} \tag{2.12}$$

is single-valued and Lipschitz continuous on some neighborhood of $0 \in X^*$ with $M_\gamma(0) = \bar{x}$.

We also consider in what follows a *quantitative* version of this notion that specifies the modulus of tilt stability. Namely, $\bar{x}$ is a tilt-stable minimizer of f with *modulus* $\kappa > 0$ if the mapping M_γ is Lipschitz continuous with constant κ in the framework of Definition 2.2.

As mentioned above, Poliquin and Rockafellar fully characterize tilt-stable minimizers of extended-real valued l.s.c. functions in finite dimensions via the *second-order subdifferential* (generalized Hessian) by Mordukhovich [20]. Let us present two infinite-dimensional extensions of the second-order construction of [20] crucial for the major results of the paper.

Definition 2.3 (second-order subdifferentials). *Let $f : X \to \overline{I\!R}$ with $\bar{x} \in \mathrm{dom}\, f$, and let $\bar{x}^* \in \partial f(\bar{x})$. Then we say that:*

(i) *The* COMBINED SECOND-ORDER SUBDIFFERENTIAL *of f at $\bar{x}$ relative to $\bar{x}^*$ is the set-valued mapping $\breve{\partial}^2 f(\bar{x}, \bar{x}^*) : X^{**} \rightrightarrows X^*$ with the values*

$$\breve{\partial}^2 f(\bar{x}, \bar{x}^*)(u) := \big(\widehat{D}^* \partial f\big)(\bar{x}, \bar{x}^*)(u) \quad \textit{for all} \quad u \in X^{**}. \tag{2.13}$$

(ii) *The* MIXED SECOND-ORDER SUBDIFFERENTIAL *of f at $\bar{x}$ relative to $\bar{x}^*$ is the set-valued mapping $\partial^2_M f(\bar{x}, \bar{x}^*) : X^{**} \rightrightarrows X^*$ with the values*

$$\partial^2_M f(\bar{x}, \bar{x}^*)(u) := \big(D^*_M \partial f\big)(\bar{x}, \bar{x}^*)(u) \quad \textit{for all} \quad u \in X^{**}. \tag{2.14}$$

Both constructions (2.13) and (2.14) reduce to that of [20] in finite dimensions. The mixed second-order subdifferential is introduced in [21, Definition 1.118] (together with the *normal* one not used in the paper) while the combined second-order subdifferential seems to be new in literature. Note however that its finite-dimensional version with the normal cone $\partial f(\cdot) = N(\cdot; \Omega)$ in (2.14) has been recently used in [13, 14] for different purposes.

When f is $\mathcal{C}^2$ around $\bar{x}$ with $\bar{x}^* = \nabla f(\bar{x})$, both $\breve{\partial}^2 f(\bar{x}, \bar{x}^*)(u)$ and $\partial^2_M f(\bar{x}, \bar{x}^*)(u)$ reduce to the classical single-valued Hessian operator:

$$\breve{\partial}^2 f(\bar{x}, \bar{x}^*)(u) = \partial^2_M f(\bar{x}, \bar{x}^*)(u) = \big\{\nabla^2 f(\bar{x})^* u\big\} \quad \text{for all} \quad u \in X^{**},$$

where $\nabla^2 f(\bar{x})^* = \nabla^2 f(\bar{x})$ in the Hilbert space setting.

3 Second-Order Characterizations of Strong Metric Regularity for Subgradient Mappings

In this section we characterize the property of strong metric regularity for the subdifferential (2.2) of l.s.c. convex functions in general Banach spaces and the limiting subdifferential (2.4) of arbitrary l.s.c. functions in the Asplund space setting. We begin with the convex case.

Theorem 3.1 (strong metric regularity of the subdifferential of convex analysis in Banach spaces). *Let X be a Banach space, and let $f : X \to \overline{I\!R}$ be a l.s.c. convex function. Then for any $(\bar{x}, \bar{x}^*) \in \mathrm{gph}\, \partial f$ the following assertions are equivalent:*

(i) *The subgradient mapping* $\partial f\colon X \rightrightarrows X^*$ *is strongly metrically regular around the point* $(\bar{x},\bar{x}^*)$ *with modulus* $\kappa>0$.

(ii) *There are neighborhoods* U *of* $\bar{x}$ *and* U^* *of* $\bar{x}^*$ *such that the mapping* $(\partial f)^{-1}$ *admits a single-valued localization* $\vartheta\colon U^*\to U$ *around* $(\bar{x}^*,\bar{x})$ *and that for any pairs* $(u^*,u)\in\text{gph}\,\vartheta=\text{gph}\,(\partial f)^{-1}\cap(U^*\times U)$ *we have the* SECOND-ORDER GROWTH CONDITION

$$f(x)\ge f(u)+\langle u^*,x-u\rangle+\frac{1}{2\kappa}\|x-u\|^2 \quad whenever \quad x\in U. \tag{3.1}$$

Proof. First we justify implication (ii)$\Longrightarrow$(i). Supposing that (ii) holds, take any (u^*,u) and (x^*,x) from the graph of ϑ. It follows from definition (2.2) and the second-order growth condition (3.1) that

$$\begin{aligned}\|x^*-u^*\|\cdot\|x-u\| &\ge \langle x^*-u^*,x-u\rangle=\langle x^*,x-u\rangle+\langle u^*,u-x\rangle\\ &\ge \Big[f(x)-f(u)+\frac{1}{2\kappa}\|u-x\|^2\Big]+\Big[f(u)-f(x)+\frac{1}{2\kappa}\|x-u\|^2\Big]\\ &=\frac{1}{\kappa}\|x-u\|^2 \ \text{ around } \ (\bar{x},\bar{x}^*),\end{aligned}$$

which implies that $\|\vartheta(x^*)-\vartheta(u^*)\|=\|x-u\|\le\kappa\|x^*-u^*\|$. Hence ϑ is Lipschitz continuous localization of ∂f with modulus κ that justifies the strong metric regularity of the subdifferential mapping ∂f with modulus κ claimed in (i).

To justify the converse implication (i)$\Longrightarrow$(ii), suppose that the subdifferential mapping ∂f is strongly metrically regular around $(\bar{x},\bar{x}^*)$ with modulus κ and thus find neighborhoods U of $\bar{x}$ and U^* of $\bar{x}^*$ such that $(\partial f)^{-1}$ has a single-valued and Lipschitz continuous localization $\vartheta\colon U^*\to U$ with modulus κ and $\vartheta(\bar{x}^*)=\bar{x}$. Moreover, the monotonicity of the subdifferential ∂f for convex functions implies that the mapping ϑ is monotone as well, which ensures its maximal monotonicity on $U^*\times U$ due to its local continuity.

Let us now define g as the Fenchel conjugate of $f+\delta(\cdot;\overline{\text{co}}\,U)$, i.e., $g:=(f+\delta(\cdot;\overline{\text{co}}\,U))^*$ on X^*. It is well known from convex analysis that g is a proper convex function with $\partial g=(\partial(f+\delta(\cdot;\overline{\text{co}}\,U)))^{-1}$. Note further that for any $x^*\in U^*$ we have the inclusions

$$x^*\in\partial f\big(\vartheta(x^*)\big)\subset\partial f\big(\vartheta(x^*)\big)+N\big(\vartheta(x^*);\overline{\text{co}}\,U\big)\subset\partial\big(f+\delta(\cdot;\overline{\text{co}}\,U)\big)\big(\vartheta(x^*)\big),$$

which imply that $\vartheta(x^*)\in(\partial(f+\delta(\cdot;\overline{\text{co}}\,U)))^{-1}(x^*)=\partial g(x^*)$ and hence $\text{gph}\,\vartheta\subset\text{gph}\,\partial g$. Since ∂g is monotone and ϑ is maximal monotone on $U^*\times U$, we have that $\text{gph}\,\partial g\cap(U^*\times U)=\text{gph}\,\vartheta$. Thus ∂g is single-valued and Lipschitz continuous on U^* with modulus κ, and furthermore it follows for any $x^*,u^*\in U^*$ that

$$\begin{aligned}0\le g(x^*)-g(u^*)-\langle\partial g(u^*),x^*-u^*\rangle &\le \langle\partial g(x^*),x^*-u^*\rangle-\langle\partial g(u^*),x^*-u^*\rangle\\ &= \langle\partial g(x^*)-\partial g(u^*),x^*-u^*\rangle\\ &\le \kappa\|x^*-u^*\|\cdot\|x^*-u^*\|,\end{aligned}$$

which ensure that g is Fréchet differentiable on the neighborhood U^* and that the mapping $\nabla g:U^*\to U$ is Lipschitz continuous on U^* with modulus κ.

Select next $\alpha>0$ such that $\widetilde{U}:=I\!B_{\kappa\alpha}(\bar{x})\subset U$ and $I\!B_{3\alpha}(\bar{x}^*)\subset U^*$, which ensures that $\nabla g(\widetilde{U}^*)\subset\widetilde{U}$ with $\widetilde{U}^*:=I\!B_\alpha(\bar{x})^*$. Pick further any $(u^*,u)\in\text{gph}\,\vartheta\cap(\widetilde{U}^*\times\widetilde{U})=(\text{gph}\,\nabla g)\cap(\widetilde{U}^*\times\widetilde{U})$. Since ∇g is Lipschitz continuous with modulus κ on U^*, for every $v^*\in U^*$ we have the relationships

$$\begin{aligned}g(v^*)-g(u^*)-\langle u,v^*-u^*\rangle &=\int_0^1\big\langle\nabla g\big(u^*+t(v^*-u^*)\big)-\nabla g(u^*),v^*-u^*\big\rangle\,dt\\ &\le\int_0^1\kappa t\|v^*-u^*\|\cdot\|v^*-u^*\|\,dt=\frac{\kappa}{2}\|v^*-u^*\|^2.\end{aligned} \tag{3.2}$$

Moreover, it follows from the *biconjugate formula* in [31, Theorem 2.3.4] that

$$\langle u,u^*\rangle-g(u^*)=g^*(u)=\big(f+\delta(\cdot;\overline{\rm co}\,U)\big)^{**}(u)=f(u)+\delta(u;\overline{\rm co}\,U)=f(u).$$

This together with (3.2) implies the estimate

$$g(v^*)\le h(v^*):=-f(u)+\langle u,v^*\rangle+\frac{\kappa}{2}\|v^*-u^*\|^2+\delta(v^*;U^*)\ \text{ for all }\ v^*\in X^*.$$

For any $x\in\widetilde{U}$ we get from the biconjugate formula and the estimate above that

$$\begin{aligned}f(x)&=\big(f+\delta(\cdot;\overline{\rm co}\,U)\big)^{**}(x)=g^*(x)\ge h^*(x)=\sup_{v^*\in X^*}\big\{\langle v^*,x\rangle-h(v^*)\big\}\\&=\sup_{v^*\in U^*}\Big\{\langle v^*,x\rangle-\langle v^*,u\rangle-\frac{\kappa}{2}\|v^*-u^*\|^2+f(u)\Big\}\\&=\sup_{v^*\in U^*}\Big\{\langle v^*-u^*,x-u\rangle-\frac{\kappa}{2}\|v^*-u^*\|^2\Big\}+\langle u^*,x-u\rangle+f(u).\end{aligned}\tag{3.3}$$

Furthermore, it is easy to observe the inequality

$$\sup_{v^*\in U^*}\Big\{\langle v^*-u^*,x-u\rangle-\frac{\kappa}{2}\|v^*-u^*\|^2\Big\}\le\frac{1}{2\kappa}\|x-u\|^2.\tag{3.4}$$

Next we consider the *duality mapping* $J:X\rightrightarrows X^*$ given by $J(v):=\frac{1}{2}\partial(\|\cdot\|^2)(v)$. Employing the well-known subdifferential formula of convex analysis gives us

$$J(v)=\big\{v^*\in X^*|\ \langle v^*,v\rangle=\|v\|^2=\|v^*\|^2\big\}\ne\emptyset\ \text{ for all }\ v\in X.\tag{3.5}$$

Pick any $z^*\in J\big(\frac{1}{\kappa}(x-u)\big)$ and get from (3.5) that

$$\langle z^*,x-u\rangle-\frac{\kappa}{2}\|z^*\|^2=\frac{1}{\kappa}\|x-u\|^2-\frac{1}{2\kappa}\|x-u\|^2=\frac{1}{2\kappa}\|x-u\|^2,\tag{3.6}$$

which readily implies the relationships

$$\begin{aligned}\|z^*+u^*-\bar x^*\|&\le\|z^*\|+\|u^*-\bar x^*\|=\frac{1}{\kappa}\|x-u\|+\|u^*-\bar x^*\|\le\frac{1}{\kappa}\Big(\|x-\bar x\|+\|u-\bar x\|\Big)+\alpha\\&\le\frac{1}{\kappa}\Big(\kappa\alpha+\kappa\alpha\Big)+\alpha=3\alpha,\end{aligned}$$

and hence $z^*+u^*\in I\!\!B_{3\alpha}(\bar x^*)\subset U^*$ by the above choice of α. This together with (3.6) shows that the equality holds in (3.4). Combining it with (3.3) gives us that

$$f(x)\ge f(u)+\langle u^*,x-u\rangle+\frac{1}{2\kappa}\|x-u\|^2\ \text{ for all }\ x\in\widetilde{U},$$

which clearly justifies (3.1) and thus completes the proof of the theorem. △

Note that the equivalence of Theorem 3.1 extends to the general Banach spaces the one from [1, Corollary 3.9] established in Hilbert spaces with the proof essentially used specific properties of Hilbert spaces. Another significant feature of Theorem 3.1 in comparison with [1] is that we obtain a precise *quantitative* relationship between the constant of quadratic growth in (3.1) and the modulus of strong metric regularity of the subdifferential. Note that our proof of implication (i)$\Longrightarrow$(ii) in Theorem 3.1 is inspired by some ideas from the proof of [17, Lemma 5.2] given in a rather different finite-dimensional setting under the assumption that the conjugate f^* is smooth with the Lipschitz continuous derivative.

The next theorem concerns strong metric regularity of the limiting subdifferential (2.4) for arbitrary l.s.c. functions in Asplund spaces and provides somewhat different information even for convex functions in comparison with Theorem 3.1.

Theorem 3.2 (strong metric regularity of the limiting subdifferential of l.s.c. functions in Asplund spaces). *Let X be an Asplund space, and let $f : X \to \overline{I\!R}$ be an arbitrary l.s.c. function. Then for any $(\bar{x}, \bar{x}^*) \in \operatorname{gph} \partial f$ in the case of the limiting subdifferential* (2.4) *we have that assertion* (ii) *of Theorem* 3.1 *is equivalent to the following:*

(iii) *The subgradient mapping ∂f is strongly metrically regular around $(\bar{x}, \bar{x}^*)$ with modulus $\kappa > 0$ and there exist numbers $r \in (0, \kappa^{-1})$ and $\varepsilon > 0$ such that*

$$f(x) \geq f(\bar{x}) + \langle \bar{x}^*, x - \bar{x} \rangle - \frac{r}{2}\|x - \bar{x}\|^2 \quad \textit{whenever} \quad x \in I\!B_\varepsilon(\bar{x}). \tag{3.7}$$

Proof. Observe first that the proof of (ii)$\Longrightarrow$(i) in Theorem 3.1 based merely on inequality (3.1) holds in the nonconvex case under consideration. Furthermore, condition (3.7) is an immediate consequence of (3.1). Thus we get implication (ii)$\Longrightarrow$(iii) of the theorem.

To justify the converse implication, let assertion (iii) be satisfied. By strong metric regularity of ∂f we find neighborhoods U of $\bar{x}$ and U^* of $\bar{x}^*$ such that $(\partial f)^{-1}$ admits a single-valued and Lipschitz continuous localization $\vartheta : U^* \to U$ with modulus κ. Now we show that condition (3.7) ensures the existence of constants $\alpha > 0$ and $\nu > 0$ such that

$$f(x) \geq f(\bar{x}) + \langle \bar{x}^*, x - \bar{x} \rangle + \frac{\alpha}{2}\|x - \bar{x}\|^2 \quad \text{for all} \quad x \in I\!B_\nu(\bar{x}). \tag{3.8}$$

Arguing by contradiction, suppose that such α, ν do not exist and then prove that there is some number $\eta > 0$ for which

$$f(x) \geq f(\bar{x}) + \langle \bar{x}^*, x - \bar{x} \rangle + \frac{1 - 3r\kappa}{4\kappa}\|x - \bar{x}\|^2 \quad \text{whenever} \quad x \in I\!B_\eta(\bar{x}). \tag{3.9}$$

Indeed, the negation of (3.9) implies the existence of a sequence $\{x_k\} \subset X$ converging to the reference point $\bar{x}$ as $k \to \infty$ and such that

$$f(x_k) < f(\bar{x}) + \langle \bar{x}^*, x_k - \bar{x} \rangle + \frac{1 - 3r\kappa}{4\kappa}\|x_k - \bar{x}\|^2. \tag{3.10}$$

Combining (3.10) with (3.7) gives us the inequality

$$\inf_{x \in I\!B_\varepsilon(\bar{x})} \Big\{ f(x) + \frac{r}{2}\|x - \bar{x}\|^2 - \langle \bar{x}^*, x - \bar{x} \rangle \Big\} > f(x_k) + \frac{r}{2}\|x_k - \bar{x}\|^2 - \langle \bar{x}^*, x_k - \bar{x} \rangle - \varepsilon_k$$

with $\varepsilon_k := \Big(\frac{r}{2} + \frac{1 - 3r\kappa}{4\kappa} \Big) \|x_k - \bar{x}\|^2 - \delta_k = \frac{1 - r\kappa}{4\kappa}\|x_k - \bar{x}\|^2 - \delta_k > 0$ along some sequence $\delta_k \downarrow 0$ as $k \to \infty$. By Ekeland's variational principle with the parameter ε_k applied for each $k \in I\!N$ to the function written above in $\{\cdot\}$, which is obviously l.s.c. and bounded from below, we find a new sequence $\{\widehat{x}_k\} \subset I\!B_\varepsilon(\bar{x})$ satisfying the conditions

$$\|\widehat{x}_k - x_k\| \leq \lambda_k := \sqrt{\frac{\kappa \varepsilon_k}{1 - r\kappa}} \to 0 \quad \text{as} \quad k \to \infty \quad \text{and}$$

$$\inf_{x \in I\!B_\varepsilon(\bar{x})} \Big\{ f(x) + \frac{r}{2}\|x - \bar{x}\|^2 - \langle \bar{x}^*, x - \bar{x} \rangle + \frac{\varepsilon_k}{\lambda_k}\|x - \widehat{x}_k\| \Big\} \geq f(\widehat{x}_k) + \frac{r}{2}\|\widehat{x}_k - \bar{x}\|^2 - \langle \bar{x}^*, \widehat{x}_k - \bar{x} \rangle.$$

Since $x_k \to \bar{x}$ as $k \to \infty$, we have that $\widehat{x}_k \in \operatorname{int} I\!B_\varepsilon(\bar{x})$ for all $k \in I\!N$ without loss of generality. Applying now the generalized Fermat rule to the local minimizer $\widehat{x}_k$ of the function written under the infimum sign above for each $k \in I\!N$ and then using the subdifferential sum rule valid for the limiting subdifferential in Asplund spaces give us the inclusions

$$\begin{aligned} 0 &\in \partial\Big(f + \frac{r}{2}\|\cdot - \bar{x}\|^2 - \langle \bar{x}^*, \cdot - \bar{x}\rangle + \frac{\varepsilon_k}{\lambda_k}\|\cdot - \widehat{x}_k\|\Big)(\widehat{x}_k) \\ &\subset \partial f(\widehat{x}_k) + rJ(\widehat{x}_k - \bar{x}) - \bar{x}^* + \frac{\varepsilon_k}{\lambda_k} I\!B^*, \end{aligned}$$

where $J(\cdot)$ is the duality mapping computed in (3.5). Hence for each $k \in I\!N$ there are dual elements $u_k^* \in J(\widehat{x}_k - \bar{x})$ and $v_k^* \in I\!B^*$ such that

$$\bar{x}^* - ru_k^* - \frac{\varepsilon_k}{\lambda_k} v_k^* \in \partial f(\widehat{x}_k).$$

From (3.5), the choice of ε_k, and the formula for λ_k we get the estimates

$$\Big\|ru_k^* + \frac{\varepsilon_k}{\lambda_k} v_k^*\Big\| \le r\|u_k^*\| + \frac{\varepsilon_k}{\lambda_k}\|v_k^*\| \le r\|\widehat{x}_k - \bar{x}\| + \sqrt{\frac{(1-r\kappa)\varepsilon_k}{\kappa}} \to 0 \ \text{ as } \ k \to \infty.$$

Moreover, the Lipschitz property of the localization ϑ of $(\partial f)^{-1}$ with constant κ yields

$$\|\widehat{x}_k - \bar{x}\| \quad \le \quad \kappa\Big\|ru_k^* + \frac{\varepsilon_k}{\lambda_k} u_k^*\Big\| \le r\kappa\|\widehat{x}_k - \bar{x}\| + \sqrt{\kappa(1-r\kappa)\varepsilon_k},$$

and hence we arrive at the relationships

$$\begin{aligned} (1-r\kappa)\|x_k - \bar{x}\| &\le (1-r\kappa)\|\widehat{x}_k - \bar{x}\| + (1-r\kappa)\|\widehat{x}_k - x_k\| \le \sqrt{\kappa(1-r\kappa)\varepsilon_k} + (1-r\kappa)\lambda_k \\ &= \sqrt{4\kappa(1-r\kappa)\varepsilon_k} = \sqrt{(1-r\kappa)^2\|x_k - \bar{x}\|^2 - 4\kappa(1-r\kappa)\delta_k}, \end{aligned}$$

which contradicts the choice of $\delta_k > 0$ above and thus justifies (3.9) with some $\eta > 0$.

Our assumption on the absence of constants $\alpha, \nu > 0$ satisfying (3.7) readily implies that $1 - 3r\kappa \le 0$, and so $r\kappa \ge \frac{1}{3}$. Define further $r_1 := \frac{3r\kappa - 1}{2\kappa} \in (0, \kappa^{-1})$ and observe by (3.9) that an estimate of type (3.7) holds with replacing r by r_1 and ε by η. Employing again (3.9) gives us $\eta_1 > 0$ such that

$$f(x) \ge f(\bar{x}) + \langle \bar{x}^*, x - \bar{x}\rangle + \frac{1 - 3r_1\kappa}{4\kappa}\|x - \bar{x}\|^2 \ \text{ whenever } \ x \in I\!B_{\eta_1}(\bar{x}).$$

Similarly we obtain that $r_1\kappa \ge \frac{1}{3}$, which clearly implies that $3r\kappa - 1 \ge \frac{2}{3}$. We get $r\kappa \ge \frac{1}{3} + \frac{2}{3^2}$. Defining then $r_2 := \frac{3r_1\kappa - 1}{2\kappa} \in (0, \kappa^{-1})$ and arguing in the same way lead us to the estimates

$$r_2\kappa \ge \frac{1}{3}, \quad r_1\kappa \ge \frac{1}{3} + \frac{2}{3^2}, \ \text{ and } \ r\kappa \ge \frac{1}{3} + \frac{2}{3^2} + \frac{4}{3^2}.$$

By induction we have the relationships

$$r\kappa \ge \frac{1}{3} + \frac{2}{3^2} + \ldots + \frac{2^k}{3^{k+1}} = 1 - \frac{2^{k+1}}{3^{k+1}} \ \text{ for all } \ k \in I\!N,$$

which show that $r\kappa \ge 1$ as $k \to \infty$. Thus we arrive at a contradiction with the assumption that $r\kappa < 1$ of the theorem and so justify the existence of $\alpha, \nu > 0$ for which (3.8) holds.

To proceed further, suppose without loss of generality that $\widehat{U} := I\!B_\nu(\bar{x}) \subset U$ and then find a number $\gamma > 0$ such that $\widehat{U}^* := \operatorname{int} I\!B_\gamma(\bar{x}^*) \subset \vartheta^{-1}(\widehat{U})$ for the inverse subdifferential localization $\vartheta\colon U^* \to U$ with $4\gamma < \alpha\nu$ and $I\!B_{2\gamma}(\bar{x}^*) \subset U^*$. We claim that for any $x^* \in \widehat{U}^*$ the following optimization problem attains its *unique minimum* at the point $\vartheta(x^*)$:

$$\text{minimize } \; f(x) - \langle x^*, x\rangle \; \text{ subject to } \; x \in I\!B_\nu(\bar{x}). \tag{3.11}$$

Indeed, fix $k \in I\!N$ and take any (approximate) k^{-2}-minimizer u_k of (3.11), i.e., such that

$$\inf\big\{\big(f + \delta(\cdot; I\!B_\nu(\bar{x}))\big)(x) - \langle x^*, x\rangle\big\} + k^{-2} \ge f(u_k) - \langle x^*, u_k\rangle.$$

Employing Ekeland's variational principle in this setting allows us to find a new sequence $\{\widehat{u}_k\} \subset I\!B_\nu(\bar{x})$ satisfying the relationships $\|\widehat{u}_k - u_k\| \le k^{-1}$ and

$$\inf\Big\{\big(f + \delta(\cdot; I\!B_\nu(\bar{x}))\big)(x) - \langle x^*, x\rangle + k^{-1}\|x - \widehat{u}_k\|\Big\} \ge f(\widehat{u}_k) - \langle x^*, \widehat{u}_k\rangle. \tag{3.12}$$

Combining (3.12) with inequality (3.8) shows that

$$f(\bar{x}) - \langle x^*, \bar{x}\rangle + k^{-1}\|\bar{x} - u_k\| \ge f(\widehat{u}_k) - \langle x^*, \widehat{u}_k\rangle \ge f(\bar{x}) + \langle \bar{x}^*, \widehat{u}_k - \bar{x}\rangle + \frac{\alpha}{2}\|\widehat{u}_k - \bar{x}\|^2 - \langle x^*, \widehat{u}_k\rangle,$$

and hence we have the estimates

$$\frac{\alpha}{2}\|\widehat{u}_k - \bar{x}\|^2 \le \langle x^* - \bar{x}^*, \widehat{u}_k - \bar{x}\rangle + k^{-1}\|\widehat{u}_k - \bar{x}\| \le \big(\|x^* - \bar{x}^*\| + k^{-1}\big)\|\widehat{u}_k - \bar{x}\|.$$

This yields in turn that

$$\|\widehat{u}_k - \bar{x}\| \le \frac{2}{\alpha}\Big(\|x^* - \bar{x}^*\| + k^{-1}\|\Big) \le \frac{2}{\alpha}\Big(\gamma + k^{-1}\Big) < \nu$$

for $k \in I\!N$ sufficiently large, and thus we get without loss of generality that $\widehat{u}_k \in \operatorname{int} I\!B_\nu(\bar{x})$ for all $k \in I\!N$. Applying the generalized Fermat rule to the minimizer $\widehat{u}_k$ of (3.12) and then the subdifferential sum rule again gives us

$$0 \in \partial\big(f + \delta(\cdot; I\!B_\nu(\bar{x}))\big)(\widehat{u}_k) - x^* + k^{-1}I\!B^* = \partial f(\widehat{u}_k) - x^* + k^{-1}I\!B^*, \tag{3.13}$$

which ensures the existence of $u_k^* \in k^{-1}I\!B^*$ with $\widehat{u}_k \in (\partial f)^{-1}(x^* + u_k^*)$. Since $x^* + u_k^* \in I\!B_{\eta+k^{-1}}(\bar{x}^*) \subset U^*$ and $\widehat{u}_k \in I\!B_\nu(\bar{x}) \subset U$, we get $\vartheta(x^* + u_k^*) = \widehat{u}_k$. It follows from the continuity of ϑ that $\widehat{u}_k = \vartheta(x^* + u_k^*) \to \vartheta(x^*)$ and so $u_k \to \vartheta(x^*)$ as $k \to \infty$. Combining this with (3.7) shows that problem (3.11) attains its minimum at $\vartheta(x^*)$ for any $x^* \in \widehat{U}^*$. By the above choice of u_k as an arbitrary k^{-2}-minimizer of (3.11), we justify that $\vartheta(x^*)$ is the unique optimal solution to this problem.

Now define the convex function $g := (f + \delta(\cdot; I\!B_\nu(\bar{x})))^* : X^* \to \overline{I\!R}$ and claim that

$$\operatorname{gph}\vartheta \cap (\widetilde{U}^* \times \widetilde{U}) = \operatorname{gph}\partial g \cap (\widetilde{U}^* \times \widetilde{U}) \tag{3.14}$$

for any neighborhoods $\widetilde{U} \subset \widehat{U}$ of $\bar{x}$ and $\widetilde{U}^* := \vartheta^{-1}(\widetilde{U})$ of $\bar{x}^*$. Indeed, note first that

$$\langle x^*, \vartheta(x^*)\rangle - g^*(\vartheta(x^*)) \ge \langle x^*, \vartheta(x^*)\rangle - f(\vartheta(x^*)) = \sup_{x \in \widehat{U}}\big\{\langle x^*, x\rangle - f(x)\big\} = g(x^*).$$

for every $x^* \in \widehat{U}^*$, which implies that $\vartheta(x^*) \in \partial g(x^*)$. Moreover, since $\vartheta(x^*)$ is the unique solution of problem (3.11), it is easy to check that ϑ is monotone. Thus ϑ is maximal monotone on $\widetilde{U}^* \times \widetilde{U}$. Due to the monotonicity of ∂g we ensure the equality in (3.14).

Considering next the conjugate $h := g^* : X \to \overline{I\!R}$ of g and following the proof of the corresponding part in Theorem 3.1 guarantee that for $(u^*, u) \in (\mathrm{gph}\,\vartheta) \cap (\widetilde{U}^* \times \widetilde{U}) = (\mathrm{gph}\,(\partial f)^{-1}) \cap (\widetilde{U}^* \times \widetilde{U})$ we get the estimate

$$h(x) \geq h(u) + \langle u^*, x - u\rangle + \frac{1}{2\kappa}\|x - u\|^2 \quad \text{whenever} \quad x \in \widetilde{U}. \tag{3.15}$$

Lemma 2.1 tells us that $(\partial h)^{-1}(u^*) = \nabla g(u^*) = \vartheta(u^*) = u$. Since $\vartheta(x^*)$ is the unique minimizer of problem (3.11), we have that

$$\langle u^*, u\rangle - h(u) = h^*(u^*) = g^{**}(u^*) = g(u^*) = \langle u^*, \vartheta(u^*)\rangle - f\big(\vartheta(u^*)\big) = \langle u^*, u\rangle - f(u),$$

which implies that $f(u) = h(u)$, and furthermore

$$h(x) = \big(f + \delta(\cdot; I\!B_\nu(\bar{x}))\big)^{**}(x) \leq \big(f + \delta(\cdot; I\!B_\nu(\bar{x}))\big)(x) = f(x) \quad \text{for all} \quad x \in \widetilde{U} \subset I\!B_\nu(\bar{x}).$$

This together with (3.15) gives us (3.1) and completes the proof of the theorem. $\triangle$

When $\bar{x}$ is a *local minimizer* of f in Theorem 3.2, we get the following characterization of strong metric regularity of the limiting subdifferential of f given in the form of Theorem 3.1.

Corollary 3.3 (characterizing strong metric regularity of the limiting subdifferential at minimum points). *Let X be a Asplund space, and let $f : X \to \overline{I\!R}$ be a l.s.c. function. Assume that $\bar{x}$ is a local minimizer of f. The following are equivalent:*

(i) *∂f is strongly metrically regular with modulus $\kappa > 0$ around $(\bar{x}, 0)$.*

(ii) *There are neighborhoods U of $\bar{x}$ and U^* of 0 such that $(\partial f)^{-1}$ has a single-valued localization $\vartheta : U^* \to U$ and that for any $(u^*, u) \in \mathrm{gph}\,\vartheta = \mathrm{gph}\,(\partial f)^{-1} \cap U^* \times U$ we have*

$$f(x) \geq f(u) + \langle u^*, x - u\rangle + \frac{1}{2\kappa}\|x - u\|^2 \quad \textit{for all} \quad x \in U. \tag{3.16}$$

Proof. It follows from the generalized Fermat rule that $0 \in \partial f(\bar{x})$ if $\bar{x}$ is an arbitrary local minimizer of the function f. Then condition (3.7) holds automatically with $\bar{x}^* = 0$ and get the claimed characterization from Theorem 3.2. $\triangle$

Some comments on the results of Theorem 3.2 and Corollary 3.3 are now in order.

Remark 3.4 (comments on characterizations of strong metric regularity for the limiting subdifferential). We have the following observations:

(i) The assumption on "$r\kappa < 1$" is essential in the conclusion of Theorem 3.2. If this assumption fails, condition (3.7) does not guarantee the validity of assertion (ii) in Theorem 3.1. To illustrate it, take $f(\cdot) := -\frac{1}{2\kappa}\|\cdot\|^2$ in the Hilbert space X with an arbitrary number $\kappa > 0$, $\bar{x} = 0$, and $\bar{x}^* = 0$ and observe that (3.7) holds for all $r \geq \kappa^{-1}$ and that the subgradient mapping ∂f is strongly metrically regular at $(0, 0)$ with modulus κ. On the other hand, the second-order growth condition (3.1) is not satisfied at $(u, u^*) = (0, 0)$.

(ii) If $X = I\!R^n$ and ∂f is subdifferentially continuous at $\bar{x}$ for 0, the result of Corollary 3.3 has been recently obtained in [9, Theorem 3.3] without mentioning the relationship for the modulus κ in (i) and (ii). It has been also shown in [9, Theorem 3.3] that these conditions are equivalent to the property of $\bar{x}$ to be so-called "stable strong local minimizer." Furthermore, [9, Example 3.4] demonstrates that the latter equivalence fails without the subdifferential continuity of ∂f. Note to this end that our results in Corollary 3.3 do not require the subdifferential continuity of f while neither (i) nor (ii) of Corollary 3.3 holds in

[9, Example 3.4], where $\bar{x}$ is a stable strong local minimizer of f. It is worth mentioning that the "stable strong local minimizer" requirement of [9] is similar to the so-called "uniform second-order growth condition with respect to the tilt parametrization" first used in [6, Theorem 5.36] for characterizing tilt-stable local minimizers for conic programs with twice continuously differentiable data in Banach spaces. The latter result can be easily deduced from Corollary 3.3 when the space in question is Asplund.

(iii) It follows from the proof above that the characterizations in Theorem 3.2 and Corollary 3.3 are valid for strong metric regularity of *abstract subdifferentials* satisfying the sum rule in the framework of (3.13) with just one non-Lipschitzian term in the corresponding Banach space settings: see [21, Section 2.5] and the references therein for more discussions on such constructions and their specifications.

4 Second-Order Characterizations of Tilt Stability

We start this section with characterizing tilt-stable minimizers of a convex function via strong metric regularity of its subdifferential in general Banach space.

Proposition 4.1 (tilt-stable local minimizers of convex functions in Banach space). *Let X be a Banach space, and let $f : X \to \overline{I\!R}$ be a l.s.c. convex function with $\bar{x} \in \operatorname{dom} f$. The following assertions are equivalent:*

(i) *The point $\bar{x}$ is a tilt-stable local minimizer of f with modulus $\kappa > 0$.*

(ii) *The point $\bar{x}$ is a global minimizer of f and the subgradient mapping ∂f is strongly metrically regular around $(\bar{x}, 0)$ with modulus κ.*

Proof. Assuming first that assertion (i) holds, we find numbers $\gamma, \eta > 0$ such that the function M_γ in (2.12) is single-valued and Lipschitz continuous with modulus κ on the set $U^* := I\!B_\eta(0) \subset X^*$. It is obvious that $\bar{x}$ is a local minimizer of the convex function f and thus it is a global minimizer as well. Define $g := \big(f + \delta(\cdot; U)\big)^*$ with $U := I\!B_\gamma(\bar{x})$. It follows from Lemma 2.1 that $\partial g(x^*) = M_\gamma(x^*)$ for all $x^* \in U^*$. Hence the subdifferential ∂g is single-valued and Lipschitz continuous with modulus κ on U^*. It is shown in the proof of Theorem 3.1 that g is Fréchet differentiable and its derivative $\nabla g : U^* \to I\!B_\gamma(\bar{x})$ is Lipschitz continuous on U^*. Defining further $\widetilde{U} := I\!B_{\frac{\gamma}{2}}(\bar{x})$ and $\widetilde{U}^* := (\nabla g)^{-1}(\widetilde{U})$, we have

$$\nabla g(x^*) = \partial\big(f + \delta(\cdot; U)\big)^{-1}(x^*) = (\partial f)^{-1}(x^*) \cap \widetilde{U} \text{ for all } x^* \in \widetilde{U}^*.$$

This shows that $\operatorname{gph} \nabla g \cap (\widetilde{U}^* \times \widetilde{U}) = \operatorname{gph} (\partial f)^{-1} \cap (\widetilde{U}^* \times \widetilde{U})$ and thus proves (ii).

Conversely, suppose that (ii) holds. By Theorem 3.1 there are neighborhoods $U \subset X$ of $\bar{x}$ and $U^* \subset X^*$ of $\bar{x}^* := 0$ as well as a single-valued and Lipschitz continuous localization $\vartheta : U^* \to U$ of $(\partial f)^{-1}$ around $(0, \bar{x})$ such that the second-order growth condition (3.1) holds. Take further any $\gamma > 0$ with $\widetilde{U} := I\!B_\gamma(\bar{x}) \subset U$ and $\widetilde{U}^* := \vartheta^{-1}(\widetilde{U})$. It easily follows from (3.1) that whenever $x^* \in \widetilde{U}^*$ we have the estimate

$$f(x) - \langle x^*, x\rangle \ge f\big(\vartheta(x^*)\big) - \langle x^*, \vartheta(x^*)\rangle + \frac{1}{2\kappa}\|x - \vartheta(x^*)\|^2 \text{ for all } x \in \widetilde{U} = I\!B_\gamma(\bar{x}).$$

This yields that $\vartheta(x^*) = M_\gamma(x^*)$ and thus justifies the claimed tilt stability of $\bar{x}$ with the prescribed modulus κ completing the proof of the proposition. △

Next we establish equivalent relationships between tilt stability of local minimizers and strong metric regularity of the limiting subdifferential for nonconvex extended-real-valued functions in Asplund spaces.

Theorem 4.2 (tilt stability via strong metric regularity of the limiting subdifferential for nonconvex functions in Asplund spaces). *Let X be an Asplund space, and let $f : X \to \overline{I\!R}$ be a l.s.c. function with $\bar{x} \in \operatorname{dom} f$. Suppose that $0 \in \partial f(\bar{x})$ and that f is subdifferentially continuous at $\bar{x}$ for $\bar{x}^* = 0$. Then the following assertions are equivalent:*

(i) *The function f is prox-regular at $\bar{x}$ for 0 with some constant $r < \kappa^{-1}$ in (2.11) and $\bar{x}$ is a tilt-stable minimizer of f with modulus $\kappa > 0$.*

(ii) *The subdifferential ∂f is strongly metrically regular around $(\bar{x}, 0)$ with modulus κ and $\bar{x}$ is a local minimizer of f.*

If X is a Hilbert space, then equivalence (i)$\Longleftrightarrow$ (ii) *is still valid without imposing the condition of $r < \kappa^{-1}$ in* (i)*. Finally, under the validity of the equivalent conditions* (i) *and* (ii) *the exact Lipschitzian bound of the function M_γ in* (2.12) *is calculated by the formula*

$$\operatorname{lip} M_\gamma(\bar{x}) = \inf_{\eta>0} \sup \Big\{ \|u\| \Big| u^* \in \breve{\partial}^2 f(x, x^*)(u), (x, x^*) \in \operatorname{gph} \partial f \cap I\!B_\eta(\bar{x}, 0), \|u^*\| \le 1 \Big\} \quad (4.1)$$

for $\gamma > 0$ sufficiently small. If $X = I\!R^n$, then (4.1) *reduces to the pointbased formula*

$$\operatorname{lip} M_\gamma(\bar{x}) = \max \Big\{ \|u\| \Big| \ u^* \in \partial^2 f(\bar{x}, 0)(u), \ \|u^*\| \le 1 \Big\}. \quad (4.2)$$

Proof. Suppose that (i) holds, and thus $\bar{x}$ is obviously a local minimizer of f. To justify (ii), we need to show that the subdifferential ∂f is strongly metrically regular around $(\bar{x}, 0)$. By the assumed tilt stability, find $\gamma > 0$ such that the mapping M_γ from (2.12) is single-valued and Lipschitz continuous on a neighborhood U^* of 0. Define $g := (f + \delta(\cdot; U))^*$ with $U := I\!B_\gamma(\bar{x})$. Similarly to the proof of (3.14) we have that $M_\gamma(x^*) = \partial g(x^*)$ for all $x^* \in U^*$. Hence g is Fréchet differentiable and ∇g is Lipschitz continuous with modulus κ on U^*. Letting $h := g^*$ and following the last part in the proof of Theorem 3.2 allow us to find neighborhoods $\widetilde{U}$ of $\bar{x}$ with $\widetilde{U} \subset \operatorname{int} U$ and $\widetilde{U}^*$ of 0 in X^* such that for $(u^*, u) \in \operatorname{gph} M_\gamma \cap (\widetilde{U}^* \times \widetilde{U})$ we have the growth condition (3.15) and furthermore $h(u) = f(u)$ with $h(x) \le f(x)$ for $x \in \widetilde{U}$. This gives

$$f(x) \ge f(u) + \langle u^*, x - u \rangle + \frac{1}{2\kappa} \|x - u\|^2 \ \text{ for all } \ x \in \widetilde{U}, \quad (4.3)$$

which implies in turn that $u^* \in \partial f(u) \cap \widetilde{U}^*$, i.e., $u \in (\partial f)^{-1}(u^*) \cap \widetilde{U}$. Since f is subdifferentially continuous at $\bar{x}$ for 0, it is possible to assume that $\widetilde{U} \subset I\!B_\varepsilon(\bar{x})$ and $\widetilde{U}^* \subset I\!B_\varepsilon(0)$ with $|f(u) - f(\bar{x})| \le \varepsilon$, where ε is taken from (2.11). Picking now an arbitrary element $v \in (\partial f)^{-1}(u^*) \cap \widetilde{U}$, we get from (2.11) that

$$f(u) - f(v) \ge \langle u^*, u - v \rangle - \frac{r}{2} \|u - v\|^2.$$

Choosing $x = v$ in (4.3) and then combining that inequality with the latter give us that $\frac{r}{2}\|v - u\|^2 \ge \frac{1}{2\kappa}\|v - u\|^2$ and thus $v = u$ due to the assumption of $r\kappa < 1$. It follows therefore that $M_\gamma(u^*) = (\partial f)^{-1}(u^*) \cap \widetilde{U}$ for all $u^* \in \widetilde{U}^*$, which ensures the strong metric regularity of ∂f around $(\bar{x}, 0)$ and thus justifies (ii) in the Asplund space setting.

If the space X is Hilbert, we can proceed in justifying (ii) without the assumption that $r < \kappa^{-1}$ in (i). Indeed, in this case we get from the prox-regularity (2.11) with an arbitrary constant $r > 0$ that the operator $T + rI$ is monotone, where T is a localization of ∂f relative to $\widetilde{U}$ and $\widetilde{U}^*$. Let S be a maximal monotone extension of $(M_\gamma)^{-1}$, which is maximal monotone relative to $\widetilde{U}$ and $\widetilde{U}^*$; see [3, Theorem 20.11]. Hence

$$\operatorname{gph} S \cap (\widetilde{U} \times \widetilde{U}^*) = \operatorname{gph} (M_\gamma)^{-1} \cap (\widetilde{U} \times \widetilde{U}^*) \subset \operatorname{gph} T \cap (\widetilde{U} \times \widetilde{U}^*).$$

It follows from the proof of [26, Lemma 3.1] valid in Hilbert spaces that the sets $\mathrm{gph}\, S$, $\mathrm{gph}\, M_\gamma^{-1}$, and $\mathrm{gph}\, T$ coincide locally around $(\bar{x}, 0)$. This implies the strong metric regularity of ∂f around $(\bar{x}, 0)$ by the Lipschitz property of M_γ around $(\bar{x}^*, 0)$ imposed in (i).

Next we suppose that (ii) holds and get from Corollary 3.3 that the growth condition (3.16) is satisfied. Proceeding similarly to the proof of Proposition 4.1, find $\gamma > 0$ such that the mapping M_γ is single-valued and Lipschitz continuous on some neighborhood U^* of $0 \in X^*$. Taking into account that the prox-regularity of f at $\bar{x}$ for $\bar{x}^* = 0$ is straightforward from (3.16), we arrive at all the statements in (i) with an arbitrary constant $r > 0$.

To complete the proof of the theorem, it remains to justify the exact bound formulas in (4.1) and (4.2) under the validity of the equivalent assertions (i) and (ii). Indeed, we get from the proof above that $\mathrm{lip}\, M_\gamma(\bar{x}) = \mathrm{lip}\,(\partial f)^{-1}(0, \bar{x})$, and so applying [21, Theorem 4.7] to $(\partial f)^{-1}$ gives us the expressions

$$\begin{aligned}\mathrm{lip}\, M_\gamma(\bar{x}) &= \inf_{\eta>0} \sup \Big\{ \|u\| \Big| \; u \in (\widehat{D}^* \partial f^{-1})(x^*, x)(u^*), \|u^*\| \le 1, (x^*, x) \in \mathrm{gph}\,(\partial f)^{-1} \cap I\!B_\eta(0, \bar{x}) \Big\} \\ &= \inf_{\eta>0} \sup \Big\{ \|u\| \Big| \; u^* \in (\widehat{D}^* \partial f)(x, x^*)(u), \|u^*\| \le 1, (x, x^*) \in (\mathrm{gph}\, \partial f) \cap I\!B_\eta(\bar{x}, 0) \Big\}.\end{aligned}$$

The proof of (4.2) is similar by employing [21, Theorem 4.10] instead of [21, Theorem 4.7] in the case of finite-dimensional spaces with taking into account that the supremum is attained in (4.2) due to the outer semicontinuity and locally boundedness of the coderivative positively homogeneous mapping. This completes the proof of the theorem. $\triangle$

Observe that the proof of implication (ii)$\Longrightarrow$(i) in Theorem 4.2 does not require the subdifferential continuity of f at $\bar{x}$ for 0. When X is finite-dimensional, equivalence (i)$\Longleftrightarrow$(ii) in the above theorem has been recently derived in [9, Theorem 3.3] by employing the deep result of [26, Theorem 1.3] (not used here) and without the quantitative relationships between the moduli. The exact bound formula (4.1) is new while its finite-dimensional counterpart (4.2) is given in the form different but equivalent to that obtained in [17, Theorem 2.3] in the case of tilt stability. Note finally that the requirement $r < \kappa^{-1}$ in Theorem 4.2(i) holds automatically when the function f is convex. In this case the equivalence of Theorem 4.2 reduces to the result of Proposition 4.1.

Next we present a major result of this paper, which fully characterizes tilt-stable local minimizers of prox-regular and subdifferentially continuous extended-real-valued functions in Hilbert spaces via the combined second-order subdifferential (2.13).

Theorem 4.3 (second-order subdifferential characterizations of tilt-stable minimizers in Hilbert spaces). *Let X be a Hilbert space, and let $f : X \to \overline{I\!R}$ be a l.s.c. function having $0 \in \partial f(\bar{x})$. Assume that f is both prox-regular and subdifferentially continuous at $\bar{x}$ for $\bar{x}^* = 0$. Then the following assertions are equivalent:*

(i) *The point $\bar{x}$ is a tilt-stable local minimizer of the function f with modulus $\kappa > 0$.*

(ii) *There is a constant $\eta > 0$ such that for all $u \in X$ we have*

$$\langle u^*, u\rangle \ge \frac{1}{\kappa}\|u\|^2 \;\; \text{whenever} \;\; u^* \in \breve{\partial}^2 f(x, x^*)(u) \;\; \text{with} \;\; (x, x^*) \in \mathrm{gph}\, \partial f \cap I\!B_\eta(\bar{x}, 0) \tag{4.4}$$

via the combined second-order subdifferential (2.13) *of f around $(\bar{x}, 0)$. Furthermore, the equivalent assertions* (i) *and* (ii) *imply that the exact Lipschitzian bound of the mapping M_γ from* (2.12) *for all $\gamma > 0$ sufficiently small is computed by the formula*

$$\mathrm{lip}\, M_\gamma(\bar{x}) = \inf_{\eta>0} \sup \Big\{ \frac{\|u\|^2}{\langle u^*, u\rangle} \Big| \; u^* \in \breve{\partial}^2 f(x, x^*)(u), \; (x, x^*) \in \mathrm{gph}\, \partial f \cap I\!B_\eta(\bar{x}, 0) \Big\}$$

(*with the convention that* $0/0 = 0$) *and that the condition*

$$\langle u^*, u\rangle \ge \frac{1}{\kappa}\|u\|^2 \quad \textit{whenever} \quad u^* \in \partial^2_M f(\bar{x}, 0)(u), \quad u \in X \tag{4.5}$$

is satisfied in terms of the mixed second-order subdifferential (2.14) *of* f *at* $(\bar{x}, 0)$.

Proof. Suppose first that assertion (i) holds. Then Theorem 4.2 tells us that the subdifferential ∂f is strongly metrically regular around $(\bar{x}, 0)$ and that $\bar{x}$ is a local minimizer of f. Thus Corollary 3.3 allows us to find neighborhoods U of $\bar{x}$ and U^* of $0 \in X^*$ such that the mapping $(\partial f)^{-1}$ admits a single-valued Lipschitzian localization $\vartheta : U^* \to U$ and the second-order growth condition (3.1) is satisfied. This gives us the estimate

$$\langle x^* - u^*, \vartheta(x^*) - \vartheta(u^*)\rangle \ge \frac{1}{\kappa}\|\vartheta(x^*) - \vartheta(u^*)\|^2 \text{ for all } x^*, u^* \in U^*,$$

which implies that the mappings ϑ and $\vartheta^{-1} - \kappa^{-1}I$ are monotone. Since ϑ is continuous, it is maximal monotone and so is its inverse ϑ^{-1}. Denoting by Θ the maximal monotone *extension* of the mapping $\vartheta^{-1} - \kappa^{-1}I$, we get that the mapping $\Theta + \kappa^{-1}I$ is a monotone with $\operatorname{gph}\vartheta^{-1} \subset \operatorname{gph}(\Theta + \kappa^{-1}I) \cap (U \times U^*)$; see [3, Theorem 20.11]. Since ϑ^{-1} is maximal monotone relative to U and U^*, it follows that

$$\operatorname{gph}\partial f \cap (U \times U^*) = \operatorname{gph}\vartheta^{-1} = \operatorname{gph}(\Theta + \kappa^{-1}I) \cap (U \times U^*).$$

Let $\eta > 0$ be such that $I\!B_{2\eta}(\bar{x}, 0) \subset U \times U^*$. For any $(x, x^*) \in \operatorname{gph}\partial f \cap I\!B_\eta(\bar{x}, 0)$ we get from definition (2.12) and [21, Theorem 1.62] that

$$\breve{\partial}^2 f(x, x^*) = \widehat{D}^*\partial f(x, x^*) = \widehat{D}^*\vartheta^{-1}(x, x^*) = \widehat{D}^*\Theta(x, x^* - \kappa^{-1}x) + \kappa^{-1}I. \tag{4.6}$$

Picking now any $u^* \in \breve{\partial}^2 f(x, x^*)(u)$ with $u \in X$ and using (4.6) give us the inclusion $u^* - \kappa^{-1}u \in \widehat{D}^*\Theta(x, x^* - \kappa^{-1}x)(u)$. It follows from the maximal monotonicity of Θ and from Lemma 6.2 in Section 6 that $\langle u^* - \kappa^{-1}u, u\rangle \ge 0$, i.e., $\langle u^*, u\rangle \ge \kappa^{-1}\|u\|^2$. This fully justifies condition (4.4) as well as the inequality "$\ge$" in the formula for $\operatorname{lip} M_\gamma(\bar{x})$. The mixed second-order subdifferential condition (4.5) is obtained by passing to the limit in (4.4) by using the strong convergence of the arguments by the definitions in (2.8) and (2.14).

Conversely, assume that (ii) holds. Without loss of generality suppose further that $\bar{x} = 0$ and $f(\bar{x}) = 0$; otherwise we may replace the function f by $f(\cdot + \bar{x}) - f(\bar{x})$. It follows from the definition of prox-regular functions in (2.11) that there are some positive constants r and $\varepsilon > 0$ satisfying $f(x) \ge -\frac{r}{2}\|x\|^2$ for all $x \in I\!B_\varepsilon(0)$. Substituting f by $f + \delta(\cdot; I\!B_\varepsilon(0))$ if necessary, we may assume that

$$f(x) \ge -\frac{r}{2}\|x\|^2 \text{ for all } x \in X,$$

which means that f is minorized by a quadratic function. By Lemma 6.1 there is some number $\lambda_0 > 0$ such that the operator $T + \lambda_0 I$ is locally maximal monotone around $(0, 0)$, where T is a localization of ∂f at $(\bar{x}, 0)$ relative to neighborhoods U of $\bar{x}$ and U^* of 0 satisfying $U \times U^* \subset I\!B_\eta(\bar{x}, 0)$. Let R be a maximal monotone extension of $T + \lambda_0 I$, and let $J_1 : X \times X \to X \times X$ be defined by $J_1(x, y) := (x, y + \lambda_0 x)$ for all $(x, y) \in X \times X$. Employing the classical open mapping theorem, we get that the set $V := J_1(U \times U^*)$ is open in $X \times X$. Observe the relationship

$$\operatorname{gph}(T + \lambda_0 I) \cap V = \operatorname{gph} R \cap V \tag{4.7}$$

and then for any $\lambda > 0$ define the resolvent of R by $R_\lambda := (I + \lambda R)^{-1}$ and the mapping $J_2 : X \times X \to X \times X$ by $J_2(x, y) := (x + \lambda y, x)$ for all $(x, y) \in X \times X$. The celebrated Minty theorem tells us that $\operatorname{dom} R_\lambda = X$. Note further that $(0, 0) = J_2(0, 0) \in \operatorname{gph} R_\lambda = J_2(\operatorname{gph} R)$. Using again the open mapping theorem ensures that $W := J_2(V)$ is a neighborhood of $(0, 0)$. Picking any $u^* \in \widehat{D}^* R_\lambda(x, x^*)(u)$ with $u \in X$ and $(x, x^*) \in W$, we obtain that $-u \in \widehat{D}^*(I + \lambda R)(x^*, x)(-u^*)$. It follows from (4.7) and [21, Theorem 1.62] that

$$\begin{aligned} -u &\in -u^* + \widehat{D}^* R\Big(x^*, \frac{x - x^*}{\lambda}\Big)(-\lambda u^*) = -u^* + \widehat{D}^*(\lambda_0 I + T)\Big(x^*, \frac{x - x^*}{\lambda}\Big)(-\lambda u^*) \\ &= -u^* - \lambda\lambda_0 u^* + \widehat{D}^* T\Big(x^*, \frac{x - x^*}{\lambda} - \lambda_0 x^*\Big)(-\lambda u^*) \end{aligned} \tag{4.8}$$

for the elements (u, u^*, x, x^*) chosen above. Observe that

$$\Big(x^*, \frac{x - (1 + \lambda\lambda_0)x^*}{\lambda}\Big) = J_1^{-1}(J_2^{-1}(x, x^*)) \in J_1^{-1}\big(J_2^{-1}(W)\big) = J_1^{-1}(V) = U \times U^* \subset I\!B_\eta(\bar{x}, 0),$$

which implies together with (4.4) and (4.8) that $\langle u^*, u - (1 + \lambda\lambda_0)u^*\rangle \geq \lambda\mu\|u^*\|^2$, where $\mu := \kappa^{-1}$. Thus we arrive at the two-sided estimates

$$[1 + \lambda(\lambda_0 + \mu)]\|u^*\|^2 \leq \langle u^*, u\rangle \leq \|u^*\| \cdot \|u\| \;\text{ whenever }\; u^* \in \widehat{D}^* R_\lambda(x, x^*)(u). \tag{4.9}$$

Since $\operatorname{gph} T$ is closed in norm$\times$norm topology around $(0, 0)$ due to the prox-regularity and continuity of the subdifferential ∂f at 0 for 0, the sets $\operatorname{gph} R$ and $\operatorname{gph} R_\lambda$ are also closed around this point. By Lemma 6.4 below the estimates in (4.9) show that the mapping R_λ^{-1} is metrically regular with modulus $[1 + \lambda(\lambda_0 + \mu)]^{-1}$, or equivalently the single-valued mapping R_λ is Lipschitz continuous on some small ball $I\!B_\alpha(0)$ with the same modulus.

To proceed further, find two neighborhoods $U_\alpha \subset U$ and $U_\alpha^* \subset U^*$ of 0 satisfying $(1 + \lambda\lambda_0)x + x^* \in I\!B_\alpha(0)$ for $(x, x^*) \in U_\alpha \times U_\alpha^*$. Picking any $(x_1, x_1^*), (x_2, x_2^*) \in \operatorname{gph} T \cap (U_\alpha \times U_\alpha^*)$, it follows from (4.7) and the representation $R_\lambda = (I + \lambda R)^{-1}$ that

$$x_i \in R_\lambda\big((1 + \lambda_0\lambda)x_i + \lambda x_i^*\big) \;\text{ for }\; i = 1, 2.$$

Hence we have the relationships

$$\begin{aligned} &[1 + \lambda(\lambda_0 + \mu)]^2\|x_1 - x_2\|^2 \leq \|(1 + \lambda_0\lambda)(x_1 - x_2) + \lambda(x_1^* - x_2^*)\|^2 \leq \\ &\leq (1 + \lambda_0\lambda)^2\|x_1 - x_2\|^2 + 2\lambda(1 + \lambda_0\lambda)\langle x_1 - x_2, x_1^* - x_2^*\rangle + \lambda^2\|x_1^* - x_2^*\|^2, \end{aligned} \tag{4.10}$$

which imply in turn the inequalities

$$\begin{aligned} 0 &\geq \lambda\mu[2 + 2\lambda\lambda_0 + \lambda\mu]\|x_1 - x_2\|^2 - 2\lambda(1 + \lambda_0\lambda)\|x_1 - x_2\|\|x_1^* - x_2^*\| - \lambda^2\|x_1^* - x_2^*\|^2 \\ &\geq \lambda(\mu\|x_1 - x_2\| - \|x_1^* - x_2^*\|)([2 + 2\lambda\lambda_0 + \lambda\mu]\|x_1 - x_2\| + \lambda\|x_1^* - x_2^*\|). \end{aligned}$$

Summarizing the above gives us the estimate

$$\|x_1 - x_2\| \leq \mu^{-1}\|x_1^* - x_2^*\| = \kappa\|x_1^* - x_2^*\|,$$

which means that the mapping T^{-1} is single-valued and Lipschitz continuous with constant κ on $\operatorname{dom} T^{-1} \cap U_\alpha^*$ relative to U_α. Since T is a localization of ∂f, we find neighborhoods $\widetilde{U} := \operatorname{int} I\!B_\nu(\bar{x})$ of $\bar{x}$ with some $\nu > 0$, $\widetilde{U}^*$ of $0 \in X^*$, and a single-valued localization ϑ of $(\partial f)^{-1}$ relative to $\widetilde{U}^*$ and $\widetilde{U}$ such that ϑ is Lipschitz continuous with constant κ on its domain. By going back to the proof of Theorem 3.2, observe that this property of ϑ still ensures that the optimization problem (3.11) attains its minimum at $\vartheta(x^*)$ for all x^* from some neighborhood of $0 \in X^*$ since $\bar{x}$ is the local minimizer of f; see below. This shows that the set $\mathrm{int}(\mathrm{dom}\,\vartheta)$ is a neighborhood of the origin and thus justifies the strong metric regularity of ∂f around $(\bar{x}, 0)$.

To justify assertion (ii), it remains to show by Theorem 4.2 that $\bar{x}$ is the local minimizer of f. Observe to this end that (4.10) implies the inequality

$$\langle x_1 - x_2, x_1^* - x_2^* \rangle + \frac{\lambda}{2(1+\lambda_0\lambda)} \|x_1^* - x_2^*\|^2 \geq 0 \text{ for all } (x_1, x_1^*), (x_2, x_2^*) \in \mathrm{gph}\, T \cap (U_\alpha \times U_\alpha^*),$$

which yields in turn that the mapping $\xi I + T^{-1}$ with $\xi := \frac{\lambda}{2(1+\lambda_0\lambda)}$ is monotone around $(0,0)$. When $\lambda > 0$ is sufficiently small, we obtain from [5, Proposition 4.4 and Proposition 5.3] that $e_\xi f$ is $\mathcal{C}^{1,1}$ on a neighborhood of the origin and that $\nabla e_\xi f = \xi^{-1}(I - (I + \xi T)^{-1}) = (\xi I + T^{-1})^{-1}$ is monotone around $(0,0)$, where $e_\xi f$ is the *Moreau envelope* of f defined by

$$e_\xi f(x) := \inf_{u \in X} \Big\{ f(u) + \frac{1}{2\xi}\|u - x\|^2 \Big\}.$$

It is well known that $e_\xi f$ is locally convex around $x = 0$. Since $0 = \nabla e_\xi(0)$, we get $f(x) \geq e_\xi f(x) \geq e_\xi f(0) = 0$ for x around $\bar{x} = 0$, and thus $\bar{x}$ is a local minimizer of f. This ensures the validity of assertion (i) and of the inequality "$\leq$" in the formula for $\mathrm{lip}\, M_\gamma(\bar{x})$.

Finally, the necessary condition (4.5) for tilt-stable minimizers follows from (4.4) by passing to the limit as $\eta \downarrow 0$ and using definition (2.14) of the mixed second-order subdifferential. This completes the proof of the theorem. $\triangle$

Next we present several consequences of Theorem 4.3 and other results of this section important for their own sake. The first corollary concerns the finite-dimensional setting and provides both qualitative and quantitative characterizations of tilt-stable minimizers.

Corollary 4.4 (tilt stability in finite dimensions). *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a l.s.c. function having $0 \in \partial f(\bar{x})$ and such that f is prox-regular and subdifferentially continuous at $\bar{x}$ for 0. Consider the following statements:*

(i) *The point $\bar{x}$ is a tilt-stable local minimizer of f with modulus $\kappa > 0$.*

(ii) *$\partial^2 f(\bar{x}, 0)$ is* POSITIVE-DEFINITE WITH MODULUS *$\mu > 0$ in the sense that*

$$\langle u^*, u \rangle \geq \mu \|u\|^2 \quad \text{whenever} \quad u^* \in \partial^2 f(\bar{x}, 0)(u). \tag{4.11}$$

Then implication (i) $\Longrightarrow$ (ii) *holds with $\mu = \kappa^{-1}$ while implication* (ii) $\Longrightarrow$ (i) *is satisfied with any $\kappa > \mu^{-1}$. Furthermore, the validity of* (i) *with some modulus $\kappa > 0$ is equivalent to* POSITIVE-DEFINITENESS *of $\partial^2 f(\bar{x}, 0)$ in the sense that*

$$\langle u^*, u \rangle > 0 \quad \text{whenever} \quad u^* \in \partial^2 f(\bar{x}, 0)(u),\ u \neq 0. \tag{4.12}$$

Finally, the exact Lipschitzian bound of the mapping M_γ in (2.12) *is calculated by*

$$\mathrm{lip}\, M_\gamma(\bar{x}) = \max \Big\{ \frac{\|u\|^2}{\langle u^*, u \rangle} \Big|\ u^* \in \partial^2 f(\bar{x}, 0)(u) \Big\}. \tag{4.13}$$

(*with the convention that* $0/0 = 0$) *for all $\gamma > 0$ sufficiently small, provided that $\bar{x}$ is a tilt-stable minimizer of f.*

Proof. Implication (i)$\Longrightarrow$(ii) follows directly from Theorem 4.3. To justify the converse implication (ii)$\Longrightarrow$(i), pick any $u^* \in D^*R_\lambda(0,0)(u)$ and observe from the proof of Theorem 4.3 that $[1+\lambda(\lambda_0+\mu)]\|u^*\| \le \|u\|$. It follows from [21, Theorem 4.10] that $\operatorname{lip} R_\lambda(0) \le [1+\lambda(\lambda_0+\mu)]^{-1}$, which allows us to find a constant $\alpha > 0$ as well as neighborhoods $U_\alpha \subset U$ and $U^*_\alpha \subset U^*$ of the origin such that R_λ is Lipschitz continuous on $I\!B_\alpha(0)$ with modulus $[1+\lambda(\lambda_0+\mu)]^{-1}+\lambda^2$ and that $(1+\lambda\lambda_0)x + x^* \in I\!B_\alpha(0)$ for all $(x,x^*) \in U_\alpha \times U^*_\alpha$. Taking further any pairs $(x_1,x_1^*),(x_2,x_2^*) \in \operatorname{gph} T \cap (U_\alpha \times U^*_\alpha)$, we get that $x_i \in R_\lambda((1+\lambda_0\lambda)x_i + \lambda x_i^*)$ for $i=1,2$ and hence

$$\|x_1-x_2\|^2 \le \big\{[1+\lambda(\lambda_0+\mu)]^{-1}+\lambda^2]\big\}^2 \|(1+\lambda_0\lambda)(x_1-x_2)+\lambda(x_1^*-x_2^*)\|^2,$$

which implies in turn that

$$\begin{aligned} 2\lambda[\mu+O(\lambda)]\|x_1-x_2\|^2 &= \Big[1-\frac{(1+\lambda_0\lambda)^2}{[1+\lambda(\lambda_0+\mu)]^2}+o(\lambda)\Big]\|x_1-x_2\|^2 \\ &\le 2\lambda[1+O(\lambda)]\langle x_1^*-x_2^*, x_1-x_2\rangle + o(\lambda)\|x_1^*-x_2^*\|^2 \\ &\le 2\lambda[1+O(\lambda)]\|x_1^*-x_2^*\|\cdot\|x_1-x_2\| + o(\lambda)\|x_1^*-x_2^*\|^2. \end{aligned}$$

Dividing both sides of the obtained estimate by $\lambda > 0$ gives us

$$[\mu+O(\lambda)]\|x_1-x_2\|^2 - [1+O(\lambda)]\|x_1^*-x_2^*\|\cdot\|x_1-x_2\| - O(\lambda)\|x_1^*-x_2^*\|^2 \le 0,$$

and then we arrive at the norm estimate

$$\begin{aligned} \|x_1-x_2\| &\le \frac{1+O(\lambda)+\sqrt{[1+O(\lambda)]^2-4O(\lambda)[\mu+O(\lambda)]}}{2\mu+2O(\lambda)}\|x_1^*-x_2^*\| \\ &= [\mu^{-1}+O(\lambda)]\|x_1^*-x_2^*\|. \end{aligned} \tag{4.14}$$

For any given modulus $\kappa > \mu^{-1}$ choose positive numbers α and $\lambda > 0$ sufficiently small and such that $\mu^{-1}+O(\lambda) < \kappa$. Then it follows from estimate (4.14) that

$$\|x_1-x_2\| \le \kappa\|x_1^*-x_2^*\| \ \text{ whenever } \ (x_i,x_i^*) \in \operatorname{gph} T \cap (U_\alpha \times U^*_\alpha) \ \text{ as } \ i=1,2.$$

This implies by the arguments in Theorem 4.3 that the subdifferential ∂f is strongly metrically regular around $(\bar{x},0)$ and that $\bar{x}$ is a local minimizer of f. Employing Theorem 4.2 shows that assertion (i) and the inequality "$\le$" in (4.13) hold. Combining the latter with (4.5) and gives us the equality in the exact bound formula (4.13), where the supremum is attained by the arguments similar to those in the proof of (4.2) in Theorem 4.2.

It remains to justify that the validity of condition (4.12) ensures that $\bar{x}$ is a tilt-stable local minimizer of f with *some* modulus $\kappa > 0$, which is proved to be equivalent to the fulfillment of (4.11) with some $\mu > 0$. Arguing by contradiction, suppose that there is no $\mu > 0$ for which (4.11) holds. Then invoke [17, Lemma 4.4] and find a sequence $\{(u_k,\beta_k)\} \subset I\!R^n \times I\!R$ such that $\|u_k\| \ne 0$, $\beta_k u_k \in \partial^2 f(\bar{x},0)(u_k)$, and $\beta_k \le k^{-1}$ for all $k \in I\!N$. Assuming without loss of generality that $\|u_k\| = 1$, we get that $u_k \to \bar{u}$ with some unit vector $\bar{u}$ along a subsequence, and thus $0 \in \partial^2 f(\bar{x},0)(\bar{u})$ by passing to the limit as $k \to \infty$. But this clearly contradicts condition (4.12) and thus completes the proof of the corollary. △

Note that characterization (4.12) of tilt-stable minimizers is the main result of [26, Theorem 1.3] obtained by a different way. The quantitative second-order characterization in (4.11) involving the modulus κ as well as the exact bound formula (4.13) are new. The equivalence of the latter to that in (4.2) for tilt-stable local minimizers can be derived by using condition (4.5) of Theorem 4.3 in finite dimensions.

The next corollary presents new characterizations of strong metric regularity of the limiting subdifferential in terms of the second-order subdifferential conditions in Hilbert and finite-dimensional spaces.

Corollary 4.5 (second-order characterizations of strong metric regularity). *Let X be a Hilbert space, and let $f\colon X \to \overline{I\!R}$ be a l.s.c. function with $(\bar{x}, \bar{x}^*) \in \operatorname{gph} \partial f$. Suppose that f is prox-regular and subdifferentially continuous at $\bar{x}$ for $\bar{x}^*$. Then condition* (iii) *of Theorem* 3.2 *with the prescribed modulus $\kappa > 0$ is equivalent to the following one:*

$$\langle u^*, u\rangle \geq \frac{1}{\kappa}\|u\|^2 \;\; \textit{whenever} \;\; u^* \in \breve{\partial}^2 f(x, x^*)(u), \; (x, x^*) \in \operatorname{gph} \partial f \cap I\!B_\eta(\bar{x}, \bar{x}^*), \;\; \textit{and} \;\; u \in X.$$

Furthermore, for $X = I\!R^n$ condition (iii) *of Theorem* 3.2 *holds with some modulus $\kappa > 0$ if and only if there is a constant $\mu > 0$ such that*

$$\langle u^*, u\rangle \geq \mu\|u\|^2 \;\; \textit{whenever} \;\; u^* \in \partial^2 f(\bar{x}, \bar{x}^*)(u) \;\; \textit{and} \;\; u \in I\!R^n. \tag{4.15}$$

Proof. Define $g(x) := f(x) - \langle \bar{x}^*, x\rangle$ for all $x \in X$. By Theorem 3.2 it is easy to check that condition (iii) is equivalent to the fact that the subdifferential ∂g is strongly metrically regular around $(\bar{x}, 0)$ and that $\bar{x}$ is a local minimizer of g. Combining this with Theorem 4.2, Theorem 4.3, and Corollary 4.4 justifies all the statements of the corollary. △

To conclude this section, let us summarize the obtained *qualitative* (i.e., without relationships for moduli and formulas for the exact Lipschitzian bound) characterizations of tilt-stable minimizers of extended-real-valued functions in Hilbert spaces.

Corollary 4.6 (tilt stability in Hilbert spaces). *Let X be a Hilbert space, and let $f : X \to \overline{I\!R}$ be a l.s.c. function with $\bar{x} \in \operatorname{dom} f$ and $0 \in \partial f(\bar{x})$. Suppose that f is subdifferentially continuous at $\bar{x}$ for $\bar{x}^* = 0$. The following assertions are equivalent:*

(i) *f is prox-regular at $\bar{x}$ for 0 and $\bar{x}$ is a tilt-stable local minimizer of f.*

(ii) *∂f is strongly metrically regular around $(\bar{x}, 0)$ and $\bar{x}$ is a local minimizer of f.*

(iii) *There are neighborhoods U of $\bar{x}$, U^* of $0 \in X^*$, and a number $r > 0$ such that the mapping $(\partial f)^{-1}$ admits a single-valued localization $\vartheta : U^* \to U$ and that for any $(u^*, u) \in \operatorname{gph} \vartheta = \operatorname{gph}(\partial f)^{-1} \cap (U^* \times U)$ we have the second-order growth condition*

$$f(x) \geq f(u) + \langle u^*, x - u\rangle + r\|x - u\|^2 \;\; \textit{for all} \;\; x \in U.$$

(iv) *f is prox-regular at $\bar{x}$ for 0 and we have the second-order subdifferential condition*

$$\inf_{\eta>0} \sup \left\{ \frac{\|u\|^2}{\langle u^*, u\rangle} \,\middle|\, u^* \in \breve{\partial}^2 f(x, x^*)(u), \; (x, x^*) \in \operatorname{gph} \partial f \cap I\!B_\eta(\bar{x}, 0) \right\} < \infty.$$

Proof. This is a combination of the corresponding results established in Corollary 3.3, Theorem 4.2, and Theorem 4.3. △

5 Applications to Tilt Stability and Strong Regularity for Nonlinear Programs in Hilbert Spaces

This section is devoted to applications of the general results obtained in the previous sections as well as further developments to problems of nonlinear programming (NLP) given by:

$$\begin{cases} \text{minimize } \ \varphi_0(x) \ \text{ subject to } x \in X, \\ \varphi_i(x) \le 0 \ \text{ for } \ i = 1, \ldots, m, \\ \varphi_i(x) = 0 \ \text{ for } \ i = m+1, \ldots, m+r, \end{cases} \tag{5.1}$$

where X is a *Hilbert space*, and where all the functions φ_i are $\mathcal{C}^2$ around the reference point $\bar{x}$. Note that the major results derived below are also new even in finite dimensions.

Define the set of feasible solutions to (5.1) by

$$\Omega := \big\{x \in X \big|\ \varphi(x) \in I\!R^m_- \times \{0_r\}\big\} \ \text{ with } \ \varphi(x) := \big(\varphi_1(x), \ldots, \varphi_{m+r}(x)\big) \tag{5.2}$$

and observe that problem (5.1) can be written in the equivalent unconstrained format:

$$\text{minimize } \ f(x) := \varphi_0(x) + \delta(x; \Omega) \text{ with } \ x \in X. \tag{5.3}$$

If $\bar{x} \in \Omega$ is a local minimizer of (5.1), it satisfies the following first-order optimality condition via the normal cone to the feasible set Ω (see, e.g., [21, Proposition 5.1]):

$$0 \in \partial f(\bar{x}) = \nabla\varphi_0(\bar{x}) + N(\bar{x}; \Omega). \tag{5.4}$$

We say that $\bar{x}$ is a *tilt-stable local minimizer* of the nonlinear program (5.1) with *modulus* $\kappa > 0$ if it satisfies all the requirements of Definition 2.2 with respect to the extended-real-valued function f in (5.3) and the Lipschitz constant κ for the mapping M_γ therein.

Let us now recall some well-known qualification conditions used in this section; see [21, 29] for more details. The first one is the *linear independent constraint qualification* (LICQ) for (5.1) at $\bar{x} \in \Omega$, which means that the gradients $\big\{\nabla\varphi_i(\bar{x})\big|\ i \in I(\bar{x})\big\}$ are linearly independent in $X^* = X$ along the set of active constraint indices $I(\bar{x}) := \{i \in \{1, \ldots, m + r\}|\ \varphi_i(\bar{x}) = 0\}$. The second condition strictly weaker than LICQ is the *Mangasarian-Fromovitz constraint qualification* (MFCQ) for (5.1) at $\bar{x} \in \Omega$ meaning that the gradients $\big\{\nabla\varphi_i(\bar{x})\big|\ i \in \{m+1, \ldots, m+r\}\big\}$ are linearly independent and there is $d \in X$ such that

$$\begin{cases} \langle\nabla\varphi_i(\bar{x}), d\rangle < 0 \ \text{ for } \ i \in \big\{1, \ldots, m\big\} \cap I(\bar{x}), \\ \langle\nabla\varphi_i(\bar{x}), d\rangle = 0 \ \text{ for } \ i \in \big\{m+1, \ldots, m+r\big\}. \end{cases} \tag{5.5}$$

It is worth noting that both LICQ and MFCQ are *robust* in the sense that if either MFCQ or LICQ holds at $\bar{x}$, then it must be satisfied at all x in a neighborhood $\mathcal{O}$ of $\bar{x}$. In these cases the normal cone to Ω at $x \in \mathcal{O} \cap \Omega$ is equivalently calculated by the formulas

$$N(x; \Omega) = \begin{cases} \nabla\varphi(x)^* N\big(\varphi(x); \Theta\big) \ \text{ with } \ \Theta := I\!R^m_- \times \{0_r\} \\ \big\{\nabla\varphi(x)^*\lambda\big|\ \langle\lambda, \varphi(x)\rangle = 0, \ \lambda \in I\!R^m_+ \times I\!R^r \ \text{ for } \ i \in I(x)\big\}. \end{cases} \tag{5.6}$$

Let us further consider the standard *Lagrange function*

$$L(x, \lambda) := \varphi_0(x) + \sum_{i=1}^{m+r} \lambda_i\varphi_i(x) \ \text{ with } \ x \in X \ \text{ and } \ \lambda \in I\!R^{m+r}$$

and then define the set-valued mapping $\Psi : X \rightrightarrows X$ by

$$\Psi(x) := \big\{\nabla_x L(x,\lambda)\big|\ \lambda \in N\big(\varphi(x);\Theta\big)\big\}. \tag{5.7}$$

It is well known that if $\bar{x}$ is a local minimizer of problem (5.1), then we have the *stationary condition* $0 \in \Psi(\bar{x})$ written equivalently in the form of the *KKT system*

$$0 \in \nabla\varphi_0(\bar{x}) + \nabla\varphi(\bar{x})^*\lambda = \nabla_x L(\bar{x},\lambda) \ \text{ with some } \ \lambda \in N\big(\varphi(\bar{x});\Theta\big) \tag{5.8}$$

provided that MFCQ holds at $\bar{x}$. Taking into account the explicit form of Θ in (5.6) allows us to describe the set of *Lagrange multipliers* satisfying (5.8) as

$$\Lambda(\bar{x}) := \big\{\lambda \in I\!R^m_+ \times I\!R^r\big|\ 0 \in \nabla_x L(\bar{x},\lambda),\ \langle\lambda,\varphi(\bar{x})\rangle = 0\big\}. \tag{5.9}$$

Thus the pairs $(\bar{x},\lambda)$ with $\lambda \in \Lambda(\bar{x})$ are solutions of the following *generalized equation* (GE) in the sense of Robinson [27]:

$$0 \in \begin{bmatrix} \nabla_x L(x,\lambda) \\ -\varphi(x) \end{bmatrix} + \begin{bmatrix} 0 \\ N(\lambda;\Theta^+) \end{bmatrix} \ \text{ with } \ \Theta^+ := I\!R^m_+ \times I\!R^r. \tag{5.10}$$

Based on (5.9), we introduce the *parameterized* set of multipliers useful in the sequel:

$$\Lambda(x,x^*) := \big\{\lambda \in I\!R^m_+ \times I\!R^r\big|\ x^* \in \nabla_x L(x,\lambda),\ \langle\lambda,\varphi(x)\rangle = 0\big\} \ \text{ with } \ x^* \in \Psi(x), \tag{5.11}$$

where the mapping Ψ is defined in (5.7). It is clear that $\Lambda(x) = \Lambda(x,0)$ and that $\Lambda(x,x^*)$ is singleton for any $x^* \in \Psi(x)$ with $x \in \mathcal{O}$ provided that LICQ holds at $\bar{x}$.

In the second-order framework, a well-recognized condition for NLP (5.1) was introduced by Robinson [27] under the name of *strong second-order sufficient condition* (SSOSC). Recall that SSOSC holds at $\bar{x}$ if for all $\lambda \in \Lambda(\bar{x})$ we have

$$\langle u, \nabla^2_{xx} L(\bar{x},\lambda)u\rangle > 0 \ \text{ whenever } \ \langle\nabla\varphi_i(\bar{x}),u\rangle = 0 \ \text{ for } \ i \in I_+(\bar{x},\lambda),\ u \neq 0, \tag{5.12}$$

where $I_+(\bar{x},\lambda) := \big\{i \in \{1,\ldots,m\}\big|\ \lambda_i > 0\big\} \cup \{m+1,\ldots,m+r\}$. On the other hand, the following *perturbed* version of SSOSC is often used in the Hilbert space setting (see, e.g., [6, 15] and the references therein): there is $\alpha > 0$ such that for all $\lambda \in \Lambda(\bar{x})$ we have

$$\langle u, \nabla^2_{xx} L(\bar{x},\lambda)u\rangle \geq \alpha\|u\|^2 \ \text{ whenever } \ \langle\nabla\varphi_i(\bar{x}),u\rangle = 0,\ i \in I_+(\bar{x},\lambda). \tag{5.13}$$

The next simple proposition confirms that conditions (5.12) and (5.13) are equivalent in finite dimensions under the validity of MFCQ at $\bar{x}$. It allows us to refer to both of them as to SSOSC for (5.1) at $\bar{x}$ in this setting.

Proposition 5.1 (equivalent forms of SSOSC in finite dimensions). *Let $X = I\!R^n$, and let MFCQ hold for* (5.1) *at $\bar{x}$. Then forms* (5.12) *and* (5.13) *of SSOSC are equivalent.*

Proof. It is obvious that (5.13) implies (5.12). We need to show that (5.12) is sufficient for (5.13) under the validity of MFCQ. Indeed, suppose that (5.12) holds while (5.13) does not. Hence there exist two sequence $\{u_k\} \subset I\!R^n$ and $\{\lambda_k\} \subset \Lambda(\bar{x})$ satisfying

$$\langle u_k, \nabla^2_{xx} L(\bar{x},\lambda_k)u_k\rangle \leq k^{-1} \text{ with } \|u_k\| = 1,\ \langle\nabla\varphi_i(\bar{x}),u_k\rangle = 0, \text{ and } i \in I_+(\bar{x},\lambda_k) \tag{5.14}$$

for all $k \in I\!N$. Since MFCQ holds at $\bar{x}$, it is well known that $\Lambda(\bar{x})$ is closed and bounded in $I\!R^{m+r}$. Without loss of generality we assume that the sequence (u_k,λ_k) converges to some $(u,\lambda)\in I\!R^n\times\Lambda(\bar x)$ as $n\to\infty$. Note further that $I_+(\bar x,\lambda)\subset I_+(\bar x,\lambda_k)$ for all $k\in I\!N$ sufficiently large. Letting $k\to\infty$ in (5.14) gives us

$$\langle u,\nabla^2_{xx}L(\bar x,\lambda)u\rangle\le 0\ \text{ with }\ \|u\|=1,\ \langle\nabla\varphi_i(\bar x),u\rangle=0,\ \text{ and }\ i\in I_+(\bar x,\lambda),$$

which contradicts (5.12) and thus completes the proof of the proposition. △

Another second-order condition for the classical nonlinear programs (5.1) in finite dimensions, labeled as the *standard second-order sufficient condition* (standard SOSC), is formulated as follows: for all $\lambda\in\Lambda(\bar x)$ we have $\langle u,\nabla^2_{xx}L(\bar x,\lambda)u\rangle>0$ (or, equivalently, $\langle u,\nabla^2_{xx}L(\bar x,\lambda)u\rangle\ge\alpha\|u\|^2$ with some constant $\alpha>0$) whenever $u\ne 0$ satisfies

$$\langle\nabla\varphi_i(\bar x),u\rangle=0\ \text{ for }\ i\in I_+(\bar x,\lambda)\text{ and }\langle\nabla\varphi_i(\bar x),u\rangle\ge 0\ \text{ for }\ i\in I(\bar x)\setminus I_+(\bar x,\lambda).$$

It has been recognized that SSOSC (5.12) in finite dimension is stronger than the standard SOSC; see [27]. Now we introduce a new condition in the Hilbert space setting that is a uniform version of the standard SOSC, being stronger than the latter, while playing a crucial role in the characterization of tilt stability in both finite and infinite dimensions.

Definition 5.2 (uniform second-order sufficient condition) *We say that the* UNIFORM SECOND-ORDER SUFFICIENT CONDITION (USOSC) *holds for* (5.1) *at $\bar x$ with modulus $\ell>0$ if there is a constant $\eta>0$ such that*

$$\begin{array}{l}\langle u,\nabla^2_{xx}L(x,\lambda)u\rangle\ge\ell\|u\|^2\ \ \textit{whenever}\ \ (x,x^*)\in\operatorname{gph}\Psi\cap I\!B_\eta(\bar x,0),\lambda\in\Lambda(x,x^*),\\ \langle\nabla\varphi_i(x),u\rangle=0\ \ \textit{for}\ \ i\in I_+(x,\lambda)\ \ \textit{and}\ \ \langle\nabla\varphi_i(x),u\rangle\ge 0\ \ \textit{for}\ \ i\in I(x)\setminus I_+(x,\lambda),\end{array}\tag{5.15}$$

where the mapping Ψ and the set $\Lambda(x,x^*)$ are defined in (5.7) *and* (5.11)*, respectively.*

The next proposition shows that under the validity of MFCQ at $\bar x$ the introduced USOSC is implied by SSOSC (5.12) in finite dimensions.

Proposition 5.3 (SSOC implies USOSC under MFCQ). *Let $X=I\!R^n$, and let $\bar x$ be a feasible solution to* (5.1) *satisfying the first-order optimality condition* (5.4) *under the validity of MFCQ at $\bar x$. Assume also that SSOSC* (5.12) *holds at this point. Then USOSC from Definition* 5.2 *is satisfied $\bar x$ with some modulus $\ell>0$.*

Proof. Having SSOSC at $\bar x$, we argue by contradiction and assume that there is no number $\ell>0$ such that USOSC holds at $\bar x$ with modulus ℓ. This allows us to find a sequence $\{x_k^*,x_k,u_k,\lambda_k\}\subset X^*\times X\times X\times I\!R^{m+r}$ satisfying

$$\begin{array}{l}x_k\to\bar x,\ x_k^*\to 0,\ \lambda_k\in\Lambda(x_k,x_k^*),\ \|u_k\|=1,\ \langle u_k,\nabla^2_{xx}L(x_k,\lambda_k)u_k\rangle\le k^{-1},\ \text{ and}\\ \langle\nabla\varphi_i(x_k),u_k\rangle=0\text{ for }i\in I_+(x_k,\lambda_k),\ \langle\nabla\varphi_i(x_k),u_k\rangle\ge 0\text{ for }i\in I(x_k)\setminus I_+(x_k,\lambda_k).\end{array}\tag{5.16}$$

It follows from the inclusion $\lambda_k\in\Lambda(x_k,x_k^*)$ in (5.16) and construction (5.11) that

$$x_k^*=\nabla\varphi_0(x_k)+\sum_{i=1}^{m+r}\lambda_{k_i}\nabla\varphi_i(x_k)\ \text{ with }\ \lambda_k\in I\!R^m_+\times I\!R^r\ \text{ and }\ \langle\lambda_k,\varphi(x_k)\rangle=0,\tag{5.17}$$

where λ_{k_i} signifies the i^{th} component of the vector λ_k. Define further the number

$$\alpha:=\sup\big\{\nabla\varphi_i(x_k)\big|\ i\in\{0,\ldots,m+r\},\ k\in I\!N\big\}<\infty$$

and observe that the linear independence of the gradient vectors $\nabla\varphi_{m+1}(\bar x),\ldots,\nabla\varphi_{m+r}(\bar x)$ allows us to find a constant $\beta>0$ such that

$$
\begin{aligned}
\|x_k^*\|+\alpha+\alpha\sum_{i=1}^m\lambda_{k_i} &\ge \Big\|x_k^*-\nabla\varphi_0(x_k)-\sum_{i=1}^m\lambda_{k_i}\nabla\varphi_i(x_k)\Big\|=\Big\|\sum_{i=m+1}^{m+r}\lambda_{k_i}\nabla\varphi_i(x_k)\Big\|\\
&\ge \Big\|\sum_{i=m+1}^{m+r}\lambda_{k_i}\nabla\varphi_i(\bar x)\Big\|-\Big\|\sum_{i=m+1}^{m+r}\lambda_{k_i}\big(\nabla\varphi_i(\bar x)-\nabla\varphi_i(x_k)\big)\Big\| \qquad (5.18)\\
&\ge \beta\sum_{i=m+1}^{m+r}|\lambda_{k_i}|-O(k)\sum_{i=m+1}^{m+r}|\lambda_{k_i}|=(\beta-O(k))\sum_{i=m+1}^{m+r}|\lambda_{k_i}|.
\end{aligned}
$$

It also follows from MFCQ (5.5) at $\bar x$ that there are $d\in I\!\!R^n$ and $\gamma>0$ such that

$$
\begin{cases}\langle\nabla\varphi_i(\bar x),d\rangle<-2\gamma \ \text{ for } \ i\in I(\bar x)\cap\{1,\ldots,m\},\\ \langle\nabla\varphi_i(\bar x),d\rangle=0 \ \text{ for } \ i\in\{m+1,\ldots,m+r\}.\end{cases} \qquad (5.19)
$$

Since $x_k\to\bar x$ as $k\to\infty$, we suppose without loss of generality that $\langle\nabla\varphi_i(x_k),d\rangle<-\gamma$ for all $i\in I(x_k)\cap\{1,\ldots,m\}\subset I(\bar x)\cap\{1,\ldots,m\}$ and $k\in I\!\!N$. Note further that $\lambda_{k_i}=0$ for $i\in\{1,\ldots,m\}\setminus I(x_k)$. Hence we get $\lambda_{k_i}\langle\nabla\varphi_i(x_k),d\rangle\le-\gamma\lambda_{k_i}$ whenever $i\in\{1,\ldots,m\}$. Combining this with the relationships in (5.17) and (5.19) gives us the estimates

$$
\begin{aligned}
-\|x_k^*\|\|d\| &\le \langle x_k^*,d\rangle=\langle\nabla\varphi_0(x_k),d\rangle+\sum_{i=1}^m\lambda_{k_i}\langle\nabla\varphi_i(x_k),d\rangle+\sum_{i=m+1}^{m+r}\lambda_{k_i}\langle\nabla\varphi_i(x_k),d\rangle\\
&\le \alpha\|d\|-\gamma\sum_{i=1}^m\lambda_{k_i}+O(k)\sum_{i=m+1}^{m+r}|\lambda_{k_i}|.
\end{aligned} \qquad (5.20)
$$

It follows from (5.18) and (5.20) that

$$
(\beta-O(k))\sum_{i=m+1}^{m+r}|\lambda_{k_i}|\le\|x_k^*\|+\alpha+\frac{\alpha}{\gamma}\Big(\|x_k^*\|\cdot\|d\|+\alpha\|d\|+O(k)\sum_{i=m+1}^{m+r}|\lambda_{k_i}|\Big),
$$

which ensures that the sequence $(\lambda_{k_{m+1}},\ldots,\lambda_{k_{m+r}})$ is uniformly bounded in $I\!\!R^r$ and so is the sequence $(\lambda_{k_1},\ldots,\lambda_{k_m})$ due to (5.20). We may assume that $\lambda_k\to\lambda\in I\!\!R^m_+\times I\!\!R^r$ and $u_k\to u\in I\!\!R^n$ as $k\to\infty$ with $\|u\|=1$. Noting that $I_+(\bar x,\lambda)\subset I_+(x_k,\lambda_k)$ for sufficiently large $k\in I\!\!N$, we get from (5.16) that

$$
\lambda\in\Lambda(\bar x),\ \langle u,\nabla^2_{xx}L(\bar x,\lambda)u\rangle\le 0,\ \text{ and }\ \langle\nabla\varphi_i(\bar x),u\rangle=0\ \text{ for }\ i\in I_+(\bar x,\lambda),
$$

which contradicts (5.12) and thus completes the proof of the proposition. △

In what follows we show (see Theorem 5.6 and Example 5.8) that the introduced USOSC is in fact *strictly weaker* than its SSOSC counterpart (5.12) in finite dimensions even under the simultaneous fulfillment of MFCQ and the well-known *constant rank* constraint qualification formulated below.

The next theorem reveals the role of USOSC in characterizing tilt stability of local minimizers for (5.1) in the general Hilbert space setting.

Theorem 5.4 (second-order characterization of tilt-stable local minimizers for NLP in Hilbert spaces). *Let X be a Hilbert space, and let $\bar x$ be a feasible solution to* (5.1) *satisfying* (5.4). *Consider the following assertions:*

(i) *The point $\bar{x}$ is a tilt-stable local minimizer of problem* (5.1) *with modulus $\kappa > 0$.*

(ii) *The point $\bar{x}$ is a local minimizer of* (5.1) *and the mapping Ψ from* (5.7) *is strongly metrically regular at $(\bar{x}, 0)$ with modulus κ.*

(iii) *The USOSC from Definition* 5.2 *holds at $\bar{x}$ with modulus $\ell = \kappa^{-1}$.*

Then assertions (i) *and* (ii) *are equivalent to each other provided that the MFCQ condition holds at $\bar{x}$. If in addition the LICQ condition is satisfied at $\bar{x}$, then assertion* (iii) *is equivalent to those in* (i) *and* (ii).

Proof. To justify the first equivalence (i)$\Longleftrightarrow$(ii), observe that by [5, Proposition 2.4] the MFCQ condition ensures that the function $\delta(\cdot;\Omega) = \delta(\varphi(\cdot);\Theta)$ is prox-regular and subdifferentially continuous at $\bar{x}$ for $-\nabla\varphi_0(\bar{x})$. Hence the function $f = \varphi_0 + \delta(\cdot;\Omega)$ is also prox-regular and subdifferentially continuous at $\bar{x}$ for 0 by taking into account that φ_0 is $\mathcal{C}^2$ around $\bar{x}$. Moreover, we have that

$$\partial f(x) = \nabla\varphi_0(x) + N(x;\Omega) = \Psi(x) \ \text{ for all } \ x \in \mathcal{O},$$

where $\mathcal{O}$ is the neighborhood of $\bar{x}$ on which the MFCQ condition holds. Thus the equivalence between (i) and (ii) follows from Theorem 4.2.

Suppose further that the LICQ condition holds at $\bar{x}$. Since in our local analysis inactive inequality constraints can be dropped out from the constraint system (5.1), we assume without loss of generality that $\varphi(\bar{x}) = 0_{m+r}$. Then the LICQ condition is equivalent to the surjectivity of $\nabla\varphi(\bar{x})$. The robustness of the surjectivity condition allows us to find a neighborhood $\mathcal{O}$ of $\bar{x}$ such that $\nabla\varphi(x)$ is surjective for all $x \in \mathcal{O}$. It suffices to prove the equivalence between (i) and (iii) under the latter condition.

To proceed, choose $\eta > 0$ sufficiently small such that $I\!B_\eta(\bar{x}) \subset \mathcal{O}$ and pick any element $u^* \in \breve{\partial} f(x, x^*)(u)$ with $(x, x^*) \in \text{gph}\,\partial f \cap I\!B_\eta(\bar{x}, 0)$. It follows from [21, Theorem 1.62] and the symmetricity of the Hessian operator for $\mathcal{C}^2$ functions in Hilbert spaces that

$$u^* \in \widehat{D}^*\big(\nabla\varphi_0 + N(\cdot;\Omega)\big)(x, x^*)(u) = \nabla^2\varphi_0(x)u + \widehat{D}^*N(\cdot;\Omega)\big(x, x^* - \nabla\varphi_0(x)\big)(u) \quad (5.21)$$

for all $u \in X$. By Lemma 6.3 proved in Section 6 we get that

$$u^* - \nabla^2\varphi_0(x)u - \nabla^2\langle\lambda(x), \varphi\rangle(x)u \in \nabla\varphi(x)^*\widehat{D}^*N(\cdot;\Theta)\big(\varphi(x), \lambda(x)\big)\big(\nabla\varphi(x)u\big), \quad (5.22)$$

where $\lambda(x) \in N(\varphi(x);\Theta)$ is the unique multiplier satisfying $x^* - \nabla\varphi_0(x) = \nabla\varphi(x)^*\lambda(x)$. Since Θ is a convex polyhedron, it follows from the proof of [10, Theorem 2] that

$$\widehat{N}\big((\varphi(x), \lambda(x)); \text{gph}\,N(\cdot;\Theta)\big) = K\big(\varphi(x), \lambda(x)\big)^* \times K\big(\varphi(x), \lambda(x)\big), \quad (5.23)$$

where $K(\varphi(x), \lambda(x)) := N(\varphi(x);\Theta)^* \cap \lambda(x)^\perp$ is the corresponding *critical cone*. This together with (5.22) gives us $v^* \in K(\varphi(x), \lambda(x))^*$ and $v \in K(\varphi(x), \lambda(x))$ satisfying

$$u^* - \nabla^2_{xx}L\big(x, \lambda(x)\big)u = \nabla\varphi(x)^*v^* \ \text{ and } \ \nabla\varphi(x)u = -v. \quad (5.24)$$

It implies in turn the relationships

$$\langle u^* - \nabla^2_{xx}L\big(x, \lambda(x)\big)u, u\rangle = \langle\nabla\varphi(x)^*v^*, u\rangle = \langle v^*, \nabla\varphi(x)u\rangle = \langle v^*, -v\rangle \ge 0. \quad (5.25)$$

using further the classical expression

$$N\big(\varphi(x);\Theta\big)^* = \big\{w \in I\!R^{m+r} \,|\, w_i = 0 \text{ for } i \in \{m+1, \ldots, m+r\} \text{ and } w_i \le 0 \text{ for } i \in I(x)\big\},$$

we arrive at the critical cone representation

$$K\big(\varphi(x),\lambda(x)\big)=\left\{\begin{array}{l} w\in I\!R^{m+r}\Big| w_i=0\ \text{ for }\ i\in I_+(x,\lambda(x)),\\ w_i\le 0\ \text{ for }\ i\in I(x)\setminus I_+(x,\lambda(x)).\end{array}\right. \tag{5.26}$$

Assuming now that (iii) holds and combining the relationships in (5.24)–(5.26) give us

$$\langle u^*,u\rangle\ge\langle\nabla^2_{xx}L(x,\lambda(x))u,u\rangle\ge\frac{1}{\kappa}\|u\|^2\ \text{ for all }\ u^*\in\breve{\partial}^2 f(x,x^*)(u),$$

which ensures the validity of assertion (i) by the corresponding implication of Theorem 4.3. Conversely, let (i) hold and pick any $u\in X$ satisfying

$$\nabla\varphi_i(x)u=0\ \text{ for }\ i\in I_+(x,\lambda(x))\ \text{ and }\ \nabla\varphi_i(x)u\ge 0\ \text{ for }\ i\in I(x)\setminus I_+(x,\lambda(x)).$$

Then we find $v^*\in\{v\}^\perp$ with $v:=-\nabla\varphi(x)u$ and get from (5.21)–(5.24) and (5.26) that $v\in K(\varphi(x),\lambda(x))$ and $v^*\in K(\varphi(x),\lambda(x))^*$. Denoting

$$u^*:=\nabla^2_{xx}L\big(x,\lambda(x)\big)+\nabla\varphi(x)^*v^*\in\breve{\partial}^2 f(x,x^*)(u)$$

and using condition (4.4) in Theorem 4.3 ensure that

$$\langle\nabla^2_{xx}L(x,\lambda(x))u,u\rangle=\langle u^*,u\rangle+\langle v^*,v\rangle\ge\frac{1}{\kappa}\|u\|^2,$$

which justifies the validity of (5.15) and thus completes the proof of the theorem. △

It is well known that the classical *strong regularity* of the generalized equation (5.10) in the sense of Robinson [27] (i.e., the local inverse to its linearization is locally single-valued and Lipschitz continuous around the point in question) is fully characterized by the simultaneous fulfillment of LICQ and SSOSC in finite dimensions; see [6, 11] for more details and references. It is questioned by Robinson [27, p. 56] whether the validity of the standard SOSC instead of its strong version (5.12) can be used to derive strong regularity of (5.10) under LICQ. The answer is negative; see detail in the example presented in [27, pp. 56–57]. Our next result shows that the new USOSC from Definition 5.2, a uniform counterpart of the standard SOSC, characterizes such a regularity under LICQ even in Hilbert spaces, being furthermore equivalent to the validity of SSOSC in this setting.

Corollary 5.5 (characterizations of strong regularity in Hilbert spaces). *Let X be a Hilbert space, let $\bar{x}$ be a feasible solution to* (5.1) *satisfying* (5.4)*, and let LICQ hold at $\bar{x}$. Then the following assertions are equivalent:*

(i) *The point $\bar{x}$ is a tilt-stable local minimizer of problem* (5.1)*.*

(ii) *The point $\bar{x}$ is a local minimizer of* (5.1) *and the mapping Ψ from* (5.7) *is strongly metrically regular at $(\bar{x},0)$.*

(iii) *There are numbers $\ell,\eta>0$ such that USOSC* (5.15) *holds at $\bar{x}$.*

(iv) *There is a number $\alpha>0$ such that SSOSC* (5.13) *holds at $\bar{x}$.*

(v) *The generalized equation* (5.10) *is strongly regular at $(\bar{x},\bar{\lambda})$ with the unique Lagrange multiplier $\Lambda(\bar{x})=\{\bar{\lambda}\}$ in* (5.9)*.*

Proof. By Theorem 5.4 it suffices to prove the chain of implications (i)⟹(iv)⟹(v)⟹(ii). Assume first that (i) holds and use the mixed second-order necessary condition (4.5) for tilt-stable minimizers from Theorem 4.3. Applying it to the function f in (5.3), we find a number $\alpha > 0$ such that

$$\langle u^*, u\rangle \geq \alpha\|u\|^2 \ \text{ whenever } \ u^* \in \partial^2_M f(\bar{x},0)(u) \ \text{ and } \ u \in X.$$

This together with [21, Theorem 1.62] gives us that

$$\langle u^*, u\rangle \geq \alpha\|u\|^2, \ u^* \in \partial^2_M f(\bar{x},0)(u) = \nabla^2\varphi_0(\bar{x})u + D^*_M N(\cdot;\Omega)\big(\bar{x}, -\nabla\varphi_0(\bar{x})\big)(u). \quad (5.27)$$

Since $N(x;\Omega) = \nabla\varphi(x)^* N(\varphi(x);\Theta)$ and we have LICQ, it follows from the second-order subdifferential chain rule in [21, Theorem 1.127] that

$$\begin{array}{l} D^*_M N(\cdot;\Omega)\big(\bar{x}, -\nabla\varphi_0(\bar{x})\big)(u) = \nabla^2\langle\bar{\lambda},\varphi\rangle(\bar{x})^* u \\ +\nabla\varphi(\bar{x})^* D^* N(\cdot;\Theta)\big(\varphi(\bar{x}),\bar{\lambda}\big)\big(\nabla\varphi(\bar{x})u\big), \end{array} \quad (5.28)$$

where $\bar{\lambda}$ is the unique Lagrange multiplier from (5.9). Furthermore, it follows from the proof of [23, Theorem 5.2] that

$$\left\{\begin{array}{l} \nabla\varphi(\bar{x})u \in \operatorname{dom} D^*N(\cdot;\Theta)(\varphi(\bar{x}),\bar{\lambda}) \Longleftrightarrow \langle\nabla\varphi_i(\bar{x}),u\rangle = 0, \ i \in I_+(\bar{x},\lambda), \\ \min\big\{\langle v^*, \nabla\varphi(\bar{x})u\rangle \big|\ v^* \in D^*N(\cdot;\Theta)\big(\varphi(\bar{x}),\bar{\lambda}\big)\big(\nabla\varphi(\bar{x})u\big)\big\} = 0. \end{array}\right. \quad (5.29)$$

For any $u \in X$ satisfying $\langle\nabla\varphi_i(\bar{x}),u\rangle = 0$ as $i \in I_+(\bar{x},\bar{\lambda})$, we get from (5.27) and (5.28) that $u^* = \nabla^2_{xx}L(\bar{x},\bar{\lambda})u + \nabla\varphi(\bar{x})^* v^* \in \partial^2_M f(\bar{x},0)(u)$ for $v^* \in D^*N(\cdot;\Theta)(\varphi(\bar{x}),\bar{\lambda})(\nabla\varphi(\bar{x})u)$. Then the relationships in (5.27) and (5.29) imply that

$$\langle\nabla^2_{xx}L(\bar{x},\bar{\lambda})u,u\rangle \geq \alpha\|u\|^2 \ \text{ whenever } \ \langle\nabla\varphi_i(\bar{x}),u\rangle = 0, \ i \in I_+(\bar{x},\bar{\lambda}),$$

which is (5.13) due to $\Lambda(\bar{x}) = \{\bar{\lambda}\}$. This justifies implication (i)$\Longrightarrow$(iv). Implication (iv)$\Longrightarrow$(v) follows from the proof of [15, Theorem 2.16] since the assumed LICQ and SSOSC ensure the fulfillment of all the hypotheses (H1)–(H4) therein.

It remains to verify implication (v)$\Longrightarrow$(ii). It follows from the first part of [6, Theorem 5.20] that the strong regularity in (v) ensures validity of the uniform quadratic growth condition in the sense of [6, Definition 5.16] at $\bar{x}$ for the nonlinear program (5.1). From this quadratic growth it is easy to deduce inequality (3.16) with some positive constant $\kappa > 0$. Employing finally Corollary 3.3 and the fact that $\Psi(x) = \partial f(x)$ for $x \in \mathcal{O}$, we arrive at (ii) and thus complete the proof of the corollary. $\triangle$

When $\dim X < \infty$, the SSOSC characterization of tilt-stable minimizers of (5.1) from Corollary 5.5 was first proved under LICQ [23, Theorem 5.2] by a different method. Furthermore, the equivalence between (i) and (ii) in finite dimensions has been recently established in [23, Theorem 5.2] under the validity of merely MFCQ at the reference point.

Paper [22] employs also the combination of MFCQ and the *constant rank constraint qualification* (CRCQ) to derive separately necessary conditions and sufficient conditions (but not characterizations) for tilt-stable minimizers in finite-dimensional NLP. The conditions obtained in [22] reduce to the SSOSC characterization of Corollary 5.5 under LICQ, which ensures the validity of both MFCQ and CRCQ while not vice versa. Recall that *CRCQ* holds at $\bar{x}$ if there is a neighborhood $\mathcal{W}$ of $\bar{x}$ such that the gradient system $\{\nabla\varphi_i(x)|\ i \in J\}$ has the same rank in $\mathcal{W}$ for any index $J \subset I(\bar{x})$; see, e.g., [13, 19, 22] and the references therein for historical remarks and recent developments. Note that $I(x) \subset I(\bar{x})$ provided that x is sufficiently close to $\bar{x}$, i.e., the CRCQ condition is robust.

It is worth mentioning that MFCQ and CRCQ are independent in the sense that one can not imply another. It is proved in [22, Theorem 3.5] that *SSOSC* (5.12) is *sufficient* for tilt-stability of local minimizers of (5.1) in $I\!R^n$ under the simultaneous validity of MFCQ and CRCQ. Our next result shows that the new *USOSC* have a complete *characterization* of tilt-stable minimizers in the same setting. For simplicity we omit (as in [22]) the equality constraints of (5.1) in the rest of this section.

Theorem 5.6 (USOSC characterization of tilt-stable minimizers under MFCQ and CRCQ). *Let $X = I\!R^n$, and let $\bar{x}$ be a feasible solution to* (5.1) *satisfying* (5.4)*. Assume that both MFCQ and CRCQ hold at $\bar{x}$. Then the following assertions are equivalent:*

(i) *The point $\bar{x}$ is a tilt-stable local minimizer of* (5.1) *with modulus $\kappa > 0$.*

(ii) *The USOSC from Definition* 5.2 *holds at $\bar{x}$ with modulus $\ell = \kappa^{-1}$.*

Proof. Let $\eta > 0$ be sufficiently small so that both MFCQ and CRCQ hold at all $x \in I\!B_\eta(\bar{x})$. Pick any $u^* \in \breve{\partial}^2 f(x, x^*)(u)$ with $(x, x^*) \in \operatorname{gph} \partial f \cap I\!B_\eta(\bar{x}, 0) = \operatorname{gph} \Psi \cap I\!B_\eta(\bar{x}, 0)$ and any $\lambda \in \Lambda(x, x^*)$. It follows from (5.21) and [13, Theorem 6] that

$$u^* - \nabla^2 L(x, \lambda)u \in K(x, x^* - \nabla\varphi_0(x))^* \quad \text{and} \quad -u \in K(x, x^* - \nabla\varphi_0(x)) \tag{5.30}$$

for some $\lambda \in \Lambda(x, x^*)$, where $K(x, x^* - \nabla\varphi_0(x)) := \widehat{N}(x; \Omega)^* \cap \{x^* - \nabla\varphi_0(x)\}^\perp$ for the critical cone to the feasible set (5.2). It is well known that the assumed MFCQ ensures the representation

$$\widehat{N}(x; \Omega)^* = \big\{w \in X \big|\ \langle\nabla\varphi_i(x), w\rangle \le 0,\ i \in I(x)\big\}.$$

Using this formula and the fact that $x^* - \nabla\varphi_0(\bar{x}) = \sum_{i \in I(x)} \lambda_i \nabla\varphi_i(x)$, we conclude similarly to the proof of Theorem 5.4 that

$$-u \in K(x, x^* - \nabla\varphi_0(x)) \Longleftrightarrow \left\{ \begin{array}{l} \langle\nabla\varphi_i(x), u\rangle = 0 \ \text{ for } \ i \in I_+(x, \lambda), \\ \langle\nabla\varphi_i(x), u\rangle \ge 0 \ \text{ for } \ i \in I(x) \setminus I_+(x, \lambda). \end{array} \right. \tag{5.31}$$

If (ii) holds, then the imposed USOSC together with (5.15) and (5.30) gives us that

$$\langle u^*, u\rangle \ge \langle\nabla^2_{xx} L(x, \lambda)u, u\rangle \ge \frac{1}{\kappa}\|u\|^2 \ \text{ for all } \ u^* \in \breve{\partial}^2 f(x, x^*)(u),\ u \in I\!R^n,$$

which implies (i) by Theorem 4.3. Conversely, suppose that (i) is satisfied. Pick u with

$$\langle\nabla\varphi_i(x), u\rangle = 0 \ \text{ as } \ i \in I_+(x, \lambda) \ \text{ and } \ \langle\nabla\varphi_i(x), u\rangle \ge 0 \ \text{ as } \ i \in I(x) \setminus I_+(x, \lambda)$$

with any $\lambda \in \Lambda(x, x^*)$ and select $v^* \in \{u\}^\perp$; hence $v^* \in K(x, x^* - \nabla\varphi_0(x))^*$ by the equivalence in (5.31) above. It follows from (5.30) that $u^* := \nabla^2 L(x, \lambda)u + v^* \in \breve{\partial}^2 f(x, x^*)(u)$, which yields by combining with (4.4) that

$$\langle\nabla^2_{xx} L(x, \lambda)u, u\rangle = \langle u^*, u\rangle - \langle v^*, u\rangle = \langle u^*, u\rangle \ge \frac{1}{\kappa}\|u\|^2.$$

This ensures the validity of USOSC with modulus $\ell = \kappa^{-1}$ and completes the proof. △

Now we recover the aforementioned sufficient condition from [22, Theorem 3.5] derived by a different approach.

Corollary 5.7 (sufficiency of SSOSC for tilt stability under MFCQ and CRCQ). *Let $X = I\!R^n$, let $\bar{x} \in \Omega$ satisfy* (5.4)*, and let both MFCQ and CRCQ hold at $\bar{x}$. Then the validity of SSOSC at $\bar{x}$ ensures that $\bar{x}$ is a tilt-stable local minimizer of* (5.1)*.*

Proof. We know from Proposition 5.3 that SSOSC at $\bar{x}$ implies the fulfillments of USOSC at this point with some modulus $\ell > 0$. Thus the result of the corollary is an immediate consequence of Theorem 5.6. $\triangle$

The next example shows that SSOSC is *not necessary* for tilt stability of local minimizers in finite dimensions under the validity of both MFCQ and CRCQ.

Example 5.8 (SSOSC is not a necessary condition for tilt stability under MFCQ and CRCQ). Consider the following nonlinear problem in $I\!R^3$:

$$\begin{cases} \text{minimize } \ \varphi_0(x) := x_3 + \frac{1}{4}x_1 + x_3^2 - x_1x_2 \ \text{ subject to} \\ \varphi_1(x) := x_1 - x_3 \le 0 \\ \varphi_2(x) := -x_1 - x_3 \le 0 \\ \varphi_3(x) := x_2 - x_3 \le 0 \\ \varphi_4(x) := -x_2 - x_3 \le 0, \\ x = (x_1, x_2, x_3) \in X = I\!R^3. \end{cases} \tag{5.32}$$

It is easy to check that both MFCQ and CRCQ hold at $\bar{x} = (0, 0, 0)$. Taking any vector $a = (a_1, a_2, a_3)$ around $0 \in I\!R^3$ and writing the function f from (5.4) for all feasible points $x \in \Omega$, we have in this case the expression

$$\begin{aligned} f(x) - \langle a, x\rangle \ &= x_3 + \frac{1}{4}x_1 - a_1x_1 - a_2x_2 - a_3x_3 + x_3^2 - x_1x_2 \\ &\ge \frac{1}{3}x_3 + \Big(\frac{1}{4} - a_1\Big)x_1 + \frac{1}{3}x_3 - a_2x_2 + \Big(\frac{1}{3} - a_3\Big)x_3 \\ &\ge \frac{1}{3}|x_1| + \Big(\frac{1}{4} - a_1\Big)x_1 + \frac{1}{3}|x_2| - a_2x_2 + \Big(\frac{1}{3} - a_3\Big)x_3 \ge 0. \end{aligned}$$

It follows that $M_\gamma(a) = \{\bar{x}\}$ whenever a is around $0 \in I\!R^3$. Thus $\bar{x}$ is a tilt-stable local minimizer of program (5.32), and we only need to check that SSOSC (5.12) does not hold at $\bar{x}$. It is easy to see that $\big(\frac{3}{8}, \frac{5}{8}, 0, 0\big) \in \Lambda(\bar{x})$, and hence $u = (0, 1, 0) \ne 0$ satisfies the equation

$$\langle \nabla\varphi_i(\bar{x}), u\rangle = 0 \ \text{ for } \ i \in I_+(\bar{x}, \lambda) = \{1, 2\}.$$

At the same time $\langle u, \nabla^2_{xx}L(\bar{x}, \lambda)u\rangle = 0$, which shows that SSOSC does not hold at $\bar{x}$.

Note that the generalized equation/KKT system (5.10) associated with problem (5.32) is *not strongly regular* at the tilt-stable minimizer $\bar{x}$ and the corresponding Lagrange multiplier in Example 5.8. Indeed, the converse assertion ensures LICQ and thus contradicts Corollary 5.5. Observe also that we do *not* have *strong stability* in the sense of Kojima [16] in this example. Indeed, it has been well recognized (see the original version in [16, Theorem 7.2] and the improved one in [6, Proposition 5.37] with the references therein) that strong stability of NLP can be characterized, under the validity of MFCQ, via a uniform quadratic growth condition equivalent in this case to SSOSC. As shown in Example 5.8, SSOSC does not hold at the tilt-stable minimizer $\bar{x}$ in problem (5.32) while MFCQ is satisfied. Thus strong stability fails in this setting.

6 Appendix: Some Technical Lemmas

The final section presents four new results of variational analysis and generalized differentiation, which are of their own interest while are employed in this paper as technical lemmas in the proofs of some major results. The first lemma is particularly useful for justifying several results in Section 4. Its finite-dimensional version is derived in [25, Proposition 4.8] by somewhat different arguments.

Lemma 6.1 (maximal monotonicity from prox-regularity and subdifferential continuity). *Let $f : X \to \overline{I\!R}$ be a l.s.c. and quadratically minorized function defined on the Hilbert space X. Suppose that f is both prox-regular and subdifferentially continuous at $\bar{x} \in \operatorname{dom} f$ for $\bar{x}^* \in \partial f(\bar{x})$. Then there is some $\xi > 0$ such that the operator $\partial f + \xi I$ is locally maximal monotone around $(\bar{x}, \bar{x}^* + \xi\bar{x})$.*

Proof. Since f is both prox-regular and subdifferentially continuous at $\bar{x} \in \operatorname{dom} f$ for $\bar{x}^* \in \partial f(\bar{x})$, estimate (2.11) holds with some $r, \varepsilon > 0$ sufficiently small and without the condition of $|f(u) - f(\bar{x})| < \varepsilon$. It follows from [5, Proposition 4.4] that there exist $\lambda \in (0, r^{-1})$ and $\varepsilon_0 \in (0, \varepsilon)$ for which we can find a neighborhood U_λ of $\bar{x} + \lambda\bar{x}^*$ such that the mapping $P_\lambda f := (I + \lambda T)^{-1}$ is single-valued and Lipschitz continuous on U_λ, where T is a localization of ∂f relative to $I\!B_{\varepsilon_0}(\bar{x})$ and $I\!B_{\varepsilon_0}(\bar{x}^*)$. Hence the inverse mapping $(T + \xi I)^{-1}$ is also single-valued and Lipschitz continuous on $V_\lambda := \lambda U_\lambda$ with $\xi := \lambda^{-1}$. We may shrink U_λ if necessary by choosing a neighborhood W_λ of $\bar{x}$ such that $(T + \xi I)^{-1}(V_\lambda) \subset W_\lambda \subset I\!B_{\varepsilon_0}(\bar{x})$. Note that the monotonicity of $T + rI$ follows from the definition of prox-regular functions. Since $\xi = \lambda^{-1} > r$, the operator $T + \xi I = T + rI + (\xi - r)I$ is also monotone.

Let S be any monotone operator with $\operatorname{gph}(T + \xi I) \cap (W_\lambda \times V_\lambda) \subset \operatorname{gph} S$. To prove that $\partial f + \xi I$ is locally maximal monotone around $(\bar{x}, \bar{x}^* + \xi\bar{x})$, we need to check that

$$\operatorname{gph}(T + \xi I) \cap (W_\lambda \times V_\lambda) = \operatorname{gph} S \cap (W_\lambda \times V_\lambda). \tag{6.1}$$

To proceed, take any $(u, u^*) \in \operatorname{gph} S \cap (W_\lambda \times V_\lambda)$ and thus get that

$$\langle u^* - v^*, u - v\rangle \geq 0 \ \text{ for all } \ (v, v^*) \in \operatorname{gph}(T + \xi I) \cap (W_\lambda \times V_\lambda). \tag{6.2}$$

Since $(T + \xi I)^{-1}$ is continuous on V_λ and since $(T + \xi I)^{-1}(V_\lambda) \subset W_\lambda \subset I\!B_{\varepsilon_0}(\bar{x})$, for any $x \in X$ we deduce from (6.2) that

$$\langle u^* - (u^* + tx), u - (T + \xi I)^{-1}(u^* + tx^*)\rangle \geq 0$$

whenever $t > 0$ is sufficiently small. This implies the relationship

$$\langle x, u - (T + \xi I)^{-1}(u^*)\rangle = \limsup_{t \downarrow 0} \langle x, u - (T + \xi I)^{-1}(u^* + tx)\rangle \leq 0 \ \text{ for all } \ x \in X,$$

which clearly yields that $u - (T + \xi I)^{-1}(u^*) = 0$, i.e., $(u, u^*) \in \operatorname{gph}(T + \xi I) \cap (W_\lambda \times V_\lambda)$. The latter justifies our claim (6.1) and completes the proof of the lemma. △

The next result extends that of [26, Theorem 2.1] to the Hilbert space setting. In fact, the proof of [26, Theorem 2.1] can be easily modified for this case; see, e.g., [8, Lemma 5.2]). Here we present a new and simple proof in Hilbert space whose idea is also used in the proof of Theorem 4.3 and Corollary 4.4.

Lemma 6.2 (coderivatives of maximal monotone operators). *Let X be a Hilbert space, and $T : X \rightrightarrows X$ be a maximal monotone operator. Then we have for any pair $(\bar{x}, \bar{x}^*) \in \operatorname{gph} T$ we have that*

$$\langle u^*, u\rangle \geq 0 \ \ \textit{whenever} \ \ u^* \in \widehat{D}^* T(\bar{x}, \bar{x}^*)(u). \tag{6.3}$$

Consequently, $\langle u^*, u\rangle \geq 0$ whenever $u^* \in D^*_M T(\bar{x}, \bar{x}^*)(u)$.

Proof. It is well known that for any $\lambda > 0$ the resolvent $R_\lambda = (I + \lambda T)^{-1}$ is nonexpansive with $\operatorname{dom} R_\lambda = H$ by the classical Minty theorem. Pick an arbitrary pair $(u, u^*) \in \operatorname{gph} \widehat{D}^*T(\bar{x}, \bar{x}^*)$ and deduce from [21, Theorem 1.62] that

$$-\lambda^{-1}u \in \widehat{D}^*R_\lambda(\bar{x} + \lambda\bar{x}^*, \bar{x})(-u^* - \lambda^{-1}u).$$

Since R_λ is nonexpansive, it follows from [21, Theorem 1.43] that $\|-\lambda^{-1}u\| \le \|-u^* - \lambda^{-1}u\|$, which clearly implies that

$$\lambda^{-2}\|u\|^2 \le \| - u^* - \lambda^{-1}u\|^2 = \|u^*\|^2 + 2\lambda^{-1}\langle u^*, u\rangle + \lambda^{-2}\|u\|^2$$

and yields in turn that $0 \le \lambda\|u^*\|^2 + 2\langle u^*, u\rangle$ for all $\lambda > 0$. Letting $\lambda \downarrow 0$ gives us that $\langle u^*, u\rangle \ge 0$, which is the claimed relationship (6.3). Similarly, by replacing $(\bar{x}, \bar{x}^*)$ with any point $(x, x^*) \in \operatorname{gph} T$, we also have

$$\langle u^*, u\rangle \ge 0 \ \text{ whenever } \ u^* \in \widehat{D}^*T(x, x^*)(u).$$

This fact easily implies the second conclusion of the lemma by passing to the limit as $(x, x^*) \to (\bar{x}, \bar{x}^*)$ and using definition (2.8) of the mixed coderivative. △

Our next lemma is a new calculus rule for the combined second-order subdifferential (2.13) involving the indicator functions for inverse images of sets under $\mathcal{C}^2$ mappings between arbitrary Banach spaces. It is not needed in this paper, but the reader can easily observe that the proof holds true if we use ε-enlargements of (2.3) in the construction (2.4) of the limiting subdifferential, which is more appropriate in general Banach spaces; see Section 2.

Lemma 6.3 (combined second-order subdifferential for indicators of inverse images). *Let $f : X \to Y$ be a mapping between arbitrary Banach spaces, let Θ be an arbitrary subset of Y with $\bar{y} := f(\bar{x}) \in \Theta$, and let $\bar{x}^* \in N(\bar{x}; f^{-1}(\Theta))$. Assume that f is $\mathcal{C}^2$ around $\bar{x}$ and that the derivative $\nabla f(\bar{x})\colon X \to Y$ is surjective. Then for all $u \in X^{**}$ we have*

$$\breve{\partial}^2\delta\big(\cdot; f^{-1}(\Theta)\big)(\bar{x}, \bar{x}^*)(u) = \nabla^2\langle \bar{y}^*, f\rangle(\bar{x})^*u + \nabla f(\bar{x})^*\breve{\partial}^2\delta(\cdot; \Theta)(\bar{y}, \bar{y}^*)\big(\nabla f(\bar{x})^{**}u\big), \tag{6.4}$$

where $\bar{y}^* \in N(\bar{y}; \Theta)$ is a unique element satisfying $\bar{x}^* = \nabla f(\bar{x})^*\bar{y}^*$.

Proof. Note first that the existence and uniqueness of $\bar{y}^*$ in the lemma follows from [21, Theorem 1.17]. Define further $G : X \rightrightarrows Y^*$ by $G(x) := N(f(x); \Omega)$ and deduce from [21, Theorem 1.66] and the surjectivity of $\nabla f(\bar{x})$ that

$$\widehat{D}^*G(\bar{x}, \bar{y}^*)\big(\nabla f(\bar{x})^{**}u\big) = \nabla f(\bar{x})^*\widehat{D}^*N(\cdot; \Theta)\big(\nabla f(\bar{x})^{**}u\big) \ \text{ for all } \ u \in X^{**}.$$

This allows us to reduce justifying (6.4) to the proof of the equality

$$\breve{\partial}^2\delta(\cdot; f^{-1}(\Theta))(\bar{x}, \bar{x}^*)(u) = \nabla^2\langle \bar{y}^*, f\rangle(\bar{x})^*u + \widehat{D}^*G(\bar{x}, \bar{y}^*)(\nabla f(\bar{x})^{**}u), \quad u \in X^{**}. \tag{6.5}$$

Let us first show that the inclusion "$\subset$" holds in (6.5). Pick any $u^* \in \widehat{D}^*N(\cdot; \Omega)(\bar{x}, \bar{x}^*)(u)$ with $\Omega := f^{-1}(\Theta)$ and for any $\varepsilon > 0$ find $\eta > 0$ such that

$$\langle u^*, x - \bar{x}\rangle - \langle u, x^* - \bar{x}^*\rangle \le \varepsilon(\|x - \bar{x}\| + \|x^* - \bar{x}^*\|) \tag{6.6}$$

with $(x, x^*) \in \operatorname{gph} N(\cdot; \Omega) \cap I\!B_\eta(\bar{x}, \bar{x}^*)$. Since $\nabla f(\bar{x})$ is surjective, we find a neighborhood $\mathcal{O}$ of $\bar{x}$ such that $\nabla f(x)$ is surjective for all $x \in \mathcal{O}$. Choose a sufficiently small $\gamma > 0$

satisfying $\|\nabla f(x)^*y^* - \nabla f(\bar{x})^*\bar{y}^*\| \le \eta$ for all $(x, y^*) \in \operatorname{gph} G \cap I\!B_\gamma(\bar{x}, \bar{y}^*)$. Note from [21, Theorem 1.17] that $\nabla f(x)^*y^* \in N(x; f^{-1}(\Theta))$ and that

$$\nabla f(x)^*y^* - \nabla f(\bar{x})^*\bar{y}^* = \big[\nabla f(x) - \nabla f(\bar{x})\big]^*\bar{y}^* + \nabla f(\bar{x})^*(y^* - \bar{y}^*) + \big[\nabla f(x) - \nabla f(\bar{x})\big]^*(y^* - \bar{y}^*).$$

Combining this with (6.6) gives us that

$$\begin{aligned}&\langle u^* - \nabla^2\langle \bar{y}^*, f\rangle(\bar{x})u, x - \bar{x}\rangle + o(\|x - \bar{x}\|) - \langle \nabla f(\bar{x})^{**}u, y^* - \bar{y}^*\rangle - O(\|x - \bar{x}\|)\|y^* - \bar{y}^*\|\\ &\le \varepsilon(\|x - \bar{x}\| + \|y^* - \bar{y}^*\| + O(\|x - \bar{x}\| + \|y^* - \bar{y}^*\|),\end{aligned}$$

which implies in turn that

$$\limsup_{(x,y^*)\overset{\operatorname{gph} G}{\to}(\bar{x},\bar{y}^*)} \frac{\langle u^* - \nabla^2\langle \bar{y}^*, f\rangle(\bar{x})u, x - \bar{x}\rangle - \langle \nabla f(\bar{x})^{**}u, y^* - \bar{y}^*\rangle}{\|x - \bar{x}\| + \|y^* - \bar{y}^*\|} \le K\varepsilon \quad \text{for all} \quad \varepsilon > 0,$$

where K is some positive constant not depending on ε. This ensures that $u^* \in \nabla^2\langle \bar{y}^*, f\rangle(\bar{x})u + \widehat{D}^*G(\bar{x}, \bar{y}^*)(\nabla f(\bar{x})^{**}u)$ and thus justifies the inclusion "$\subset$" in (6.5).

To prove the converse inclusion in (6.5), fix an arbitrary element $u^* \in \nabla^2\langle \bar{y}^*, f\rangle(\bar{x})u + \widehat{D}^*G(\bar{x}, \bar{y}^*)(\nabla f(\bar{x})^{**}u)$ and deduce that for any $\varepsilon > 0$ there is $\eta > 0$ such that

$$\langle u^* - \nabla^2\langle \bar{y}^*, f\rangle(\bar{x})u, x - \bar{x}\rangle - \langle \nabla f(\bar{x})^{**}u, y^* - \bar{y}^*\rangle \le \varepsilon(\|x - \bar{x}\| + \|y^* - \bar{y}^*\|) \tag{6.7}$$

whenever $(x, y^*) \in \operatorname{gph} G \cap I\!B_\eta(\bar{x}, \bar{y}^*)$. For any number $\gamma \in (0, \eta)$ satisfying $I\!B_\gamma(\bar{x}) \subset \mathcal{O}$, pick $(x, x^*) \in \operatorname{gph} N(\cdot; \Omega) \cap I\!B_\gamma(\bar{x}, \bar{x}^*)$ and by employing [21, Theorem 1.17] find a unique normal $y^* \in N(f(x); \Theta) = G(x)$ such that $x^* = \nabla f(x)^*y^*$. Defining further the number $\kappa := \inf\big\{\|\nabla f(\bar{x})^*z^*\| \;\big|\; \|z^*\| = 1,\ z^* \in Y^*\big\} \in (0, \infty)$, we get that

$$\begin{aligned}\|x^* - \bar{x}^*\| &= \|\nabla f(x)^*y^* - \nabla f(\bar{x})^*\bar{y}^*\|\\ &= \|\nabla f(\bar{x})^*(y^* - \bar{y}^*) + \big[\nabla f(x) - \nabla f(\bar{x})\big]^*\bar{y}^* + \big[\nabla f(x) - \nabla f(\bar{x})\big]^*(y^* - \bar{y}^*)\| \\ &\ge \kappa\|y^* - \bar{y}^*\| - O(\|x - \bar{x}\|)\|\bar{y}^*\| - O(\|x - \bar{x}\|)\|y^* - \bar{y}^*\|.\end{aligned} \tag{6.8}$$

Thus the number $\gamma > 0$ can be chosen sufficiently small so that $\|y^* - \bar{y}^*\| \le \eta$. Similarly to the proof in the first part of the theorem we conclude that the left-hand side of (6.7) can be replaced by the expression

$$\langle u^*, x - \bar{x}\rangle - \langle u, \nabla f(x)y^* - \nabla f(\bar{x})^*\bar{y}^*\rangle + o(\|x - \bar{x}\|) + O(\|x - \bar{x}\|)\|y^* - \bar{y}^*\|.$$

This together with (6.7) and (6.8) gives us that

$$\limsup_{(x,x^*)\overset{\operatorname{gph} N(\cdot; f^{-1}(\Theta))}{\to}(\bar{x},\bar{x}^*)} \frac{\langle u^*, x - \bar{x}\rangle - \langle u, x^* - \bar{x}^*\rangle}{\|x - \bar{x}\| + \|x^* - \bar{x}^*\|} \le L\varepsilon \quad \text{for all} \quad \varepsilon > 0,$$

where $L > 0$ is some constant that does not depend on ε. This justifies the inclusion "$\supset$" in (6.5) and thus completes the proof of the lemma. $\triangle$

The last result of this section provides a *quantitative* characterization of metric regularity of set-valued mappings between Asplund space with a *prescribed modulus*. This seems to be new; cf. [21, Chapter 4] and the references therein for other coderivative characterizations.

Lemma 6.4 (quantitative coderivative characterization of metric regularity with prescribed modulus). *Let $F : X \rightrightarrows Y$ be a set-valued mapping between two Asplund spaces, and let its graph $\operatorname{gph} F$ be locally closed around $(\bar{x}, \bar{y}) \in \operatorname{gph} F$. Then F is metrically regular around $(\bar{x}, \bar{y})$ with modulus $\mu > 0$ if and only if there is some $\eta > 0$ such that*

$$\inf\Big\{\|x^*\| \;\Big|\; x^* \in \widehat{D}^*(x, y)(y^*),\ \|y^*\| = 1,\ (x, y) \in \operatorname{gph} F \cap I\!B_\eta(\bar{x}, \bar{y})\Big\} \ge \mu^{-1}. \tag{6.9}$$

Proof. If F is metrically regular around $(\bar{x},\bar{y})$, then we get (6.9) from [21, Theorem 1.54]. It remains to prove the converse implication. Arguing by contradiction, suppose that F is not metrically regular around $(\bar{x},\bar{y})$ with modulus μ while (6.9) holds for some $\eta>0$. This allows us to find $(\widehat{x},\widehat{y})\in\text{int}\,I\!\!B_\eta(\bar{x},\bar{y})$ satisfying $\text{dist}\,(\widehat{x};F^{-1}(\widehat{y}))>\mu\,\text{dist}\,(\widehat{y};F(\widehat{x}))$ and such that $(2\mu+1)\varepsilon<\frac{\eta}{4}$, where $\varepsilon:=\text{dist}\,(\widehat{y};F(\widehat{x}))>0$. To proceed, pick any $\nu\in(\mu,2\mu)$ with

$$\text{dist}\,\big(\widehat{x};F^{-1}(\widehat{y})\big)>\nu\varepsilon>\mu\,\text{dist}\,\big(\widehat{y};F(\widehat{x})\big)$$

and for any $\alpha>0$ find some $\widetilde{y}\in F(\widehat{x})$ satisfying

$$\|\widetilde{y}-\widehat{y}\|\le\text{dist}\,\big(\widehat{y};F(\widehat{x})\big)+\alpha=\varepsilon+\alpha. \tag{6.10}$$

For the l.s.c. and bounded from below function $\varphi(x,y):=\|y-\widehat{y}\|+\delta((x,y);\text{gph}\,F)$ on the Asplund space $X\times Y$ we have that

$$\inf_{(x,y)\in X\times Y}\varphi(x,y)+\varepsilon+\alpha\ge\varphi(\widehat{x},\widetilde{y}).$$

Applying the seminal Ekeland variational principle (see, e.g., [21, Theorem 2.26]) to the function φ with the new norm $\|(x,y)\|_\xi:=\|x\|+\xi\|y\|$, $\xi>0$ on $X\times Y$ gives us $(x_0,y_0)\in\text{gph}\,F$ satisfying

$$\begin{cases}\|x_0-\widehat{x}\|+\xi\|y_0-\widetilde{y}\|\le\nu\varepsilon,\\ \|y_0-\widehat{y}\|=\varphi(x_0,y_0)\le\varphi(\widehat{x},\widetilde{y})=\|\widetilde{y}-\widehat{y}\|,\\ \displaystyle\inf_{(x,y)\in X\times Y}\varphi(x,y)+\frac{\varepsilon+\alpha}{\nu\varepsilon}(\|x-x_0\|+\xi\|y-y_0\|)\ge\varphi(x_0,y_0)=\|y_0-\widehat{y}\|.\end{cases} \tag{6.11}$$

Consider further the l.s.c. functions on $X\times Y$ defined by

$$\varphi_1(x,y):=\|y-\widehat{y}\|,\ \varphi_2(x,y):=\delta((x,y);\text{gph}\,F),\ \text{ and }\ \varphi_3(x,y):=\frac{\alpha+\varepsilon}{\nu\alpha}(\|x-x_0\|+\xi\|y-y_0\|),$$

where two of them are Lipschitz continuous. Then for any $0<\beta<\text{dist}\,(\widehat{x};F^{-1}(\widehat{y}))-\nu\varepsilon$ we employ [21, Lemma 2.32] (the basic fuzzy sum rule or subgradient description of the extremal principle) to the optimization problem in (6.11) and thus find $(x_i,y_i)\in I\!\!B_\beta(x_0,y_0)$ as $i=1,2,3$ such that $(x_2,y_2)\in\text{gph}\,F$ and that

$$\begin{aligned}0\ &\in\widehat{\partial}\varphi_1(x_1,y_1)+\widehat{\partial}\varphi_2(x_2,y_2)+\widehat{\partial}\varphi_3(x_3,y_3)+\frac{\beta}{\nu\varepsilon}I\!\!B_{X^*}\times\xi I\!\!B_{Y^*}\\ &\subset\{0\}\times\widehat{\partial}\|\cdot-\widehat{y}\|(y_1)+\widehat{N}\big((x_2,y_2);\text{gph}\,F\big)+\frac{\varepsilon+\alpha+\beta}{\nu\varepsilon}I\!\!B_{X^*}\times\xi I\!\!B_{Y^*}.\end{aligned} \tag{6.12}$$

It follows from (6.11) that

$$\|x_1-\widehat{x}\|\le\|x_1-x_0\|+\|x_0-\widehat{x}\|<\beta+\nu\varepsilon<\text{dist}\,\big(\widehat{x};F^{-1}(\widehat{y})\big),$$

and hence $x_1\notin F^{-1}(\widehat{y})$ and $y_1\ne\widehat{y}$. This together (6.12) allows us to find some $y_1^*\in Y^*$ with $\|y_1^*\|=1$ and $(x_3^*,y_3^*)\in\frac{\varepsilon+\alpha+\beta}{\nu\varepsilon}I\!\!B_{X^*}\times\xi I\!\!B_{Y^*}$ such that $(-x_3^*,-y_1^*-y_3^*)\in\widehat{N}((x_2,y_2);\text{gph}\,F)$. Defining next $x_2^*:=-x_3^*\|y_1^*-y_3^*\|^{-1}$ and $y_2^*:=(y_1^*-y_3^*)\|y_1^*-y_3^*\|^{-1}$, we get that

$$\begin{cases}x_2^*\in\widehat{D}^*F(x_2,y_2)(y_2^*),\\ \displaystyle\|y_2^*\|=1,\ \|x_2^*\|\le\frac{\varepsilon+\alpha+\beta}{\nu\varepsilon}\Big(1-\frac{\varepsilon+\alpha+\beta}{\nu\varepsilon}\xi\Big)^{-1}=\nu^{-1}+O(\alpha,\beta,\xi).\end{cases} \tag{6.13}$$

Furthermore, it follows from (6.10), (6.11), and the choice of ν that

$$\begin{aligned}\|x_2-\bar{x}\|+\|y_2-\bar{y}\| &\le \|x_2-x_0\|+\|x_0-\widehat{x}\|+\|\widehat{x}-\bar{x}\|+\|y_2-y_0\|+\|y_0-\widehat{y}\|+\|\widehat{y}-\bar{y}\|\\ &\le (\|x_2-x_0\|+\|y_2-y_0\|)+(\|x_0-\widehat{x}\|+\|\widehat{y}-\bar{y}\|)+\nu\varepsilon+\|\widetilde{y}-\widehat{y}\|\\ &\le \beta+\frac{\eta}{2}+\nu\varepsilon+\varepsilon+\alpha\le\frac{\eta}{2}+(2\mu+1)\varepsilon+\alpha+\beta\le\frac{3\eta}{4}+\alpha+\beta.\end{aligned}$$

This together with (6.13) gives us that $(x_2,y_2)\in I\!B_\eta(\bar{x},\bar{y})$ and $\|x_2^*\|<\mu^{-1}$ when the positive numbers α,β,ξ are chosen to be sufficiently small. Thus we arrive at a contradiction with (6.9) and (6.13) and complete the proof of the lemma. $\triangle$

References

[1] F. J. Aragón Artacho and M. H. Geoffroy, Charaterizations of metric regularity of subdifferentials, *J. Convex Anal.* **15** (2008), 365–380.

[2] M. Bačák, J. M. Borwein, A. C. Eberhard and B. S. Mordukhovich, Infimal convolutions and Lipschitzian properties of subdifferentials for prox-regular functions in Hilbert spaces, *J. Convex Anal.* **17** (2010), 737–763.

[3] H. H. Bauschke and P. L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, Springer, New York, 2011.

[4] F. Bernard and L. Thibault, Prox-regular functions and sets in Banach spaces, *Set-Valued Anal.* **12** (2004), 25–47.

[5] F. Bernard and L. Thibault, Prox-regular functions in Hilbert spaces, *J. Math. Anal. Appl.* **303** (2005), 1–14.

[6] J. F. Bonnans and A. Shapiro, *Perturbation Analysis of Optimization Problems*, Springer, New York, 2000.

[7] J. M. Borwein and Q. J. Zhu, *Techniques of Variational Analysis*, Springer, New York, 2005.

[8] N. H. Chieu and N. Q. Huy, Second-order subdifferentials and convexity of real-valued functions, *Nonlinear Anal.* **74** (2011), 154–160.

[9] D. Drusvyatskiy and A. S. Lewis, Tilt stability, uniform quadratic growth, and strong metric regularity of the subdifferential, preprint (2012).

[10] A. L. Dontchev and R. T. Rockafellar, Characterizations of strong regularity for variational inequalities over polyhedral convex sets, *SIAM J. Optim.* **6** (1996), 1087–1105.

[11] A. L. Dontchev and R. T. Rockafellar, *Implicit Functions and Solution Mappings. A View from Variational Analysis*, Springer, Dordrecht, 2009.

[12] A. C. Eberhard and R. Wenczel, A study of tilt-stable optimality and sufficient conditions, *Nonlinear Anal.* **75** (2012), 1260–1281.

[13] R. Henrion, A. Y. Kruger and J. V. Outrata, Some remarks on stability of generalized equations, *J. Optim. Theory Appl.*, to appear.

[14] R. Henrion, J. V. Outrata and T. Surowiec, On regular coderivatives in parametric equilibria with non-unique multipliers, *Math. Program.*, to appear (2012).

[15] K. Ito and K. Kunisch, *Lagrange Multiplier Approach to Variational Problems and Applications*, SIAM, Philadelphia, PA, 2008.

[16] M. Kojima, Strongky stable stationary solutions in nonlinear programming, in *Analysis and Computation of Fixed Points* (S. M. Robinson, ed.), pp. 93–138, Academic Press, New York, 1980.

[17] A. B. Levy, R. A. Poliquin and R. T. Rockafellar, Stability of locally optimal solutions, *SIAM J. Optim.* **10** (2000), 580–604.

[18] A. S. Lewis and S. Zhang, Partial smoothness, tilt stability, and generalized Hessians, *SIAM J. Optim.*, to appear.

[19] S. Lu, Implications of the constant rank constraint qualification, *Math. Program.* **126** (2011), 365–392.

[20] B. S. Mordukhovich, Sensitivity analysis in nonsmooth optimization, in *Theoretical Aspects of Industrial Design* (D. A. Field and V. Komkov, eds.), SIAM Proc. Applied Math., Vol. 58, pp. 32–46, SIAM, Philadelphia, PA, 1992.

[21] B. S. Mordukhovich, *Variational Analysis and Generalized Differentiation, I: Basic Theory, II: Applications* Springer, Berlin, 2006.

[22] B. S. Mordukhovich and J. V. Outrata, Tilt stability in nonlinear programming under Mangasarian-Fromovitz constraint qualification, *Kybernetika*, to appear (2012).

[23] B. S. Mordukhovich and R. T. Rockafellar, Second-order subdifferential calculus with applications to tilt stability in optimization, *SIAM J. Optim.* **22** (2012).

[24] B. S. Mordukhovich, R. T. Rockafellar and M. E. Sarabi, Characterizations of full stability in constrained optimization, preprint (2012).

[25] R. A. Poliquin and R. T. Rockafellar, Prox-regular functions in variational analysis, *Trans. Amer. Math. Soc.* **348** (1996), 1805–1838.

[26] R. A. Poliquin and R. T. Rockafellar, Tilt stability of a local minimum, *SIAM J. Optim.* **8** (1998), 287–299.

[27] S. M. Robinson, Strongly regular generalized equations, *Math. Oper. Res.* **5** (1980), 43–62.

[28] R. T. Rockafellar, On the maximality of sums of nonlinear monotone operators, *Trans. Amer. Math. Soc.* **149** (1970), 75–88.

[29] R. T. Rockafellar and R. J-B Wets, *Variational Analysis*, Springer, Berlin, 1998.

[30] W. Schirotzek, *Nonsmooth Analysis*, Springer, Berlin, 2007.

[31] C. Zălinescu, *Convex Analysis in General Vector Spaces*, World Scientific, Singapore, 2002.