Uniqueness of Kusuoka Representations

Alois Pichler*
Norwegian University of Science and Technology (NTNU)
Norway

Alexander Shapiro†
School of Industrial and Systems Engineering
Georgia Institute of Technology
Atlanta, GA 30332-0205

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Abstract

This paper addresses law invariant coherent risk measures and their Kusuoka representations. By elaborating the existence of a minimal representation we show that every Kusuoka representation can be reduced to its minimal representation. Uniqueness – in a sense specified in the paper – of the risk measure's Kusuoka representation is derived from this initial result. The Kusuoka representation is usually given for nonatomic probability spaces. We also discuss Kusuoka representations for spaces with atoms.

Further, stochastic order relations are employed to identify the minimal Kusuoka representation. It is shown that measures in the minimal representation are extremal with respect to the order relations. The tools are finally employed to provide the minimal representation for important practical examples.

Keywords: Law invariant coherent measure of risk, Fenchel-Moreau theorem, Kusuoka representation, stochastic order relations.

1 Introduction

This paper addresses Kusuoka representations [10] of law invariant, coherent risk measures. In a sense such representations are natural and useful in various applications of risk measures. Original derivations were performed in [10, 9] in $L_\infty(\Omega, \mathcal{F}, P)$ spaces. For an analysis in $L_p(\Omega, \mathcal{F}, P)$, $p \in [1, \infty)$, spaces we may refer to Pflug and Römisch [16] (cf. also [5, 8]). It is known that Kusuoka representations are not unique (cf. [16]). This raises the question of in some sense minimality of such representations. It was shown in [21] that indeed for spectral risk measures, the Kusuoka representation with a single probability measure is unique. It was

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also posed a question whether such minimal representation is unique in general? In this paper we give a positive answer to this question and show how such minimal representations can be derived in a constructive way. We also discuss Kusuoka representations for spaces which are not nonatomic.

Throughout the paper we work with spaces $\mathcal{Z} := L_p(\Omega, \mathcal{F}, P)$, $p \in [1, \infty)$, rather than $L_\infty(\Omega, \mathcal{F}, P)$ spaces. That is, $Z \in \mathcal{Z}$ can be viewed as a random variable with finite p-th order moment with respect to the reference probability distribution P. The space $\mathcal{Z}$ equipped with the respective norm is a Banach space with the dual space of continuous linear functionals $\mathcal{Z}^* := L_q(\Omega, \mathcal{F}, P)$, where $q \in (1, \infty]$ and $1/p + 1/q = 1$ (cf. [17] for an extended discussion on the supporting Banach space).

It is said that a (real valued) functional $\rho : \mathcal{Z} \to \mathbb{R}$ is a *coherent* risk measure[1] if it satisfies the following axioms (Artzner et al. [1]):

(A1) *M*onotonicity: If $Z, Z' \in \mathcal{Z}$ and $Z \succeq Z'$, then $\rho(Z) \geq \rho(Z')$.

(A2) *C*onvexity:

$$\rho(tZ + (1-t)Z') \leq t\rho(Z) + (1-t)\rho(Z')$$

for all $Z, Z' \in \mathcal{Z}$ and all $t \in [0, 1]$.

(A3) *T*ranslation Equivariance: If $a \in \mathbb{R}$ and $Z \in \mathcal{Z}$, then $\rho(Z + a) = \rho(Z) + a$.

(A4) *P*ositive Homogeneity: If $t \geq 0$ and $Z \in \mathcal{Z}$, then $\rho(tZ) = t\rho(Z)$.

It is said that risk measure ρ is *convex* if it satisfies axioms (A1)-(A3). The notation $Z \succeq Z'$ means that $Z(\omega) \geq Z'(\omega)$ for a.e. $\omega \in \Omega$. For a thorough discussion of coherent (convex) risk measures we refer to Föllmer and Schied [6].

We say that $Z_1, Z_2 \in \mathcal{Z}$ are *distributionally equivalent* if $F_{Z_1} = F_{Z_2}$, where $F_Z(z) := P(Z \leq z)$ denotes the cumulative distribution function (cdf) of $Z \in \mathcal{Z}$. It is said that risk measure $\rho : \mathcal{Z} \to \mathbb{R}$ is *law invariant* (with respect to the reference probability measure P) if for any distributionally equivalent $Z_1, Z_2 \in \mathcal{Z}$ it follows that $\rho(Z_1) = \rho(Z_2)$. *Unless stated otherwise we assume that ρ is a (real valued) law invariant coherent risk measure.* An important example of a law invariant coherent risk measure is the (upper) Average Value-at-Risk

$$\mathsf{AV@R}_\alpha(Z) := \inf_{t \in \mathbb{R}} \left\{ t + (1-\alpha)^{-1} \mathbb{E}[Z - t]_+ \right\} = (1-\alpha)^{-1} \int_\alpha^1 F_Z^{-1}(\tau) d\tau, \qquad (1.1)$$

where $\alpha \in [0, 1)$ and $F_Z^{-1}(\tau) := \sup\{t : F_Z(t) \leq \tau\}$ is the right side quantile function. Note that $F_Z^{-1}(\cdot)$ is a monotonically nondecreasing right side continuous function. We denote by $\mathfrak{P}$ the set of probability measures on $[0, 1]$ having zero mass at 1. When talking about topological properties of $\mathfrak{P}$ we use the *weak topology* of probability measures (e.g., [2]).

We say that a set $\mathfrak{M}$ of probability measures on $[0, 1]$ is a *Kusuoka set* if the representation

$$\rho(Z) = \sup_{\mu \in \mathfrak{M}} \int_0^1 \mathsf{AV@R}_\alpha(Z) d\mu(\alpha), \quad Z \in \mathcal{Z} \qquad (1.2)$$

[1]In a financial context the term *coherent risk measure* is often used for mapping $\mathcal{R}(Z) = \rho(-Z)$, or the concave mapping $\mathcal{R}(Z) = -\rho(-Z)$ instead. The axioms then change accordingly.

holds. Note that since $\rho(Z)$ is finite valued for every $Z \in L_p(\Omega, \mathcal{F}, P)$, with $p \in [1, \infty)$, every measure $\mu \in \mathfrak{M}$ in representation (1.2) has zero mass at $\alpha = 1$ and hence $\mathfrak{M} \subset \mathfrak{P}$. Note also that if $\mathfrak{M}$ is a Kusuoka set, then its topological closure is also a Kusuoka set (cf. [21, Proposition 1]).

The paper is organized as follows. In the next section we introduce the notation and discuss minimality and uniqueness of the Kusuoka representations. In Section 3 we consider Kusuoka representations on general, not necessarily nonatomic, probability spaces. In Section 4 we investigate maximality of Kusuoka representations with respect to order, or dominance relations. In Section 5 we discuss some examples, while Section 6 is devoted to conclusions.

2 Uniqueness of Kusuoka sets

It is known that the Kusuoka representation is not unique in general. In this section we elaborate that there is minimal Kusuoka representation in the sense outlined below. To obtain the result we shall relate the Kusuoka representation to spectral risk measures first and then outline the results.

We can view the integral in the right hand side of (1.2) as the Lebesgue-Stieltjes integral with $\mu : \mathbb{R} \to [0, 1]$ being a right side continuous monotonically nondecreasing function (distribution function) such that $\mu(t) = 0$ for $t < 0$ and $\mu(t) = 1$ for $t \geq 1$. For example, take $\mu(t) = 0$ for $t < 0$ and $\mu(t) = 1$ for $t \geq 0$. This is a distribution function corresponding to a measure of mass one at $\alpha = 0$. Thus

$$\int_0^1 \mathsf{AV@R}_\alpha(Z) d\mu(\alpha) = \mathsf{AV@R}_0(Z) = \mathbb{E}[Z], \tag{2.1}$$

where the integral is understood as taken from 0^- to 1.

Note: When considering integrals of the form $\int_0^\gamma h(\alpha) d\mu(\alpha)$, $\gamma \geq 0$, with respect to a distribution function $\mu(\cdot)$ we always assume that the integral is taken from 0^-.

> Assume that the space $(\Omega, \mathcal{F}, P)$ is nonatomic. Then, without loss of generality, we can take this space to be the interval $\Omega = [0, 1]$ equipped with its Borel sigma algebra and the uniform probability measure P. In this paper we refer to this space as the *uniform* probability space. Unless stated otherwise we assume that $(\Omega, \mathcal{F}, P)$ is the uniform probability space.

Recall that it is said that a transformation $T : \Omega \to \Omega$ is measurable if $T^{-1}(A) \in \mathcal{F}$ for any $A \in \mathcal{F}$.

Definition 2.1 (Measure-preserving transformation) *We say that $T : \Omega \to \Omega$ is a* measure-preserving transformation *if T is one-to-one, onto, measurable and for any $A \in \mathcal{F}$ it follows that $P(A) = P(T^{-1}(A))$. We denote by $\mathfrak{F}$ the group of measure-preserving transformations.*

This definition of measure-preserving transformations is slightly stronger than the standard one in that we also assume that T is one-to-one and onto; hence T is invertible and for any $A \in \mathcal{F}$ it follows that $T(A) \in \mathcal{F}$ and $P(A) = P(T(A))$.

Remark 1 Note that two random variables $Z_1, Z_2 : \Omega \to \mathbb{R}$ have the same probability distribution iff there exists a measure-preserving transformation $T \in \mathfrak{F}$ such that[2] $Z_2 = Z_1 \circ T$ (e.g., [9]). Also a random variable $Z : \Omega \to \mathbb{R}$ is distributionally equivalent to F_Z^{-1}, i.e., there exists $T \in \mathfrak{F}$ such that a.e. $F_Z^{-1} = Z \circ T$ (e.g., [21]).

Definition 2.2 (Spectral risk functions) *We say that* $\sigma : [0,1) \to \mathbb{R}_+$ *is a* spectral function *if* $\sigma(\cdot)$ *is right side continuous, monotonically nondecreasing and such that* $\int_0^1 \sigma(t)dt = 1$. *The set of spectral functions will be denoted by* $\mathfrak{S}$.

Consider the linear mapping $\mathbb{T} : \mathfrak{P} \to \mathfrak{S}$ defined as

$$(\mathbb{T}\mu)(\tau) := \int_0^\tau (1-\alpha)^{-1} d\mu(\alpha),\ \tau \in [0,1). \tag{2.2}$$

This mapping is onto, one-to-one with the inverse $\mu = \mathbb{T}^{-1}\sigma$ given by

$$\mu(\alpha) = \left(\mathbb{T}^{-1}\sigma\right)(\alpha) = (1-\alpha)\sigma(\alpha) + \int_0^\alpha \sigma(\tau)d\tau, \quad \alpha \in [0,1] \tag{2.3}$$

(cf. [21, Lemma 2]). In particular, let $\mu := \sum_{i=1}^n c_i \delta_{\alpha_i}$, where δ_α denotes the probability measure of mass one at α, c_i are positive numbers such that $\sum_{i=1}^n c_i = 1$, and $0 \le \alpha_1 < \alpha_2 < \cdots < \alpha_n < 1$. Then

$$\mathbb{T}\mu = \sum_{i=1}^n \frac{c_i}{1-\alpha_i} \mathbf{1}_{[\alpha_i,1]}, \tag{2.4}$$

where $\mathbf{1}_A$ denotes the indicator function of set A.

The (real valued) coherent risk measure $\rho : \mathcal{Z} \to \mathbb{R}$ is continuous (in the norm topology of $\mathcal{Z} = L_p(\Omega, \mathcal{F}, P)$) (cf. [18]), and by the Fenchel-Moreau theorem has the following dual representation

$$\rho(Z) = \sup_{\zeta \in \mathfrak{A}} \langle \zeta, Z \rangle, \quad Z \in \mathcal{Z}, \tag{2.5}$$

where $\mathfrak{A} \subset \mathcal{Z}^*$ is a convex and weakly* compact set of density functions and the scalar product on $\mathcal{Z}^* \times \mathcal{Z}$ is defined as $\langle \zeta, Z \rangle := \int_0^1 \zeta(\tau) Z(\tau) d\tau$. Since ρ is law invariant, the dual set $\mathfrak{A}$ is invariant under the group $\mathfrak{F}$ of measure preserving transformations, i.e., if $\zeta \in \mathfrak{A}$, then $\zeta \circ T \in \mathfrak{A}$ for any $T \in \mathfrak{F}$ (cf. [21]).

Note that if

$$\rho(Z) = \sup_{\zeta \in \mathfrak{C}} \langle \zeta, Z \rangle, \quad Z \in \mathcal{Z}, \tag{2.6}$$

holds for some set $\mathfrak{C} \subset \mathcal{Z}^*$, then $\mathfrak{C} \subset \mathfrak{A}$. For a set $\Upsilon \subset \mathcal{Z}^*$ we denote

$$\mathcal{O}(\Upsilon) := \{\zeta \circ T : T \in \mathfrak{F},\ \zeta \in \Upsilon\} \tag{2.7}$$

its orbit with respect to the group $\mathfrak{F}$. Consider the following definition (cf. [21, Definition 1]).

Definition 2.3 (Generating sets) *We say that a set* $\Upsilon \subset \mathfrak{S}$ *of spectral functions is a* generating set *if the representation* (2.6) *holds for* $\mathfrak{C} := \mathcal{O}(\Upsilon)$. *That is,*

$$\rho(Z) = \sup_{\sigma \in \Upsilon} \int_0^1 \sigma(\tau) F_Z^{-1}(\tau) d\tau, \quad Z \in \mathcal{Z}. \tag{2.8}$$

[2]The notation $(Z \circ T)(\omega)$ stands for the composition $Z(T(\omega))$.

It follows that if Υ is a generating set, then $\mathcal{O}(\Upsilon) \subset \mathfrak{A}$.

Proposition 2.1 *A set $\mathfrak{M} \subset \mathfrak{P}$ is a Kusuoka set iff the set $\Upsilon := \mathbb{T}(\mathfrak{M})$ is a generating set.*

Proof. By using the integral representation (1.1) of $\mathsf{AV@R}$ we can write

$$\int_0^1 \mathsf{AV@R}_\alpha(Z) d\mu(\alpha) = \int_0^1 \int_\alpha^1 (1-\alpha)^{-1} F_Z^{-1}(\tau) d\tau d\mu(\alpha) = \int_0^1 (\mathbb{T}\mu)(\tau) F_Z^{-1}(\tau) d\tau. \tag{2.9}$$

Therefore representation (1.2) can be written as

$$\rho(Z) = \sup_{\mu \in \mathfrak{M}} \int_0^1 (\mathbb{T}\mu)(\tau) F_Z^{-1}(\tau) d\tau. \tag{2.10}$$

Together with (2.8) this shows that $\mathfrak{M}$ is a Kusuoka set iff $\mathbb{T}(\mathfrak{M})$ is a generating set. ∎

Definition 2.4 (Exposed points) *It is said that $\bar{\zeta}$ is a* weak* exposed point *of $\mathfrak{A} \subset \mathcal{Z}^*$ if there exists $Z \in \mathcal{Z}$ such that $g_Z(\zeta) := \langle \zeta, Z \rangle$ attains its maximum over $\zeta \in \mathfrak{A}$ at the unique point $\bar{\zeta}$. In that case we say that Z* exposes *$\mathfrak{A}$ at $\bar{\zeta}$. We denote by* $\mathrm{Exp}(\mathfrak{A})$ *the set of exposed points of $\mathfrak{A}$.*

A result going back to Mazur [12] says that if X is a separable Banach space and $K \subset X^*$ is a nonempty weakly* compact subset of X^*, then the set of points $x \in X$ which expose K at some point $x^* \in K$ is a dense (in the norm topology) subset of X (see, e.g., [11] for a discussion of these type results). Since the space $\mathcal{Z} = L_p(\Omega, \mathcal{F}, P)$, $p \in [1, \infty)$, is *separable* we have the following theorem[3]:

Theorem 2.1 *Let $\mathfrak{A}$ be the dual set in* (2.5). *Then the set*

$$\mathcal{D} := \{Z \in \mathcal{Z} : Z \text{ exposes } \mathfrak{A} \text{ at a point } \bar{\zeta}\} \tag{2.11}$$

is a dense (in the norm topology) subset of $\mathcal{Z}$.

This allows to proceed towards a minimal representation of a risk measure as follows.

Proposition 2.2 *Let* $\mathrm{Exp}(\mathfrak{A})$ *be the set of exposed points of $\mathfrak{A}$. Then the representation* (2.6) *holds with $\mathfrak{C} :=$* $\mathrm{Exp}(\mathfrak{A})$. *Moreover, if the representation* (2.6) *holds for some weakly* closed set $\mathfrak{C}$, then* $\mathrm{Exp}(\mathfrak{A}) \subset \mathfrak{C}$.

Proof. Consider the set $\mathcal{D}$ defined in (2.11). By Theorem 2.1 this set is dense in $\mathcal{Z}$. So for $Z \in \mathcal{Z}$ fixed, let $\{Z_n\} \subset \mathcal{D}$ be a sequence of points converging (in the norm topology) to Z. Let $\{\zeta_n\} \subset \mathrm{Exp}(\mathfrak{A})$ be a sequence of the corresponding maximizers, i.e., $\rho(Z_n) = \langle \zeta_n, Z_n \rangle$. Since $\mathfrak{A}$ is bounded, we have that $\|\zeta_n\|^*$ is uniformly bounded. Since $\rho : \mathcal{Z} \to \mathbb{R}$ is real valued it is continuous (cf. [18]), and thus $\rho(Z_n) \to \rho(Z)$. We also have that

$$|\rho(Z_n) - \langle \zeta_n, Z \rangle| = |\langle \zeta_n, Z_n \rangle - \langle \zeta_n, Z \rangle| \le \|\zeta_n\|^* \|Z_n - Z\| \to 0.$$

[3]We need here for the dual set $\mathfrak{A}$ to be compact, this is why we assume in this section that ρ is real valued.

It follows that

$$\rho(Z) = \sup\{\langle \zeta_n, Z\rangle : n = 1, ...\},$$

and hence the representation (2.6) holds with $\mathfrak{C} = \mathrm{Exp}(\mathfrak{A})$.

Let $\mathfrak{C}$ be a weakly* closed set such that the representation (2.6) holds. It follows that $\mathfrak{C}$ is a subset of $\mathfrak{A}$ and is weakly* compact. Consider a point $\zeta \in \mathrm{Exp}(\mathfrak{A})$. By the definition of the set $\mathrm{Exp}(\mathfrak{A})$, there is $Z \in \mathcal{D}$ such that $\rho(Z) = \langle \zeta, Z\rangle$. Since $\mathfrak{C}$ is weakly* compact, the maximum in (2.6) is attained and hence $\rho(Z) = \langle \zeta', Z\rangle$ for some $\zeta' \in \mathfrak{C}$. By the uniqueness of the maximizer ζ it follows that $\zeta' = \zeta$. This shows that $\mathrm{Exp}(\mathfrak{A}) \subset \mathfrak{C}$. ∎

By the above proposition, the weak* closure $\overline{\mathrm{Exp}(\mathfrak{A})}$ coincides with the intersection of all weakly* closed sets $\mathfrak{C}$ satisfying representation (2.6).

Note: For a set $A \subset \mathcal{Z}^*$ we denote by $\overline{A}$ the topological closure of A in the weak* topology of the space $\mathcal{Z}^*$.

Proposition 2.3 *The sets* $\mathrm{Exp}(\mathfrak{A})$ *and* $\overline{\mathrm{Exp}(\mathfrak{A})}$ *are invariant under the group* $\mathfrak{F}$ *of measure preserving transformations.*

Proof. For $T \in \mathfrak{F}$ we have that

$$\langle \zeta \circ T, Z\rangle = \int_0^1 \zeta(T(\tau))Z(\tau)d\tau = \int_0^1 \zeta(\tau)Z(T^{-1}(\tau))d\tau = \langle \zeta, Z \circ T^{-1}\rangle.$$

Thus $\bar\zeta \in \mathfrak{A}$ is a maximizer of $\langle \zeta, Z\rangle$ over $\zeta \in \mathfrak{A}$, iff $\bar\zeta \circ T$ is a maximizer of $\langle \zeta, T^{-1} \circ Z\rangle$ over $\zeta \in \mathfrak{A}$. Consequently we have that if $\bar\zeta \in \mathrm{Exp}(\mathfrak{A})$, then $\bar\zeta \circ T \in \mathrm{Exp}(\mathfrak{A})$, i.e., $\mathrm{Exp}(\mathfrak{A})$ is invariant under $\mathfrak{F}$.

Now consider a point $\bar\zeta \in \overline{\mathrm{Exp}(\mathfrak{A})}$. Then there is a sequence $\zeta_n \in \mathrm{Exp}(\mathfrak{A})$ such that $\langle \zeta_n, Z\rangle$ converges to $\langle \bar\zeta, Z\rangle$ for any $Z \in \mathcal{Z}$. For any $T \in \mathfrak{F}$ we have that $\zeta_n \circ T \in \mathrm{Exp}(\mathfrak{A})$, and hence $\langle \zeta_n \circ T, Z\rangle = \langle \zeta_n, Z \circ T^{-1}\rangle$ converges to $\langle \bar\zeta, Z \circ T^{-1}\rangle = \langle \bar\zeta \circ T, Z\rangle$ for any $Z \in \mathcal{Z}$. It follows that $\bar\zeta \circ T \in \overline{\mathrm{Exp}(\mathfrak{A})}$. ∎

By the above proposition we have that for any exposed point there is a distributional equivalent which is a monotonically nondecreasing function. We employ this observation in the following Corollary 2.1 to reduce the generating set.

Definition 2.5 *Denote by* $\mathfrak{E}$ *the subset of* $\mathrm{Exp}(\mathfrak{A})$ *formed by right side continuous monotonically nondecreasing functions, i.e.,* $\mathfrak{E} = \mathfrak{S} \cap \mathrm{Exp}(\mathfrak{A})$.

It follows by Proposition 2.2 that the topological closure $\bar{\mathfrak{E}}$ (of the set $\mathfrak{E}$) is the unique minimal generating set in the following sense.

Corollary 2.1 *The set* $\mathfrak{E}$ *is a generating set. Moreover, if* Υ *is any weakly* closed generating set, then the set* $\bar{\mathfrak{E}}$ *is a subset of* Υ*, i.e.,* $\bar{\mathfrak{E}}$ *coincides with the intersection of all weakly* closed generating sets.*

As before, let $(\Omega, \mathcal{F}, P)$ be the uniform probability space and consider the space $\mathcal{Z} := L_p(\Omega, \mathcal{F}, P)$, $p \in [1, \infty)$, its dual $\mathcal{Z}^* = L_q(\Omega, \mathcal{F}, P)$, and the set

$$\mathfrak{P}_q := \{\mu \in \mathfrak{P} : \mathbb{T}\mu \in \mathcal{Z}^*\}. \tag{2.12}$$

Proposition 2.4 *The set $\mathfrak{P}_q$ is closed and the mapping $\mathbb{T}$ is continuous on the set $\mathfrak{P}_q$ with respect to the weak topology of measures and the weak* topology of $\mathcal{Z}^*$.*

Proof. Let $\{\mu_k\} \subset \mathfrak{P}_q$ be a sequence of measures converging in the weak topology to $\mu \in \mathfrak{P}$, and define $\zeta_k := \mathbb{T}\mu_k$ and $\zeta := \mathbb{T}\mu$. For $Z \in \mathcal{Z}$ we have that

$$\int_0^1 Z(t)\zeta_k(t)dt = \int_0^1 \int_0^t Z(t)(1-\alpha)^{-1} d\mu_k(\alpha)dt = \int_0^1 (1-\alpha)^{-1} h_Z(\alpha) d\mu_k(\alpha),$$

where $h_Z(\alpha) := \int_\alpha^1 Z(t)dt$. Note that $|h_Z(\alpha)| \leq \|Z\|_q$ for $\alpha \in [0,1]$, and the function $\alpha \mapsto h_Z(\alpha)$ is continuous. Let us choose a dense set $\mathcal{V} \subset \mathcal{Z}$ such that for any $Z \in \mathcal{V}$ the corresponding function $(1-\alpha)^{-1}h_Z(\alpha)$ is bounded on [0,1]. For example, we can take functions $Z \in \mathcal{Z}$ such that $Z(t) = 0$ for $t \in [\gamma, 1]$ with $\gamma \in (0,1)$. Then for any $Z \in \mathcal{V}$ we have that

$$\lim_{k\to\infty} \int_0^1 Z(t)\zeta_k(t)dt = \lim_{k\to\infty} \int_0^1 (1-\alpha)^{-1} h_Z(\alpha) d\mu_k(\alpha) = \int_0^1 (1-\alpha)^{-1} h_Z(\alpha) d\mu(\alpha),$$

and hence

$$\lim_{k\to\infty} \int_0^1 Z(t)\zeta_k(t)dt = \int_0^1 Z(t)\zeta(t)dt, \tag{2.13}$$

with the integral in the right hand side of (2.13) being finite. This shows that $\mathbb{T}$ is continuous at μ and $\zeta \in \mathcal{Z}^*$, and hence $\mu \in \mathfrak{P}_q$. ∎

It follows that if $\mathfrak{C} \subset \mathfrak{A}$ is a weakly* closed set of right side continuous monotonically nondecreasing functions, then $\mathbb{T}^{-1}(\mathfrak{C})$ is a closed subset of $\mathfrak{P}$.

Definition 2.6 (Minimality) *We say that a Kusuoka set $\mathfrak{M} \subset \mathfrak{P}$ is* minimal *if $\mathfrak{M}$ is closed and for any closed Kusuoka set $\mathfrak{M}'$ it holds that $\mathfrak{M} \subset \mathfrak{M}'$. We say that a generating set Υ is* minimal *if Υ is weakly* closed and for any weakly* closed generating set Υ' it holds that $\Upsilon \subset \Upsilon'$.*

Note that it follows from the above definition that *if minimal Kusuoka set exists, then it is unique.*

By Propositions 2.1 and 2.4 we have that a Kusuoka set $\mathfrak{M}$ is minimal iff the set $\mathbb{T}(\mathfrak{M})$ is a minimal generating set. Together with Theorem 2.1 and Proposition 2.2 this implies the following result.

Theorem 2.2 *The set $\mathfrak{M} := \mathbb{T}^{-1}(\mathfrak{E})$ is a Kusuoka set and its closure $\overline{\mathfrak{M}} = \overline{\mathbb{T}^{-1}(\mathfrak{E})} = \mathbb{T}^{-1}(\bar{\mathfrak{E}})$ is the minimal Kusuoka set.*

This shows that the minimal Kusuoka set indeed exists and, as it was mentioned before, by the definition is unique. In particular we obtain the following result (cf. [21]).

Corollary 2.2 *Let $\rho_\sigma : \mathcal{Z} \to \mathbb{R}$ be a spectral measure with spectral function $\sigma \in \mathcal{Z}^* \cap \mathfrak{S}$, i.e.,*

$$\rho_\sigma(Z) = \int_0^1 \sigma(t) F_Z^{-1}(t)dt. \tag{2.14}$$

Then its minimal Kusuoka set is given by the singleton $\{\mathbb{T}^{-1}\sigma\}$.

3 Kusuoka representation on general spaces

So far we considered cases when the reference probability space is nonatomic. Of course this rules out, for example, discrete probability spaces. So what can be said about law invariant risk measures defined on probability spaces with atoms? Let us consider the following construction. Suppose that the reference probability space can be embedded into the uniform probability space. That is, let $(\Omega, \mathcal{F}, P)$ be the uniform probability space and $\mathcal{G}$ be a sigma subalgebra of $\mathcal{F}$ such that the considered reference probability space is equivalent (isomorphic) to $(\Omega, \mathcal{G}, P)$. So we can view $(\Omega, \mathcal{G}, P)$ as the reference probability space. For example, let the reference probability space be discrete with a countable (finite) number of elementary events and respective probabilities $p_1, p_2, \ldots$. Let us partition $\Omega = [0, 1]$ into intervals $A_1, A_2, \ldots$, of respective lengths $p_1, p_2, \ldots$, and consider the sigma algebra $\mathcal{G} \subset \mathcal{F}$ generated by these intervals. This will give the required embedding of the discrete probability space into the uniform probability space.

Consider the space $\hat{\mathcal{Z}} := L_p(\Omega, \mathcal{G}, P)$, $p \in [1, \infty)$, and a law invariant risk measure $\varrho : \hat{\mathcal{Z}} \to \mathbb{R} \cup \{+\infty\}$. Note that ϱ is supposed to be law invariant with respect to the *reference* probability space. That is, if $Z_1, Z_2 \in \hat{\mathcal{Z}}$ are two $\mathcal{G}$-measurable distributionally equivalent random variables, then $\varrho(Z_1) = \varrho(Z_2)$. Note that $\mathcal{G}$-measurability of $Z \in \hat{\mathcal{Z}}$ does not necessarily imply $\mathcal{G}$-measurability of F_Z^{-1}. Note also that the space $\hat{\mathcal{Z}}$ is a subspace of the space $\mathcal{Z} := L_p(\Omega, \mathcal{F}, P)$. Therefore a relevant question is whether it is possible to extend ϱ to a law invariant risk measure on the space $\mathcal{Z}$. For a risk measure $\rho : \mathcal{Z} \to \overline{\mathbb{R}}$ we denote by $\rho|_{\hat{\mathcal{Z}}}$ the restriction of ρ to the space $\hat{\mathcal{Z}}$, i.e., $\hat{\rho} = \rho|_{\hat{\mathcal{Z}}}$ is defined on the space $\hat{\mathcal{Z}}$ and $\hat{\rho}(Z) = \rho(Z)$ for $Z \in \hat{\mathcal{Z}}$. Note that in this section we consider risk measures which can take $+\infty$ value.

Definition 3.1 (Regularity) *We say that a proper lower semicontinuous law invariant coherent (convex) risk measure $\varrho : \hat{\mathcal{Z}} \to \mathbb{R} \cup \{+\infty\}$ is* regular *if there exists a proper lower semicontinuous law invariant coherent (convex) risk measure $\rho : \mathcal{Z} \to \mathbb{R} \cup \{+\infty\}$ such that $\rho|_{\hat{\mathcal{Z}}} = \varrho$.*

Similar definition was given in Noyan and Rudolf [14, Definition 6.1].

For example, the Average Value-at-Risk and mean-semideviation measures are regular risk measures irrespective of the reference probability space. We denote by $\mathfrak{G}$ the set of measure-preserving transformations of the *reference* space $(\Omega, \mathcal{G}, P)$. Note that $\mathfrak{G}$ is a subgroup of the group $\mathfrak{F}$ of measure-preserving transformations of the uniform space $(\Omega, \mathcal{F}, P)$.

For a regular law invariant coherent risk measure $\varrho = \hat{\rho}$ we can write for $Z \in \hat{\mathcal{Z}}$ the dual representation

$$\varrho(Z) = \sup_{\sigma \in \Upsilon} \int_0^1 \sigma(t) F_Z^{-1}(t) dt, \tag{3.1}$$

where Υ is a generating set of the corresponding risk measure $\rho : \mathcal{Z} \to \mathbb{R} \cup \{+\infty\}$, and the respective Kusuoka representation

$$\varrho(Z) = \sup_{\mu \in \mathfrak{M}} \int_0^1 \mathsf{AV@R}_\alpha(Z) d\mu(\alpha), \tag{3.2}$$

where $\mathfrak{M}$ is the set of probability measures corresponding to the risk measure $\rho : \mathcal{Z} \to \mathbb{R} \cup \{+\infty\}$. Conversely, if the representation (3.1) or the Kusuoka representation (3.2) holds and the right hand side of (3.1) (or (3.2)) is well defined and finite for every $Z \in \mathcal{Z}$, then ϱ is a regular law invariant coherent risk measure. Hence we have the following.

Proposition 3.1 *A proper lower semicontinuous law invariant coherent risk measure $\varrho : \hat{\mathcal{Z}} \to \mathbb{R} \cup \{+\infty\}$ is regular iff there exists a set $\Upsilon \subset L_q(\Omega, \mathcal{F}, P)$ of spectral functions such that the representation* (3.1) *holds, or equivalently iff the Kusuoka representation* (3.2) *holds.*

Let us give now sufficient conditions for regularity of every coherent risk measure defined on a reference probability space. Consider the following condition.

(B) For any $\mathcal{G}$-measurable random variable $Z : [0, 1] \to \mathbb{R}$ there exists measure preserving transformation $T \in \mathfrak{G}$, of the reference space $(\Omega, \mathcal{G}, P)$, such that $Z \circ T$ is monotonically nondecreasing.

Proposition 3.2 *Suppose that condition* (B) *holds. Then every proper lower semicontinuous law invariant coherent risk measure $\varrho : \hat{\mathcal{Z}} \to \mathbb{R} \cup \{+\infty\}$ is regular.*

Proof. Let $\varrho : \hat{\mathcal{Z}} \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous law invariant coherent risk measure. It has the dual representation

$$\varrho(Z) = \sup_{\zeta \in \hat{\mathfrak{A}}} \int_0^1 \zeta(t) Z(t) dt, \ Z \in \hat{\mathcal{Z}}, \tag{3.3}$$

where $\hat{\mathfrak{A}} \subset \hat{\mathcal{Z}}^*$ is the corresponding dual set of density functions. Since ϱ is law invariant, the set $\hat{\mathfrak{A}}$ is invariant with respect to measure-preserving transformations $T \in \mathfrak{G}$ (e.g., [20]).

Suppose that condition (B) holds. Consider an element $Y \in \hat{\mathcal{Z}}$. By condition (B) there exists $T \in \mathfrak{G}$ such that $Z = Y \circ T$ is monotonically nondecreasing. It follows that $Z \in \hat{\mathcal{Z}}$ and $Z \stackrel{\mathcal{D}}{\sim} Y$, and hence $\varrho(Z) = \varrho(Y)$, and that $Z = F_Y^{-1}$. So let $Z \in \hat{\mathcal{Z}}$ be monotonically nondecreasing and consider an element $\zeta \in \hat{\mathfrak{A}}$. By condition (B) there exists $T \in \mathfrak{G}$ such that $\eta = \zeta \circ T$ is a monotonically nondecreasing function. Since $\hat{\mathfrak{A}}$ is invariant with respect to transformations of the group $\mathfrak{G}$, it follows that $\eta \in \hat{\mathfrak{A}}$. Also by monotonicity of Z we have that $\int_0^1 \zeta(t) Z(t) dt \leq \int_0^1 \eta(t) Z(t) dt$, and hence it suffices to take the maximum in (3.3) with respect to $\zeta \in \hat{\Upsilon}$, where $\hat{\Upsilon}$ is the subset of $\hat{\mathfrak{A}}$ formed by monotonically nondecreasing $\eta \in \hat{\mathfrak{A}}$. Define now

$$\rho(Z) := \sup_{\eta \in \hat{\Upsilon}} \int_0^1 \eta(t) F_Z^{-1}(t) dt, \ Z \in \mathcal{Z}. \tag{3.4}$$

We have that $\rho : \mathcal{Z} \to \mathbb{R} \cup \{+\infty\}$ is a proper lower semicontinuous law invariant coherent risk measure and $\rho|_{\hat{\mathcal{Z}}} = \varrho$. This shows that ϱ is regular. ∎

Suppose now that the reference probability space $\{\omega_1, ..., \omega_n\}$ is finite. It can be embedded into the uniform probability space. The corresponding space $\hat{\mathcal{Z}}$ consists of all functions $Z : \{\omega_1, ..., \omega_n\} \to \mathbb{R}$. If all respective probabilities $p_1, ..., p_n$ are equal to each other, then the associated group $\mathfrak{G}$ of measure preserving transformations is given by the set of permutations of $\{\omega_1, ..., \omega_n\}$. It follows that condition (B) of Proposition 3.2 holds and hence we have the following result (cf., [15]).

Corollary 3.1 *Let the reference probability space $\{\omega_1, ..., \omega_n\}$ be finite, equipped with equal probabilities $p_i = 1/n$, $i = 1, ..., n$. Then every law invariant coherent risk measure $\varrho : \hat{\mathcal{Z}} \to \mathbb{R}$ is regular, and hence has a Kusuoka representation.*

In Pflug and Römisch [16, p.63] is given an example of a law invariant coherent risk measure defined on space of random variables $Z : \hat{\Omega} \to \mathbb{R}$, with $\hat{\Omega} = \{\omega_1, \omega_2\}$ having two elements with unequal probabilities $p_1 \neq p_2$, for which the Kusuoka representation does not hold. We give now an example of law invariant coherent risk measures which are not regular, and hence do not have Kusuoka representation.

Example 1 Consider the above framework where the reference probability space is embedded into the uniform probability space, and $\hat{\mathcal{Z}} = L_p(\Omega, \mathcal{G}, P)$. Suppose that the reference probability space $(\Omega, \mathcal{G}, P)$ has atoms. Thus there exists $r > 0$ such that the set $A^r := \{\omega \in \Omega : P(\{\omega\}) = r\}$, of all atoms having probability r, is nonempty. Clearly the set A^r is finite, let $N := |A^r|$ be the cardinality of A^r (the set A^r could be a singleton, i.e., it could be that $N = 1$). The set $A^r = \cup_{i=1}^N A_i$, where $A_i \subset \Omega$, $i = 1, ..., N$, are intervals of length r. Note that since $Z \in \hat{\mathcal{Z}}$ is $\mathcal{G}$-measurable, $Z(\cdot)$ is constant on each A_i; we denote by $Z(\omega)$ this constant when $\omega \in A_i$.

Assume that the reference probability space is not a finite set equipped with equal probabilities, so that $P(\Omega \setminus A^r) > 0$. Define

$$\varrho(Z) := \frac{1}{N} \sum_{\omega \in A^r} Z(\omega), \ Z \in \hat{\mathcal{Z}}. \tag{3.5}$$

Clearly $\varrho : \hat{\mathcal{Z}} \to \mathbb{R}$ is a coherent risk measure. Suppose further that the following condition holds.

(C) If $Z, Z' \in \hat{\mathcal{Z}}$ are distributionally equivalent, then there exists a permutation π of the set A^r such that $Z'(\omega) = Z(\pi(\omega))$ for any $\omega \in A^r$.

Under this condition, the risk measure ϱ is law invariant as well. Let us show that it is not regular.

Let us argue by a contradiction. Indeed, suppose that $\varrho = \hat{\rho}$ for some proper lower semicontinuous law invariant coherent risk measure $\rho : \mathcal{Z} \to \mathbb{R} \cup \{+\infty\}$ and $\hat{\rho} := \rho|_{\hat{\mathcal{Z}}}$. Note that $\varrho(Z)$ depends only on values of $Z(\cdot)$ on the set A^r, i.e., for any $Z, Z' \in \hat{\mathcal{Z}}$ we have that $\rho(Z) = \rho(Z')$ if $Z(\omega) = Z'(\omega)$ for all $\omega \in A^r$. In particular, if $Z(\omega) = 0$ for all $\omega \in A^r$, then $\rho(Z) = 0$. It follows that $\rho(\mathbf{1}_B) = 0$ for any $\mathcal{G}$-measurable set $B \subset \Omega \setminus A^r$. Note that the set $\Omega \setminus A^r$ has a nonempty interior since the set A^r is a union of a finite number of intervals and $P(\Omega \setminus A^r) > 0$, and hence we can take $B \subset \Omega \setminus A^r$ to be an open interval. Since ρ is law invariant it follows that $\rho(\mathbf{1}_{T(B)}) = 0$ for any measure-preserving transformation $T : \Omega \to \Omega$ of the uniform probability space. Hence there is a family of sets $B_i \in \mathcal{F}$, $i = 1, ..., m$, such that $\rho(\mathbf{1}_{B_i}) = 0$, $i = 1, ..., m$, and $[0, 1] = \cup_{i=1}^m B_i$. It follows that for any bounded $Z \in \mathcal{Z}$ there are $c_i > 0$, $i = 1, ..., m$, such that $Z \leq \sum_{i=1}^m c_i \mathbf{1}_{B_i}$. Consequently

$$\rho(Z) \leq \rho\left(\sum_{i=1}^m c_i \mathbf{1}_{B_i}\right) \leq \sum_{i=1}^m c_i \rho(\mathbf{1}_{B_i}) = 0,$$

this clearly is a contradiction.

It follows that the risk measure ϱ, defined in (3.5), does not have a Kusuoka representation. Note that it was only essential in the above construction that the restriction of the risk measure $\varrho : \hat{\mathcal{Z}} \to \mathbb{R}$ to the set A^r is a law invariant coherent risk measure. So, for example, under the above assumptions the risk measure $\varrho(Z) := \max\{Z(\omega) : \omega \in A^r\}$ is also not regular.

As far as condition (C) is concerned, suppose for example that the reference space is finite, equipped with respective probabilities $p_1, ..., p_n$. For some $r \in \{p_1, ..., p_n\}$ let $\mathcal{I}_r := \{i : p_i = r,\ i = 1, ..., n\}$. Suppose that

$$\sum_{i\in I} p_i \neq \sum_{j\in J} p_j, \quad \forall I \subset \mathcal{I}_r,\ \forall J \subset \{1, ..., n\} \setminus \mathcal{I}_r. \tag{3.6}$$

Then condition (C) holds for the set $A^r := \{\omega_i : i \in \mathcal{I}_r\}$. That is, condition (3.6) ensures existence of a nonregular risk measure ϱ.

4 Maximality of Kusuoka sets

In this section we discuss Kusuoka representations with respect to stochastic dominance relations. As before we consider probability measures μ supported on the interval $[0, 1]$ and continue identifying the measure μ with its cumulative distribution function $\mu(\alpha) = \mu\{(-\infty, \alpha]\}$, $\alpha \in \mathbb{R}$. Recall that since μ is supported on [0,1], it follows that $\mu(\alpha) = 0$ for $\alpha < 0$ and $\mu(\alpha) = 1$ for $\alpha \geq 1$.

Definition 4.1 *It is said that μ_1 is* dominated in first stochastic order *by μ_2, denoted $\mu_1 \preceq \mu_2$, if $\mu_1(\alpha) \geq \mu_2(\alpha)$ for all $\alpha \in \mathbb{R}$. If moreover, $\mu_1 \neq \mu$ we write $\mu_1 \prec \mu_2$.*

A set of measures $(\mathfrak{M}, \preceq)$, equipped with the first order stochastic dominance relation, is a partially ordered set.

In the space of functions $\sigma \in \mathcal{Z}^*$ we consider the following dominance relation.

Definition 4.2 *For $\sigma_1, \sigma_2 \in \mathcal{Z}^*$ it is said that σ_1 is* majorized *by σ_2, denoted $\sigma_1 \preccurlyeq \sigma_2$, if*

$$\textstyle\int_\gamma^1 \sigma_1(t)dt \leq \int_\gamma^1 \sigma_2(t)dt \quad \text{for all } \gamma \in [0, 1], \text{ and} \tag{4.1}$$

$$\textstyle\int_0^1 \sigma_1(t)dt = \int_0^1 \sigma_2(t)dt. \tag{4.2}$$

Remark 2 For monotonically nondecreasing functions $\sigma_1, \sigma_2 : [0, 1] \to \mathbb{R}$ the above concept of majorization is closely related to the concept of dominance in the convex order. That is, if σ_1 and σ_2 are monotonically nondecreasing right side continuous functions, then they can be viewed as right side quantile functions $\sigma = F_{Z_1}^{-1}$ and $\sigma_2 = F_{Z_2}^{-1}$ of some respective random variables Z_1 and Z_2. It is said that Z_1 dominates Z_2 in the convex order if $\mathbb{E}[u(Z_1)] \geq \mathbb{E}[u(Z_2)]$ for all convex functions $u : \mathbb{R} \to \mathbb{R}$ such that the expectations exist. Equivalently this can written as (see, e.g., Müller and Stoyan [13])

$$\int_\gamma^1 F_{Z_1}^{-1}(t)dt \geq \int_\gamma^1 F_{Z_2}^{-1}(t)dt \ \text{ for all } \ \gamma \in [0, 1], \text{ and } \mathbb{E}[Z_1] = \mathbb{E}[Z_2].$$

The dominance in the convex (concave) order was used in studying risk measures in Föllmer and Schied [6] and Dana [3], for example.

Note that if $\sigma_1, \sigma_2 \in \mathcal{Z}^* \cap \mathfrak{S}$ and $\sigma_1 \preccurlyeq \sigma_2$, then

$$\int_0^1 \sigma_1(t) F_Z^{-1}(t)dt \le \int_0^1 \sigma_2(t) F_Z^{-1}(t)dt, \;\; Z \in \mathcal{Z}. \tag{4.3}$$

Example 2 Let ρ be given by maximum of a finite number of spectral risk measures, i.e.,

$$\rho(Z) := \max_{1\le i\le n} \int_0^1 \sigma_i(t) F_Z^{-1}(t)dt, \;\; Z \in \mathcal{Z}, \tag{4.4}$$

for some $\sigma_i \in \mathcal{Z}^* \cap \mathfrak{S}$, $i = 1, ..., n$. Then the set $\Upsilon := \{\sigma_1, ..., \sigma_n\}$ is a generating set and the convex hull of $\mathcal{O}(\Upsilon)$ is the dual set of ρ. An element σ_i of $\{\sigma_1, ..., \sigma_n\}$ is an exposed point of the dual set iff σ_i can be strongly separated from the other σ_j, i.e., iff there exists $Z \in \mathcal{Z}$ such that $\langle \sigma_i, Z\rangle \ge \langle \sigma_j, Z\rangle + \varepsilon$, for some $\varepsilon > 0$ and all $j \ne i$. So the generating set Υ is minimal iff every σ_i can be strongly separated from the other σ_j.

If there is σ_i which is majorized by another σ_j, then it follows by (4.3) that removing σ_i from the set Υ will not change the corresponding maximum, and hence $\Upsilon \setminus \{\sigma_i\}$ is still a generating set. Therefore the condition: "*every σ_i is not majorized by any other σ_j*" is necessary for the set Υ to be minimal. For $n = 2$ this condition is also sufficient. However, it is not sufficient already for $n = 3$. For example let σ_1 and σ_2 be such that σ_1 is not majorized by σ_2, and σ_2 is not majorized by σ_1, and let $\sigma_3 := (\sigma_1 + \sigma_2)/2$. Then clearly σ_3 can be removed from Υ, while σ_3 is not majorized by σ_1 or σ_2. Indeed, if σ_3 is majorized say by σ_1, then

$$\int_\gamma^1 \sigma_3(t)dt = \tfrac{1}{2}\left(\int_\gamma^1 \sigma_1(t)dt + \int_\gamma^1 \sigma_2(t)dt\right) \le \int_\gamma^1 \sigma_1(t)dt, \; \gamma \in [0,1],$$

and hence

$$\int_\gamma^1 \sigma_2(t)dt \le \int_\gamma^1 \sigma_1(t)dt, \; \gamma \in [0,1],$$

a contradiction with the condition that σ_2 is not majorized by σ_1.

We show now that first order dominance of measures is transformed by the operator $\mathbb{T}$ into majorization order in the sense of Definition 4.2. The converse implication,

$$\{\sigma_1 \preccurlyeq \sigma_2\} \Rightarrow \{\mathbb{T}^{-1}\sigma_1 \preceq \mathbb{T}^{-1}\sigma_2\}$$

does not hold in general. A simple counterexample is provided by the measures $\mu_1 := 0.1\delta_{0.2} + 0.9\delta_{0.6}$ and $\mu_2 := 0.2\delta_{0.5} + 0.8\delta_{0.9}$. This shows that the first order stochastic dominance indeed is a stronger concept.

Proposition 4.1 *For $\mu_1, \mu_2 \in \mathfrak{P}_q$ it holds that if $\mu_1 \preceq \mu_2$, then $\mathbb{T}\mu_1 \preccurlyeq \mathbb{T}\mu_2$.*

Proof. By definition of $\mathbb{T}$ and reversing the order of integration we can write for $\gamma \in (0,1)$,

$$\int_0^\gamma (\mathbb{T}\mu_1)(t)dt = \int_0^\gamma \int_0^t \frac{1}{1-\alpha} d\mu_1(\alpha)\, dt = \int_0^\gamma \int_\alpha^\gamma \frac{1}{1-\alpha} dt\, d\mu_1(\alpha) = \int_0^\gamma \frac{\gamma-\alpha}{1-\alpha} d\mu_1(\alpha).$$

Now by Riemann-Stieltjes integration by parts

$$\int_0^\gamma (\mathbb{T}\mu_1)(t)dt = \frac{\gamma-\alpha}{1-\alpha}\mu_1(\alpha)\Big|_{\alpha=0}^{\gamma} - \int_0^\gamma \mu_1(\alpha) d\left(\frac{\gamma-\alpha}{1-\alpha}\right) = \int_0^\gamma \mu_1(\alpha)\frac{1-\gamma}{(1-\alpha)^2} d\alpha,$$

where we used that $\mu_1(0^-) = \mu_2(0^-) = 0$. Since $\mu_1 \preceq \mu_2$ we have that $\mu_1(\cdot) \geq \mu_2(\cdot)$ and hence

$$\int_0^\gamma (\mathbb{T}\mu_1)(t)dt = \int_0^\gamma \mu_1(\alpha)\frac{1-\gamma}{(1-\alpha)^2}d\alpha \geq \int_0^\gamma \mu_2(\alpha)\frac{1-\gamma}{(1-\alpha)^2}d\alpha = \int_0^\gamma (\mathbb{T}\mu_2)(t)dt.$$

Because of $\int_0^1 (\mathbb{T}\mu)(t)dt = 1$, this implies that

$$\int_\gamma^1 (\mathbb{T}\mu_1)(t)dt \leq \int_\gamma^1 (\mathbb{T}\mu_2)(t)dt,$$

and hence $\mathbb{T}\mu_1 \preccurlyeq \mathbb{T}\mu_2$. ∎

Let $\mathfrak{M}$ be a Kusuoka set and $\mu_1, \mu_2 \in \mathfrak{M}$. As it was pointed above, it follows from (4.3) that if $\mathbb{T}\mu_1 \preccurlyeq \mathbb{T}\mu_2$, then the measure μ_1 can be removed from $\mathfrak{M}$. Hence it follows by Proposition 4.1 that if $\mu_1 \preceq \mu_2$, then the measure μ_1 can be removed from $\mathfrak{M}$.

Definition 4.3 *A measure $\mu \in \mathfrak{M} \subset \mathfrak{P}$ is called a* maximal element*, a* maximal measure *or* non-dominated measure *of $(\mathfrak{M}, \preceq)$, if there is no $\nu \in \mathfrak{M}$ satisfying $\mu \prec \nu$.*

The measures δ_0 and δ_1 are extremal in the sense that

$$\delta_0 \preceq \mu \preceq \delta_1 \qquad \text{for all } \mu \in \mathfrak{P},$$

that is δ_1 (δ_0, resp.) is always a maximal (minimal, resp.) measure.

The next theorem elaborates that it is sufficient to consider the non-dominated measures within the closure of any Kusuoka set.

Theorem 4.1 *Let $\mathfrak{M}$ be a Kusuoka set of law invariant coherent risk measure $\rho : \mathcal{Z} \to \mathbb{R}$, and $\overline{\mathfrak{M}}$ be its topological closure. Then the augmented set $\{\mu' \in \mathfrak{P} \colon \mu' \preccurlyeq \mu \text{ for some } \mu \in \mathfrak{M}\}$ is a Kusuoka set as well. Furthermore, the set of extremal measures of $\overline{\mathfrak{M}}$, i.e., $\mathfrak{M}' := \{\mu \in \overline{\mathfrak{M}} : \nu \not\succ \mu \text{ for all } \nu \in \mathfrak{M}\}$ is a Kusuoka set.*

Proof. Let $\mu' \preceq \mu \in \mathfrak{M}$. As $\alpha \mapsto \mathsf{AV@R}_\alpha(Y)$ is a nondecreasing function and as $\mu(\cdot) \leq \mu'(\cdot)$ it follows by Riemann-Stieltjes integration by parts that

$$\begin{aligned}\int_0^1 \mathsf{AV@R}_\alpha(Y)d\mu(\alpha) &= \mu(\alpha)\mathsf{AV@R}_\alpha(Y)\big|_{\alpha=0_-}^1 - \int_0^1 \mu(\alpha)d\mathsf{AV@R}_\alpha(Y)\\ &\geq \mu'(\alpha)\mathsf{AV@R}_\alpha(Y)\big|_{\alpha=0_-}^1 - \int_0^1 \mu'(\alpha)d\mathsf{AV@R}_\alpha(Y)\\ &= \int_0^1 \mathsf{AV@R}_\alpha(Y)\mu'(d\alpha),\end{aligned}$$

which is the first assertion.

For the second assertion recall that $(\mathfrak{M}, \preceq)$ is a partially ordered set and so is $(\overline{\mathfrak{M}}, \preceq)$. Consider a chain $\mathfrak{C} \subset \overline{\mathfrak{M}}$, that is for every μ, ν it holds that $\mu \preceq \nu$ or $\nu \preceq \mu$ (totality). Then the chain $\mathfrak{C}$ has an upper bound in $\overline{\mathfrak{C}} \subset \overline{\mathfrak{M}}$: to accept this (cf. the proof of Helly's Lemma in [22]) define

$$\mu_{\mathfrak{C}}(x) := \inf_{\mu \in \mathfrak{C}} \mu(x)$$

(the upper bound), which is a positive, non-decreasing function satisfying $\mu_{\mathfrak{C}}(1)=1$. As any $\mu\in\mathfrak{C}$ is right side continuous it is upper semi-continuous, thus $\mu_{\mathfrak{C}}$, as an infimum, is upper semi-continuous as well, hence $\mu_{\mathfrak{C}}$ is right side continuous and $\mu_{\mathfrak{C}}$ thus represents a measure, $\mu_{\mathfrak{C}}\in\mathfrak{P}$. To show that $\mu_{\mathfrak{C}}\in\overline{\mathfrak{C}}$ let x_i be a dense sequence in $[0,1]$ and choose $\mu_{i,n}\in\mathfrak{C}$ such that $\mu_{i,n}(x_i)<\mu_{\mathfrak{C}}(x_i)+2^{-n}$. As $\mathfrak{C}$ is a chain one may define $\mu_n:=\min\{\mu_{i,n}: i=1,2,\dots n\}$. It holds that $\mu_n(x_i)<\mu_{\mathfrak{C}}(x_i)+2^{-n}$ for all $i=1,2,\dots n$. As x_i is dense, and as μ_n, as well as $\mu_{\mathfrak{C}}$ are right side continuous it follows that $\mu_n\to\mu_{\mathfrak{C}}$ uniformly, hence $\mu_{\mathfrak{C}}\in\overline{\mathfrak{C}}$.

By Zorn's Lemma there is at least one maximal element μ^* in $\overline{\mathfrak{M}}$, that is there is no element $\nu\in\overline{\mathfrak{M}}$ such that $\nu\succ\mu^*$. Hence

$$\begin{aligned}\mu^* &\in \left\{\mu\in\overline{\mathfrak{M}}:\ \neg\exists\nu\in\mathfrak{M}:\nu\succ\mu\right\}\\ &= \left\{\mu\in\overline{\mathfrak{M}}:\ \forall\nu\in\mathfrak{M}:\nu\not\succ\mu\right\}=\mathfrak{M}'\end{aligned}$$

and $\mathfrak{M}'$ thus is a non-empty Kusuoka set. Recall that $\mathfrak{M}'\subset\overline{\mathfrak{M}}$, hence

$$\begin{aligned}\rho(Y) &= \sup_{\mu\in\mathfrak{M}}\int_0^1 \mathsf{AV@R}_\alpha(Y)\,\mu(\mathrm{d}\alpha)\\ &= \sup_{\mu\in\overline{\mathfrak{M}}}\int_0^1 \mathsf{AV@R}_\alpha(Y)\,\mu(\mathrm{d}\alpha)\geq \sup_{\mu\in\mathfrak{M}'}\int_0^1 \mathsf{AV@R}_\alpha(Y)\,\mu(\mathrm{d}\alpha).\end{aligned}$$

To establish equality assume that $\rho(\cdot)\neq\sup_{\mu\in\mathfrak{M}'}\int_0^1\mathsf{AV@R}_\alpha(\cdot)\,\mu(\mathrm{d}\alpha)$. Then there is a random variable Y satisfying

$$\rho(Y)>\sup_{\mu'\in\mathfrak{M}'}\int_0^1 \mathsf{AV@R}_\alpha(Y)\,\mu'(\mathrm{d}\alpha). \tag{4.5}$$

For this Y and for some $\varepsilon>0$ choose $\mu\in\overline{\mathfrak{M}}$ such that $\rho(Y)<\varepsilon+\int_0^1\mathsf{AV@R}_\alpha(Y)\,\mu(\mathrm{d}\alpha)$.

Consider the cone $\mathcal{M}_\mu:=\left\{\nu\in\overline{\mathfrak{M}}:\mu\preceq\nu\right\}$. Notice that $(\mathfrak{M}_\mu,\preceq)$ again is a partially ordered set, for which Zorn's Lemma implies that there is a maximal element $\bar\mu\in\mathfrak{M}_\mu$ with respect to $\preceq$, and it holds that $\mu\preceq\bar\mu$ by construction.

We claim that $\bar\mu\in\mathfrak{M}'$. Indeed, if it were not, then there is $\nu'\in\mathfrak{M}$ with $\nu'\succ\bar\mu$. But this means $\mu\preceq\bar\mu\prec\nu'$, so $\nu'\in\mathfrak{M}_\mu$ and hence $\bar\mu\prec\nu'$, contradicting the fact that $\bar\mu$ is a maximal measure in $\mathfrak{M}_\mu$, and hence $\bar\mu\in\mathfrak{M}'$.

Now by Riemann-Stieltjes integration by parts

$$\begin{aligned}\rho(Y)-\varepsilon &< \int_0^1 \mathsf{AV@R}_\alpha(Y)\mathrm{d}\mu(\alpha)\\ &= \mu(\alpha)\mathsf{AV@R}_\alpha(Y)\big|_{\alpha=0}^1-\int_0^1\mu(\alpha)\mathrm{d}\mathsf{AV@R}_\alpha(Y)\\ &\leq \bar\mu(\alpha)\mathsf{AV@R}_\alpha(Y)\big|_{\alpha=0}^1-\int_0^1\bar\mu(\alpha)\mathrm{d}\mathsf{AV@R}_\alpha(Y)\\ &= \int_0^1 \mathsf{AV@R}_\alpha(Y)\bar\mu(\mathrm{d}\alpha),\end{aligned}$$

as $\alpha\mapsto\mathsf{AV@R}_\alpha(Y)$ is a non-decreasing function and as $\mu\preceq\bar\mu$, that is $\mu(\cdot)\geq\bar\mu(\cdot)$.

But as $\overline{\mu}\in\mathfrak{M}'$ this is a contradiction to (4.5), such that the assertion holds indeed. ∎

5 Examples

In order to demonstrate the minimal Kusuoka representation we have chosen two examples. The first is taken from [4]. The second example elaborates the Kusuoka representation of the higher order semideviations.

5.1 Higher order measures

This first example generalizes the Average Value-at-Risk as introduced in (1.1).

Proposition 5.1 *The minimal Kusuoka representation of the risk measure*

$$\rho(Z) := \inf_{t\in\mathbb{R}} \left\{t + c\left\|(Z-t)_+\right\|_p\right\},\ Z \in L_p(\Omega,\mathcal{F},P), \tag{5.1}$$

($c>1$ and $1<p<\infty$) is given by

$$\rho(Z) = \sup\left\{\rho_\sigma(Z):\ \sigma\in\mathfrak{S} \text{ and } \|\sigma\|_q = c\right\}, \tag{5.2}$$

where ρ_σ is the spectral risk measure associated with the spectrum σ (cf. (2.14)) and $\frac{1}{p}+\frac{1}{q}=1$.

Remark 5.1 Equation (5.2) is not a Kusuoka representation in its genuine form, but it is a concise way of writing

$$\rho_\sigma(Z) = \int_0^1 \sigma(u) F_Z^{-1}(u)du = \int_0^1 \mathsf{AV@R}_\alpha(Z)d\mu(\alpha),$$

where $\mu = \mathbb{T}^{-1}\sigma$ according to (2.2) and (2.14).

Remark 5.2 It is obvious that (5.2) represents the Average Value-at-Risk, whenever $c=\frac{1}{1-\alpha}$ and $p=1$, such that Proposition 5.1 generalizes the Average Value-at-Risk.

Proof. The Kusuoka representation of (5.1) is given in [4] in the form

$$\rho(Z) = \sup\left\{\rho_\sigma(Z):\ \sigma\in\mathfrak{S} \text{ and } \|\sigma\|_q \le c\right\}, \tag{5.3}$$

i.e., the dual set of ρ is $\mathfrak{A} = \mathcal{O}\left\{\sigma\in\mathfrak{S} : \|\sigma\|_q\le c\right\}$ (of course, the condition $\|\sigma\|_q \le c$ implies that the set $\mathfrak{A}$ is a subset of the dual space $L_q(\Omega,\mathcal{F},P)$). We prove first that it is enough to consider functions σ satisfying $\|\sigma\|_q = c$. Indeed, for σ satisfying $\|\sigma\|_q < c$ define $\sigma_\Delta(u) := \mathbf{1}_{[\Delta,1]}(u)\cdot\left(\sigma(u)+\frac{1}{1-\Delta}\int_0^\Delta\sigma(u')du'\right)$ ($\Delta\in[0,1)$). It is evident that $\Delta\mapsto\|\sigma_\Delta\|_q$ is a continuous and unbounded function, hence there is a Δ_0 such that $\|\sigma_{\Delta_0}\| = c$. Moreover $\sigma \preccurlyeq \sigma_\Delta$ by construction, such that one may pass – according to (4.3) – to σ_{Δ_0} without changing the objective in (5.3) to get the representation (5.2).

The set in the dual corresponding to (5.2) is

$$\mathfrak{C} = \mathcal{O}\left(\left\{\zeta\in\mathfrak{S}:\ \|\zeta\|_q = c\right\}\right).$$

Now let $\zeta\in\mathfrak{C}\subset\mathfrak{A}$ be chosen, and consider the random variable $Z := \zeta^{q-1}$. The function $g_Z(\cdot) := \langle\cdot, Z\rangle$ (defined in Definition 2.4) attains its maximum over $\mathfrak{A}$ at ζ, because

$$g_Z(\zeta') \le \|\zeta'\|_q\cdot\|Z\|_p \le c\|Z\|_p = c\|\zeta\|_q^{p/q} = c^q = \|\zeta\|_q^q = g_Z(\zeta)$$

by Hölder's inequality whenever $\zeta'\in\mathfrak{A}$. Moreover ζ is the unique maximizer, as $1<p<\infty$. This shows, using the notation from Definition 2.4, that $\mathfrak{C}$ collects just maximizers, that is $\mathfrak{C} = \operatorname{Exp}(\mathfrak{A})$. This completes the proof. ∎

5.2 Higher order semideviation

Our second example addresses the $p-$semideviation risk measure. We elaborate its Kusuoka representation, as well as the minimal Kusuoka representation.

Proposition 5.2 *For $p \geq 1$ the Kusuoka representation of the $p-$semideviation*

$$\rho(Z) := \mathbb{E}[Z] + \lambda \left\|(Z - \mathbb{E}Z)_+\right\|_p, \; Z \in L_p(\Omega, \mathcal{F}, P),$$

($0 \leq \lambda \leq 1$) is

$$\rho(Z) = \sup_{\sigma \in \mathfrak{S}} \left(1 - \tfrac{\lambda}{\|\sigma\|_q}\right) \mathbb{E}[Z] + \tfrac{\lambda}{\|\sigma\|_q} \rho_\sigma(Z). \tag{5.4}$$

The representation (5.4) *is moreover the* minimal *Kusuoka representation whenever $p > 1$.*

Remark 5.3 As in the first example the formula presented in (5.4) does not appear as a Kusuoka representation in its traditional form. To make the Kusuoka representation evident we rewrite it as

$$\rho(Z) = \sup_{\mu \in \mathfrak{P}_q} \left\{ \left(1 - \tfrac{\lambda}{\|\mathbb{T}\mu\|_q}\right) \mathbb{E}[Z] + \tfrac{\lambda}{\|\mathbb{T}\mu\|_q} \int_0^1 \mathsf{AV@R}_\alpha(Z) d\mu(\alpha) \right\}, \tag{5.5}$$

such that the supremum in the representation (5.5) is among all measures of the form

$$\left(1 - \tfrac{\lambda}{\|\mathbb{T}\mu\|_q}\right) \delta_0 + \tfrac{\lambda}{\|\mathbb{T}\mu\|_q} \mu, \quad \mu \in \mathfrak{P}_q,$$

which finally provides a Kusuoka representation.

Remark 5.4 Notice that $\|\sigma\|_q \geq \|\sigma\|_1 = \int_0^1 \sigma(u) \mathrm{d}u = 1$, such that $1 - \frac{\lambda}{\|\sigma\|_q} \geq 0$ and (5.4) is indeed a positive quantity whenever $0 \leq \lambda \leq 1$.

Proof. By Hölder's duality $\mathbb{E}[Z\zeta] \leq (\mathbb{E}|Z|^p)^{\frac{1}{p}} (\mathbb{E}|\zeta|^q)^{\frac{1}{q}}$ and $(\mathbb{E}|Z|^p)^{\frac{1}{p}} = \sup\left\{\mathbb{E}[Z\zeta] \colon \|\zeta\|_q \leq 1\right\}$. Clearly

$$\left(\mathbb{E}[Z_+^p]\right)^{\frac{1}{p}} = \sup\left\{\mathbb{E}[Z\zeta] \colon \zeta \geq 0, \|\zeta\|_q \leq 1\right\},$$

the supremum being attained at $\zeta = \frac{Z_+^{p-1}}{\left\|Z_+^{p-1}\right\|_q}$. It follows that

$$\left\|(Z - \mathbb{E}[Z])_+\right\|_p = \sup\left\{\mathbb{E}[(Z - \mathbb{E}[Z])\,\zeta] \colon \zeta \geq 0, \|\zeta\|_q \leq 1\right\}. \tag{5.6}$$

Now

$$\begin{aligned}
\rho(Z) &= \mathbb{E}[Z] + \lambda \left\|(Z - \mathbb{E}[Z])_+\right\|_p \\
&= \sup\left\{\mathbb{E}[Z] + \lambda\, \mathbb{E}[(Z - \mathbb{E}[Z])\,\zeta] \colon \zeta \geq 0, \|\zeta\|_q \leq 1\right\} \\
&= \sup\left\{(1 - \lambda\, \mathbb{E}[\zeta])\, \mathbb{E}[Z] + \lambda\, \mathbb{E}[Z\zeta] \colon \zeta \geq 0, \|\zeta\|_q \leq 1\right\} && (5.7) \\
&= \sup\left\{\left(1 - \lambda \tfrac{\mathbb{E}[\zeta]}{\|\zeta\|_q}\right) \mathbb{E}[Z] + \lambda\, \mathbb{E}\left[Z \tfrac{\zeta}{\|\zeta\|_q}\right] \colon \zeta \geq 0\right\} \\
&= \sup\left\{\left(1 - \tfrac{\lambda}{\|\zeta\|_q}\right) \mathbb{E}[Z] + \tfrac{\lambda}{\|\zeta\|_q} \mathbb{E}[Z\zeta] \colon \zeta \geq 0,\ \mathbb{E}[\zeta] = 1\right\}. && (5.8)
\end{aligned}$$

For ζ feasible in (5.8) define the function $\sigma_\zeta(u) := \mathsf{V@R}_u(\zeta)$ and observe that

$$\|\sigma_\zeta\|_q^q = \int_0^1 \sigma_\zeta^q(\alpha)\mathrm{d}\alpha = \mathbb{E}[\zeta^q] = \|\zeta\|_q^q .$$

Now notice that every random variable ζ in (5.8) is given by $\zeta = \sigma(U)$ for a uniform U where σ is non-negative, non-decreasing and $\int_0^1 \sigma(u)\,\mathrm{d}u = 1$ – that is, σ is a spectral function. It follows that

$$\begin{aligned}\rho(Z) &= \sup_{\sigma\in\mathfrak{S}}\sup\left\{\left(1-\tfrac{\lambda}{\|\zeta\|_q}\right)\mathbb{E}[Z] + \tfrac{\lambda}{\|\zeta\|_q}\mathbb{E}[Z\zeta] \colon \zeta\geq 0,\, \mathbb{E}[\zeta]=1,\, \mathsf{V@R}(\zeta)=\sigma\right\}\\ &= \sup_{\sigma\in\mathfrak{S}}\sup\left\{\left(1-\tfrac{\lambda}{\|\sigma\|_q}\right)\mathbb{E}[Z] + \tfrac{\lambda}{\|\sigma\|_q}\mathbb{E}[Z\zeta] \colon \zeta\geq 0,\, \mathbb{E}[\zeta]=1,\, \mathsf{V@R}(\zeta)=\sigma\right\} \quad (5.9)\\ &= \sup_{\sigma\in\mathfrak{S}}\left(1-\tfrac{\lambda}{\|\sigma\|_q}\right)\mathbb{E}[Z] + \tfrac{\lambda}{\|\sigma\|_q}\rho_\sigma(Z).\end{aligned}$$

This completes the proof of the Kusuoka representation.

In order to verify that this is the minimal Kusuoka representation consider the corresponding set $\mathfrak{C}\subset\mathfrak{A}$ in the dual, which is in view of (5.7)

$$\mathfrak{C} = \{\zeta = (1-\lambda\,\mathbb{E}[\zeta'])\,\mathbf{1} + \lambda\zeta' : \|\zeta'\|_q = 1,\, \zeta'\geq 0\}$$

(cf. [19, Example 6.20]). For $Z\in L_p(\Omega,\mathcal{F},P)$ consider the function $g_Z(\zeta) := \langle\zeta, Z\rangle$ as in Definition 2.4. It holds that

$$\sup_{\zeta\in\mathfrak{A}} g_Z(\zeta) = \sup_{\zeta'\geq 0,\,\|\zeta'\|_q\leq 1}(1-\lambda\,\mathbb{E}[\zeta'])\mathbb{E}[Z] + \lambda\mathbb{E}[\zeta' Z] = \mathbb{E}[Z] + \lambda\sup_{\|\zeta'\|_q\leq 1}\mathbb{E}[\zeta'(Z-\mathbb{E}[Z])].$$

For $1<p<\infty$ the latter supremum is *uniquely* attained at

$$\bar{\zeta}' = \frac{(Z-\mathbb{E}[Z])_+^{p-1}}{\|\,(Z-\mathbb{E}[Z])_+^{p-1}\,\|_q},$$

such that g_Z attains its *unique* maximum at

$$\bar{\zeta} = (1-\lambda\mathbb{E}[\bar{\zeta}'])\mathbf{1} + \lambda\bar{\zeta}'. \quad (5.10)$$

It follows that $\mathfrak{C}$ consists of exposed points only, that is $\mathfrak{C} = \mathrm{Exp}(\mathfrak{A})$. This proves that $\mathfrak{C}$ is the minimal Kusuoka representation, and thus (5.4). ∎

Corollary 5.1 (cf. [21]) *For $p=1$ the minimal Kusuoka representation of the absolute semideviation $\rho(Z) := \mathbb{E}[Z] + \lambda\,\mathbb{E}\left[(Z-\mathbb{E}[Z])_+\right]$, with $\lambda\in[0,1]$, is*

$$\rho(Z) = \sup_{\kappa\in[0,1]}\{(1-\lambda\kappa)\,\mathsf{AV@R}_0 + \lambda\kappa\,\mathsf{AV@R}_{1-\kappa}(Z)\}. \quad (5.11)$$

Proof. Note that the supremum in (5.6) is attained at $\zeta = \frac{(Z-\mathbb{E}[Z])_+^0}{\left\|(Z-\mathbb{E}[Z])_+^0\right\|_\infty}$, and in (5.8) thus at $\zeta = \frac{(Z-\mathbb{E}[Z])_+^0}{\left\|(Z-\mathbb{E}[Z])_+^0\right\|_1}$, which is a function of type $\zeta = \frac{\mathbf{1}_B}{P(B)}$ (choose $B = \{Z > \mathbb{E}[Z]\}$ to accept the correspondence). Next observe that $\sigma_\zeta(u) = \mathsf{V@R}_u(\zeta) = \frac{1}{1-\alpha}\mathbf{1}_{[\alpha,1]}(u)$ (for $\alpha = P\left(B^{\complement}\right)$), which is the spectral function of the Average Value-at-Risk of level α. It follows that

$$\begin{aligned}
\rho(Z) &= \sup_{\sigma=\frac{1}{1-\alpha}\mathbf{1}_{[\alpha,1]}} \left\{\left(1-\frac{\lambda}{\|\sigma\|_\infty}\right)\mathbb{E}[Z] + \frac{\lambda}{\|\sigma\|_\infty}\rho_\sigma(Z)\right\}, \\
&= \sup_\alpha \left\{(1-\lambda(1-\alpha))\,\mathbb{E}[Z] + \lambda(1-\alpha)\,\mathsf{AV@R}_\alpha(Y)\right\}, \\
&= \sup_{\kappa\in(0,1)} \left\{(1-\lambda\kappa)\,\mathbb{E}[Z] + \lambda\kappa\,\mathsf{AV@R}_{1-\kappa}(Z)\right\}.
\end{aligned}$$

That is, the set $\mathfrak{M} := \cup_{\kappa\in(0,1)}\{(1-\lambda\kappa)\delta_0 + \lambda\kappa\delta_{1-\kappa}\}$ is a Kusuoka set and its closure is the minimal Kusuoka set. ∎

6 Conclusion

This paper addresses the Kusuoka representation of law invariant coherent risk measures, introduced by S. Kusuoka in [10]. In general many representations may describe the same risk measure, but we demonstrate that there is a minimal representation available, and this minimal representation is moreover unique.

The minimal representation can be extracted by identifying the set of exposed points of the dual set and then applying the corresponding one-to-one transformation. Moreover, it turns out that the minimal representation only consists of measures which are nondominated in first order stochastic dominance. This relation, as well as the convex order stochastic dominance relation can be employed to identify the necessary measures. The general results are elaborated for two dedicated, important examples to demonstrate this fact.

We also consider Kusuoka representations on a class of general probability spaces, which potentially contain atoms. The gap for law invariant risk measures on spaces, which contain – or do not contain – atoms, is clarified and sufficiently described.

We want to note that many of the results can be extended to convex risk measures, which are not necessarily positively homogeneous. This is in line with the results obtained by Schachermayer et al. in [9], and preceding work by Fratelli et al. in [7]. Besides the fact that the notation gets significantly more involved, we have postponed corresponding investigations for future research.

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