The Slater condition is generic in linear conic programming

Mirjam Dür∗ Bolor Jargalsaikhan† Georg Still‡

November 9, 2012

Abstract

We call a property generic if it holds for almost all problem instances. For linear conic problems, it has been shown in the literature that properties like uniqueness, strict complementarity or nondegeneracy of the optimal solution are generic under the assumption that Slater’s condition is fulfilled. The possibility that Slater’s condition generically fails has not been excluded. In this paper, we close this gap by proving that Slater’s condition is generic in linear conic programs. We also summarize genericity results of other properties and discuss connections between them.

Keywords: Conic programs; Slater’s condition; generic properties
Mathematical Subject Classification 2010: 90C25, 90C31

∗University of Trier, Department of Mathematics, 54286 Trier, Germany. Email: duer@uni-trier.de
†University of Groningen, Johann Bernoulli Institute for Mathematics and Computer Science, P.O. Box 407, 9700 AK Groningen, The Netherlands. Email: B.Jargalsaikhan@rug.nl
‡University of Twente, Department of Applied Mathematics, P.O. Box 217, 7500 AE Enschede, The Netherlands. Email: g.still@math.utwente.nl
1 Introduction

We consider the following pair of conic problems in what we might call “standard form”:

\[
\begin{align*}
\text{max} & \quad c^T x \\
\text{s.t.} & \quad X := B - \sum_{i=1}^n x_i A_i \in \mathcal{K} \\
\end{align*}
\]

where \( c \in \mathbb{R}^n \), \( B, A_i \) (\( i = 1, \ldots, n \)) are matrices in \( \mathcal{S}_k \), the set of symmetric \( k \times k \)-matrices, and \( \mathcal{K} \subseteq \mathcal{S}_k \) is a pointed full-dimensional closed convex cone. The dual problem of \((P_0)\) is

\[
\begin{align*}
\text{min} & \quad \langle B, Y \rangle \\
\text{s.t.} & \quad \langle A_i, Y \rangle = c_i, \quad i = 1, \ldots, n, \\
& \quad Y \in \mathcal{K}^*,
\end{align*}
\]

where \( \mathcal{K}^* \) is the dual cone of \( \mathcal{K} \) with respect to the inner product \( \langle \cdot, \cdot \rangle \). Note that \( \mathcal{S}_k \equiv \mathbb{R}^m \) with \( m := \frac{1}{2}(k+1)k \).

A variety of problems such as linear programs, semidefinite programs, second order cone programs and copositive/completely positive programs can be formulated as conic problems for appropriate cones \( \mathcal{K} \).

We say that Slater’s condition holds for \((P_0)\) if there exists \( X = B - \sum_{i=1}^n x_i A_i \in \text{int} \mathcal{K} \). Likewise, we say that Slater’s condition holds for \((D_0)\) if there exists \( Y \in \text{int} \mathcal{K}^* \) which fulfills the constraints \( \langle A_i, Y \rangle = c_i \) for all \( i \).

We call a property generic, if it holds for almost all instances. More precisely, assume that the instances of a certain problem class can be parametrized in some way. Then we say that a property is generic if the set of parameters describing problem instances where the property fails has measure zero.

Genericity of properties like nondegeneracy, strict complementarity and uniqueness of solutions of linear conic programs have been discussed before. Alizadeh, Haeberly, and Overton [1] as well as Shapiro [14] specifically discuss generic properties of SDPs. Pataki and Tunçel [12] obtain genericity results on strict complementarity, uniqueness, and nondegeneracy for general linear conic programs.

However, the results in [1] have been proven under the assumption that the Slater condition is satisfied, and in [12], the genericity results are restricted to gap-free problems (i.e., problems with finite optimal value and zero duality gap). The possibility that these assumptions generically fail has not been excluded, so these genericity results lack a foundation. Merely for the SDP case, it is indicated in [14, p. 310] that the Slater condition (resp. the Mangasarian-Fromovitz condition) is generic.

The aim of this article is to close this gap and prove genericity of Slater’s condition for linear problems over arbitrary proper cones. More specifically, we show that in linear conic programming for almost all feasible pairs of problems the primal and dual Slater conditions
hold. Note that the Slater condition also ensures a zero duality gap.

Usually, genericity results are proven by applying transversality theory from differential topology. We refer to [9] for such genericity results in smooth nonlinear finite programming, and to [1] for results for semidefinite programming. However, to obtain genericity results for general conic programming these transversality theorems cannot be used since the boundaries of general convex cones cannot be described by smooth manifolds. One needs genericity results for convex functions (Lipschitz functions), a theory that has been developed in the area of geometric measure theory. Founding work for this was done by Federer and others (see e.g. [11] for an overview). The results in [12] are based on this theory.

In this paper, instead of using transversality theory, we will use results from measure theory as outlined in Section 2.1.

Throughout the paper, we make the following assumption:

Assumption 1.1. The matrices $A_1, \ldots, A_n$ are linearly independent, i.e., they span an $n$-dimensional linear space in $S_k$.

Clearly, this assumption poses no loss of generality. Moreover, a standard result in differential topology (see e.g. [9]) states that for almost all sets $\{A_i\}_{i=1}^n$, the matrices $A_i$ are linearly independent. Therefore, Assumption 1.1 is satisfied generically.

Note that this assumption implies that $X := B - \sum_{i=1}^n x_i A_i$ is uniquely determined by $x$.

2 Auxiliary results

2.1 A useful result on the boundary of convex sets

Roughly speaking, Slater’s condition says that the feasible set of the problem is not contained in the boundary of the convex cone. For this reason, it is intuitive that the proof of our main result should be based on properties of this boundary. More specifically, we will use the fact that the boundary of a convex set has measure zero, a result which is known but for which we give a proof here to make the paper self-contained:

Lemma 2.1. Let $\mathcal{T}$ be a full-dimensional closed convex set in $\mathbb{R}^s$. Then the boundary of $\mathcal{T}$ has $s$-dimensional Lebesgue measure zero.

Proof. We repeat here the elegant proof of [10]. Consider an open ball $B_\varepsilon(p)$ with center $p \in \text{bd} \mathcal{T}$ and radius $\varepsilon > 0$. Since there exists a hyperplane supporting the convex set $\mathcal{T}$ at $p$, at least half of the ball does not contain points of $\mathcal{T}$. Therefore,

$$\lim_{\varepsilon \to 0} \frac{\mu(\mathcal{T} \cap B_\varepsilon(p))}{\mu(B_\varepsilon(p))} < 1.$$
Now Lebesgue’s density theorem (see e.g. [4]), says that for almost all points \( p \) of the Lebesgue measurable set \( T \) we have that
\[
\lim_{\varepsilon \to 0} \frac{\mu(T \cap B_\varepsilon(p))}{\mu(B_\varepsilon(p))} = 1.
\]
This immediately implies that \( \text{bd} \ T \) has measure zero. \( \square \)

Note that the genericity results in [1] have been formulated with respect to the Lebesgue measure, whereas those in [12] use the Hausdorff measure. However, it is known that the \( n \)-dimensional Lebesgue and Hausdorff measures in \( \mathbb{R}^n \) coincide (see, e.g., [11]).

### 2.2 An often used reasoning

In this paper, we will use the following argument several times. Suppose some property depends on a parameter \( Q \in \mathbb{R}^M \). As usual, we say that the property holds for almost all parameters, if the set
\[
\mathcal{Q} := \{ Q \in \mathbb{R}^M \mid \text{the property does not hold for } Q \}
\]
has measure zero. Assume that for some \( N \) with \( 0 < N < M \) we have a decomposition of \( Q \in \mathbb{R}^M \) of the form
\[
Q = (Q_1, Q_2) \quad \text{with} \quad Q_1 \in \mathbb{R}^N, Q_2 \in \mathbb{R}^{M-N}.
\]
Suppose further that for each fixed \( Q_1 \in \mathbb{R}^N \) in \( Q = (Q_1, Q_2) \) the property holds for almost all parameters \( Q_2 \in \mathbb{R}^{M-N} \). Then the property holds for almost all \( Q \in \mathbb{R}^M \). Indeed, the assumption means that the set \( \mathcal{Q} \) of parameters where the property does not hold is a set \( \mathcal{Q} := \mathbb{R}^N \times \mathcal{Q}^2 \) with a set \( \mathcal{Q}^2 \subset \mathbb{R}^{M-N} \) of measure zero in \( \mathbb{R}^{M-N} \). But then Fubini’s theorem (see, e.g., [3 Appendix 1]) implies that the set \( \mathcal{Q} \) has measure zero in \( \mathbb{R}^M \).

### 3 Genericity of Slater’s condition

In this section, we show that the Slater condition is a generic property for linear conic problems.

#### 3.1 Slater’s condition for problems in self-dual form

In order to prove a genericity result, we first need to parametrize all problem instances. For this purpose, it is more appropriate to consider conic problems in so-called self-dual form:

\[
\begin{align*}
\max \quad & \langle C, X \rangle - \langle C, B \rangle \\
\text{s.t.} \quad & X \in (B + \mathcal{L}) \cap \mathcal{K}.
\end{align*}
\]

(\( P \))

and

\[
\begin{align*}
\min \quad & \langle B, Y \rangle \\
\text{s.t.} \quad & Y \in (\mathcal{L}^\perp - C) \cap \mathcal{K}^*.
\end{align*}
\]

(\( D \))
where $C, B \in \mathbb{R}^m$, and $\mathcal{L}$ is a linear subspace of $\mathbb{R}^m$. We will sometimes write $\mathcal{F}_P$ and $\mathcal{F}_D$ to denote the feasible sets of $(P)$, $(D)$ respectively.

It is easy to see that the problems $(P_0)$, $(D_0)$ are equivalent to $(P)$, $(D)$, respectively. Indeed, choose $C \in S_k$ satisfying $\langle A_i, C \rangle = -c_i$ ($i = 1, \ldots, n$) and put $\mathcal{L} := \text{span}\{A_i, \ i = 1, \ldots, n\}$. Then the feasible sets of $(P_0)$ and $(P)$ are clearly equal to each other, and their objective function values are the same since for $X = B - \sum_{i=1}^n x_i A_i$ we have

$$\langle C, X \rangle - \langle C, B \rangle = \langle C, X - B \rangle = \langle C, (-\sum_{i=1}^n x_i A_i) \rangle = -\sum_{i=1}^n x_i \langle C, A_i \rangle = c^T x.$$ 

Problems $(D_0)$ and $(D)$ have the same objective function, and a point $Y$ is feasible for $(D)$ iff $Y \in K^*$ and

$$Y + C \in \mathcal{L}^\perp \iff \langle Y + C, A_i \rangle = 0 \text{ for all } i \iff \langle Y, A_i \rangle = c_i \text{ for all } i$$

by the relation between $C$ and $c$. So the feasible sets of $(D_0)$ and $(D)$ are equal as well, and the problems are equivalent.

**Definition 3.1.** We say that Slater’s condition holds for $(P)$, if there exists $X \in (B + \mathcal{L}) \cap \text{int } K$. Analogously, we say that Slater’s condition holds for $(D)$, if there exists $Y \in (\mathcal{L}^\perp - C) \cap \text{int } K^*$.

We can parameterize instances of $(P)$ by pairs $(B, \mathcal{L})$ with $B \in \mathbb{R}^m$ and $\mathcal{L} \equiv \{A_i, i = 1, \ldots, n\} \in \mathbb{R}^{m \times n}$. We say that Slater’s condition holds generically for problem $(P)$, if it holds for almost all feasible instances of $(P)$, i.e., the set of pairs $(B, \mathcal{L})$ parametrizing feasible instances of $(P)$ which fail to fulfill Slater’s condition has measure zero. Analogous definitions apply for $(D)$.

The next theorem shows that Slater’s condition is indeed generic:

**Theorem 3.2.** For almost all problem instances of $(P)$, we have that

- either the feasible set of $(P)$ is empty, i.e., $(B + \mathcal{L}) \cap K = \emptyset$, or
- Slater’s condition holds for $(P)$, i.e., $(B + \mathcal{L}) \cap \text{int } K \neq \emptyset$.

An analogous result holds for the dual program $(D)$.

**Proof.** We will show the theorem for the primal problem; the proof for $(D)$ is analogous. To prove the theorem, we will show that for any fixed $\mathcal{L}$ the statement holds for almost all $B \in \mathbb{R}^m$. By the reasoning outlined in Section 2.2, it will then hold for all instances of $(P)$.

So let us fix $\mathcal{L}$. Then we can decompose $B \in \mathbb{R}^m$ uniquely as $B = B_1 \oplus B_2$ such that $B_1 \in \mathcal{L}$ and $B_2 \in \mathcal{L}^\perp$. By construction, we have $B_2 + \mathcal{L} = B + \mathcal{L}$ and $B_1 + \mathcal{L} = \mathcal{L}$. Therefore, the feasibility of $(P)$ and the validity of the Slater condition is determined by $B_2$. Consider the projection $\mathcal{P} := \text{proj}_{\mathcal{L}^\perp} K$ of $K$ onto $\mathcal{L}^\perp$. Clearly, $\mathcal{P}$ is a convex set. We distinguish the following three cases.
Case 1: $B_2 \in \text{rint } \mathcal{P}$. We will show that in this case Slater’s condition holds. Since $B_2 \in \mathcal{P}$, there exists a pre-image $\tilde{B} \in \text{proj}^{-1}_{\mathcal{L}}(B_2) = (B + \mathcal{L}) \cap \mathcal{K}$. By construction, $\tilde{B} + \mathcal{L} = B_2 + \mathcal{L} = B + \mathcal{L}$, so Slater’s condition is equivalent to the condition $(\tilde{B} + \mathcal{L}) \cap \text{int } \mathcal{K} \neq \emptyset$. We will show that this always holds. To this end, suppose by contradiction that $(\tilde{B} + \mathcal{L}) \cap \text{int } \mathcal{K} = \emptyset$ and recall the assumption that $\text{int } \mathcal{K} \neq \emptyset$. Since $\tilde{B} \in \mathcal{K}$, we clearly must have $\tilde{B} \in \text{bd } \mathcal{K}$. Moreover, there must exist a hyperplane supporting $\mathcal{K}$ at the boundary point $\tilde{B}$ and containing $\tilde{B} + \mathcal{L}$, i.e., there exists $N \in \mathcal{S}_k$ with

$$\langle N, Z \rangle \geq \langle N, \tilde{B} \rangle \text{ for all } Z \in \mathcal{K} \quad \text{and} \quad \langle N, L \rangle = 0 \text{ for all } L \in \mathcal{L}. \quad (3.1)$$

Since $B_2 \in \text{rint } \mathcal{P}$, there exists an open neighborhood $\mathcal{N}_\delta(B_2) \subseteq \text{rint } \mathcal{P}$. By continuity of the projection, there exists an open neighborhood $\mathcal{N}_\delta(\tilde{B})$ such that $\text{proj}_{\mathcal{L}}^{-1} \mathcal{N}_\delta(\tilde{B}) \subseteq \mathcal{N}_\delta(B_2)$. The supporting hyperplane divides $\mathcal{N}_\delta(\tilde{B})$ into two parts. Take a point $Q \in \mathcal{N}_\delta(\tilde{B})$ such that $\langle N, Q \rangle < \langle N, \tilde{B} \rangle$. By construction, $R := \text{proj}_{\mathcal{L}}^{-1}(Q) \in \mathcal{N}_\delta(B_2) \subset \text{rint } \mathcal{P}$, so there must exist a pre-image $\tilde{R} \in \text{proj}_{\mathcal{L}}^{-1}(R) = (Q + \mathcal{L}) \cap \mathcal{K}$. Since $\tilde{R} \in \mathcal{K}$, we have from (3.1) that

$$\langle N, \tilde{R} \rangle \geq \langle N, \tilde{B} \rangle. \quad (3.2)$$

On the other hand, since $\tilde{R} \in (Q + \mathcal{L})$, we can write $\tilde{R} = Q + L$ for some $L \in \mathcal{L}$. From (3.1) we have $\langle N, L \rangle = 0$, which gives

$$\langle N, \tilde{R} \rangle = \langle N, Q + L \rangle = \langle N, Q \rangle < \langle N, \tilde{B} \rangle,$$

contradicting (3.2). Hence, our assumption must have been wrong, and therefore Slater’s condition is indeed true in this case.

Case 2: $B_2 \notin \text{cl } \mathcal{P}$. Then by definition of the projection, we have $\emptyset = (B_2 + \mathcal{L}) \cap \mathcal{K} = (B + \mathcal{L}) \cap \mathcal{K}$ and the feasible set of $(P)$ is empty.

Case 3: $B_2 \in \text{bd } \mathcal{P}$. We will show that the set of parameters $B$ for which this happens has measure zero. By Lemma 2.1, the set $\text{bd } \mathcal{P} \subseteq \mathcal{L}^\perp$ has zero measure in $\mathbb{R}^{m-n}$. Therefore, its pre-image $\text{proj}^{-1}_{\mathcal{L}}(\text{bd } \mathcal{P}) = \{ B \in \mathbb{R}^m \mid \text{proj}_{\mathcal{L}}^{-1}(B) = B_2 \in \text{bd } \mathcal{P} \}$ must have zero measure in $\mathbb{R}^m$ by the arguments from Section 2.2.

Summarizing, we have shown that if we fix $\mathcal{L}$, then for almost all $B$ problem $(P)$ either fulfills $(B + \mathcal{L}) \cap \mathcal{K} = \emptyset$, i.e., $(P)$ is infeasible, or $(B + \mathcal{L}) \cap \text{int } \mathcal{K} \neq \emptyset$, i.e., $(P)$ fulfills the Slater condition. The argument from Section 2.2 now completes the proof.

3.2 Genericity of Slater’s condition for problems in standard form

So far, we have shown genericity of Slater’s condition for the self-dual problems $(P)$ and $(D)$. Next, we translate these results to the problems $(P_0)$ and $(D_0)$ in standard form.

For the primal problems $(P)$ and $(P_0)$ there is nothing to show, since the feasible sets of $(P)$
and \((P_0)\) depend on \(B\) in the same way. Therefore, the genericity result for \((P)\) in Theorem 3.2 directly translates to the same genericity of Slater’s condition for \((P_0)\).

When we consider the dual problems \((D)\) and \((D_0)\), we have to transform the \(C\)-space \(S_k \equiv \mathbb{R}^m\) into the \(c\)-space \(\mathbb{R}^n\), and the genericity result for \((D_0)\) requires a bit more work:

**Theorem 3.3.** For almost all \((c, \{A_i\}_{i=1}^n)\) we have: if \((D_0)\) is feasible, then it satisfies the Slater condition.

**Proof.** We will show that for any fixed set \(\{A_i : i = 1, \ldots, n\}\) the statement is true for almost all \(c \in \mathbb{R}^n\). The reasoning of Section 2.2 will then complete the proof.

So fix \(\{A_i : i = 1, \ldots, n\}\) and put \(L = \text{span}\{A_i : i = 1, \ldots, n\}\). Then we can write any \(C \in \mathbb{R}^m = L^\perp \oplus L\) as

\[
C = C_1 + C_2 \quad \text{with } C_1 \in L^\perp \equiv \mathbb{R}^{m-n} \text{ and } C_2 = \text{proj}_L(C) \in L \equiv \mathbb{R}^n,
\]

(3.3)

where \(\text{proj}_L\) denotes the projection onto \(L\). Now choose \(C\) such that \(c_i = -\langle A_i, C \rangle\) for all \(i\), and note that for \(Y \in S_k\) we have \(Y \in (L^\perp - C)\) if and only if \(\langle A_i, Y \rangle = -\langle A_i, C \rangle = -\langle A_i, C_2 \rangle\), and consequently

\[
\langle A_i, Y \rangle = c_i \quad \text{for all } i \quad \iff \quad Y \in (L^\perp - C).
\]

(3.4)

Observe that since \(\{A_i\}_{i=1}^n\) are linearly independent, the relation \(-\langle A_i, C_2 \rangle = c_i\) for \(i = 1, \ldots, n\) defines a linear bijection between \(C_2\) and \(c\):

\[
T : L \to \mathbb{R}^n, \quad T(C_2) = c.
\]

(3.5)

Combining (3.5) and (3.4), we conclude that

\[
\text{Slater fails for } (D_0) \text{ wrt. } c \quad \iff \quad \text{Slater fails for } (D) \text{ wrt. all } C = C_1 + C_2,
\]

where \(C_1 \in L^\perp\) and \(C_2 = T^{-1}(c)\).

It remains to show that the set

\[
C_2 := \{C_2 \in L \mid \text{Slater fails for } (D) \text{ wrt. } C_2 \text{ and } \mathcal{F}_D \neq \emptyset\}
\]

has zero measure in \(L \equiv \mathbb{R}^n\). To this end, recall that the genericity result for \((D)\) in Theorem 3.2 implies that the set

\[
C := \{C \in \mathbb{R}^m \mid \text{Slater fails for } (D) \text{ and } \mathcal{F}_D \neq \emptyset\}
\]

has zero measure in \(\mathbb{R}^m\). Using (3.4), we see that \(C = \{(C_1, C_2) \in L^\perp \times L \mid C_2 \in C_2\}\). Invoking the argument from Section 2.2 for \(C\), we conclude that \(C_2\) has zero measure in \(L \equiv \mathbb{R}^n\), as desired. \[\square\]
4 Other generic properties

In this section, we first discuss the concept of nondegenerate solutions, and show how these relate to Slater’s condition. Then we discuss strict complementarity and uniqueness of optimal solutions, and finally we summarize genericity results that follow from our earlier results as well as from the literature.

4.1 Nondegeneracy

Let us denote the minimal face of $\mathcal{K}$ containing $X$ and the minimal face of $\mathcal{K}^*$ containing $Y$, respectively, by $J(X) = \text{face}(X, \mathcal{K})$ and $G(Y) = \text{face}(Y, \mathcal{K}^*)$.

Observe that for each feasible $X$ we have $X \in \text{rint } J(X)$.

For a face $F$ of $\mathcal{K}$, we define the complementary face as $F^\Delta = \{ Q \in \mathcal{K}^* | \langle Q, S \rangle = 0 \text{ for all } S \in F \}$.

Clearly, $F^\Delta$ is a closed convex cone. Moreover, it is not difficult to see that if $X \in \text{rint } F$, then $F^\Delta = \{ Q \in \mathcal{K}^* | \langle Q, X \rangle = 0 \}$.

This immediately implies that the complementary face of $J(X)$ is equivalently given by

$$J^\Delta(X) = \{ Q \in \mathcal{K}^* | \langle Q, X \rangle = 0 \}.$$  (4.1)

Analogous definitions and results apply to $G^\Delta(Y)$, the complementary face of $G(Y)$.

Definition 4.1. The extreme points of $\mathcal{F}_P$ (resp. $\mathcal{F}_D$) are called primal (resp. dual) basic feasible solutions.

The following characterization of basic solutions is given in [12, Theorem 1]:

Lemma 4.2. Let $X$ be feasible for $(P)$. Then $X$ is a basic feasible solution if and only if

$$\text{span}(J(X)) \cap \mathcal{L} = \{0\}. \tag{4.2}$$

Imposing the analogous condition for the dual problem gives the concept of (primal) nondegeneracy:

Definition 4.3. A primal feasible solution $X$ is called nondegenerate, if

$$\text{span}(J^\Delta(X)) \cap \mathcal{L}^\perp = \{0\}. \tag{4.3}$$

Nondegeneracy of the dual feasible solution $Y$ is defined analogously.
We next discuss the connection between nondegeneracy and Slater’s condition, and we will actually show that the existence of a nondegenerate solution implies Slater’s condition. We first need a technical lemma:

**Lemma 4.4.** Let $X$ be a nondegenerate feasible solution of $(P)$, i.e., $X \in \mathcal{F}_P$ and $\mathcal{L}^\perp \cap \text{span}(J^\Delta(X)) = \{0\}$. Then there exists $L \in \mathcal{L}$ such that:

$$\langle S, L \rangle > 0 \text{ for all } S \in J^\Delta(X) \cap B_1,$$

where $B_1 := \{ S \mid ||S|| = 1 \}$ is the unit sphere in $\mathbb{R}^m$.

**Proof.** We will make use of the following theorem of the alternative (see [5, p.68]): Let $\emptyset \neq I, J$ be (possibly infinite) index sets, and let $b_i, a_j \in \mathbb{R}^m$ for $i \in I, j \in J$. Suppose the set $\text{conv}\{a_j \mid j \in J\}$ is closed. Then precisely one of the following alternatives is true:

(I) $\left\{ \begin{array}{l} (a_j, L) > 0 \text{ for all } j \in J \\ (b_i, L) = 0 \text{ for all } i \in I \end{array} \right\}$ has a solution $L$

(II) $0 \notin \text{conv}\{a_j \mid j \in J\} + \text{span}\{b_i \mid i \in I\}$

For our purposes, let $J := J^\Delta(X) \cap B_1$, and let $a_S := S$ for $S \in J$. Then the set $\text{conv}\{S \mid S \in J\}$ is compact and $0 \notin \text{conv}\{S \mid S \in J\}$. Let further $\{B_i \mid i \in I\}$ be a basis of $\mathcal{L}^\perp$. Then obviously the nondegeneracy assumption for $X$ implies $\mathcal{L}^\perp \cap \text{conv}(J) = \emptyset$, and thus

$$0 \notin \text{span}\{B_i \mid i \in I\} + \text{conv}\{S \mid S \in J\}.$$

Therefore, system (I) must be true, i.e., there exist some $L$ such that $\langle S, L \rangle > 0$ for all $S \in J$ and $\langle B_i, L \rangle = 0$ for all $i \in I$, i.e., $L \in \mathcal{L}$, as desired. $\square$

**Theorem 4.5.** Let $X$ be a nondegenerate feasible solution of $(P)$. Then Slater’s condition holds for $(P)$, i.e., $(B + \mathcal{L}) \cap \text{int} \mathcal{K} \neq \emptyset$. Analogous results are true for the problems $(D)$, $(P_0)$, and $(D_0)$.

**Proof.** We will show that there exist $L \in \mathcal{L}$ and $\alpha > 0$ such that $(X + \alpha L) \in (B + \mathcal{L}) \cap \text{int} \mathcal{K}$. Since $X$ is nondegenerate, take $L$ satisfying (4.4) as in the Lemma 4.4. Then clearly $(X + \alpha L) \in B + \mathcal{L}$ since $X \in B + \mathcal{L}$ and $L \in \mathcal{L}$. To prove $(X + \alpha L) \in \text{int} \mathcal{K}$, we have to show that

$$\langle X + \alpha L, S \rangle > 0 \text{ for all } S \in \mathcal{K}^* \cap B_1$$

where $B_1 := \{ S \mid ||S|| = 1 \}$ is again the unit sphere. We will now construct an $\alpha > 0$ such that this holds.

It follows from (4.4) and the compactness of the set $J^\Delta(X) \cap B_1$ that there exists some $\varepsilon > 0$ such that $\langle L, S \rangle \geq 2\varepsilon > 0$ for all $S \in J^\Delta(X) \cap B_1$. By continuity of the linear function $\langle L, \cdot \rangle$,
there exists some $\delta > 0$ such that

$$\langle L, S \rangle \geq \varepsilon \text{ for all } S \in J^\alpha_\delta(X) \cap B_1,$$

(4.6)

where $J^\alpha_\delta(X) := \{ S \in K^* \mid |S - S| < \delta \text{ for some } S \in J^\alpha(X) \}$. Since $X \in K$, we have $\langle X, S \rangle \geq 0$ for all $S \in K^*$, and by the definition of $J^\alpha(X)$ in (4.1) we have that $\langle X, S \rangle > 0$ for all $S \in (K^* \setminus J^\alpha_\delta(X)) \cap B_1$. By compactness of this set, there exists some $T$ such that

$$\langle X, S \rangle \geq T > 0 \text{ for all } S \in (K^* \setminus J^\alpha_\delta(X)) \cap B_1.$$  

(4.7)

Let $M := \min\{ \langle L, S \rangle \mid S \in (K^* \setminus J^\alpha_\delta(X)) \cap B_1 \}$. We claim that $X + \alpha L \in \text{int} K$ for all $0 < \alpha < T/M$.

We have the following two cases:

- $S \in (K^* \setminus J^\alpha_\delta(X)) \cap B_1$: then $\langle X + \alpha L, S \rangle = \langle X, S \rangle + \langle \alpha L, S \rangle \geq T + \alpha M > 0$.

- $S \in J^\alpha_\delta(X) \cap B_1$: using $\langle X, S \rangle \geq 0$ and (4.6), we have $\langle X + \alpha L, S \rangle = \langle X, S \rangle + \langle \alpha L, S \rangle \geq \alpha \varepsilon > 0$.

By combining these two cases, we have shown that (4.5) holds, and the result follows.

For the case of semidefinite programming, it has been shown implicitly in [1, Proof of Theorem 14] that given $L$, for almost all $B$ all feasible points of $FP$ are nondegenerate. Hence, by applying Theorem 4.5 it follows for the SDP case that given $L$, for almost all $B$ we have: if $FP$ is nonempty, then $FP$ has Slater points.

We wish to mention that in geometric measure theory, transversality results have been proven which roughly speaking assert that generically all intersection points of two convex sets are nondegenerate. In [8, Lemma 3.1], e.g., the following result has been shown:

**Lemma 4.6.** (8, Lemma 3.1) Let $K, L \subset \mathbb{R}^m$ be compact convex sets with nonempty interior. Then for almost all $B \in \mathbb{R}^m$ (wrt. to Hausdorff measure) the sets $K$ and $B + L$ intersect almost transversally, i.e., for all $X \in \text{bd} K \cap \text{bd} L$ it holds

$$N(K, X) \cap N(L, X) = \{0\} \quad \text{and} \quad N(K, X) \cap -N(L, X) = \{0\}$$

where $N(K, X)$ denotes the normal cone of $K$ at $X$.

A similar result has been given in [16, Theorem 2]. In combination with Theorem 4.5 also these results could be used to show that generically the Slater condition holds.

### 4.2 Strict complementarity and uniqueness

For optimal solutions $\overline{X}$ of $(P)$ and $\overline{Y}$ of $(D)$, we denote $J := \text{face}(\overline{X}, K)$ and $G := \text{face}(\overline{Y}, K^*)$. 

10
Definition 4.7. The optimal solutions $\bar{X}$ of $(P)$ and $\bar{Y}$ of $(D)$ are called strictly complementary, if

$$\bar{X} \in \text{rint} J \quad \text{and} \quad \bar{Y} \in \text{rint} J^\Delta.$$  \hfill (4.8)

In [12], a slightly different definition is given: there the optimal solutions $\bar{X}$ and $\bar{Y}$ are called strictly complementary if

$$\bar{X} \in \text{rint} F \quad \text{and} \quad \bar{Y} \in \text{rint} F^\Delta$$  \hfill (4.9)

holds for some face $F$ of $K$. It is clear that (4.8) implies (4.9). Conversely, let (4.9) be satisfied. We always have $\bar{X} \in \text{rint} J$, and for $\bar{X} \in \text{rint} F$, we have $F^\Delta = J^\Delta$ by (4.1). Therefore, (4.8) and (4.9) are equivalent.

The following lemma shows the relation between nondegeneracy, strict complementarity, basic solutions and uniqueness.

Lemma 4.8. [12, Theorem 2] Let $X$ be a feasible solution of $(P)$. Then the following hold.

(a) If $X$ is nondegenerate, then an arbitrary complementary solution of $(D)$ must be basic. Moreover, if there is a complementary dual solution, it must be unique.

(b) Suppose that $Y$ is a dual solution and $X$ and $Y$ are strictly complementary. Then $Y$ is basic if and only if $X$ is nondegenerate.

Another closely related definition of strict complementarity has been given in [13], where strict complementarity is defined as

$$J^\Delta = G.$$  \hfill (4.10)

It can be shown that (4.10) and (4.8) are equivalent, see the proof of [12, Theorem 2]. Similarly, considering the dual problem, strict complementarity can be defined as

$$\bar{Y} \in \text{rint} G \quad \text{and} \quad \bar{X} \in \text{rint} G^\Delta.$$  \hfill (4.11)

Neither of the conditions (4.8) or (4.11) implies the other unless $K$ or $K^*$ are facially exposed as noted in [13, Remark 3.3.2]. An illustrating example of these “asymmetric” strict complementarity definitions is given in [2, Example 1]. In [2, Definition 3.1], yet another strict complementarity definition is given. Note that not all cones normally used in optimization are facially exposed: it is well known that the semidefinite cone is facially exposed, but for instance the copositive cone is not, see [3].

We remark that strict complementarity plays a crucial role for the numerical performance of optimization algorithms. One example where this can be seen is [2], where an augmented primal-dual algorithm is developed to solve problems over the doubly nonnegative matrix cone. Strict complementarity is fundamental for this algorithm, since regularity of the augmented primal-dual function depends on this.
4.3 Genericity of strict complementarity, nondegeneracy, and uniqueness

This section is devoted to results asserting that for almost all conic program instances certain regularity conditions hold, such as uniqueness, nondegeneracy and strict complementarity of solutions as well as the Slater condition.

By combining the genericity result for the Slater condition in Theorem 3.2 with well known optimality conditions for conic programs, we obtain the following:

**Corollary 4.9.** Let $L$ be given. For almost all $(B, C) \in \mathbb{R}^{2m}$, the following holds:

(a) If both $(P)$ and $(D)$ are feasible, then both have optimal solutions $\bar{X}, \bar{Y}$.

(b) For each such pair the complementarity condition $\langle \bar{X}, \bar{Y} \rangle = 0$ is satisfied. In particular the problems are gap free, i.e. $v_P = v_D$.

For completeness we also recall the genericity results from [12] with respect to uniqueness and strict complementarity of optimal solutions. Genericity of uniqueness follows as argued in [12] e.g. by applying the following result from geometric measure theory for convex bodies.

First, let us introduce some notation. For a closed convex set $\emptyset \neq T \subset \mathbb{R}^\ell$ we recall the support function $h(T, \cdot)$ of $T$: for $u \in \mathbb{R}^\ell$ we define

$$h(T, u) := \sup_{x \in T} \langle x, u \rangle$$  \hspace{1cm} (4.12)

For $u \in \text{dom} h(T, \cdot) \setminus \{0\}$ the corresponding supporting hyperplane is given by

$$H(T, u) := \{x \in \mathbb{R}^n \mid \langle x, u \rangle = h(T, u)\}$$

with support set $F(T, u) := H(T, u) \cap T$. Note that for given $0 \neq u$ a solution of the convex problem in (4.12) is unique if and only if $\dim F(T, u) = 0$. The next theorem precisely asserts that for almost all objective vectors $u \in \mathbb{R}^\ell$ a solution of the convex program is unique.

**Theorem 4.10.** [13, Theorem 2.2.9] Let $\emptyset \neq T \subset \mathbb{R}^\ell$ be a compact, convex set. Then the set of all vectors $u$ in the $(\ell - 1)$-dimensional unit sphere $B_1 := \{u \in \mathbb{R}^\ell \mid \|u\| = 1\}$ which fulfill $\dim F(T, u) > 0$ has $(\ell - 1)$-dimensional Hausdorff measure zero.

By combining this theorem with our previous results we obtain the following:

**Corollary 4.11.** Let $L$ be given. Then for almost all $(C, B) \in \mathbb{R}^{2m}$ the following holds: If the corresponding conic programs $(P)$ and $(D)$ are feasible, then the optimal solutions $\bar{X}, \bar{Y}$ of $(P)$ and $(D)$ are unique and satisfy $\langle \bar{X}, \bar{Y} \rangle = 0$.

**Proof.** For almost all $C$, we have that if $(D)$ is feasible, then by Theorem 3.2 the Slater condition holds for $(D)$. Thus, by well known results, the level sets $F_P(\alpha) = \{X \in (B + L) \cap K \mid \langle C, X \rangle \geq \alpha\}$ are compact. Applying Theorem 4.10 to a nonempty compact set $T := F_P(\alpha)$, the result follows. \qed
In [12] the genericity of strict complementarity of solutions of \((P)\) and \((D)\) has been shown under the assumption that the problems are gap-free. For fixed \(L\) the authors consider the sets (see [12, p. 455 and Proposition 1])

\[
\mathcal{D}(L) = \{(C, B) \mid \text{the corresponding problems } (P) \text{ and } (D) \text{ are feasible with } v_P = v_D \}
\]

and

\[
\mathcal{D}(L) := \{(C, B) \in \mathcal{D}(L) \mid \text{some optimal solutions } \bar{X} \text{ of } (P) \text{ and } \bar{Y} \text{ of } (D) \text{ are strictly complementary}\}.
\]

**Theorem 4.12.** [12, Proposition 2] For fixed \(L\), the set \(\overline{\mathcal{D}(L)} \setminus \mathcal{D}(L)\) has \(\dim(\overline{\mathcal{D}(L)})\)-dimensional Hausdorff measure zero.

As in [12, p.456], to prove that the nondegeneracy condition holds generically at solutions of \((P)\), we combine Theorem 3.2, Corollary 4.9, Theorem 4.12, Corollary 4.11 and Lemma 4.8.

Note that Lemma 4.8(b) does not hold for \(X\) and \(Y\) interchanged unless \(\mathcal{K}\) is facially exposed. However, if we define strict complementarity as (4.11), then Lemma 4.8(b) holds for \(X\) and \(Y\) interchanged. Analogously, we can show that (4.11) is a generic property. Thus, using the same arguments, generically at solutions of \((D)\) the nondegeneracy condition holds. Finally, combining everything, we have

**Corollary 4.13.** Let \(L\) be given arbitrarily. Then for almost all \((C, B) \in \mathbb{R}^{2m}\) the following is true: If the corresponding programs \((P), (D)\) are both feasible, then there exist unique optimal solutions \(\bar{X}\) of \((P)\) and \(\bar{Y}\) of \((D)\). These solutions are nondegenerate and satisfy the strict complementarity condition.

Let us finish this section by translating the previous results to the pair of primal/dual conic programs \((P_0)\) and \((D_0)\) in standard form.

Recalling that under Slater’s condition the upper/lower level sets of \((P_0)\), \((D_0)\) are compact, we can apply Theorem 4.10 as before and conclude that for any fixed \(\{A_i\}_{i=1}^n\) the following holds for almost all \((c, B) \in \mathbb{R}^n \times \mathcal{S}_k\): if \((P_0)\), \((D_0)\) are both feasible, then the solutions \(\bar{x}, \bar{y}\) exist and are unique.

Furthermore, with the same arguments as before in Theorem 3.3 by using the decompositions

\[
C = \text{proj}_L(C) + \text{proj}_{L^\perp}(C), \quad B = \text{proj}_L(B) + \text{proj}_{L^\perp}(B)
\]

the genericity results with respect to strict complementarity in Theorem 4.12 translates from \((P)\), \((D)\) to \((P_0)\), \((D_0)\) as well. Summarizing we have obtained the following.

**Corollary 4.14.** For almost all \((c, B, \{A_i\}_{i=1}^n)\) we have: If \((P_0)\), \((D_0)\) are both feasible, then there exist unique optimal solutions \(\bar{X}\) of \((P_0)\) and \(\bar{Y}\) of \((D_0)\). Moreover, \(\bar{X} := B - \sum_{i=1}^n \pi_i A_i\) and \(\bar{Y}\) are both nondegenerate and satisfy the strict complementarity condition.
5 Conclusion

In this paper we have proven that for almost all problem data \((C, B, \{A_i\}_{i=1}^n)\) we have that if the corresponding conic programs are feasible, then they also have Slater points. Together with the earlier results in [1] and [12], this implies that the set of “nice” feasible conic programs with unique, nondegenerate and strictly complementary solutions is dense. However, from a practical viewpoint it is also important to know whether this set of nice problems is open, i.e., whether around a given nice problem the regularity conditions such as Slater’s condition, uniqueness, nondegeneracy, strict complementarity of solutions are stable wrt. small perturbations of the problem data.

We emphasize that in the theory of generic properties of nonlinear programs (founded by Jongen, Jonker, and Twilt, see e.g. [9]), usually a property is called generic if it holds for a subset of the data set which is dense (or stronger a set whose complement has measure zero) and open. Such stability questions in general conic programming will be discussed in a forthcoming paper.

Here, we only wish to mention that the Slater condition is trivially stable. Indeed, assume that \(X = B - \sum_{i=1}^n x_i A_i\) is a Slater point for \((P)\) wrt. to the data \((C, B, \{A_i\}_{i=1}^n)\). Then also the point \(X := B - \sum_{i=1}^n \bar{x}_i \bar{A}_i\) is a Slater point after any sufficiently small perturbation \((C, B, \{A_i\}_{i=1}^n)\) of \((C, B, \{A_i\}_{i=1}^n)\).

Acknowledgements

We would like to thank Jorge A. León for pointing out reference [10]. We also gratefully acknowledge support by the Netherlands Organisation for Scientific Research (NWO) through Vici grant no.639.033.907.

References


