On the Rank of Cutting-Plane Proof Systems

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Abstract

We introduce a natural abstraction of propositional proof systems that are based on cutting planes. This leads to a new class of proof systems that includes many well-known methods, such as Gomory-Chvátal cuts, lift-and-project cuts, Sherali-Adams cuts, or split cuts. The rank of a proof system corresponds to the number of rounds that is needed to show the nonexistence of integral solutions. We exhibit a family of polytopes without integral points contained in the n-dimensional $0/1$-cube that has rank $\Omega(n/\log n)$ for *every* proof system in our class. In fact, we show that whenever some cutting-plane based proof system has (maximal) rank n on a particular family of instances, then any cutting-plane proof system in our class has rank $\Omega(n/\log n)$ for this family. This shows that the rank complexity of worst-case instances is intrinsic to the problem; it does not depend on specific cutting-plane proof systems, except for log factors. We also construct a new cutting-plane proof system that has worst-case rank $O(n/\log n)$ for any polytope without integral points, implying that our universal lower bound is essentially tight.

1 Introduction.

Cutting planes are a fundamental, theoretically and practically relevant tool in combinatorial optimization and integer programming. Cutting planes help to eliminate irrelevant fractional solutions from polyhedral relaxations while preserving the feasibility of integer solutions. There are several well-known procedures to systematically derive valid inequalities for the integer hull P_I of a rational polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\} \subseteq [0,1]^n$ (see, e.g., Cornuéjols [2008], Cornuéjols and Li [2001]). These include Gomory-Chvátal cuts Chvátal [1973], Gomory [1958], the lift-and-project cuts of Balas, Ceria and Cornuéjols Balas et al. [1993], Sherali-Adams cuts Sherali and Adams [1990], the matrix cuts of Lovász and Schrijver Lovász and Schrijver [1991], and split cuts Cook et al. [1990]. Repeated application of these operators is guaranteed to yield a linear description of the integer hull, and the question naturally arises of how many rounds are, in fact, necessary. This gives rise to the notion of rank. For example, it is known that the Gomory-Chvátal rank of a polytope contained in the n-dimensional $0/1$-cube is at most $O(n^2 \log n)$ Eisenbrand and Schulz [2003], whereas the rank of all other methods mentioned before is bounded above by n, which is known to be tight (see, e.g., Cook and Dash [2001], Cornuéjols [2008]). These convexification procedures can also be viewed as propositional proof systems (e.g., Chvátal et al. [1989], Dantchev [2007], Dash [2005]), each using its own set of rules to prove that a system of linear inequalities with integer coefficients

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does not have a 0/1-solution. While exponential lower bounds on the lengths of the proofs were obtained for specific systems (e.g., Bonet et al. [1997], Dash [2005], Pudlák [1999]), there is no general technique available that would work for all propositional proof systems (which would actually prove that NP $\neq$ co-NP). We formalize the concept of an "admissible" cutting-plane proof system (see Definition 2.1 below for details) and provide a generic framework that comprises all proof systems based on cutting planes mentioned above and allows us to make general statements on the rank of these proof systems.

Our main contributions are as follows. The introduction of admissible cutting-plane procedures exposes the commonalities of several well-known convexification methods and helps to explain several of their properties on a higher level. It also allows us to uncover much deeper connections. In particular, in the context of cutting-plane procedures as refutation systems in propositional logic, we will show that if an arbitrary admissible cutting-plane procedure has maximal rank n, then so does the Gomory-Chvátal procedure. In addition, the rank of the Sherali-Adams cuts (for fixed level d), the rank of the linear matrix cuts of Lovász and Schrijver, the rank of the lift-and-project cuts by Balas, Ceria, and Cornuéjols, and the rank of the split cut operator is at least $n-1$. In this sense, we show that some of the better known procedures belong to the weakest members in the class of admissible cutting-plane procedures. However, we also provide a family of instances, i.e., polytopes $P \subseteq [0,1]^n$ with empty integer hull, for which the rank of *every* admissible cutting-plane procedure is $\Omega(n/\log n)$. In fact, we show that the rank of any admissible cutting-plane procedure is $\Omega(n/\log n)$ whenever there is some admissible cutting-plane procedure that has maximal rank n. Last not least, we introduce a new cutting-plane procedure whose rank is bounded by $O(n/\log n)$, which implies that our universal lower bound is virtually tight. In the course of our proofs, we also exhibit several interesting structural properties of integer-empty polytopes $P \subseteq [0,1]^n$ with maximal Gomory-Chvátal rank, maximal matrix cut rank, or maximal split rank.

We briefly discuss some related work. A (tight) lower bound of n for the rank of the Gomory-Chvátal procedure for polytopes $P \subseteq [0,1]^n$ with $P_I = \emptyset$ was established in Chvátal et al. [1989]. This was later used to obtain a lower bound of $(1+\epsilon)n$ on the Gomory-Chvátal rank of arbitrary polytopes $P \subseteq [0,1]^n$, showing that in contrast to most other cutting-plane procedures, the Gomory-Chvátal procedure does not have an upper bound of n if $P_I \neq \emptyset$ Eisenbrand and Schulz [2003]. The currently best lower bound is $\Omega(n^2)$ Rothvoss and Sanità [2013]. The upper bound is known to be n if $P_I = \emptyset$ Bockmayr et al. [1999]. A complete characterization of all polytopes $P \subseteq [0,1]^n$ with $P_I = \emptyset$ and maximal Gomory-Chvátal rank was given in Pokutta and Schulz [2011]. Lower bounds of n for the matrix cut operators N_0, N, and N_+ of Lovász and Schrijver Lovász and Schrijver [1991] were given in Cook and Dash [2001], Cornuéjols and Li [2002b], Goemans and Tuncel [2001]. Lower bounds for the split cut operator SC were obtained in Cornuéjols and Li [2002a]. We refer to Dash [2005] for a superb summary of the literature on lower bounds on the size and length (as opposed to the rank) of cutting-plane proofs based on these operators. These operators (and some strengthenings thereof) have recently regained attention Georgiou et al. [2007], Lasserre [2001], Pokutta and Schulz [2009], mostly due to an interesting connection between the inapproximability of certain combinatorial optimization problems and the integrality gaps of their linear and semi-definite programming relaxations. For example, it was shown in Schoenebeck et al. [2007] that the integrality gaps of the natural LP relaxations of the vertex cover and the max cut problem remain at least $2-\epsilon$ after $\epsilon\, n$ rounds of the Sherali-Adams operator. A related result for the stronger Lovász-Schrijver operator established an integrality gap of $2-\epsilon$ after $\Omega(\sqrt{\log n/\log\log n})$ rounds Georgiou et al. [2007]. In Schoenebeck [2008] it was shown that even for the stronger Lasserre hierarchy Lasserre [2001] one cannot expect to be able to prove the unsatisfiability of certain k-CSP formulas within $\Omega(n)$ rounds. As a result, a $7/6-\epsilon$ integrality gap for the vertex cover problem after $\Omega(n)$ rounds of the Lasserre hierarchy follows. In Charikar et al. [2009], the strength of the Sherali-Adams operator is studied in terms of integrality gaps for well-known problems like max cut, vertex cover, and sparsest cut, and in Mathieu and Sinclair [2009] integrality gaps for the fractional matching polytope, which has Gomory-Chvátal rank 1, are provided, showing that although the matching problem can be solved in polynomial time, it cannot be approximated well with a small number of rounds of the Sherali-Adams operator. In addition, it was shown that for certain tautologies that can be expressed in first-order logic, the Lovász-Schrijver N_+ rank can be constant, whereas the Sherali-Adams rank grows poly-logarithmically Dantchev [2007]. A link between the Sherali-Adams closure and border bases, and hence algebraic geometry, has been established in Pokutta and Schulz [2009].

Our work complements these results in a variety of ways. On the one hand, we provide a basic framework that allows us to show that in the case of polytopes $P \subseteq [0,1]^n$ with $P_I = \emptyset$ all admissible cutting-plane procedures exhibit a similar behavior in terms of maximal rank, as long as log factors are omitted. On the other hand, we define a new cutting-plane procedure that is optimal with respect to the lower bound, i.e., it establishes $P_I = \emptyset$ in $\mathrm{O}(n/\log n)$ rounds and, therefore, outperforms well-known, classical cutting-plane procedures in terms of maximal rank. We believe that our approach may also be used to establish general bounds on integrality gaps, *independent* of the specific cutting-plane procedure used.

The paper is organized as follows. In Section 2 we introduce the notion of admissible cutting-plane procedures. We also prove admissibility of the the proof systems based on Sherali-Adams cuts (for fixed level d), linear matrix cuts, lift-and-project cuts, Gomory-Chvátal cuts, or split cuts. We derive basic results and upper bounds on the rank for generic admissible cutting-plane procedures in Section 3. In Section 4, we characterize polytopes $P \subseteq [0,1]^n$ with $P_I = \emptyset$ and maximal rank. These results are used in Section 5 to show that several well-established cutting-plane proof systems are relatively weak, in the sense that whenever any other admissible cutting-plane procedure requires n rounds to certify $P_I = \emptyset$, then their own rank is n (e.g., Gomory-Chvátal) or $n-1$ (e.g., matrix cuts). The generic lower bounds for the rank of admissible cutting-plane procedures are constructed in Section 6. In Section 7 we provide a new admissible cutting-plane procedure that is rank optimal in the sense that it realizes the lower bound $\Omega(n/\log n)$ for polytopes in the n-dimensional 0/1-cube without integral points. We conclude by establishing a universal lower bounds for the rank of the subtour elimination relaxation of the traveling salesman problem in Section 8.

2 Admissible Cutting-Plane Proof Systems.

Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a rational polyhedron that is contained in the n-dimensional 0/1-cube; i.e., we assume that $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$, and $P \subseteq [0,1]^n$. We use a_i to denote row i of A, and b_i is the corresponding entry on the right-hand side. The integer hull, P_I, of P is the convex hull of all integer points in P, $P_I = \mathrm{conv}(P \cap \{0,1\}^n)$. If F is a face of the n-dimensional unit cube, $[0,1]^n$, then $P \cap F$ can be viewed as the set of those points in P for which certain coordinates have been fixed to 0 or 1. We define $\varphi_F(P)$ as the projection of P onto the space of variables that are not fixed by F. If $P, Q \subseteq [0,1]^n$ are polytopes, we say that $P \cong Q$ if there exists a face F of the n-dimensional unit cube such that $\varphi_F(P) = Q$. We also let $[n] = \{1, 2, \ldots, n\}$.

A cutting-plane procedure consists of an operator M that maps P to a polytope $M(P)$, which we call the M-closure of P. A linear inequality that is valid for $M(P)$ is called an M-cut.

Definition 2.1. *We say that a cutting-plane procedure M is* admissible *if it has the following properties:*

1. *M* ***strengthens*** *P* ***and keeps*** *P_I* ***intact****: $P_I \subseteq M(P) \subseteq P$.*
2. ***Preservation of inclusion****: If $P \subseteq Q$, then $M(P) \subseteq M(Q)$, for all polytopes $P, Q \subseteq [0,1]^n$.*
3. ***Homogeneity****: $M(F \cap P) = F \cap M(P)$, for all faces F of $[0,1]^n$.*
4. ***Single coordinate rounding****: If $x_i \leq \epsilon < 1$ (or $x_i \geq \epsilon > 0$) is valid for P, then $x_i \leq 0$ (or $x_i \geq 1$) is valid for $M(P)$.*

5. ***Commuting with coordinate flips and coordinate duplication**: Let $\tau_i : [0,1]^n \to [0,1]^n$ with $\tau_i(x)_j = x_j$ and $\tau_i(x)_i = 1 - x_i$ be a coordinate flip, then $\tau_i(M(P)) = M(\tau_i(P))$. Similarly, if $\lambda_i : [0,1]^n \to [0,1]^{n+1}$ with $\lambda_i(x)_{n+1} = x_i$ and $\lambda_i(x)_j = x_j$ for $j \in [n]$ is a coordinate duplication, then $\lambda_i(M(P)) = M(\lambda_i(P))$.*

6. ***Short verification**: There exists a polynomial p such that for any inequality $cx \leq \delta$ that is valid for $M(P)$ there is a set $I \subseteq [m]$ with $|I| \leq p(n)$ such that $cx \leq \delta$ is valid for $M(\{x : a_i x \leq b_i, i \in I\})$. We call $p(n)$ the* verification degree *of M.*

We believe that these conditions are quite natural and capture the essence of cutting-plane proof systems. In fact, all cutting-plane procedures mentioned above satisfy these conditions, as we will show below. Condition 1 ensures that $M(P)$ is a relaxation of P_I that is not worse than P itself. Condition 2 establishes the monotonicity of the procedure; because any inequality that is valid for Q is also valid for P, the same should hold for the corresponding M-cuts. Condition 3 states that the order in which we fix certain variables to 0 or 1 and apply the operator should not matter. Condition 4 makes sure that an admissible procedure is able to derive the most basic conclusions, while Condition 5 makes certain that the natural symmetry of the 0/1-cube is maintained. Finally, Condition 6 guarantees that admissible cutting-plane procedures cannot be too powerful; otherwise even $M(P) = P_I$ would be included, and the class of admissible procedures would become too broad to derive interesting results. Note also that (6) is akin to an axiom on the “independence of irrelevant alternatives."

A cutting-plane operator can be applied iteratively; we define $M^{(i+1)}(P) := M(M^{(i)}(P))$ for $i \in \mathbb{Z}_+$, where $M^{(0)}(P) := P$. Obviously, $P_I \subseteq M^{(i+1)}(P) \subseteq M^{(i)}(P) \subseteq \cdots \subseteq M^{(1)}(P) \subseteq M^{(0)}(P) = P$. In general, it is necessarily true that there exists a finite $k \in \mathbb{Z}_+$ such that $P_I = M^{(k)}(P)$. However, we will see that for polytopes $P \subseteq [0,1]^n$ with $P_I = \emptyset$ this follows from Properties 3 and 4. In this sense, every admissible cutting-plane procedure can be viewed as a system for proving the unsatisfiability of propositional formulas in conjunctive normal form (which can be naturally represented as systems of integer inequalities), which is the setting considered here. The rank of P with respect to M is the smallest $k \in \mathbb{Z}_+$ such that $P_I = M^{(k)}(P)$. We write $\text{rk}_M(P) = k$ (and drop the index M if it is clear from the context).

The following lemmas follow readily from the definition of admissible cutting-plane procedures.

Lemma 2.2. *Let M be an admissible cutting-plane procedure, and let $P \subseteq [0,1]^n$ be a polytope and $(i,l) \in [n] \times \{0,1\}$ such that $P \cap \{x_i = l\} = \emptyset$. Then $M(P) \cap \{x_i = 1 - l\} = M(P \cap \{x_i = 1 - l\}) = M(P)$.*

Lemma 2.3. *Let M be admissible and consider two polytopes $Q \subseteq P \subseteq [0,1]^n$. Then $rk(Q) \leq rk(P)$ if $Q_I = P_I$.*

In particular, if M is admissible, then $\text{rk}(Q) \leq \text{rk}(P)$ for $Q \subseteq P \subseteq [0,1]^n$ with $P_I = \emptyset$.

In the remainder of this section we recall the definition of some classical cutting-plane procedures and show that they are admissible.

Gomory-Chvátal cuts. Let $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$, and let $P = \{x : Ax \leq b\} \subseteq [0,1]^n$ be a polytope. The *Gomory-Chvátal closure* of P is defined as

$$P' := \bigcap_{\lambda \in \mathbb{R}^m_+, \lambda A \in \mathbb{Z}^n} \{x : \lambda A x \leq \lfloor \lambda b \rfloor\}.$$

It is well known that P' is again a polytope Chvátal [1973] (in fact, it suffices to consider vectors $\lambda \in [0,1]^m$), and we can apply the operator iteratively by setting $P^{(i+1)} := (P^{(i)})'$ for $i \geq 0$. Here, $P^{(0)} := P$.

It is straightforward to show that the Gomory-Chvátal procedure satisfies Properties 1, 2 and 4 of Definition 2.1. LP duality and Carathéodory's Theorem imply that any valid inequality for P' is (dominated by) the conic combination of at most n Gomory-Chvátal cuts, each of which can, in turn, be derived from at most n inequalities of the original system $Ax \leq b$. Hence, Property 6 holds with $p(n) = n^2$. The following well-known lemma (see, e.g., [Cook et al., 1998, Lemma 6.33]) establishes Property 3:

Lemma 2.4. *Let P be a polytope and let F be a face of P. Then $(P \cap F)' = P' \cap F$.*

In [Eisenbrand and Schulz, 2003, Lemma 4.3], it was shown that the Gomory-Chvátal closure commutes with unimodular transformations. As coordinate flips $x_i \mapsto 1 - x_i$ with $i \in [n]$ are unimodular transformations, the first part of (5) is established as well. It remains to prove that the Gomory-Chvátal procedure commutes with coordinate duplications. An implicit proof can be found in Chvátal et al. [1989] (see [Chvátal et al., 1989, Lemma 2.2] and [Chvátal et al., 1989, Lemma 8.2]). We include a proof here, for completeness.

Lemma 2.5. *Let $P \subseteq [0,1]^n$ be a polytope and let $\tau : [0,1]^n \to [0,1]^{n+1}$ be a coordinate duplication, i.e., $\tau(x)_i = x_i$ for all $i \in [n]$ and $\tau(x)_{n+1} = x_k$ for some $k \in [n]$. Then $\tau(P)' = \tau(P')$.*

Proof. Without loss of generality, let $k = n$. We first show that

$$\tau(P)' = \bigcap \{(c,0)x \leq \lfloor \delta \rfloor : P \subseteq \{cx \leq \delta\}, c \in \mathbb{Z}^n, \delta \in \mathbb{R}\} \cap \{x_n = x_{n+1}\}.$$

Obviously, $\tau(P)' \subseteq \{x_n = x_{n+1}\}$ and $\tau(P)' \subseteq \bigcap \{(c,0)x \leq \lfloor \delta \rfloor : P \subseteq \{cx \leq \delta\}, c \in \mathbb{Z}^n, \delta \in \mathbb{R}\}$, so it remains to show the other inclusion. Consider an arbitrary $\tilde{x} \in [0,1]^n \setminus \tau(P)'$. We claim that

$$\tilde{x} \notin \bigcap \{(c,0)x \leq \lfloor \delta \rfloor : P \subseteq \{cx \leq \delta\}, c \in \mathbb{Z}^n, \delta \in \mathbb{R}\} \cap \{x_n = x_{n+1}\},$$

which would imply the missing direction. If $\tilde{x}_n \neq \tilde{x}_{n+1}$ the statement is immediate. Thus, suppose that $\tilde{x}_n = \tilde{x}_{n+1}$. As $\tilde{x} \notin \tau(P)'$ there exists an inequality $cx \leq \delta$ valid for $\tau(P)$ with $c \in \mathbb{Z}^{n+1}$, $\delta \in \mathbb{R}$ such that $c\tilde{x} > \lfloor \delta \rfloor$. Let $\tilde{c} = c + c_{n+1}(e_n - e_{n+1})$ and observe that $\tilde{c}x = cx$ for all $x \in \{x_{n+1} = x_n\}$. Thus we have $\tilde{c}\tilde{x} = c\tilde{x} > \lfloor \delta \rfloor$. Further, $\tilde{c}$ is of the form $\tilde{c} = (\tilde{c}_1, \ldots, \tilde{c}_n, 0)$. It therefore remains to show that $P \subseteq \{\bar{c}x \leq \delta\}$, where $\bar{c} = (\tilde{c}_1, \ldots, \tilde{c}_n)$. This, however, is a consequence of $cx \leq \delta$ being valid for $\tau(P)$. Hence, the claim follows. We therefore obtain

$$\begin{aligned}\tau(P)' &= \bigcap \{(c,0)x \leq \lfloor \delta \rfloor : P \subseteq \{cx \leq \delta\}, c \in \mathbb{Z}^n, \delta \in \mathbb{R}\} \cap \{x_n = x_{n+1}\} \\ &= \tau\left(\bigcap \{cx \leq \lfloor \delta \rfloor : P \subseteq \{cx \leq \delta\}, c \in \mathbb{Z}^n, \delta \in \mathbb{R}\}\right) = \tau(P').\end{aligned}$$

□

It follows that the Gomory-Chvátal procedure is indeed admissible.

Matrix cuts. We now consider the cutting-plane procedures N_0 and N, introduced by Lovász and Schrijver Lovász and Schrijver [1991] (see also Balas et al. [1993] for N_0 and Sherali and Adams [1990] for N), which are commonly called matrix-cut operators.[1] Let $P = \{x : Ax \leq b\} \subseteq [0,1]^n$ with $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$. We assume that the system $Ax \leq b$ includes the inequalities $x_j \geq 0$ and $x_j \leq 1$ for all $j \in [n]$. Then the $N(P)$ closure of P can be obtained by the following procedure:

Step 1: Generate the nonlinear system $x_j(b - Ax) \geq 0$, $(1 - x_j)(b - Ax) \geq 0$, for all $j \in [n]$.
Step 2: Linearize the system by substituting y_{ij} for $x_i x_j$ and $x_j x_i$ with $i \neq j$ and x_j for x_j^2. Call the resulting polyhedron Q.

[1] To avoid some technical subtleties, we pass over their strongest operator, N_+, here because it generally leads to non-polyhedral closures.

Step 3: Let $N(P) := \text{proj}_x(Q)$.

We obtain the lift-and-project closure of Balas, Ceria and Cornuéjols Balas et al. [1993] for coordinate j by restricting the multipliers in Step 1 to x_j and $(1 - x_j)$ for a fixed $j \in [n]$. If we then set $P_j := \text{proj}_x(Q)$, we can define $N_0(P) := \bigcap_{j \in [n]} P_j$. There are actually several equivalent ways of defining the matrix-cut operators (see, e.g., Cook and Dash [2001], Cornuéjols [2008], Goemans and Tuncel [2001], Sherali and Adams [1990]). In general, $N_0(P) \neq N(P)$ because $N_0(P)$ does not take advantage of the fact that $x_i x_j = x_j x_i$. By definition, the matrix-cut operators satisfy the following relation, which implies Property 1 of Definition 2.1.

Lemma 2.6. *Let $P \subseteq [0,1]^n$ be a polytope. Then*

$$P_I \subseteq N(P) \subseteq N_0(P) \subseteq P.$$

Property 2 follows from the definition of N_0 resp. N as well. Properties 3 and 5 were shown in Cook and Dash [2001] (Lemmas 2.2 and 2.1, resp.). For Property 4, i.e., single coordinate rounding, consider $x_i \geq \epsilon > 0$. Step 1 implies that we get the following inequality: $(1 - x_i)x_i \geq (1 - x_i)\epsilon$. Step 2 yields $0 \geq \epsilon - \epsilon x_i$, which is equivalent to $x_i \geq 1$. Hence, the latter inequality is valid for $N_0(P)$ and $N(P)$. One can derive $x_i \leq 0$ in a similar manner from $x_i \leq 1 - \epsilon$. It remains to show Property 6 for N_0 and N. This essentially follows from [Dash, 2005, Lemma 3.1]. We include a proof here, again for completeness.

Lemma 2.7. *Both N_0 and N satisfy Property 6 with verification degree $p(n) = O(n^3)$.*

Proof. Let $P \subseteq [0,1]^n$ be a polytope. We formulate the proof for $N(P)$. The proof for $N_0(P)$ is essentially identical. We may assume that the polytope that results from Step 2 is represented as $Q = \{(x,y) : \tilde{A}x + Fy \geq \tilde{b}\}$. Here, each inequality of the system $\tilde{A}x + Fy \geq \tilde{b}$ is the result of multiplying some inequality of the original system $b - Ax \geq 0$ with x_j or $(1 - x_j)$, for some $j \in [n]$, and linearization. We then have $N(P) = \{x : \lambda^h \tilde{A}x \leq \lambda^h \tilde{b} \text{ for all } h \in H\}$, where $\{\lambda^h\}_{h \in H}$ is the set of extreme rays of the projection cone $C := \{\lambda \leq 0 : \lambda F = 0\}$. Simple counting shows that each λ^h has at most $\mathrm{O}(n^2)$ non-zero entries. Now, let $cx \leq \delta$ be an inequality that is valid for $N(P)$. By the observation above, $cx \leq \delta$ is dominated by some conic combination of at most n inequalities of the type $\lambda^h \tilde{A}x \leq \lambda^h \tilde{b}$. If we consider the set of inequalities of $\tilde{A}x + Fy \geq \tilde{b}$ that correspond to non-zero entries of λ^h, for all $h \in H$, and, in turn, collect all the inequalities of the original system $Ax \leq b$ on which this set is based, we obtain a subsystem $A_I x \leq b_I$ of $Ax \leq b$ such that $cx \leq \delta$ is valid for $N(\{x : A_I x \leq b_I\})$, and $|I| = \mathrm{O}(n^3)$. □

The Sherali-Adams operator, introduced in Sherali and Adams [1990], is defined similarly to N, except for not iteratively projecting after each round, but performing first several lifting operations and then projecting once. Thus the Sherali-Adams operator refines the N operator, and one can use similar arguments to show that it is admissible as well, if one assumes that the number of lifting operations (i.e., the degree of the resulting polynomials) is fixed.

Split cuts. Let $P \subseteq [0,1]^n$. Then the split closure of P, denoted by $\text{SC}(P)$, is defined as

$$\text{SC}(P) := \bigcap_{(\pi,\pi_0) \in \mathbb{Z}^{n+1}} \text{conv}((P \cap \{\pi x \leq \pi_0\}) \cup (P \cap \{\pi x \geq \pi_0 + 1\})).$$

Properties 1 and 2 follow directly from the definition of the split closure. As for Property 3, we have, for any face F of $[0,1]^n$,

$$
\begin{aligned}
F \cap \mathrm{SC}(P) =& F \cap \bigcap_{(\pi,\pi_0)\in\mathbb{Z}^{n+1}} \mathrm{conv}((P\cap\{\pi x \le \pi_0\}) \cup (P \cap \{\pi x \ge \pi_0+1\})) \\
=& \bigcap_{(\pi,\pi_0)\in\mathbb{Z}^{n+1}} \mathrm{conv}(((P\cap\{\pi x \le \pi_0\}) \cup (P \cap \{\pi x \ge \pi_0+1\}))\cap F) \\
=& \bigcap_{(\pi,\pi_0)\in\mathbb{Z}^{n+1}} \mathrm{conv}((P\cap F\cap\{\pi x \le \pi_0\}) \cup (P \cap F\cap \{\pi x \ge \pi_0+1\})) \\
=& \mathrm{SC}(P\cap F),
\end{aligned}
$$

where the second equation holds because P is contained in the half-space defined by F (see e.g., [Cornuéjols, 2008, Lemma 2]). An implicit proof of this property is also given in [Cook et al., 1990, p. 165].

Property 4 follows from the fact that $\mathrm{SC}(P) \subseteq P'$. Property 5 is a simple exercise if one uses that $\tau_i(\{x : Ax \le b\}) = \{x : \tilde{A}x \le \tilde{b}\}$ where $\tilde{A}$ is identical to A except for column i in which all signs are reversed, and $\tilde{b}$ is equal to b minus column i of A. It remains to show Property 6:

Lemma 2.8. *The split-cut operator satisfies Property 6.*

Proof. Let $P = \{x \in [0,1]^n : Ax \le b\}$ be a polytope with $A \in \mathbb{Z}^{m\times n}, b \in \mathbb{Z}^n$, and let $cx \le \delta$ be valid for $\mathrm{SC}(P)$ with $(c,\delta) \in \mathbb{Z}^{n+1}$. Without loss of generality, we may assume that there exists an inequality $\pi x \le \pi_0$ with $(\pi,\pi_0) \in \mathbb{Z}^{n+1}$ so that $cx \le \delta$ is valid for $\mathrm{conv}(P^+ \cup P^-)$ where $P^- := P \cap \{\pi x \le \pi_0\}$ and $P^+ := P \cap \{\pi x \ge \pi_0 + 1\}$. We have to show that there exists a polytope $Q := \{x \in [0,1]^n : \tilde{A}x \le \tilde{b}\}$ where $\tilde{A}x \le \tilde{b}$ is a subsystem of $Ax \le b$ with a polynomial number of inequalities, so that $cx \le \delta$ is valid for $\mathrm{SC}(Q)$.

Because $cx \le \delta$ is valid for P^- there exist $\lambda \in \mathbb{R}^m_+$ and $\tau \in \mathbb{R}_+$ such that $c = \lambda A + \tau\pi$ and $\lambda b + \tau\pi_0 \le \delta$. Using Carathéodory's Theorem, we may assume that $|\mathrm{supp}(\lambda)| \le n$. Let $\tilde{A}^- x \le \tilde{b}^-$ denote the subsystem induced by the non-zero entries of λ. Similarly, let $\tilde{A}^+ x \le \tilde{b}^+$ denote the corresponding induced subsystem for P^+. We define $Q := \{x \in [0,1]^n : \tilde{A}^- x \le \tilde{b}^-, \tilde{A}^+ x \le \tilde{b}^+\}$. Clearly, $P \subseteq Q$ and Q is given by at most $2n$ inequalities. Let Q^+ and Q^- be defined similar to P^+ and P^-. Because $cx \le \delta$ is valid for Q^+ and Q^-, it follows that $cx \le \delta$ is valid for $\mathrm{SC}(Q)$, which completes the proof. □

3 Universal Upper Bounds.

We will now show that there is a natural upper bound on $\mathrm{rk}(P)$ for any admissible cutting-plane procedure M whenever $P_I = \emptyset$, and that this upper bound is attained if and only if the rank of $P \cap F$ is maximal for all faces F of $[0,1]^n$. The proof of the first result is almost identical to that for the Gomory-Chvátal procedure [Bockmayr et al., 1999, Lemma 3].

Theorem 3.1. *Let M be an admissible cutting-plane procedure and let $P \subseteq [0,1]^n$ be a polytope of dimension d with $P_I = \emptyset$. If $d = 0$, then $M(P) = \emptyset$. If $d > 0$, then $rk(P) \le d$.*

Proof. If $d = 0$, there exists an index i such that $x_i < 1$ and $x_i > 0$ are valid for P. Property 4 of Definition 2.1 implies that $x_i \le 0$ and $x_i \ge 1$ are valid for $M(P)$ and, therefore, $M(P) = P_I = \emptyset$. If $d = 1$, P is the convex hull of two points $y, z \in [0,1]^n$, $y \ne z$. Since $P_I = \emptyset$, there exists an index i such that $0 < y_i < 1$. W.l.o.g., $y_i \le z_i$. Then $x_i \ge y_i$ is valid for P and $M(P) \subseteq \{x_i = 1\}$, by Property 4. There also exists an index j with $0 < z_j < 1$. W.l.o.g., $y_j \le z_j$. Using similar arguments as before, we obtain $M(P) \subseteq \{x_j = 0\}$. As P is a line between y and z, it follows that all points x in the relative interior of P satisfy $0 < x_i < 1$ and $0 < x_j < 1$. Consequently, $P \cap \{x_i = 1\} \cap \{x_j = 0\} = \emptyset$, and, therefore, $M(P) = \emptyset$. We now proceed by induction on n and d. If P is contained in $\{x_n = 0\}$ or in $\{x_n = 1\}$, then the result follows by induction on n. Otherwise, the dimension of $P \cap \{x_n = 0\}$ and that of $P \cap \{x_n = 1\}$ is strictly smaller than

d. By induction hypothesis and repeated application of Property 3, we get $M^{(d-1)}(P) \cap \{x_n = 0\} = M^{(d-1)}(P \cap \{x_n = 0\}) = \emptyset$ and $M^{(d-1)}(P) \cap \{x_n = 1\} = M^{(d-1)}(P \cap \{x_n = 1\}) = \emptyset$. Hence, $x_n < 1$ and $x_n > 0$ are valid for $M^{d-1}(P)$, and Property 4 yields $M^{(d)}(P) = \emptyset$. □

The following lemma, which is generalization of [Pokutta and Schulz, 2011, Theorem 3.7], states that $\mathrm{rk}(P)$ is "sandwiched" between the largest rank of P intersected with a facet of the 0/1-cube and that number plus one.

Lemma 3.2. *Let M be an admissible cutting-plane procedure, and let $P \subseteq [0,1]^n$ be a polytope with $P_I = \emptyset$. Then $k \leq rk(P) \leq k+1$, where $k = \max_{(i,l)\in[n]\times\{0,1\}} rk(P \cap \{x_i = l\})$. Moreover, if there exist $i \in [n]$ and $l \in \{0,1\}$ such that $rk(P \cap \{x_i = l\}) < k$, then $rk(P) = k$.*

Proof. Clearly, $k \leq \mathrm{rk}(P)$, by Lemma 2.3. For the right-hand side of the inequality, observe that $M^k(P) \cap \{x_i = l\} = M^k(P \cap \{x_i = l\}) = \emptyset$, by Property 3 of Definition 2.1. It follows that $x_i < 1$ and $x_i > 0$ are valid for $M^k(P)$ for all $i \in [n]$. Hence $x_i \leq 0$ and $x_i \geq 1$ are valid for $M^{k+1}(P)$ for all $i \in [n]$, and we may deduce $M^{k+1}(P) = \emptyset$, i.e., $\mathrm{rk}(P) \leq k+1$. It remains to prove that $\mathrm{rk}(P) = k$ if there exist $i \in [n]$ and $l \in \{0,1\}$ such that $\mathrm{rk}(P \cap \{x_i = l\}) =: h < k$. Without loss of generality, assume that $l = 1$; otherwise apply the corresponding coordinate flip. Then $M^h(P) \cap \{x_i = l\} = \emptyset$ and, therefore, $x_i < 1$ is valid for $M^h(P)$. Because $h < k$ we can derive that $x_i \leq 0$ is valid for $M^k(P)$. It follows now that $M^k(P) = M^k(P) \cap \{x_i = 0\} = M^k(P \cap \{x_i = 0\}) = \emptyset$, which implies $\mathrm{rk}(P) \leq k$; the claim follows. □

Interestingly, one can show that $\mathrm{rk}(P \cap \{x_i = l\}) = k$ for all $(i,l) \in [n] \times \{0,1\}$ is not sufficient for $\mathrm{rk}(P) = k+1$. Lemma 3.2 has the following corollary:

Corollary 3.3. *Let M be an admissible cutting-plane procedure and let $P \subseteq [0,1]^n$ be a polytope with $P_I = \emptyset$. Then $rk(P) = n$ if and only if $rk(P \cap F) = k$ for all k-dimensional faces F of $[0,1]^n$ with $1 \leq k \leq n$.*

Proof. One direction follows by induction from Lemma 3.2, and the other one is trivial. □

4 Polytopes with Maximal Rank.

We will now study polytopes $P \subseteq [0,1]^n$ with $P_I = \emptyset$ and $\mathrm{rk}_M(P) = n$, where M is an admissible cutting-plane procedure. We start with the two-dimensional case, which serves as the base for the general case. We use e to denote the all-ones vector of appropriate dimension. For a face F of the 0/1-cube, we define $\frac{1}{2}e^F$ to be fixed to 0 or 1 according to F, and to 1/2 on all other coordinates. $\mathrm{Int}(P)$ is the interior of P, and $\mathrm{RInt}_F(P)$ is the interior of $\varphi_F(P)$, where F is a face of the 0/1-cube and φ_F is the canonical projection that projects out the coordinates that have been fixed by F. We use F_k to denote the set of all n-dimensional points that have exactly k coordinates equal to 1/2, and the other $n-k$ coordinates are 0 or 1. The polytope B_n is defined as

$$B_n := \left\{ x \in [0,1]^n : \sum_{i\in S} x_i + \sum_{i\in[n]\setminus S} (1-x_i) \geq 1 \quad \text{for all } S \subseteq [n] \right\}$$

for all $n \in \mathbb{Z}_+$. Note that B_n contains no integer points, and if F is a k-dimensional face of $[0,1]^n$, then $B_n \cap F \cong B_k$. The following theorem is a straight-forward generalization of [Pokutta and Schulz, 2011, Theorem 3.11 and Lemma 3.12].

Theorem 4.1. *Let M be admissible and let $P \subseteq [0,1]^2$ be a polytope with $P_I = \emptyset$ so that $rk(P) = 2$. Then*

1. *$\frac{1}{2}e \in Int(P)$, and*

2. *$P \cap \{x_i = l\} \neq \emptyset$ for all $(i,l) \in [2] \times \{0,1\}$.*

Proof. The second part is a direct consequence of Lemma 3.2. We prove Part (1) by contradiction. Let $P \cap \{x_i = l\} \neq \emptyset$ for all $(i,l) \in [2] \times \{0,1\}$ and suppose that $\frac{1}{2}e \notin \text{Int}(P)$. Then there exist $\tilde{x} \in P \cap \{x_i = l\}$ with $(i,l) \in [2] \times \{0,1\}$ and $a \in \mathbb{Z}^2$ such that $ax \leq a(\frac{1}{2}e)$ is valid for P and $a\tilde{x} = a(\frac{1}{2}e)$, i.e., $a\tilde{x} = a(\frac{1}{2}e)$ is the hyperplane defined by the points $\tilde{x}$ and $\frac{1}{2}e$. Without loss of generality we can assume that $i = 1$ and $l = 0$ (otherwise we can apply coordinate permutations and flips). Then $\tilde{x}$ is of the form $\tilde{x} = (0,c)$ with $c \in (0,1)$, as $P_I = \emptyset$. It is easy to see that the hyperplanes $ax = a(\frac{1}{2}e)$ and $x_1 = 1$ intersect in the point $\tilde{y} = (1, 1-c)$. Note that $\tilde{y}$ is not necessarily in P. If we now maximize x_2 over P, we get $\max_{x \in P} x_2 \leq \max_{x \in Q} x_2 = \max_{x \in \{(1,1-c),(0,c)\}} x_2 < 1$ with $P \subseteq Q$, where $Q = [0,1]^2 \cap \{ax \leq a(\frac{1}{2}e)\}$, contradicting our assumption that $P \cap \{x_i = l\} \neq \emptyset$ for all $(i,l) \in [2] \times \{0,1\}$. □

If M is the Gomory-Chvátal operator, P', we can obtain the stronger statement that $P' = \{\frac{1}{2}e\}$. In general, $\frac{1}{2}e \notin M(P)$. More specifically, Pokutta and Schulz [2011] contains the following characterization of polytopes $P \subseteq [0,1]^n$ with $P_I = \emptyset$ and maximal Gomory-Chvátal rank.

Theorem 4.2. *Let $P \subseteq [0,1]^n$ be a polytope with $P_I = \emptyset$. Then the following are equivalent:*

1. $\text{rk}_{GC}(P) = n$.
2. $B_n = P'$.
3. *$F \cap P \neq \emptyset$ for all one-dimensional faces F of $[0,1]^n$.*

We now prove a similar, but slightly weaker version for generic admissible cutting-plane procedures. This weakening is a direct consequence of the fact that, in general, $\frac{1}{2}e \in \text{Int}(P)$ fails to imply $\frac{1}{2}e \in M(P)$.

Theorem 4.3. *Let M be admissible and let $P \subseteq [0,1]^n$ be a polytope with $P_I = \emptyset$ and $rk(P) = n$. Then*

1. *$P \cap F \neq \emptyset$ for all one-dimensional faces F of $[0,1]^n$.*
2. *For $v \in F_2$ and $F := \bigcap_{(i,l) \in [n] \times \{0,1\} : v_i = l, l \neq \frac{1}{2}} \{x_i = l\}$, $v \in RInt_F(P)$.*
3. *$B_n \subseteq P$.*

Proof. By Corollary 3.3 we have that $\text{rk}(P \cap F) = 1$ for all one-dimensional faces F of $[0,1]^n$ and, therefore, $P \cap F \neq \emptyset$, so the first statement follows. Theorem 4.1 implies that every two-dimensional face G of $[0,1]^n$ contains a "local copy" of $\frac{1}{2}e$ in its relative interior. More specifically, we have $\frac{1}{2}e^G \in \text{RInt}_G(P)$ by Theorem 4.1. Moreover, $F_2 = \{\frac{1}{2}e^I : I \subseteq [n] \times \{0,1\}, |I| = n-2\}$, and we obtain that $F_2 \subseteq P$ and, thus, $B_n = \text{conv}(F_2) \subseteq P$. □

We also obtain a result similar to [Eisenbrand and Schulz, 2003, Proposition 2.4] (for the Gomory-Chvátal procedure) and [Cook and Dash, 2001, Proposition 2.5] (for lift-and-project), bounding from below the number of inequalities that are necessary to describe a polytope $P \subseteq [0,1]^n$, $P_I = \emptyset$, with maximal rank.

Corollary 4.4. *Let M be admissible and let $P \subseteq [0,1]^n$ be a polytope with $P_I = \emptyset$ and $\text{rk}_M(P) = n$. Then every linear description of P needs at least 2^n inequalities.*

Proof. By Theorem 4.3, we have that $P \cap F \neq \emptyset$ for all one-dimensional faces F of $[0,1]^n$. Suppose there exists an inequality $cx \leq \delta$ valid for P such that this inequality cuts off more than one 0/1-point. Then it has to cut off a one-dimensional face F of $[0,1]^n$, which implies $P \cap F = \emptyset$. This is a contradiction to $P \cap F \neq \emptyset$ for all one-dimensional faces F of $[0,1]^n$. Thus $cx \leq \delta$ can only cut off at most one 0/1-point at a time. As $[0,1]^n$ contains 2^n 0/1-points, the proof is complete. □

We conclude this section with a few properties of B_n, which we will need later.

Lemma 4.5. *Let M be admissible. Then $rk(B_n) \leq n-1$.*

Proof. We prove something slightly stronger: $M^{k-1}(B_n) \cap F = \emptyset$ for all k-dimensional faces F of $[0,1]^n$ with $1 \leq k \leq n$. The proof is by induction on n and k. We first consider the case $n = 1$. Let F be a one-dimensional face of $[0,1]$. Observe that $B_1 \cap F = B_1$. As $B_1 = \emptyset$ it follows that $\text{rk}(B_n) = 0$. Now let $n \geq 2$ and $k = 1$. Then $B_n \cap F \cong B_1$ for all one-dimensional faces F of $[0,1]^n$ and, thus, $(B_n \cap F) = \emptyset$. Now let F be a k-dimensional face of $[0,1]^n$ with $k > 1$. Choose $(i,l) \in [n] \times \{0,1\}$ such that $F \cap \{x_i = l\} \subsetneq F$ for $l = 0$ and $l = 1$, and define $G_0 = F \cap \{x_i = 0\}$ and $G_1 = F \cap \{x_i = 1\}$. Then G_0 and G_1 are $(k-1)$-dimensional faces of $[0,1]^n$ and, by the induction hypothesis, the remark at the beginning of Section 4 and Property 3, it follows that $\emptyset = M^{k-2}(B_{k-1}) \cong M^{k-2}(B_n) \cap G_0 = M^{k-2}(B_n) \cap G_1$. Therefore, $x_i < 1$ and $x_i > 0$ are valid for $M^{k-2}(B_n) \cap F$ and, thus, $M^{k-1}(B_n) \cap F = \emptyset$. □

We obtain the following corollary from the proof:

Corollary 4.6. *$M^{k-1}(B_n) \cap F = \emptyset$ for all k-dimensional faces F of $[0,1]^n$.*

Next, we prove that a certain class of inequalities is valid for $M^{(k)}(B_n)$ whenever $M \in \{N_0, N\}$. The same class of inequalities can be shown to be valid for the corresponding rounds of the Gomory-Chvátal procedure (and in consequence for split cuts).

Lemma 4.7. *Let $M \in \{N_0, N\}$. The inequalities*

$$\sum_{i \in I} x_i + \sum_{i \in \tilde{I} \setminus I} (1 - x_i) \geq 1$$

with $I \subseteq \tilde{I} \subseteq [n]$ and $|\tilde{I}| = n - k$ are valid for $M^k(B_n)$.

Proof. The proof is by induction on k. Note that it suffices to prove the lemma for $M = N_0$. For $k = 0$, there is nothing to prove. So let $k > 0$, $I \subseteq \tilde{I} \subseteq [n]$ with $|\tilde{I}| = n - k$, and choose $j \in [n] \setminus \tilde{I}$. We want to show that the inequality $\sum_{i \in I} x_i + \sum_{i \in \tilde{I} \setminus I} (1 - x_i) \geq 1$ is valid for $M^k(B_n)$. Observe that the inequalities $\sum_{i \in I} x_i + \sum_{i \in \tilde{I} \setminus I} (1 - x_i) + x_j \geq 1$ and $\sum_{i \in I} x_i + \sum_{i \in \tilde{I} \setminus I} (1 - x_i) + (1 - x_j) \geq 1$ are valid for $M^{k-1}(B_n)$. Multiplying the first inequality with $1 - x_j$ and the second one with x_j, substituting $x_j^2 = x_j$, and adding them up yields $\sum_{i \in I} x_i + \sum_{i \in \tilde{I} \setminus I} (1 - x_i) \geq 1$. The claim follows. □

Lemma 4.7 (and its equivalent version for Gomory-Chvátal and split cuts) helps us to prove the following corollary:

Corollary 4.8. *Let $M \in \{N_0, N, GC, SC\}$. Then $M^{n-2}(B_n) = \{\frac{1}{2}e\}$.*

Proof. First note that $\frac{1}{2}e \in M^{n-2}(B_n)$, by Lemma 5.2 below. Suppose that there exists an $\tilde{x} \in M^{n-2}(B_n)$ with $\tilde{x} \neq \frac{1}{2}e$. Then there exists a coordinate $l \in [n]$ such that $\tilde{x}_l \neq \frac{1}{2}$. Choose $m \in [n]$ arbitrarily with $m \neq l$. Consider $\tilde{I} = \{l, m\}$ and let $I = \{i \in \tilde{I} : \tilde{x}_i < \frac{1}{2}\}$. Lemma 4.7 implies that $\sum_{i \in I} x_i + \sum_{i \in \tilde{I} \setminus I} (1 - x_i) \geq 1$ is valid for $M^{n-2}(B_n)$. On the other hand, $\sum_{i \in I} \tilde{x}_i + \sum_{i \in \tilde{I} \setminus I} (1 - \tilde{x}_i) < 1$. Therefore, $\tilde{x} \notin M^{n-2}(B_n)$ and the assertion follows. □

5 Implications for Gomory-Chvátal Cuts, Matrix Cuts and Split Cuts.

We immediately obtain the following corollary to Theorem 4.3 which shows that the Gomory-Chvátal procedure is, in some sense, weakest possible: Whenever the rank of some admissible cutting-plane procedure is maximal, then so is the Gomory-Chvátal rank. More precisely:

Corollary 5.1. *Let M be admissible and let $P \subseteq [0,1]^n$ be a polytope with $P_I = \emptyset$ and $\text{rk}_M(P) = n$. Then $\text{rk}_{GC}(P) = n$.*

Proof. By Theorem 4.3 (1) we have that $P \cap F \neq \emptyset$ for all one-dimensional faces F of $[0,1]^n$. With Theorem 4.2 we therefore obtain $\text{rk}_{GC}(P) = n$. $\square$

Note that Corollary 5.1 does not hold for polytopes $P \subseteq [0,1]^n$ with $P_I \neq \emptyset$: Let $P_n = \{x \in [0,1]^n : \sum_{i \in [n]} x_i \geq \frac{1}{2}\}$. Then $(P_n)_I = \{x \in [0,1]^n : \sum_{i \in [n]} x_i \geq 1\} \neq \emptyset$. In [Cook and Dash, 2001, Section 3] it was shown that $\text{rk}_{GC}(P_n) = 1$, but $\text{rk}_{N_0}(P_n) = n$.

We can also derive a slightly weaker relation between the rank of matrix cuts, split cuts and other admissible cutting-plane procedures. First we will establish lower bounds for the rank of B_n. The following result was proved in [Cook and Dash, 2001, Lemma 3.3].

Lemma 5.2. *Let $P \subseteq [0,1]^n$ be a polytope and let $F_k \subseteq P$. Then $F_{k+1} \subseteq N(P)$.*

The same is true for the Gomory-Chvátal procedure (see, e.g., [Chvátal et al., 1989, p.482, Lemma 7.2]) and the split cut operator (see, e.g., [Cornuéjols and Li, 2002a, Lemma 1]). This yields:

Lemma 5.3. *Let $M \in \{N_0, N, GC, SC\}$. Then $\text{rk}_M(B_n) = n - 1$.*

Proof. As $B_n = \text{conv}(F_2)$, Lemma 5.2 implies that $F_n \subseteq M^{(n-2)}(B_n)$, and, thus, $\text{rk}(B_n) \geq n-1$. Together with Lemma 4.5 it follows that $\text{rk}(B_n) = n - 1$. $\square$

We also obtain the following corollary that shows that the M-rank with $M \in \{N_0, N, SC\}$ is at least $n-1$ whenever it is n with respect to any other admissible cutting-plane procedure.

Corollary 5.4. *Let L be an admissible cutting-plane procedure, let $M \in \{N_0, N, SC\}$, and let $P \subseteq [0,1]^n$ be a polytope with $P_I = \emptyset$ and $\text{rk}_L(P) = n$. Then $\text{rk}_M(P) \geq n-1$ and, if P is half-integral, then $\text{rk}_M(P) = n$.*

Proof. If $\text{rk}_L(P) = n$, then $B_n \subseteq P$ by Theorem 4.3 and $\text{rk}_M(B_n) = n-1$ by Lemma 5.3. So the first part follows from Lemma 2.3. In order to prove the second part, observe that $P \cap F \neq \emptyset$ for all one-dimensional faces F of $[0,1]^n$. Thus $P \cap F \cong F_2$ for all two-dimensional faces F of $[0,1]^n$ and, by Lemma 5.2, $M(P) \cap F = \{\frac{1}{2}e^F\}$. Therefore, $B_n \subseteq M(P)$. The claim now follows from Lemma 5.3. $\square$

We will now consider the case in detail where $P \subseteq [0,1]^n$ is half-integral with $P_I = \emptyset$. The polytope $A_n \subseteq [0,1]^n$ was defined in Chvátal et al. [1989] as follows:

$$A_n := \Big\{x \in [0,1]^n : \sum_{i \in S} x_i + \sum_{i \in [n] \setminus S} (1 - x_i) \geq \frac{1}{2} \quad \text{for all } S \subseteq [n]\Big\}.$$

Lemma 5.5. *Let $P \subseteq [0,1]^n$ be a half-integral polytope with $P_I = \emptyset$. Then*

$$\text{rk}_{N_0}(P) = n \Leftrightarrow \text{rk}_N(P) = n \Leftrightarrow \text{rk}_{GC}(P) = n \Leftrightarrow \text{rk}_{SC}(P) = n \Leftrightarrow P = A_n.$$

Proof. It suffices to show that $P = A_n$ if and only if $\text{rk}_{GC}(P) = n$. Let $\text{rk}_{GC}(P) = n$. By Theorem 4.3 we have that $P \cap F \neq \emptyset$ for all one-dimensional faces F of $[0,1]^n$. Moreover, because $P_I = \emptyset$ and P half-integral, it follows that $P \cap F = \{\frac{1}{2}e^F\}$ and $P = A_n$. For the other direction, observe that if $P = A_n$, then $P \cap F \neq \emptyset$ for all one-dimensional faces F of $[0,1]^n$ and, by Theorem 4.2, we therefore have $\text{rk}_{GC}(P) = n$. $\square$

Hence, in the case of half-integral polytopes without integral points, there is exactly one polytope that realizes the maximal rank for the well-known cutting-plane procedures. Combining Corollary 5.1 and Lemma 5.5, we obtain:

Corollary 5.6. *Let $P \subseteq [0,1]^n$ be a half-integral polytope with $P_I = \emptyset$, and let M be an admissible cutting-plane procedure. Then $\text{rk}_M(P) = n$ implies $P = A_n$.*

For half-integral polytopes $P \subseteq [0,1]^2$ with $P_I = \emptyset$ the matrix cut operators, the split cut operator, and the Gomory-Chvátal procedure are actually identical.

Lemma 5.7. *Let $P \subseteq [0,1]^2$ be a half-integral polytope with $P_I = \emptyset$. Then $N_0(P) = N(P) = SC(P) = P'$.*

Proof. Let $M \in \{N_0, N, SC\}$. We have to distinguish three cases. If $\text{rk}_{GC}(P) = 0$, then $P = \emptyset$ and hence, trivially, $M(P) = P'$. If $\text{rk}_{GC}(P) = 2$, then, by Theorem 4.2, we have that $P \cap \{x_i = l\} \neq \emptyset$ for all $(i,l) \in [2] \times \{0,1\}$. It therefore follows that $P = \text{conv}(F_1)$ and $P' = \{\frac{1}{2}e\}$. Lemma 5.2 implies that $\frac{1}{2}e \in M(P)$, and together with Corollary 4.8 (note that $SC(P) \subseteq P'$) it follows that $\{\frac{1}{2}e\} = M(P)$. Therefore $M(P) = P'$ also in that case. Finally, let $\text{rk}_{GC}(P) = 1$. By Theorem 4.2 there exists $(i,l) \in [2] \times \{0,1\}$ such that $P \cap \{x_i = l\} = \emptyset$. Without loss of generality we can assume that $l = 1$; otherwise we apply coordinate flips. Thus $x_i < 1$ is valid for P and hence $x_i \leq 0$ is valid for P', i.e., $P' = P' \cap \{x_i = 0\} = \emptyset$. Similarly, we obtain that $x_i \leq 0$ is valid for $M(P)$ and thus $M(P) = M(P) \cap \{x_i = 0\} = \emptyset$, and the claim follows. □

Note that Lemma 5.5 and Lemma 5.7 are in strong contrast to the case where $P \subseteq [0,1]^n$ is a half-integral polytope with $P_I \neq \emptyset$: In the remark after Corollary 5.1, the polytope $P_n = \{x \in [0,1]^n : \sum_{i\in[n]} x_i \geq \frac{1}{2}\}$ has $\text{rk}_{GC}(P_n) = 1$, but $\text{rk}_{N_0}(P_n) = n$, as was shown in [Cook and Dash, 2001, Theorem 3.1]. On the other hand, the polytope $P = \text{conv}(\{(0,0),(1,0),(\frac{1}{2},1)\}) \subseteq [0,1]^2$ has $\text{rk}_{N_0}(P) = 1$, but $\text{rk}_{GC}(P) = 2$ as P is half-integral and $\frac{1}{2}e \in \text{Int}(P)$ (see [Pokutta and Schulz, 2011, Remark after Lemma 3.12]).

6 The Lower Bound.

We now establish a universal lower bound on the rank of admissible cutting-plane procedures. Our approach makes use of inequalities as certificates for non-membership:

Definition 6.1. *Let $cx \leq \delta$ with $c \in \mathbb{Z}^n$ and $\delta \in \mathbb{Z}$ be an inequality. The* violation set *$V(c,\delta) := \{x \in \{0,1\}^n : cx > \delta\}$ is the set of 0/1 points for which $cx \leq \delta$ serves as a certificate of infeasibility.*

The following observation is an essential building block in establishing the lower bound:

Lemma 6.2. *Let M be an admissible cutting-plane procedure, and let $P \subseteq [0,1]^n$ be a polytope. Let $cx \leq \delta$ with $c \in \mathbb{Z}^n$ and $\delta \in \mathbb{Z}$ be a valid inequality for $M(P)$ whose certificate of $M(P)$-validity depends only on $\{c_i x \leq \delta_i : i \in I\}$, where I is an index set and $c_i x \leq \delta_i$ with $c_i \in \mathbb{Z}^n$ and $\delta_i \in \mathbb{Z}$ is valid for P, for all $i \in I$. Then $V(c,\delta) \subseteq \bigcup_{i\in I} V(c_i,\delta_i)$.*

Proof. The proof is by contradiction. Suppose there is $x_0 \in \{0,1\}^n$ such that $x_0 \in V(c,\delta) \setminus \bigcup_{i\in I} V(c_i,\delta_i)$. We define $Q := [0,1]^n \cap \bigcap_{i\in I}\{x : c_i x \leq \delta_i\}$. Note that $x_0 \in Q_I$. On the other hand, by Property 6 of Definition 2.1, $cx \leq \delta$ is valid for $M(Q)$ as well. Thus $x_0 \notin M(Q)$ as $cx \leq \delta$ is valid for $M(Q)$ and $x_0 \in V(c,\delta)$. But then $Q_I \not\subseteq M(Q)$ and, therefore, M is not admissible, a contradiction. □

This lemma can be interpreted as follows: Taken together, a set of inequalities $c_i x \leq \delta_i$ certifies that a certain set of $0/1$ points is not contained in P. The cutting-plane procedures combines these inequalities into a new one, $cx \leq \delta$, that certifies that a (hopefully large) subset of the set of $0/1$ points is not contained in P. The fact that we will exploit in order to establish a lower bound is that an admissible cutting-plane procedure can access at most a polynomial number of inequalities in the derivation of a single new inequality. If we now had a polytope $P \subseteq [0,1]^n$ with $|V(a,\beta)|$ small for all inequalities $ax \leq \beta$ in a linear description of P, we could estimate how many rounds it takes to generate an inequality $cx \leq \delta$ so that $V(c,\delta) = \{0,1\}^n$. The following observation characterizes $P_I = \emptyset$ in terms of a certificate $V(c,\delta)$.

Lemma 6.3. *Let $P \subseteq [0,1]^n$ be a polytope. Then $P_I = \emptyset$ if and only if there exists an inequality $cx \leq \delta$ valid for P_I with $V(c,\delta) = \{0,1\}^n$.*

Proof. Clearly, if there exists an inequality $cx \leq \delta$ valid for P_I with $V(c,\delta) = \{0,1\}^n$, then $P_I = \emptyset$. For the other direction, $ex \leq -1$ is valid for $P_I = \emptyset$, and $V(e,-1) = \{0,1\}^n$. □

Next we establish an upper bound on the growth of the size of $V(c,\delta)$.

Lemma 6.4. *Let M be an admissible cutting-plane procedure with verification degree $p(n)$. Further, let $P = \{x \in \mathbb{R}^n : Ax \leq b\} \subseteq [0,1]^n$ be a polytope with $P_I = \emptyset$ and define $k := \max_{i\in[m]} |V(a_i,b_i)|$. If $cx \leq \delta$ has been derived by M from $Ax \leq b$ within ℓ rounds, then $|V(c,\delta)| \leq p(n)^\ell k$.*

Proof. The proof is by induction on the number ℓ of rounds. For $\ell = 1$, $cx \leq \delta$ can be derived with the help of at most $p(n)$ inequalities $\{a_i x \leq b_i\}$ from the original system $Ax \leq b$. By Lemma 6.2, it follows that $V(c,\delta) \subseteq \bigcup_i V(a_i,b_i)$ and, thus, $|V(c,\delta)| \leq \sum_i |V(a_i,b_i)| \leq p(n)k$. Now consider the case $\ell > 1$. The derivation of $cx \leq \delta$ involves at most $p(n)$ inequalities $\{c_i x \leq \delta_i\}$ each of which has been derived in at most $\ell - 1$ rounds. By Lemma 6.2, it follows that $V(c,\delta) \subseteq \bigcup_i V(c_i,\delta_i)$ and, therefore, $|V(c,\delta)| \leq \sum_i |V(c_i,\delta_i)| \leq p(n)(p(n)^{\ell-1}k) \leq p(n)^\ell k$. □

We are ready to prove a universal lower bound on the rank of admissible cutting-plane procedures:

Theorem 6.5. *Let $k \in \mathbb{Z}_+$ be fixed, and let M be an admissible cutting-plane procedure with verification degree $p(n)$. Further, let $P = \{x \in \mathbb{R}^n : Ax \leq b\} \subseteq [0,1]^n$ be a polytope with $P_I = \emptyset$ such that $P \cap F \neq \emptyset$ for all k-dimensional faces F of $[0,1]^n$. Then, for all $n \geq 2k$, $rk(P) \in \Omega(n/\log n)$.*

Proof. We will first show that if $P \cap F \neq \emptyset$ for all k-dimensional faces F of $[0,1]^n$ and $cx \leq \delta$ is a valid inequality for P, then $cx \leq \delta$ can cut off at most $(2n)^k$ 0/1 points, i.e., $|V(c,\delta)| \leq (2n)^k$. Without loss of generality, we may assume that $c \geq 0$ and that $c_i \geq c_j$ whenever $i \leq j$; otherwise we can apply coordinate flips and variable permutations. Define $l := \min\{j \in [n] : \sum_{i=1}^{j} c_i > \delta\}$. Suppose $l \leq n-k$. Define $F := \bigcap_{i=1}^{n-k}\{x_i = 1\}$. Observe that $\dim(F) = k$ and $cx > \delta$ for all $x \in F$ as $l \leq n-k$. Thus $P \cap F = \emptyset$, which contradicts our assumption that $P \cap F \neq \emptyset$ for all k-dimensional faces F of $[0,1]^n$. Therefore, $k \geq n - l + 1$. By the choice of l, every 0/1 point x_0 that is cut off by $cx \leq \delta$ has to have at least l coordinates equal to 1. The number ζ of 0/1 vectors of dimension n with this property is bounded by

$$\zeta \leq 2^{n-l}\binom{n}{l} \leq 2^k\binom{n}{n-l} \leq 2^k\binom{n}{k-1} \leq 2^k n^k \leq (2n)^k.$$

Note that the third inequality holds because $k \leq n/2$, by assumption. It follows that $|V(c,\delta)| \leq (2n)^k$.

As we have seen, any inequality $\pi x \leq \pi_0$ that is valid for P can cut off at most $(2n)^k$ 0/1 points. In order to prove that $P_I = \emptyset$, we have to derive an infeasibility certificate $cx \leq \delta$ with $V(c,\delta) = \{0,1\}^n$, by Lemma 6.3. Thus, $|V(c,\delta)| = 2^n$ is a necessary condition for $cx \leq \delta$ to be such a certificate. If $cx \leq \delta$ is derived in ℓ rounds by M from $Ax \leq b$ then, by Lemma 6.4, we have that $|V(c,\delta)| \leq p(n)^\ell (2n)^k$. Hence, $\ell \in \Omega(n/\log n)$ and, therefore, $\text{rk}(P) \in \Omega(n/\log n)$. □

Note that the result can be easily generalized to non-fixed k if k is growing slowly enough as a function of n, e.g., $k \in O(\log n)$. Theorem 6.5 implies that, in contrast to the case where $P_I \neq \emptyset$, when dealing with polytopes with $P_I = \emptyset$, the property of having high/maximal rank is *universal*, i.e., it is a property of the polytope and not the particular cutting-plane procedure used. We immediately obtain the following corollary:

Corollary 6.6. *Let M be admissible. Then $rk(B_n) \in \Omega(n/\log n)$ and $rk(A_n) \in \Omega(n/\log n)$.*

Proof. It is sufficient to observe that $A_n \cap F \neq \emptyset$ for all one-dimensional faces F of $[0,1]^n$ and $B_n \cap F \neq \emptyset$ for all two-dimensional faces F of $[0,1]^n$. The claim then follows from Theorem 6.5. □

For $k \in \mathbb{N}$, it is also easy to see that $\frac{1}{2}e \in M^k(B_n)$ whenever $M^k(B_n) \neq \emptyset$. This is true because B_n is symmetric with respect to coordinate flips and coordinate permutations and, therefore, $\frac{1}{2}e$ is obtained by averaging over all points in $M^k(B_n)$. The next corollary links all admissible cutting-plane procedures in terms of maximal rank:

Corollary 6.7. *Let $P \subseteq [0,1]^n$ be a polytope with $P_I = \emptyset$ and let L, M be two admissible cutting-plane procedures. If $\mathrm{rk}_L(P) = n$, then $\mathrm{rk}_M(P) \in \Omega(n/\log n)$.*

Proof. If $\mathrm{rk}_L(P) = n$, then, by Theorem 4.3, we have that $P \cap F \neq \emptyset$ for all one-dimensional faces F of $[0,1]^n$. The claim now follows from Theorem 6.5. □

In this sense, modulo log-factors, all admissible cutting-plane procedures are of similar strength, at least as far as proving 0/1-infeasibility of a system of linear inequalities is concerned. Using a slight modification of Theorem 6.5, we are able to prove an inverse of Corollary 4.4. For this, we say that an inequality description $Ax \leq b$ with $A \in \mathbb{Z}^{m\times n}$, $b \in \mathbb{Z}^n$, of P is *P_I-non-redundant* if for every $i \in [m]$ we have $V(a_i, b_i) \not\subseteq \bigcup_{j\in[m]\setminus\{i\}} V(a_j, b_j)$, i.e., removing one of the inequalities changes the integral hull.

Lemma 6.8. *Let M be an admissible cutting-plane procedure, and let $P \subseteq [0,1]^n$ be a polytope with $P_I = \emptyset$ such that any P_I-non-redundant inequality description of P needs at least an exponential number of inequalities (in n). Then $\mathrm{rk}_M(P) \in \Omega(n/\log n)$.*

Proof. Lemma 6.2 implies that each of the exponentially many inequalities has to be used in the derivation of the contradictory inequality $ex \leq -1$: Dropping one of those inequalities would imply that there exists $x \in \{0,1\}^n$ not cut off by any other inequality. Each application combines at most a polynomial number of inequalities. The result follows, *mutatis mutandis*, using the argument from the proof of Theorem 6.5. □

7 A Rank-Optimal Cutting-Plane Procedure.

As we have mentioned before, traditional convexification procedures such as Gomory-Chvátal or lift-and-project have worst-case rank n, and one might, therefore, wonder if the lower bound of $\Omega(n/\log n)$ in Theorem 6.5 is tight. We will now construct a new, admissible cutting-plane procedure that is asymptotically optimal.

Definition 7.1. *Let $P \subseteq [0,1]^n$ be a polytope. The cutting-plane procedure "+" is defined as follows. Let $\tilde{J} \subseteq [n]$ with $|\tilde{J}| \leq \lceil \log n \rceil$ and let $I \subseteq \tilde{I} \subseteq [n]$ with $\tilde{I} \cap \tilde{J} = \emptyset$. If there exists $\epsilon > 0$ such that*

$$\sum_{i\in I} x_i + \sum_{i\in \tilde{I}\setminus I} (1 - x_i) + \sum_{i\in J} x_i + \sum_{i\in \tilde{J}\setminus J} (1 - x_i) \geq \epsilon$$

is valid for P for all $J \subseteq \tilde{J}$, then we add the inequality $\sum_{i\in I} x_i + \sum_{i\in \tilde{I}\setminus I}(1 - x_i) \geq 1$, and we call this inequality a +-cut. Furthermore, P^+ is the set of points in P that satisfy all +-cuts.

Let us first prove that +-cuts are indeed valid; i.e., they do not cut off any integer points contained in P. At the same time, the proof of the following lemma helps to establish that the +-operator satisfies Property 6 of Definition 2.1.

Lemma 7.2. *Let $P = \{x : Ax \leq b\} \subseteq [0,1]^n$ be a polytope. Every +-cut is valid for P_I.*

Proof. For $\tilde{J} \subseteq [n]$ with $|\tilde{J}| \leq \lceil \log(n) \rceil$ and $I \subseteq \tilde{I} \subseteq [n]$ with $\tilde{I} \cap \tilde{J} = \emptyset$, let $\sum_{i\in I} x_i + \sum_{i\in \tilde{I}\setminus I}(1 - x_i) \geq 1$ be the corresponding +-cut. With Farkas' Lemma and Carathéodory's Theorem, one can identify a subsystem of $Ax \leq b$ of size polynomial in n that can be used to verify that

$\sum_{i\in I} x_i + \sum_{i\in \tilde{I}\setminus I}(1-x_i) + \sum_{i\in J} x_i + \sum_{i\in \tilde{J}\setminus J}(1-x_i) \geq \epsilon$ is valid for P, for all $J \subseteq \tilde{J}$. Note that there are at most $2^{\lceil \log n\rceil} \in \mathrm{O}(n)$ of these initial inequalities.

Now we round up all right-hand sides to 1, which leaves us with inequalities that are valid for P_I. By induction, we can verify that

$$\sum_{i\in I} x_i + \sum_{i\in \tilde{I}\setminus I}(1-x_i) + \sum_{i\in J} x_i + \sum_{i\in J_0\setminus J}(1-x_i) \geq 1$$

is valid for P_I with $J_0 = \tilde{J} \setminus \{i_0\}$, $i_0 \in \tilde{J}$, and $J \subseteq J_0$. For, consider

$$\begin{array}{rl} & \frac{1}{2}\Big(\sum_{i\in I} x_i + \sum_{i\in \tilde{I}\setminus I}(1-x_i) + \sum_{i\in J} x_i + \sum_{i\in J_0\setminus J}(1-x_i) + x_{i_0} \ \geq \ 1\Big) \\ + & \frac{1}{2}\Big(\sum_{i\in I} x_i + \sum_{i\in \tilde{I}\setminus I}(1-x_i) + \sum_{i\in J} x_i + \sum_{i\in J_0\setminus J}(1-x_i) + (1-x_{i_0}) \ \geq \ 1\Big) \\ \hline & \sum_{i\in I} x_i + \sum_{i\in \tilde{I}\setminus I}(1-x_i) + \sum_{i\in J} x_i + \sum_{i\in J_0\setminus J}(1-x_i) \ \geq \ \frac{1}{2} \end{array}$$

We can again round up the right-hand side and iteratively repeat this process until $|J_0| = 0$. □

The $+$-operator is indeed admissible. The proof of Lemma 7.2 yields Property 6. It is straighforward to establish Properties 1, 2, 4, and 5. It remains to prove Property 3. Let F be a k-dimensional face of $[0,1]^n$, and let $P \subseteq [0,1]^n$ be a polytope. Without loss of generality, we can assume that F fixes the last $n-k$ coordinates to 0. Clearly, $(P\cap F)^+ \subseteq P^+ \cap F$. For the other direction, let $\sum_{i\in I} x_i + \sum_{i\in \tilde{I}\setminus I}(1-x_i) \geq 1$ be a $+$-cut valid for $(P\cap F)^+$ with $I \subseteq \tilde{I} \subseteq [n]$. Then there exists $\epsilon > 0$ such that

$$\sum_{i\in I} x_i + \sum_{i\in \tilde{I}\setminus I}(1-x_i) + \sum_{i\in J} x_i + \sum_{i\in \tilde{J}\setminus J}(1-x_i) \geq \epsilon$$

is valid for $P\cap F$, with $J \subseteq \tilde{J} \subseteq [n]$, $|\tilde{J}| \leq \lceil \log(n) \rceil$, and $\tilde{I} \cap \tilde{J} = \emptyset$. By Farkas' Lemma, there exists $\tau \geq 1$ such that

$$\sum_{i\in I} x_i + \sum_{i\in \tilde{I}\setminus I}(1-x_i) + \sum_{i\in J} x_i + \sum_{i\in \tilde{J}\setminus J}(1-x_i) + \tau(\sum_{k+1\leq i\leq n} x_i) \geq \epsilon$$

with $J \subseteq \tilde{J} \subseteq [n]$, $|\tilde{J}| \leq \lceil \log(n) \rceil$ and $\tilde{I} \cap \tilde{J} = \emptyset$ is valid for P. Hence, so is the weaker inequality

$$\sum_{i\in I} x_i + \sum_{i\in \tilde{I}\setminus I}(1-x_i) + \sum_{i\in J} x_i + \sum_{i\in \tilde{J}\setminus J}(1-x_i) + \sum_{k+1\leq i\leq n} x_i \geq \frac{\epsilon}{\tau} \ .$$

By Definition 7.1,

$$\sum_{i\in I} x_i + \sum_{i\in \tilde{I}\setminus I}(1-x_i) + \sum_{k+1\leq i\leq n} x_i \geq 1$$

is valid for P^+. Restricting the inequality to the face F, we get that

$$\sum_{i\in I} x_i + \sum_{i\in \tilde{I}\setminus I}(1-x_i) \geq 1$$

is valid for $P^+ \cap F$; thus, Property 3 holds.

In the following we will show that, for any given polytope $P \subseteq [0,1]^n$ with $P_I = \emptyset$, $\mathrm{rk}_+(P) \in \mathrm{O}(n/\log n)$. This is a direct consequence of the following lemma; we use $P^{(k)}$ to denote the k-th closure of the $+$-operator.

Lemma 7.3. *Let $P \subseteq [0,1]^n$ be a polytope with $P_I = \emptyset$. Then $\sum_{i \in I} x_i + \sum_{i \in \tilde{I} \setminus I}(1 - x_i) \geq 1$ with $I \subseteq \tilde{I} \subseteq [n]$, $|\tilde{I}| \geq n - k\lceil \log n \rceil$, is valid for $P^{(k+1)}$.*

Proof. The proof is by induction on k. Let $k = 0$. As $P_I = \emptyset$, there exists $\epsilon > 0$ such that $\sum_{i \in I} x_i + \sum_{i \in \tilde{I} \setminus I}(1 - x_i) \geq \epsilon$ is valid for P for all $I \subseteq \tilde{I} = [n]$. Thus, $\sum_{i \in I} x_i + \sum_{i \in \tilde{I} \setminus I}(1 - x_i) \geq 1$ is valid for P^+ for all $I \subseteq \tilde{I} = [n]$. Consider now $k \geq 1$. Then $\sum_{i \in I} x_i + \sum_{i \in \tilde{I} \setminus I}(1 - x_i) \geq 1$ is valid for $P^{(k)}$ for all $I \subseteq \tilde{I} \subseteq [n]$ with $|\tilde{I}| \geq n - (k-1)\lceil \log n \rceil$. Now consider $I \subseteq \tilde{I} \subseteq [n]$ with $n - k\lceil \log n \rceil \leq |\tilde{I}| < n - (k-1)\lceil \log n \rceil$. Pick $\tilde{J} \subseteq [n]$ such that $|\tilde{J}| \leq \lceil \log(n) \rceil$, $\tilde{I} \cap \tilde{J} = \emptyset$, and $|\tilde{I} \cup \tilde{J}| \geq n - (k-1)\lceil \log n \rceil$. Then, for all $J \subseteq \tilde{J}$, we have that $\sum_{i \in I} x_i + \sum_{i \in \tilde{I} \setminus I}(1 - x_i) + \sum_{i \in J} x_i + \sum_{i \in \tilde{J} \setminus J}(1 - x_i) \geq 1$ is valid for $P^{(k)}$ by induction hypothesis. We may conclude that $\sum_{i \in I} x_i + \sum_{i \in \tilde{J} \setminus I}(1 - x_i) \geq 1$ is valid for $P^{(k+1)}$ by Definition 7.1. □

We are ready to establish an upper bound on the rank of the $+$-operator:

Theorem 7.4. *Let $P \subseteq [0,1]^n$ be a polytope with $P_I = \emptyset$. Then $\mathrm{rk}_+(P) \in O(n/\log n)$.*

Proof. It suffices to derive the inequalities $x_i \geq 1$ and $x_i \leq 0$. By Lemma 7.3 we have that $\sum_{i \in I} x_i + \sum_{i \in \tilde{I} \setminus I}(1 - x_i) \geq 1$ with $I \subseteq \tilde{I} \subseteq [n]$, $|\tilde{I}| \geq n - k\lceil \log n \rceil$, is valid for $P^{(k+1)}$. Thus, $\sum_{i \in I} x_i + \sum_{i \in \tilde{I} \setminus I}(1 - x_i) \geq 1$ with $I \subseteq \tilde{I} = \{i\}$ is valid for $k \geq (n-1)/\lceil \log n \rceil$. Observe that for $I = \{i\}$ and $I = \emptyset$ we obtain that $x_i \geq 1$ and $x_i \leq 0$ are valid for $P^{(k+1)}$, respectively. □

8 Universal Lower Bounds for the Traveling Salesman Polytope.

We will now provide a universal lower bound on the rank of the subtour elimination relaxation of the traveling salesman polytope for a large subclass of admissible cutting-plane procedures. More precisely, if M is admissible and additionally satisfies

7. **Substitution independence**: $\varphi_F(M(P \cap F)) = M(\varphi_F(P \cap F))$.

for all faces F of $[0,1]^n$, then we say that M is *strongly admissible*. Here, φ_F is the projection that eliminates all coordinates fixed by F. Note that all known operators, including N_0, N, GC, and SC, satisfy this property as well. We can show that the rank of the subtour elimination polytope is $\Omega(n/\log n)$, where n denotes the number of nodes, for any strongly admissible cutting-plane procedure M. For $n \in \mathbb{N}$, let $G = (V, E)$ be the complete graph on n vertices. Then the subtour elimination polytope $H_n \subseteq [0,1]^n$ of G is defined by the following inequalities:

$$\begin{aligned} x(\delta(\{v\}) &= 2 && \text{for all } v \in V \\ x(\delta(W)) &\geq 2 && \text{for all } \emptyset \subsetneq W \subsetneq V \\ x_e &\in [0,1] && \text{for all } e \in E. \end{aligned}$$

We obtain the following statement, which is similar to [Cook and Dash, 2001, Theorem 4.1].

Theorem 8.1. *Let M be strongly admissible. For $n \in \mathbb{N}$ and H_n as defined above, we have $rk_M(H_n) \in \Omega(n/\log n)$.*

Proof. In Chvátal et al. [1989] (see also [Cook and Dash, 2001, Theorem 4.1]) it was shown that there exists an embedding f consisting of coordinate flips and duplications so that $f(A_{\lfloor n/8 \rfloor}) \subseteq H_n$ and $f(\frac{1}{2}e) \notin (H_n)_I$. As M is strongly admissible, we have $f(\frac{1}{2}e) \in f(A^{(k)}_{\lfloor n/8 \rfloor}) = f(A_{\lfloor n/8 \rfloor})^{(k)} \subseteq (H_n)^{(k)}$, as long as $k \in \Omega(n/\log n)$ — by Corollary 6.6. The result follows. □

As the dimension of H_n is $\Theta(n^2)$, the lower bound provided in Theorem 8.1 is of the order of the square root of the dimension, modulo a log-factor. The same lower bound can be shown to hold for the subtour elimination relaxation of the asymmetric TSP problem, yielding results similar to the ones in Chvátal et al. [1989] and Cook and Dash [2001].

9 Conclusion.

We have introduced an abstract model for cutting-plane procedures that comprises many well-known procedures such as Gomory-Chvátal, Sherali-Adams (for fixed degree), Lovász-Schrijver, and split cuts. We have shown that there exists a family of polytopes $B_n \subseteq [0,1]^n$ with $(B_n)_I = \emptyset$ such that the rank of B_n in all these cutting-plane systems is $\Omega(n/\log n)$. Moreover, whenever the rank is maximal with respect to any one admissible cutting-plane procedure, then so it is, modulo a log-factor, for any other admissible cutting-plane procedure. It therefore makes sense to consider the *universal rank* of a polytope $P \subseteq [0,1]^n$ with $P_I = \emptyset$, defined as $\inf\{\mathrm{rk}_M(P) : M \text{ admissible}\}$. We believe that the lower bound on the worst-case universal rank can be strengthened if one considers strongly admissible cutting-plane proof systems as defined in Section 8. Loosely speaking, this condition ensures that embedding a polytope into a higher dimensional cube does not help to speed up the derivation of its integer hull. Interestingly, "$+$" is not strongly admissible. It can be shown that the worst-case universal rank of strongly admissible systems is in $\omega(n/\log n)$, and it stands to conjecture that the true lower bound may actually be $\Omega(n)$.

An admissible operator M does not guarantee convergence to the integral hull for polytopes $P \subseteq [0,1]^n$ with $P_I \neq \emptyset$. In this case, let $k \in \mathbb{N}$ be minimal with the property $M^k(P) = M^{k+1}(P)$. Then $Q := M^k(P)$ is *almost integral*, i.e., $Q \cap \{x_i = l\} = Q_I \cap \{x_i = l\}$ for all $(i,l) \in [n] \times \{0,1\}$. It can be shown that being able to derive the integral hull of an almost integral polytope is necessary and sufficient for M to converge to the integral hull of P in general. It is therefore conceivable to extend the abstract model of cutting-planes presented here to the case with $P_I \neq \emptyset$ using such an additional axiom.

We would also like to note that our technique readily applies to (almost) all closures arising from lattice-free body based multi-row cutting-plane procedures (see e.g., Andersen et al. [2007, 2009]). The non-trivial properties to verify are Property 3 and Property 6. Property 3 can be shown to hold for lattice-free body based cutting-plane procedures in general; the proof is almost identical to the one for split cuts. Property 6 follows from the fact that a valid inequality can be characterized as one that is valid over the disjunction that we obtain when removing a lattice-free body. For k-dimensional lattice-free bodies the disjunction involves at most 2^k polyhedra and hence, if k is fixed, Property 6 follows similar to split cuts.

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