Dual equilibrium problems: how a succession of aspiration points converges to an equilibrium

Felipe Garcia Moreno[*]
Department of Scientific Computing, CI
Federal University of Paraíba, Brazil
fgarcia@ci.ufpb.br

Paulo Roberto Oliveira[†]
Department of Systems Engineering and Computer Sciences, COPPE
Federal University of Rio de Janeiro, Brazil
poliveir@cos.ufrj.br

Antoine Soubeyran
GREQAM, AMSE, Aix-Marseille University, France
antoine.soubeyran@gmail.com

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Abstract

We consider an equilibrium problem defined on a convex set, whose cost bifunction may not be monotone. We show that this problem can be solved by the inexact partial proximal method with quasi distance. As an application, at the psychological level of behavioral dynamics, this paper shows two points: i) how a dual equilibrium problem offers a model of behavioral trap, "easy enough to reach, difficult enough to leave", which is both an aspiration point and an equilibrium, and ii) how a succession of aspiration points converges to an equilibrium, using worthwhile changes during the goal pursuit.

1 Introduction

Several problems arising in optimization, such as fixed-point problems, Nash equilibrium problems, minimax-saddle point problems, complementarity problems, variational inequalities, and others (see, for instance, Blum and Oettli

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(1994), Harker (1991), Patriksson (1993), Bianchi and Schaible (2004), Konnov (2005, 2006, 2007) and the references quoted therein) are formulated as the following equilibrium problem $EP(f, C)$:

$$\text{Find } x^* \in C \quad \text{such that} \quad f(x^*, y) \geq 0 \quad \forall\, y \in C. \tag{1}$$

where $C \subset \mathbf{R}^n$ is a nonempty closed convex set, and $f : C \times C \to \mathbf{R}$ is a bifunction that verifies

P1. $f(x, x) = 0$ for all $x \in C$;

P2. $f(x, \cdot\,) : C \to \mathbf{R}$ is pseudoconvex and lower semicontinuous for all $x \in C$;

P3. $f(\,\cdot\,, y) : C \to \mathbf{R}$ is upper hemicontinuous for all $y \in C$.

It is well known that $EP(f, C)$ is closely related to the dual equilibrium problem $DP(f, C)$:

$$\text{Find } y^* \in C \quad \text{such that} \quad f(x, y^*) \leq 0 \quad \forall\, x \in C. \tag{2}$$

We denote the solution set of $EP(f, C)$ and $DP(f, C)$ by $S(f, C)$ and $S^d(f, C)$ respectively.

In the literature, several approaches have already been considered for extending the proximal point method, see Martinet (1970) and Rockafellar (1976), to the realm of equilibrium problems. For example Moudafi (1999) further extended the proximal point method to nonmonotone equilibrium problems and Konnov (2003) further extended it for solving $EP(f, C)$ when f is a weakly monotone equilibrium bifunction. Together with the usual proximal point method, which involves the addition of a regularization term, its partial variants were also suggested and investigated for convex minimization problems in Bertsekas and Tseng (1994), Ha (1990) and Medhi and Ha (1996). These methods are motivated by the fact that they lead usually to simpler auxiliary problems than the usual proximal point method and are suitable for large-scale problems. In Konnov (2006) it was presented a partial proximal method for $EP(f, C)$ when both f is nonmonotone and $f(x, \cdot\,)$ is convex for all $x \in C$. Recently Iusem et al. (2009) have presented conditions for the existence of solution without the convexity condition.

At a mathematical level, we present a partial proximal method associated with a quasi distance for $EP(f, C)$, and we establish the convergence to a solution point of $EP(f, C)$. It uses a generalized regularization term, passing from the inner product $\langle \widetilde{x} - x^k, x - \widetilde{x} \rangle$ to $q^2(x^k, x) - q^2(\widetilde{x}, x) - q^2(x^k, \widetilde{x})$ where $q(x, x') \geq 0$ is a quasi distance. Therefore, this paper generalizes results of Konnov (2006).

Quasi distances (or quasi-metrics), see Definition 2.1, generalize the distances in the sense that they are not symmetric. However, a quasi distance is not necessarily a convex, a continuously differentiable and a coercive function. Then, if we use a quasi distance for the purpose of regularization we cannot proceed as in the Bregman case for the convergence analysis.

This generalization of equilibrium problems to quasi distances is important from the mathematical point of view because quasi-metric spaces represent an important area of research in Applied Mathematics. They have been extensively studied in the topology context; see, e.g., Künzi (1995) and Stojmirović (2004). Applications represent an impressive list of mathematical formulations of "similarity concepts", in a huge number of different disciplines like Economy, Management, Psychology, Biology, Linguistic, Artificial Intelligence, Computer Sciences and Complexity; see, e.g., Künzi et al. (2006) and Stojmirović (2004). For different examples in Psychophysics, (probability distance hypothesis), relativistic gravity in Physics, point set distances, Hausdorff distances, approach spaces, Thurston quasi metrics, path quasi distances on a directed graph, see Deza and Deza (2006). The choice of the quasi distance has its first motivation in the work of Romaguera and Sanchis (2003), they have observed that concepts of consumer choice, preference and utility functions, developed in economy, constitute efficient tools to explain properties of spaces naturally defined in computational theoretical science, including computer, domain and complexity theory. Due to the fact that these spaces are quasimetrizable but nonmetrizable, they did to develop their work in an environment of quasi-metric spaces. Now, similarly to the reasons of Romaguera and Sanchis (2003), we can affirm that spaces in behavioral sciences, potential games, among others, are essentially quasimetrizable but nonmetrizable, due to natural asymmetry of involved cost functions. This imposes the use of (non-symmetric) generalized distances, as Bregman or quasi distances, in those areas of research. Particularly, in Souza et al. (2010), a proximal learning model uses Bregman functions as capability costs. Recently in Garcia et al. (2011), we present a model for habit formation which examines the role of inertia and experience in repeated consumption processes to motivate the use of proximal methods with quasi distances. In Cruz Neto et al. (2012), we develop an alternated proximal method in Riemannian manifolds, with applications in Nash equilibrium and potential games models.

In this paper, from an Economic viewpoint, the main reason to generalize "dual equilibrium problems" to quasi metric space is the application to behavioral sciences. At the psychological level of behavioral dynamics, this paper offers a goal pursuit model with aspiration points, using worthwhile changes to be able to reach an equilibrium. This application can be done by using a new "variational approach" of the theories of stability and change, see Soubeyran (2009, 2010). In this approach inertia matters much and costs to change are modeled by a quasi distance because the cost to move from x to y is different from the cost to move from y to x, where x and y are different situations (states, actions, activities, etc.). In this behavioral context, the present paper shows two points: **i)?** how "dual equilibrium problems" model in a nice way "behavioral traps" as "easy enough to reach" and "difficult enough to leave" situations which are both "aspiration points" and "equilibrium points". This is really a striking result! In the annex, and in a more technical way, taking advantage of this variational point of view, we will show "how a succession of aspiration points converges to an equilibrium, using worthwhile changes during the goal pursuit". Hence we justify the title of this paper!

This paper is organized as follows: In the next section we recall some results needed in the sequel. In Section 3, the partial proximal method associated with a quasi distance is presented and its convergence on partially bounded sets is established under suitable assumptions. The case when the method is applied on unbounded sets is also considered. The last section 4 gives the application to how dual equilibrium problems model the concept of behavioral trap. The conclusions follow.

2 Preliminary Results

We now present the definition of quasi distance:

Definition 2.1. *(Stojmirović, 2004) Let X be a set. A mapping $q : X \times X \to \mathbf{R}_+$ is called a quasi distance if for all $x, y, z \in X$,*

1. $q(x, y) = q(y, x) = 0 \Leftrightarrow x = y$,
2. $q(x, z) \leq q(x, y) + q(y, z)$.

From the Definition 2.1, if q is also symmetric, that is, for all $x, y \in X$, $q(x, y) := q(y, x)$, then q is a distance. Therefore the concept of quasi distance generalizes the concept of a distance by lifting the symmetry condition. Next, we present examples of quasi distance.

Example 2.0.1. *(Künzi, 1993) Let $g : X \to \mathbf{R}_+$ be an injective function. Then for x, $y \in X$ $q_g(x, y) = \max\{g(x) - g(y), 0\}$, generates a quasi distance on X.*

Example 2.0.2. *For each $i = 1, ..., n$, we consider $c_i^-, c_i^+ > 0$ and $q_i : \mathbf{R} \times \mathbf{R} \to \mathbf{R}_+$ defined by*

$$q_i(x_i, y_i) = \begin{cases} c_i^+(y_i - x_i) & \text{if } y_i - x_i > 0, \\ c_i^-(x_i - y_i) & \text{if } y_i - x_i \leq 0, \end{cases}$$

which is a quasi distance on $\mathbf{R}$. Therefore $q(x, y) = \sum_{i=1}^n q_i(x_i, y_i)$ is a quasi distance on $\mathbf{R}^n$. On the other hand, for each $\overline{z} \in \mathbf{R}^n$ we have

$$q(x, \overline{z}) = \sum_{i=1}^n q_i(x_i, \overline{z}_i) = \sum_{i=1}^n \max\left\{c_i^+(\overline{z}_i - x_i), c_i^-(x_i - \overline{z}_i)\right\}, \quad x \in \mathbf{R}^n.$$

Thus $q(\,\cdot\,, \overline{z})$ is a convex function. By the same reasoning, $q(\overline{z}, \cdot\,)$ is convex.

Example 2.0.3. *Given a compact convex $C \subset \mathbf{R}^n$ such that the origin 0 is a interior point of C. Then, for x, $y \in \mathbf{R}^n$ we define $q(x, y) = \inf\{\alpha > 0 : y - x \in \alpha C\}$, thus q generates a quasi distance on $\mathbf{R}^n$.*

We note that, it is not true that a quasi distance is convex and coercive neither in the first argument nor in the second; see Garcia et al. (2011). We recall some basic notions which are required in the sequel.

Definition 2.2. *A function $h : C \to \mathbf{R}$ is said to be*

1. *pseudoconvex if for all x, $y \in C$ and all $t \in (0, 1)$ it holds that $h(z) \geq h(x)$ implies $h(y) \geq h(z)$, where $z := tx + (1 - t)y$.*

2. *upper hemicontinuous if it is upper semicontinuous on any segment contained on C.*

In Iusem et al. (2009), it is considered this definition of pseudoconvex function, and they mention that it is implied by the usual notion of pseudoconvexity, in the differentiable case.

Definition 2.3. *A bifunction $f : C \times C \to \mathbf{R}$ is said to be*

1. *monotone if, for each pair of points u, $v \in C$, we have $f(u, v)+f(v, u) \leq 0$;*

2. *pseudomonotone if, for each pair of points u, $v \in C$, we have $f(u, v) \geq 0$ implies $f(v, u) \leq 0$.*

Following Bertsekas and Tseng (1994), Ha (1990), Medhi and Ha (1996) and Konnov (2006), we fix the partition $\mathbf{R}^n = \mathbf{R}^{n_1} \times \mathbf{R}^{n_2}$, $n = n_1 + n_2$ of the initial space, i.e., each vector $u \in \mathbf{R}^n$ consists of two subvectors, $u = (x, y)$, where $x \in \mathbf{R}^{n_1}$ and $y \in \mathbf{R}^{n_2}$. It does not mean that C is a Cartesian product of the corresponding subsets. We consider the following sets:

$$\begin{aligned} X_C &:= \{x \in \mathbf{R}^{n_1} \, : \, \exists y \in \mathbf{R}^{n_2}, \ (x, y) \in C\}, \\ Y_C &:= \{y \in \mathbf{R}^{n_2} \, : \, \exists x \in \mathbf{R}^{n_1}, \ (x, y) \in C\}. \end{aligned}$$

Definition 2.4. *A bifunction $f : C \times C \to \mathbf{R}$ is said to be*

1. *strictly monotone with respect to x, if for each pair of points $u = (x, y)$ and $v = (x', y')$ in C such that $x \neq x'$, we have*

$$f(u, v) + f(v, u) < 0;$$

2. *strongly monotone with constant $\tau > 0$ with respect to x, if for each pair of points $u = (x, y)$ and $v = (x', y')$ in C, we have*

$$f(u, v) + f(v, u) \leq -\tau \|x - x'\|^2;$$

3. *weakly monotone with constant $\nu > 0$ with respect to x, if for each pair of points $u = (x, y)$ and $v = (x', y')$ in C, we have*

$$f(u, v) + f(v, u) \leq \nu \|x - x'\|^2.$$

A proof of following Lemma can be found in Iusem et al. (2009).

Lemma 2.1. *If* **P1** $-$ **P3** *hold, then $S^d(f, C) \subset S(f, C)$. Moreover if f is pseudomonotone, then $S^d(f, C) = S(f, C)$.*

Together with the aforementioned blanket assumptions, assumptions for the dual problem (2) and for the quasi distance $q : \mathbf{R}^{n_1} \times \mathbf{R}^{n_1} \to \mathbf{R}$

P4. $S^d(f, C) \neq \emptyset$;

P5. There exists $\alpha > 0$ and $\beta > 0$ such that for each pair of points $u = (x, y)$ and $v = (x', y')$ in C, we have

$$\alpha \|x - x'\| \leq q(x, x') \leq \beta \|x - x'\| \qquad \forall x,\, x' \in X_C.$$

We remark that, under assumption **P5**, the function $q(x, \cdot\,)$ is Lipschitzian for each $x \in \mathbf{R}^n$, see Garcia et al. (2011).

Proposition 2.1. *(Konnov, 2006) If f is strictly monotone with respect to x, then $S(f, C)$ has the following form*

$$S(f, C) = \{x^*\} \times Y^*. \tag{3}$$

In this paper, in order to guarantee that $EP(f, C)$ has solution, we use the following result due to Iusem et al. (2009).

Proposition 2.2. *If assumptions* **P1** $-$ **P3** *hold and f is pseudomonotone, then $EP(f, C)$ admits at least a solution if and only if $(\mathcal{H})$ is satisfied, where*

$(\mathcal{H})$ *For any sequence $\{x^n\}_{n\in\mathbb{N}} \subset C$ satisfying $\lim_{n\to\infty} \|x^n\| = +\infty$, there exists $u \in C$ and $n_0 \in \mathbb{N}$ such that $f(x^n, u) \leq 0$ $n \geq n_0$.* ■

The proof of the next technical result can be found, in Polyak (1987).

Lemma 2.2. *Let $\{v_k\}$, $\{\gamma_k\}$, and $\{\beta_k\}$ be nonnegative sequences of real numbers satisfying $v_{k+1} \leq (1 + \gamma_k)v_k + \beta_k$ and such that $\sum_{k=1}^{\infty} \beta_k < \infty$, $\sum_{k=1}^{\infty} \gamma_k < \infty$. Then, the sequence $\{v_k\}$ converges.*

3 The method and its convergence

Next we state the partial proximal point method with a quasi distance q for solving $EP(f, C)$ as follows:

Given a point $u^0 \in C$, construct a sequence $\{u^k\}_{k\in\mathbb{N}}$, $u^k = (x^k, y^k)$ in conformity with the following rules:

$$u^{k+1} \in C, \qquad q(x^{k+1}, \widetilde{x}^{k+1}) \leq \epsilon_{k+1}, \quad v^{k+1} = (\widetilde{x}^{k+1}, \widetilde{y}^{k+1}) \in C_{k+1}; \tag{4}$$

$$\epsilon_{k+1} \geq 0, \qquad \sum_{k=0}^{\infty} \epsilon_k < \infty; \qquad 0 < \lambda' \leq \lambda_k \leq \lambda'' < \infty; \tag{5}$$

$$C_{k+1} = \left\{ \widetilde{v} = (\widetilde{x}, \widetilde{y}) \in C : f(\widetilde{v}, u) + \frac{\lambda_k}{2}\left(q^2(x^k, x) - q^2(\widetilde{x}, x) - q^2(x^k, \widetilde{x})\right) \geq 0 \quad \forall u = (x, y) \in C \right\} \tag{6}$$

for $k = 0, 1, \ldots$. It means that each iterate u^{k+1} is an approximation of the exact solution v^{k+1} of the auxiliary problem in the partial proximal point method with the accuracy $\frac{\epsilon_{k+1}}{\beta}$ on x^{k+1}.

Remark 3.1. *Konnov (2006) considers the partial proximal point method under the assumption of convexity of $f(u,\cdot)$ with $q(x,y) = \|x - y\|$.*

3.1 The method on partially bounded sets

In this work, for any sequence $\{z^k\}_{k\in\mathbb{N}}$, we denote by $\omega(z^0)$ the set (possibly empty) of its limit points.

Theorem 3.1. *Suppose that assumptions* **P4** − **P5** *are fulfilled, Y_C is bounded, the sequence $\{u^k\}_{k\in\mathbb{N}}$ is generated by the method (4)- (6) where $C_k \neq \emptyset$ for each $k = 0, 1, \ldots$. Then,*

1. *$\omega(u^0) \neq \emptyset$ and $\omega(u^0) \subset S(f, C)$;*
2. *if*
$$S(f, C) = S^d(f, C), \tag{7}$$
then $\lim_{k\to\infty} x^k = x^*$, where $(x^*, y') \in S(f, C)$ for each limit point y'.

Proof. By **P4**, we take any point $u^* = (x^*, y^*) \in S^d(f, C)$ and consider the kth iteration of the method. By definition, we have

$$\frac{\lambda_k}{2}\left(q^2(x^k, x^*) - q^2(\widetilde{x}^{k+1}, x^*) - q^2(x^k, \widetilde{x}^{k+1})\right) \geq -f(v^{k+1}, u^*) \geq 0;$$

Hence
$$q^2(x^k, x^*) \geq q^2(\widetilde{x}^{k+1}, x^*) + q^2(x^k, \widetilde{x}^{k+1}), \tag{8}$$
and therefore
$$q(\widetilde{x}^{k+1}, x^*) \leq q(x^k, x^*). \tag{9}$$
On account of (4) and (9), we see that

$$q(x^{k+1}, x^*) \leq q(x^{k+1}, \widetilde{x}^{k+1}) + q(\widetilde{x}^{k+1}, x^*) \leq \epsilon_{k+1} + q(x^k, x^*)$$

From (5) and **P5** we have

$$0 \leq \|x^{k+1} - x^*\| \leq \frac{1}{\alpha}\sum_{s=0}^{\infty}\epsilon_s + \frac{1}{\alpha}\, q(x^0, x^*) < +\infty.$$

It follows that, $\{x^k\}_{k\in\mathbb{N}}$ is bounded, thus by assumption, $\{y^k\}_{k\in\mathbb{N}}$ is bounded too. Therefore, $\{(x^k, y^k)\}_{k\in\mathbb{N}}$ has limit points. Also, applying Lemma 2.2, we obtain
$$\lim_{k\to\infty} q(x^k, x^*) = \ell \geq 0. \tag{10}$$
Using the triangular inequality, (4) and (8) we have

$$\begin{aligned}
q^2(x^{k+1}, x^*) &\leq \left(q(x^{k+1}, \widetilde{x}^{k+1}) + q(\widetilde{x}^{k+1}, x^*)\right)^2 \\
&= q^2(x^{k+1}, \widetilde{x}^{k+1}) + 2q(x^{k+1}, \widetilde{x}^{k+1})q(\widetilde{x}^{k+1}, x^*) + q^2(\widetilde{x}^{k+1}, x^*) \\
&\leq \epsilon_{k+1}^2 + 2\epsilon_{k+1} q(\widetilde{x}^{k+1}, x^*) + q^2(x^k, x^*) - q^2(x^k, \widetilde{x}^{k+1}) \\
&\leq \epsilon_{k+1}^2 + 2\epsilon_{k+1} L + q^2(x^k, x^*) - q^2(x^k, \widetilde{x}^{k+1})
\end{aligned}$$

where $L < \infty$, as $\{\widetilde{x}^k\}_{k\in\mathbb{N}}$ is also bounded. On account of (5), we see that $\lim_{k\to\infty} q(x^k, \widetilde{x}^{k+1}) = 0$. From (9) and the triangular inequality we have

$$0 \leq q(x^k, x^*) - q(\widetilde{x}^{k+1}, x^*) \leq q(x^k, \widetilde{x}^{k+1})$$

it follows that $\lim_{k\to\infty} \left(q(x^k, x^*) - q(\widetilde{x}^{k+1}, x^*)\right) = 0$ and by (10) $\lim_{k\to\infty} q(x^k, x^*) = \lim_{k\to\infty} q(\widetilde{x}^{k+1}, x^*) = \ell$. Moreover, by the triangular inequality

$$\begin{aligned} q^2(x^k, x) &\leq \left(q(x^k, \widetilde{x}^{k+1}) + q(\widetilde{x}^{k+1}, x)\right)^2 \\ &= q^2(x^k, \widetilde{x}^{k+1}) + 2q(x^k, \widetilde{x}^{k+1})q(\widetilde{x}^{k+1}, x) + q^2(\widetilde{x}^{k+1}, x) \end{aligned}$$

or equivalently

$$q^2(x^k, x) - q^2(x^k, \widetilde{x}^{k+1}) - q^2(\widetilde{x}^{k+1}, x) \leq 2q(x^k, \widetilde{x}^{k+1})q(\widetilde{x}^{k+1}, x)$$

At the same time, by definition,

$$f(v^{k+1}, u) + \frac{\lambda_k}{2}\left(q^2(x^k, x) - q^2(\widetilde{x}^{k+1}, x) - q^2(x^k, \widetilde{x}^{k+1})\right) \geq 0 \qquad \forall u = (x, y) \in C.$$

Therefore

$$f(v^{k+1}, u) + \lambda_k\, q(x^k, \widetilde{x}^{k+1})q(\widetilde{x}^{k+1}, x) \geq 0 \qquad \forall u = (x, y) \in C.$$

If $\widetilde{u} = (\widetilde{x}, \widetilde{y})$ is an arbitrary limit point of $\{u^k = (x^k, y^k)\}_{k\in\mathbb{N}}$, then taking the limit in the aforementioned inequality gives

$$f(\widetilde{u}, u) \geq 0 \qquad \forall u = (x, y) \in C.$$

i.e., $\widetilde{u} \in S(f, C)$ and assertion (1) is true. If (7) holds, then we can set $x^* = \widetilde{x}$ in (10) and obtain $\ell = 0$. Therefore by **P5**, $\{x^k\}_{k\in\mathbb{N}}$ converges to $\widetilde{x}$ and (2) is also true. ■

We now discuss the basic assumption $C_k \neq \emptyset$ in Theorem 3.1.

Proposition 3.1. *Suppose that assumptions* **P1** – **P3**, **P5** *are fulfilled and* f *satisfies the condition* $(\mathcal{H})$*. If*

1. $f(\widetilde{v}, \cdot) + \frac{\lambda_k}{2}\left(q^2(x^k, \cdot) - q^2(\widetilde{x}, \cdot) - q^2(x^k, \widetilde{x})\right)$ *is a pseudoconvex function* $\forall\, \widetilde{v} = (\widetilde{x}, \widetilde{y}) \in C$*;*

2. $f(\widetilde{v}, u) + \frac{\lambda_k}{2}\left(q^2(x^k, x) - q^2(\widetilde{x}, x) - q^2(x^k, \widetilde{x})\right)$ *is pseudomonotone;*

Then $C_k \neq \emptyset$ *for each* $k = 0, 1, \ldots$.

Proof. It is clear that $f(\widetilde{v}, \cdot) + \frac{\lambda_k}{2}\left(q^2(x^k, \cdot) - q^2(\widetilde{x}, \cdot) - q^2(x^k, \widetilde{x})\right)$ verifies:

- $f(\widetilde{v}, \widetilde{v}) + \frac{\lambda_k}{2}\left(q^2(x^k, \widetilde{x}) - q^2(\widetilde{x}, \widetilde{x}) - q^2(x^k, \widetilde{x})\right) = 0$ for all $\widetilde{v} \in C$;
- By hypothesis $f(\widetilde{v}, \cdot) + \frac{\lambda_k}{2}\left(q^2(x^k, \cdot) - q^2(\widetilde{x}, \cdot) - q^2(x^k, \widetilde{x})\right) : C \to \mathbf{R}$ is a pseudoconvex function for all $\widetilde{v} \in C$.

- By assumption $f(\cdot, u)$ is upper hemicontinuous for all $u = (x, y) \in C$; $q^2(z, \cdot)$ and $q^2(\cdot, z)$ are locally Lipschitzian for all z. Thus $f(\cdot, u) + \frac{\lambda_k}{2}\left(q^2(x^k, x) - q^2(\cdot, x) - q^2(x^k, \cdot)\right)$ is upper hemicontinuous.

Next, we prove that $f(\widetilde{v}, \cdot) + \frac{\lambda_k}{2}\left(q^2(x^k, \cdot) - q^2(\widetilde{x}, \cdot) - q^2(x^k, \widetilde{x})\right)$ verifies the condition $(\mathcal{H})$ of Proposition 2.2. In fact,

$$q^2(x^k, x) - q^2(\widetilde{x}^n, x) - q^2(x^k, \widetilde{x}^n) \leq \beta^2\|x^k - x\|^2 - \alpha^2\left(\|\widetilde{x}^n - x\|^2 + \|x^k - \widetilde{x}^n\|^2\right)$$

when x^k, x are fixed and $\{\widetilde{x}^n\}_{n\in\mathbb{N}}$ is a sequence such that $\lim_{n\to\infty}\|\widetilde{x}^n\| = +\infty$. Then, the right side of above inequality going to $-\infty$, it exists $\overline{n}_0 \in \mathbb{N}$ such that

$$q^2(x^k, x) - q^2(\widetilde{x}^n, x) - q^2(x^k, \widetilde{x}^n) \leq 0 \qquad \forall n \geq \overline{n}_0.$$

On the other hand, the bifunction f verifies condition $(\mathcal{H})$ with $n_0 \in \mathbb{N}$, if we take $\widetilde{n}_0 = \max\{n_0, \overline{n}_0\}$ we have

$$f(\widetilde{v}^n, u) + \frac{\lambda_k}{2}\left(q^2(x^k, x) - q^2(\widetilde{x}^n, x) - q^2(x^k, \widetilde{x}^n)\right) \leq 0 \qquad \forall\, n \geq \widetilde{n}_0.$$

Now we apply Proposition 2.2, and we conclude that C_k is a nonempty set. ■

Remark 3.2. *The previous Proposition is justified by the following results. Suppose that the bifunction f is monotone or that f is strongly monotone with respect to x or weakly monotone with respect to x. Then if follows that the bifunction f_k*

$$f_k(u, v) = f(u, v) + \frac{\lambda_k}{2}\left(q^2(x^k, x) - q^2(x', x) - q^2(x^k, x')\right)$$

with $u = (x, y)$, $v = (x', y')$ is pseudomonotone. In fact

$$\begin{aligned} f_k(u, v) + f_k(v, u) &= f(u, v) + \frac{\lambda_k}{2}\left(q^2(x^k, x) - q^2(x', x) - q^2(x^k, x')\right) \\ &+ f(v, u) + \frac{\lambda_k}{2}\left(q^2(x^k, x') - q^2(x, x') - q^2(x^k, x)\right) \\ &= f(u, v) + f(v, u) - \frac{\lambda_k}{2}\left(q^2(x', x) + q^2(x, x')\right). \end{aligned} \tag{11}$$

- *If f is monotone then from (11) we obtain*

$$\begin{aligned} f_k(u, v) + f_k(v, u) &= f(u, v) + f(v, u) - \frac{\lambda_k}{2}\left(q^2(x', x) + q^2(x, x')\right) \\ &\leq -\frac{\lambda_k}{2}q^2(x, x') \\ &\leq -\frac{\lambda_k\alpha^2}{2}\|x - x'\|^2 \end{aligned}$$

- *If f is strongly monotone with respect to x with constant $\tau > 0$, from (11), we have*

$$f_k(u, v) + f_k(v, u) \leq -\tau\|x - x'\|^2 - \frac{\lambda_k}{2}\left(q^2(x', x) + q^2(x, x')\right) \leq 0.$$

- *If f is weakly monotone with respect to x with constant $\nu > 0$, f_k is strongly monotone with respect to x if $\frac{\lambda'\alpha^2}{2} > \nu$. In fact, from (11)*

$$\begin{aligned} f_k(u,v) + f_k(v,u) &= f(u,v) + f(v,u) - \frac{\lambda_k}{2}\left(q^2(x',x) + q^2(x,x')\right) \\ &\leq \nu\|x - x'\|^2 - \frac{\lambda_k}{2} q^2(x,x') \\ &\leq \nu\|x - x'\|^2 - \frac{\lambda_k \alpha^2}{2}\|x - x'\|^2 \\ &\leq \nu\|x - x'\|^2 - \frac{\lambda'\alpha^2}{2}\|x - x'\|^2 \\ &= -\left(\frac{\lambda'\alpha^2}{2} - \nu\right)\|x - x'\|^2. \end{aligned}$$

Thus in all these cases, f_k is monotone and therefore pseudomonotone.

Corollary 3.1. *Suppose that assumptions* **P1 − P5** *are fulfilled, Y_C is bounded, and $S(f,C) \neq \emptyset$ and f is monotone or f is strongly monotone with respect to x. If the sequence $\{u^k = (x^k, y^k)\}_{k\in\mathbf{N}}$ is generated by the method (4)- (6) then $\omega(u^0) \neq \emptyset$ and $\omega(u^0) \subset S(f,C)$; moreover, $\lim_{k\to\infty} x^k = x^*$, where $u^* = (x^*, y^*) \in S(f,C)$ and y^* is any point of $\omega(y^0)$.*

Proof. From Remark 3.2 it follows that f_k is strongly monotone with respect to x with constant $\frac{\lambda_k \alpha}{2}$. Therefore from Proposition 3.1, we obtain that $S(f_k, C) \neq \emptyset$ for all $k = 1, 2, \ldots$; moreover by Proposition 2.1, we have that

$$S(f_k, C) = \{x^k\} \times Y_k^*. \tag{12}$$

Now the assertion follows from Theorem 3.1. ■

Corollary 3.2. *Suppose that assumptions* **P1 − P5** *are fulfilled, Y_C is bounded and f is weakly monotone with respect to x. Then, there exists $\lambda' > 0$ such that the method (4)-(6) generates a sequence $\{u^k = (x^k, y^k)\}_{k\in\mathbf{N}}$ such that*

1. *$\omega(u^0) \neq \emptyset$ and $\omega(u^0) \subset S(f,C)$;*
2. *if $S(f,C) = S^d(f,C)$ then $\lim_{k\to\infty} x^k = x^*$, where $u^* = (x^*, y^*) \in S(f,C)$ and $y^* \in \omega(y^0)$.*

Proof. From Remark 3.2 f is weakly monotone with respect to x with constant $\nu > 0$. Then f_k is strongly monotone with respect to x if $\frac{\lambda'^2}{2} > \nu$. Therefore, using the argument as that in Proposition 3.1, we obtain that $S(f_k, C) \neq \emptyset$ if $\frac{\lambda'^2}{2} > \nu$ and (12) holds. All the assumptions of Theorem 3.1 being fulfilled, we conclude that (1) and (2) are true. ■

3.2 The method on unbounded sets

In this section, we use the following assumption considered by Konnov (2006) for the bifunction f in order to avoid the boundedness hypothesis on the set Y_C:

P6. For each $u^* = (x^*, y^*) \in S^d(f, C)$, it holds that there exists $\sigma > 0$ such that

$$f(u, u^*) \leq -\sigma \|y - y^*\|^2 \qquad \forall u = (x, y) \in C.$$

The proof of the next result is essentially analogous to that in Konnov (2006).

Lemma 3.1. *If* $\mathbf{P1} - \mathbf{P4}$ *and* $\mathbf{P6}$ *are fulfilled, then*

$$S^d(f, C) = X^d \times \{y^*\}. \tag{13}$$

Theorem 3.2. *Suppose that assumptions* $\mathbf{P1} - \mathbf{P6}$ *are fulfilled and that the sequence* $\{u^k = (x^k, y^k)\}_{k \in \mathbb{N}}$*, is generated by the method (4)- (6), where* $C_k \neq \emptyset$ *for each* $k = 0, 1, \ldots$. *Then,*

1. $\omega(x^0) \neq \emptyset$, $\lim_{k\to\infty} y^k = y^*$, *where* y^* *is defined in (13) and* $\omega(x^0) \times \{y^*\} \subset S(f, C)$.
2. *if additionally (7) holds,* $\lim_{k\to\infty} u^k = u^* = (x^*, y^*) \in S(f, C)$.

Proof. Consider any point $u^* = (x^*, y^*) \in C$, where y^* is defined in (13) and consider the kth iteration of the method. By definition, we have

$$\frac{\lambda_k}{2}\left(q^2(x^k, x^*) - q^2(\widetilde{x}^{k+1}, x^*) - q^2(x^k, \widetilde{x}^{k+1})\right) \geq -f(v^{k+1}, u^*) \geq \sigma \|\widetilde{y}^{k+1} - y^*\|^2; \tag{14}$$

due to **P6**. Therefore

$$q^2(x^k, x^*) \geq q^2(\widetilde{x}^{k+1}, x^*) + q^2(x^k, \widetilde{x}^{k+1}), \tag{15}$$

and therefore

$$q(\widetilde{x}^{k+1}, x^*) \leq q(x^k, x^*). \tag{16}$$

On account of (4), we see that

$$q(x^{k+1}, x^*) \leq q(x^{k+1}, \widetilde{x}^{k+1}) + q(\widetilde{x}^{k+1}, x^*) \leq \epsilon_{k+1} + q(x^k, x^*)$$

From (5) and **P5** we have

$$0 \leq \|x^{k+1} - x^*\| \leq \frac{1}{\alpha}\sum_{k=0}^{\infty} \epsilon_k + \frac{1}{\alpha}\, q(x^0, x^*) < +\infty,$$

i.e., $\{x^k\}_{k\in\mathbb{N}}$ is bounded; moreover applying the Lemma 2.2, we obtain

$$\lim_{k\to\infty} q(x^k, x^*) = \ell \geq 0. \tag{17}$$

Using the triangular inequality, (4) and (15) we have

$$\begin{aligned} q^2(x^{k+1},x^*) &\le \left(q(x^{k+1},\widetilde{x}^{k+1})+q(\widetilde{x}^{k+1},x^*)\right)^2 \\ &= q^2(x^{k+1},\widetilde{x}^{k+1})+2q(x^{k+1},\widetilde{x}^{k+1})q(\widetilde{x}^{k+1},x^*)+q^2(\widetilde{x}^{k+1},x^*) \\ &\le \epsilon_{k+1}^2+2\epsilon_{k+1}q(\widetilde{x}^{k+1},x^*)+q^2(x^k,x^*)-q^2(x^k,\widetilde{x}^{k+1}) \\ &\le \epsilon_{k+1}^2+2\epsilon_{k+1}L+q^2(x^k,x^*)-q^2(x^k,\widetilde{x}^{k+1}) \end{aligned}$$

where $L<\infty$, because $\{\widetilde{x}^k\}_{k\in\mathbb{N}}$ is also bounded. On account of (5), we see that

$$\lim_{k\to\infty} q(x^k,\widetilde{x}^{k+1})=0. \tag{18}$$

From (9) and the triangular inequality we have

$$0\le q(x^k,x^*)-q(\widetilde{x}^{k+1},x^*)\le q(x^k,\widetilde{x}^{k+1})$$

Then it follows that $\lim_{k\to\infty}\left(q(x^k,x^*)-q(\widetilde{x}^{k+1},x^*)\right)=0$ and by (10)

$$\lim_{k\to\infty} q(x^k,x^*)=\lim_{k\to\infty} q(\widetilde{x}^{k+1},x^*)=\ell. \tag{19}$$

Moreover, by the triangular inequality

$$\begin{aligned} q^2(x^k,x) &\le \left(q(x^k,\widetilde{x}^{k+1})+q(\widetilde{x}^{k+1},x)\right)^2 \\ &= q^2(x^k,\widetilde{x}^{k+1})+2q(x^k,\widetilde{x}^{k+1})q(\widetilde{x}^{k+1},x)+q^2(\widetilde{x}^{k+1},x) \end{aligned}$$

or equivalently

$$q^2(x^k,x)-q^2(x^k,\widetilde{x}^{k+1})-q^2(\widetilde{x}^{k+1},x)\le 2q(x^k,\widetilde{x}^{k+1})q(\widetilde{x}^{k+1},x). \tag{20}$$

From (14) and the last inequality above, we obtain

$$\lambda_k q(x^k,\widetilde{x}^{k+1})q(\widetilde{x}^{k+1},x^*)\ge\sigma\,\|\widetilde{y}^{k+1}-y^*\|^2;$$

Thus from (5), (18) and (19) we have $\lim_{k\to\infty}\widetilde{y}^k=y^*$. Hence $\lim_{k\to\infty}y^k=y^*$. By definition and (20), we have

$$f(v^{k+1},u)+\lambda_k\, q(x^k,\widetilde{x}^{k+1})q(\widetilde{x}^{k+1},x)\ge 0 \qquad \forall u=(x,y)\in C.$$

If $\widetilde{x}$ is an arbitrary limit point of $\{x^k\}_{k\in\mathbb{N}}$, then taking the limit in the above inequality gives $f(\widetilde{u},u)\ge 0 \quad \forall u=(x,y)\in C$. where $\widetilde{u}=(\widetilde{x},y^*)$, i.e., $\widetilde{u}\in S(f,C)$ and assertion (1) is true. If (7) holds, then we can set $x^*=\widetilde{x}$ in (19) and obtain $\ell=0$. Therefore, by **P5**, $\{x^k\}_{k\in\mathbb{N}}$ converges to $\widetilde{x}$ and (2) is also true. ∎

We now discuss the basic assumption $C_k\neq\emptyset$ in Theorem 3.2. First, we consider the case where f is strongly monotone with respect to y and the case where f is also weakly monotone with respect to x.

Corollary 3.3. *Suppose that Assumptions* **P1** $-$ **P3** *and* **P5** *are fulfilled,* $S(f,C) \neq \emptyset$ *and* f *is strongly monotone with respect to* y*. If the sequence* $\{u^k = (x^k, y^k)\}_{k\in\mathbb{N}}$ *is generated by the method (4)- (6) then it converges to a point* $u^* = (x^*, y^*) \in S(f,C)$*.*

Proof. From the assumptions, due to Proposition 2.1, we have that (7) holds. Next, the bifunction f_k defined in Remark 3.2 is strongly monotone. In fact, let $u = (x, y)$ and $v = (x', y') \in C$. Then we have

$$\begin{aligned} f_k(u,v) + f_k(v,u) &= f(u,v) + f(v,u) - \frac{\lambda_k}{2}\left(q^2(x',x) + q^2(x,x')\right) \\ &\leq -\nu\|y - y'\|^2 - \frac{\lambda_k}{2} q^2(x,x') \\ &\leq -\nu\|y - y'\|^2 - \frac{\lambda'}{2} q^2(x,x') \\ &\leq -\nu\|y - y'\|^2 - \frac{\lambda'\beta^2}{2}\|x - x'\|^2 \\ &\leq -\widetilde{\nu}\|u - v\|^2 \end{aligned}$$

where ν is the modulus of strong monotonicity of f in y, and $\widetilde{\nu} = \min\{\nu, \frac{\lambda'\beta^2}{2}\}$. Therefore, $C_k \neq \emptyset$ is a singleton for any $\lambda' > 0$ and for each $k = 1, 2, \ldots$. On the other hand, because (7) is satisfied and $S(f,C) \neq \emptyset$, then $S^d(f,C) \neq \emptyset$. Moreover, if we take arbitrary points $u^* = (x^*, y^*) \in S^d(f,C) = S(f,C)$ and $u = (x,y) \in C$, then $f(u,u^*) \leq f(u,u^*) + f(u^*,u) \leq -\nu\|y - y^*\|^2$, i.e., **P4** and **P6** are also fulfilled and the result now follows from Theorem 3.2 (2). ■

Corollary 3.4. *Suppose that assumptions* **P1** $-$ **P6** *are fulfilled,* f *is both weakly monotone with respect to* x *and strongly monotone with respect to* y*, i.e., for each pair of points* $u = (x,y)$*,* $v = (x', y') \in C$*, it holds that*

$$f(u,v) + f(v,u) \leq \gamma\|x - x'\|^2 - \nu\|y - y'\|^2,$$

where $\gamma > 0$ *and* $\nu > 0$*. Then, there exists* $\lambda' > 0$ *such that the method (4)- (6) generates a sequence* $\{u^k = (x^k, y^k)\}_{k\in\mathbb{N}}$ *such that*

1. $\omega(x^0) \neq \emptyset$*,* $\lim_{k\to\infty} \widetilde{y}^k = y^*$*, where* y^* *is defined in (13) and* $\omega(x^0)\times\{y^*\} \subset S(f,C)$*.*
2. *if additionally (7) holds,* $\lim_{k\to\infty} \widetilde{u}^k = u^* = (x^*, y^*) \in S(f,C)$*.*

Proof. Take arbitrary points $u = (x,y)$ and $v = (x', y')$ and set $\lambda'\alpha^2 > 2\gamma$. Then, we have

$$\begin{aligned} f_k(u,v) + f_k(v,u) &= f(u,v) + f(v,u) - \frac{\lambda_k}{2}\left(q^2(x',x) + q^2(x,x')\right) \\ &\leq f(u,v) + f(v,u) - \frac{\lambda_k}{2} q^2(x,x') \\ &\leq \gamma\|x - x'\|^2 - \nu\|y - y'\|^2 - \frac{\lambda_k}{2} q^2(x,x') \end{aligned}$$

$$
\begin{aligned}
&\leq \gamma\|x-x'\|^2 - \nu\|y-y'\|^2 - \frac{\lambda'\alpha^2}{2}\|x-x'\|^2 \\
&= -(\frac{\lambda'\alpha^2}{2} - \gamma)\|x-x'\|^2 - \nu\|y-y'\|^2 \\
&\leq -\widetilde{\nu}\|u-v\|^2
\end{aligned}
$$

where $\widetilde{\nu} = \min\{\frac{\lambda'\alpha^2}{2} - \gamma, \nu\} > 0$, i.e., f_k is strongly monotone and $C_k \neq \emptyset$ is a singleton for each $k = 1, 2, \ldots$. We see that all the assumptions of Theorem 3.2 are satisfied. The proof is complete. ■

4 Application: How dual equilibrium problems model the concept of behavioral trap

In this last section, we use the variational approach of Soubeyran (2009, 2010) to show how dual equilibrium problems offer a model of behavioral traps which appear in a lot of different disciplines. In the annex, taking advantage of this variational point of view, we will show, using more technical arguments, "how a succession of aspiration points converges to an equilibrium", using worthwhile changes during the goal pursuit.

Variational rationality Soubeyran (2009, 2010) unifies a lot of theories of changes where, each step, the problem is to know which action "to do" and how "to be able to do" it, choosing either to repeat the same action (a temporary routine) or to do a different action. Theories of stability and change include repeated choice problems, transformation problems (creation, invention, innovation), self regulation problems (goal setting, goal striving, goal pursuit), behavioral changes (moving from a "bad" to a "better" behavior), organizational change, social, political and cultural change, etc.

In this dynamic context (a succession of actions), this general "variational model of stability and change" supposes that agents, driven by a rather vague final goal, can set, each step, much more specific temporary intermediate goals. Such a goal setting-goal striving process helps them to make worthwhile changes which balance, each step, their motivation and resistance to change (i.e., the utility of their advantages to change and the disutility of their costs to change). This approach generalizes in several directions the famous but controversial "bounded rationality" theory (Simon, 1955) where, each step, agents can only improve or "improve enough" (satisfice) from time to time, being unable, each step, to always optimize. In psychology, it also offers a general model for almost all the theories of motivation and self regulation. In this variational context agents are equipped with variable preferences. They set, each step, intermediate goals, and use worthwhile changes to overcome inertia (resistance to change). This variational theory of change shows how an agent can end up in some individual behavioral trap (habit, routine) or how, in a social context (within groups, organizations), interrelated agents can end up in a social behavioral trap (a Nash equilibrium being a specific case).

Dual equilibrium problems In Applied Mathematics, several authors; see, e.g., Blum and Oettli (1994), Bianchi and Schaible (2004), Konnov (2005) consider the equilibrium problem $EP(f, C)$ which defines x^* as an optimum, a Nash equilibrium, a saddle point, etc; and its dual $DP(f, C)$, which defines $x^\$$ without any clear dual interpretation.

Behavioral traps As an application, let us consider the existence of "behavioral traps" in the context of the theories of stability and change. This concept, which is widely used in Economics, Management, Psychology, Sociology, and Political Sciences seems to have no formal definition. Informally, it seems to be a situation "easy enough" to reach, and "difficult enough" to leave. There are also liquidity, poverty, ecological and evolutionary traps, etc. Behavioral traps have three cognitive aspects (related to motivations, cognitions and affects) at the individual, organizational and social levels:

i) Motivational and inertial traps.

ii) Cognitive traps: competency traps, ability, capability, knowledge and learning traps, success and mental traps, etc. The list of decision making or psychological traps (Hammond et al., 1998) is quite long: anchoring, statu quo, sunk cost, confirming evidence, framing, estimating and forecasting, overconfidence, prudence, and recallability traps.

iii) Emotional traps traps are related to fear, confusion, guilt, shame, loneliness, resentment, self-doubt, stubbornness and addiction.

A variational definition of behavioral traps The need to modelize a behavioral trap is obvious, because, in the informal definition (a situation "easy enough" to reach, and "difficult enough" to leave) "what is enough"? This is not an easy task to answer to this question. A behavioral trap must have the two properties of stability and reachability. Recently, a rigorous definition of this fuzzy notion has been given in the "variational approach" of the theories of stability and change (Soubeyran, 2009, 2010). The first leading idea of this new analytic framework is to pose the dual questions: "should I stay" (substantive view, stability aspect) and "should I go" (variational view, reachability aspect). Substantive and variational rationality take two opposite, but complementary, views of the "target problem" on a landscape. To succeed to link the two sides of the same coin (approaching a target and gains, moving away and losses), this variational approach considers general advantages and costs to change functions. Using the original concept of "worthwhile change" which balances, each step, motivation and resistance to change (the utility of advantages to change, and the disutility of costs to change), this variational approach of the theories of stability and change model a behavioral trap as a situation which is i) "worthwhile to reach" (an aspiration point), and, ii) "not worthwhile to leave" (an optimum, an equilibrium, a rest point). For a recent application of this concept for variable preferences, in the variational context of worthwhile changes; see Flores-Bazán et al. (2012).

Aspiration problems In behavioral sciences, Soubeyran (2009, 2010) considers an advantage to change function $A : C \times C \to \mathbb{R}$ and examines, first, the *variational aspiration problem* $AP(S)$:

$$\text{Find } x^{\$} \in C \quad \text{such that} \quad A(x, x^{\$}) \geq 0 \quad \forall\, x \in S,$$

where $S \subset C$, which defines $x^{\$}$ as a local aspiration point, where an agent prefers to move, starting from any position $x \in S \subset C$. The aspiration point is global if $S = C$. More general formulations of advantage to change functions in term of motivation to change functions M will be given.

Formally the correspondence between the two approaches (the mathematical *dual equilibrium problems* and the *behavioral trap problem*) is given by the equality $f(x^{\$}, x) = A(x, x^{\$})$. Then, the variational aspiration problem $AP(S)$ becomes:

$$\text{Find } x^{\$} \in C \quad \text{such that} \quad f(x^{\$}, x) \geq 0 \quad \forall\, x \in S.$$

If $S = C$ it represents an *equilibrium problem*: if $x^{\$}$ is an aspiration point of the variational approach, then, it is an equilibrium of the mathematical approach $EP(f, C)$.

The approach problem considers how an agent can reach one of the equilibrium positions, using in the transition an acceptable path of changes (feasible or reachable, and worthwhile, hence improving). This is the "should I go" question related to an aspiration problem $AP(S)$, moving towards an aspiration point.

Approaching a target and gains. Let $x \curvearrowright x^{\$}$ be a move from some origin x to some target $x^{\$}$ (optimum, equilibrium, behavioral trap, etc.). The advantage to change payoff is $A(\text{origin,target}) = A(x, x^{\$}) \geq 0$. It represents a variation in utility or a variation in need moving from x to $x^{\$}$. An advantage to change function is separable when it can be explicitly modeled as the variation $A(x, x^{\$}) = V(x^{\$}) - V(x)$ of an *utility function* $V : C \to \mathbb{R}$, or as the variation $A(x, x^{\$}) = N(x) - N(x^{\$})$ of a *need function* $N : C \to \mathbb{R}_+$.

The aspiration concept is a leading notion in the **variational rationality** approach and helps to model bounded rationality. It considers an improving approach $x \curvearrowright x^{\$}$ to a target. In the special case where the target $x^{\$} = x^*$ is a maximum or a minimum, an approach to the optimum is a move $x \curvearrowright x^*$. Gains to move towards the optimum are always nonnegative:

$$A(x, x^*) = V(x^*) - V(x) \geq 0 \quad \text{or} \quad A(x, x^*) = N(x) - N(x^*) \geq 0 \qquad \forall\, x \in S \subset C.$$

In the more general case where the advantage to change function is not separable the target $x^{\$}$ represents a (free inertia) aspiration point $x^{\$} \in C$ such that $A(x, x^{\$}) \geq 0, \forall\, x \in S \subset C$. This means that, starting from a subset $x \in S \subset C$, it is advantageous to change from x to $x^{\$}$. The aspiration point is global if $S = C$. For normal form games aspiration points can play the role of "focal points".

Equilibrium problems In behavioral sciences, Soubeyran (2009, 2010) considers also the equilibrium problem $EP(D)$:

$$\text{Find } x^* \in C \quad \text{such that} \quad A(x^*, x) \leq 0 \quad \forall\, x \in D,$$

where $D \subset C$, which defines x^* as a local equilibrium where an agent prefers to stay than to move away. The equilibrium is global if $D = C$.

Then, the equality $A(x^*, x) = f(x, x^*)$ shows that the equilibrium problem $EP(D)$ of the variational rationality approach represents the dual equilibrium problem $DP(f, C)$ of the mathematical approach: if x^* is an equilibrium of the variational approach, then, it is a dual equilibrium point of the mathematical approach.

The stability problem considers how an agent, being in an equilibrium, can deviate from this equilibrium position and verifies that it is not rewarding to do it. This traditional "helicopter story" is curious because it does not tell us why the agent is there (an helicopter puts him there?), the equilibrium positions being unknown ex ante. This is the "should I stay" question related to an equilibrium problem $EP(D)$, leaving a stable position (optimum, equilibrium).

Moving away from a target and losses. Consider a move away $x^* \curvearrowright x$ from a target x^* to an other point x. There is a disadvantage to move from x^* to x if $A(x^*, x) \leq 0$. In the separable case $A(x^*, x) = V(x) - V(x^*) = N(x^*) - N(x) \leq 0$ represents a loss, a decrease in utility, $V(x) \leq V(x^*)$, or an increase of needs, $N(x) \geq N(x^*)$, when moving aways from the target. Then, the condition $A(x^*, x) \leq 0$, $\forall\, x \in C$ defines x^* as a maximum in utility, or a minimum in needs. The optimum concept has a substantive aspect. It describes a "moving away story" from the optimum.

For an equilibrium this is the same, using a Nikaido-Isoda function

$$A(x, y) = \sum_{j \in J} [V_j(y_j, x_{-j}) - V_j(x_j, x_{-j})]$$

related to a normal form game where the list of players is J, the action of player $j \in J$ is $x_j \in C_j$, actions of others are $x_{-j} = (x_h, h \in J, h \neq j)$ and his payoff is $V_j(x_j, x_{-j}) \in \mathbb{R}$.

A formal definition of behavioral trap as a dual equilibrium The concept of a behavioral trap (an easy to reach, but difficult to leave point) generalizes the concepts of optima and Nash equilibria which are only difficult to leave: $x^* \in C$ is a behavioral trap if,

i) it exists $S \subset C$, $S \neq \emptyset$, such that $A(x, x^*) \geq 0\ \forall\, x \in S$, i.e., there is an advantage to move to x^*, starting from any $x \in S$, where $S \subset C$ is the source set, and,

ii) it exists $D \subset C, D \neq \emptyset$, such that $A(x^*, x) \leq 0\ \forall\, x \in D \subset C$, i.e., there is always a loss to move away from x^* to any $x \in D$.

The larger the source set S and the deviation set D, the more deeply is the behavioral trap. In the present paper we will take $S = D = C$. Then, as expected, a behavioral trap x^* is both an aspiration point (point i)) and an equilibrium (point ii)). This shows that a behavioral trap is the solution of an dual equilibrium problem where the equilibrium problem and its dual have at least an identical solution.

When the advantage to change function $A(x,x')$ from x to x' is separable, it defines the difference between the utility $V(x')$ and the utility $V(x)$, or the difference between the need $N(x)$ and the need $N(x')$, i.e., $A(x,x') = V(x') - V(x) = N(x) - N(x')$. In this case the two concepts of optimum and aspiration points are equivalent:

$x^* \in \arg\min \{N(x) : x \in C\} \Longleftrightarrow A(x,x^*) = N(x) - N(x^*) \geq 0 \quad \forall x \in C$

and

$x^* \in \arg\max \{V(x) : x \in C\} \Longleftrightarrow A(x^*,x) = V(x) - V(x^*) \leq 0 \quad \forall x \in C.$

Then, the agent gains to move from any point x to the optimum x^*, and losses to move away from the optimum x^*.

This is not the case for a general advantage to change function. However consider a pseudomonotone advantage to change function, $A(x,x') \geq 0 \Longrightarrow A(x',x) \leq 0$. This means that if there is an advantage to move from x to x', it cannot be strictly advantageous to move from x' to x ($A(x',x) > 0$ is impossible). In this case any aspiration point is an optimum: if $A(x,x^*) \geq 0$ for all $x \in C$, then, $A(x^*,x) \leq 0$ for all $x \in C$.

As said before, in the present paper the function $f(x^\$,x) = A(x,x^\$)$ represents an advantage to move from some initial point x to a target $x^\$$.

5 Conclusions

We analyze the partial proximal point algorithm associated to quasi distances for equilibrium problems considering an appropriate regularization term and we extended the algorithm presented in Konnov (2006). This framework is suitable for use in environments where we do not have the symmetry property, e.g. Finsler manifolds, moreover our regularization does not depend of an inner product, this fact extends and permits us to treat in a unified way certain classes of optimization problems, like those presented by Konnov (2006) and for applications to the important case of quasi-metric spaces. We also give an application to behavioral dynamics in the context of the theories of change.

At this point it would be reasonable to discuss the convergence of the method under inexact solutions of the subproblems, with some error criteria which preserves the convergence results. Nevertheless, this is another area which deserves further research efforts for creating more effective computational procedures.

A How a succession of aspiration points converges to an equilibrium, using worthwhile changes during the goal pursuit.

To be able to define the key notion of "worthwhile changes" (used in the definition of a behavioral trap, see section 4) we need to list and briefly explain some of the main concepts of the "variational rationality" approach of the theories of change of Soubeyran (2009, 2010). Then, we must define variable reference points (statu quo points, target points), reference dependent utilities, advantages to change, costs to change, motivation and resistance to change (inertia), aspiration point, satisficing point, individual behavioral trap, and a bounded rational agent. In this context the present article defines the generalized regularization term $q^2(x^k, x) - q^2(\widetilde{x}, x) - q^2(x^k, \widetilde{x})$ as a marginal resistance to change and it points out the importance of approximate aspiration points to help to converge, using worthwhile changes, to an individual behavioral trap.

Worthwhile changes: a short list of variational concepts

Worthwhile changes. Among the list of concepts of the "variational rationality" theory of change Soubeyran (2009, 2010) we need the following:

-The starting point $x^k \in C$ is the present position (choice), a variable statu quo.

-The costs to change from the present position x^k to a new one x are $\mathbb{C}(x^k, x) \geq 0$, $x \in X$ while the ex ante resistance to move from x^k to x is the disutility $R(x^k, x) = D\left[\mathbb{C}(x^k, x)\right] \geq 0$ of costs to change, where $D[0] = 0$, $x \in X$, $D[\,\cdot\,]$ strictly increasing. Costs to change represent capability costs, i.e., the costs to be able to do the new action, x, given the ability to do again the past action x^k.

-The motivation to change from the present position x^k to a new one x are $A(x^k, x) \geq 0$, $x \in X$ while the motivation to move from x^k to x is the utility $M(x^k, x) = U\left[A(x^k, x)\right] \geq 0$ of costs to change, where $U[0] = 0$, $x \in X$, $U[\,\cdot\,]$ strictly increasing. An advantage to change function $A(x, x')$ represents how much an agent prefers to choose $x' \in C$ instead of choosing $x \in C$. It models the intensity of preferences. A natural property is $A(x'/x) \geq 0 \Longrightarrow A(x/x') \leq 0$: if the agent prefers x' to x, he does not strictly prefer x to x' ($A(x/x') > 0$ is impossible)

-The advantage to change function is separable if, i) $A(x^k, x) = V(x) - V(x^k)$, where $V(x)$ is the utility of being at x, or ii) if $A(x^k, x) = N(x^k) - N(x)$ where $N(x) \geq 0$ is the unsatisfied need at x. Then, $A(x^k, x) \geq 0$ means that the utility increases and the unsatisfied need decreases when moving from x^k to x.

- It is worthwhile to change from x^k to x iff the motivation to change is higher than some proportion $\frac{\mu_k}{2} > 0$ of the resistance to change, i.e., $M(x^k, x) \geq \frac{\mu_k}{2} R(x^k, x)$.

- The net motivation to change from x^k to x is $\Delta(x^k, x) = M(x^k, x) - \frac{\mu_k}{2} R(x^k, x)$.

- The worthwhile to change set at x^k is $W(x^k) = \left\{x \in C : \Delta(x^k, x) \geq 0\right\}$.

- The marginal motivation change functions is $M_{x^k}(x, x') = M(x^k, x') - M(x^k, x)$. If the motivation function is the identity, $M[A] = A$, and advantages to change are separable, $M_{x^k}(x, x') = \left[V(x') - V(x^k)\right] - \left[V(x) - V(x^k)\right] = V(x') - V(x) = A(x, x') = N(x) - N(x')$.

- The marginal resistance to change function is $R_{x^k}(x, x') = R(x^k, x') - R(x^k, x)$.

-The net marginal motivation to change from x^k to x' instead of from x^k to

x is

$$\Gamma_{x^k}(x,x') = \Delta(x^k,x') - \Delta(x^k,x) = M_{x^k}(x,x') - \frac{\mu_k}{2}R_{x^k}(x,x')$$

- Worthwhile to change preference: the agent prefers x' to x from the point of view of the statu quo x^k, i.e., $x' \geq_{x^k} x \Longleftrightarrow \Gamma_{x^k}(x,x') \geq 0$. This means that

$$M(x^k,x') - M(x^k,x) \geq \frac{\mu_k}{2}\left[R(x^k,x') - R(x^k,x)\right].$$

- Worthwhile aspiration point: starting from the present statu quo x^k the target point $\widetilde{x}^{k+1} \in C$ is a present aspiration point if the agent prefers $\widetilde{x}^{k+1}$ to x for all $x \in H \subset C$, i.e., $x' \gtrapprox_{x^k} x$ for all $x \in H$. The aspiration point is global if $H = C$.

- Preferences $\geq_{x^k}$ change along the process $\{x^k\}_{k\in\mathbb{N}}$. Then, each step, the agent is obliged to explore repeatedly the whole space of alternatives C if he wants to estimate his new preference (defined by changing motivation and resistance to change functions).

Strong worthwhile changes. In the present paper we add the following strong variational concepts:

- Strong worthwhile to change preference: the agent strongly prefers x' to x from the point of view of the statu quo x^k, i.e.,

$$x' \gtrapprox_{x^k} x \Longleftrightarrow \Gamma_{x^k}(x,x') \geq \frac{\mu_k}{2}R(x,x') \quad \text{for some } \mu_k > 0.$$

- Strong aspiration point where it is worthwhile to move: starting from the present statu quo x^k the target point $\widetilde{x}^{k+1} \in C$ is a present aspiration point if the agent prefers $\widetilde{x}^{k+1}$ to x for all $x \in C$, i.e.,

$$x' \gtrapprox_{x^k} x \quad \forall\, x \in C \Longleftrightarrow \Gamma_{x^k}(x,\widetilde{x}^{k+1}) \geq \frac{\mu_k}{2}R(x,\widetilde{x}^{k+1}).$$

- A present point x^{k+1} satisfies if it is "close enough" to an aspiration point $\widetilde{x}^{k+1}$, $q(x^{k+1},\widetilde{x}^{k+1}) \leq \varepsilon_{k+1}$. This means that, each step, the agent is unable to optimize because, each step, it is too costly to explore again the whole set of alternatives to discover his new preference. This is the case when the agent is bounded rational (Simon, 1955): the environment is very complex, and, each step, the agent lacks of time, resources, energy and cognitive capacities.

Strong worthwhile changes and the aspiration problem Let us show how the variational approach revises the equilibrium problem and its dual. For simplification, consider the sequence $u^k = (x^k, y^k)$ where $y^k = 0\ \forall\, k \in \mathbb{N}$. Each step, starting from the present point x^k, the problem of an agent is to choose where to move: either from x^k to $\widetilde{x} = \widetilde{x}^{k+1}$ or from x^k to any other $x \in C$. Let us show that $\widetilde{x}^{k+1}$ is a strong aspiration point where it is worthwhile to move, starting from anywhere.

Hypothesis : There exists $0 < \theta \leq 1 : q(x',x) \geq \theta q(x,x') \quad \forall\, x,x' \in C.$

This means that the costs to change from x' to x are higher than some proportion θ of the cost to change from x to x'. From a behavioral sciences perspective this is not a strong hypothesis.

The marginal resistance to change from x to $\widetilde{x}^{k+1}$ is $R_{x^k}(x, \widetilde{x}^{k+1}) = R(x^k, \widetilde{x}^{k+1}) - R(x^k, x)$. Let $f(\widetilde{x}^{k+1}, x) = M_{x^k}(x, \widetilde{x}^{k+1}) = M(x^k, \widetilde{x}^{k+1}) - M(x^k, x)$ be the motivation to change function. The mathematical condition (6) can be read as,

$$\widetilde{x}^{k+1} \in T(x^k) := f(\widetilde{x}^{k+1}, x) + \frac{\lambda_k}{2}\left[R(x^k, x) - R(x^k, \widetilde{x}^{k+1}) - R(\widetilde{x}^{k+1}, x)\right] \geq 0 \quad \forall\, x \in C,$$

i.e.,

$$f(\widetilde{x}^{k+1}, x) - \frac{\lambda_k}{2}\left[R(x^k, \widetilde{x}^{k+1}) - R(x^k, x)\right] - \frac{\lambda_k}{2} R(\widetilde{x}^{k+1}, x) \geq 0 \quad \forall\, x \in C,$$

or equivalently

$$\Gamma_{x^k}(x, \widetilde{x}^{k+1}) = M_{x^k}(x, \widetilde{x}^{k+1}) - \frac{\lambda_k}{2} R_{x^k}(x, \widetilde{x}^{k+1}) \geq \frac{\lambda_k}{2} R(\widetilde{x}^{k+1}, x) \quad \forall\, x \in C.$$

Then,

$$R(x', x) = q(x', x)^2 \geq \left[\theta q(x, x')\right]^2 = \theta^2 R(x, x') \qquad \forall\, x, x' \in C,$$

implies

$$\Gamma_{x^k}(x, \widetilde{x}^{k+1}) \geq \frac{\mu_k}{2} R(x, \widetilde{x}^{k+1}),$$

where $\mu_k = \theta^2 \lambda_k > 0$. This shows that, from the point of view of the present point x^k, it is strongly worthwhile to change from x to $\widetilde{x}^{k+1}$. Then, $\widetilde{x}^{k+1}$ is a strong aspiration point where it is worthwhile to move, starting from anywhere.

The paper supposes a constant reference dependent motivation to change function

$$M_{x^k}(x, \widetilde{x}^{k+1}) = M_e(x, \widetilde{x}^{k+1}) \quad \forall\, x^k \in C,$$

where $e \in E$ is a given reference point. This means that

$$M_{x^k}(x, x') = M(x^k, x') - M(x^k, x) = M(e, x') - M(e, x) = M_e(x, x') \qquad \forall\, x^k \in C.$$

But, even when motivation to change functions are constant, preferences are variable because resistance to move changes with the statu quo point x^k.

Remark A.1. *We can use $\overline{M}(x, x') = \sup\{M_e(x, x') : e \in C\}$ as a motivation to change function. Moreover, we can also consider the case of "no regret" (Soubeyran, 2010) where $M_{x^0}(x, x') \geq M_{x^k}(x, x')$ for all x^k. This means that if the agent prefers x' to x from the present point of view x^k, he would have prefered x' to x from the initial point of view x^0.*

Satisficing changes rather than optimal worthwhile changes Let

$$L_{x^k}(x, x') = \Gamma_{x^k}(x, x') - \frac{\mu_k}{2} R(x, x')$$

be the net marginal motivation to choose function. Then a strong aspiration point $\widetilde{x}^{k+1}$ relative to the statu quo point x^k is such that $L_{x^k}(x, \widetilde{x}^{k+1}) \geq 0$ for all $x \in C$.

If the agent is bounded rational in a complex environment, starting from each statu quo point x^k, it is too costly for the agent to find the true aspiration point $\widetilde{x}^{k+1}$. Then, the agent will be less demanding. He will try to find a "good enough" or satisficing point x^{k+1} such that $L_{x^k}(x, x^{k+1}) \geq -\nu_k$, $\nu_k > 0$ for all $x \in C$. It is an approximate aspiration point relative to the statu quo point x^k.

If, for each x^k, the function $(x, x') \in C \times C \longmapsto L_{x^k}(x, x')$ is continuous, then, for every $\varepsilon_{k+1} > 0$, it exists $\nu_{k+1} = \nu_{k+1}(\varepsilon_{k+1})$ such that

$$q(x^{k+1}, \widetilde{x}^{k+1}) \leq \varepsilon_{k+1} \Longrightarrow L_{x^k}(x, x^{k+1}) \geq -\nu_{k+1},\ \nu_{k+1} > 0 \qquad \forall\, x \in C.$$

Habits, routines and personal equilibrium as limits of strong aspiration points This paper considers a repeated choice problem with inertia (disutility of costs to change), variable preferences, worthwhile changes and goal pursuit (following a path of aspiration points). It shows when a path of aspiration points converges to an equilibrium point (an habit, a routine, a personal equilibrium where it is better to stay than to move. For a proximal approach of habits and routines in an analytical context with quasi distances see also Garcia et al. (2011). Proposition 2.1 tells us that if the motivation to change function is pseudomonotone, a very week hypothesis in the context of the "variational rationality" approach, and under hypothesis **P1** – **P3**, the sets of aspiration points and *equilibria* (stable) points coincide. They are behavioral traps, easy to reach, difficult to leave.

Part 2 of Theorem 3.1 tells us when a sequence of "strong" aspiration points towards which it is worthwhile to move converges to a "free inertia" aspiration point. Moreover, Corollary 3.1 shows that, in the context of Theorem 3.1, a monotone motivation to change function implies the existence of strong aspiration point where it is worthwhile to move. Corollary 3.2 considers the case of a weakly monotone function $f(x, y)$.

The impact of partial inertia Inertia (resistance to change) plays a major role for the convergence of the process $u^k = (x^k, y^k)$. Our paper shows how much more or less inertia matters. Costs to change on the variables x^k and no costs to change on the variables y^k implies the convergence of the path of actions $x^k \longrightarrow x^*$, $k \longrightarrow +\infty$, and the existence of limit points y^* of the sequence $\{y^k\}_{k \in \mathbb{N}}$ (under the boundedness hypothesis on Y_C), where each (x^*, y^*) is an aspiration point (as well as an equilibrium point when $f(u, v)$ is pseudomonotone). Then, the case of unbounded sets Y_C is examined, adding to compensate two hypothesis **P5** and **P6**. Hypothesis **P5** means that inertia is high for the variables x: costs to change $q(x, x')$ on the variable x are high enough in the small: $q(x, x') \geq \alpha\|x' - x\|$, $\alpha > 0$. Hypothesis **P6** supposes a sharp equilibrium point for the motivation to change function.

References

[1] Bertsekas, D., Tseng, P., 1994. Partial proximal minimization algorithms for convex programming. SIAM Journal on Optimization 4, 551–572.

[2] Bianchi, M., Schaible, S., 2004. Equilibrium problems under generalized convexity and generalized monotonicity. Journal of Global Optimization 30, 121–134.

[3] Blum, E., Oettli, W., 1994. From optimization and variational inequalities to equilibrium problems. The Mathematics Student 63, 123–145.

[4] Da Cruz Neto, J., Oliveira, P., Soares, R., Soubeyran, A., 2012. Learning how to play Nash, potential games and alternating method for structured nonconvex problems on Riemannian manifolds. To appear in Journal of Convex Analysis.

[5] Deza, E., Deza, M., 2006. Dictionary of Distances. Elsevier.

[6] Flores-Bazán, F., Luc, D. T., Soubeyran, A., 2012. Maximal elements under reference-dependent preferences with applications to behavioral traps and games. To appear in Journal of Optimization theory and Application. DOI 10.1007/s10957-012-0100-z.

[7] Garcia, F., Oliveira, P., Soubeyran, A., 2011. A proximal algorithm with quasi distance. Application to habit's formation. To appear in Optimization. DOI 10.1080/02331934.2011.564623.

[8] Ha, C., 1990. A generalization of the proximal point algorithm. SIAM Journal on Control and Optimization 28, 503–512.

[9] Hammond, J., Keeney, R., Raiffa, H., 1998. The hidden traps in decision making. Harvard Business Review 76, 3–9.

[10] Harker, P., 1991. Generalized Nash games and quasi-variational inequalities. European Journal of Operational Research 54, 81–94.

[11] Iusem, A., Kassay, G., Sosa, W., 2009. On certain conditions for the existence of solutions of equilibrium problems. Mathematical Programming 116, Ser. B, 259–273.

[12] Konnov, I., 2005. Generalized monotone equilibrium problems and variational inequalities. In Handbook of generalized convexity and generalized monotonicity, Nonconvex Optimization and Its Applications vol. 76, Springer, New York, pp. 559–618

[13] Konnov, I., 2006. Partial proximal point method for nonmonotone equilibrium problems. Optimization Methods & Software 21, 373–384.

[14] Konnov, I., 2007. Combined relaxation methods for generalized monotone variational inequalities. In Generalized convexity and related topics, Lecture Notes in Economics and Mathematical Systems vol. 583, Springer, Berlin, pp. 3–31.

[15] Künzi, H., 1995. Nonsymmetric topology. In Topology with applications, vol. 4 of Bolyai Soc. Math. Stud. János Bolyai Math. Soc., Budapest, pp. 303–338.

[16] Künzi, H., Pajoohesh, H., Schellekens, M., 2006. Partial quasi-metrics. Theoretical Computer Science 365, 237–246.

[17] Martinet, B., 1970. Régularisation d'inéquations variationnelles par approximations successives. Rev. Française Informat. Recherche Opérationnelle 4, 154–158.

[18] Medhi, D., Ha, C., 1996. Generalized proximal point algorithm for convex optimization. Journal of Optimization Theory and Applications 88, 475–488.

[19] Moudafi, A., 1999. Proximal point algorithm extended to equilibrium problems. Journal of Natural Geometry 15, 91–100.

[20] Patriksson, M., 1993. A unified description of iterative algorithms for traffic equilibria. European Journal of Operational Research 71, 154–176.

[21] Polyak, B., 1987. Introduction to optimization. Optimization Software Inc. Publications Division, New York.

[22] Rockafellar, R., 1976. Monotone operators and the proximal point algorithm. SIAM Journal on Control and Optimization 14, 877–898.

[23] Romaguera, S., Sanchis, M., 2003. Applications of utility functions defined on quasi-metric spaces. Journal of Mathematical Analysis and Applications 283, 219–235.

[24] Simon, H., 1955. A behavioral model of rational choice. The Quarterly Journal of Economics 69, 99–118.

[25] Soubeyran, A., 2009. Variational rationality, a theory of individual stability and change: worthwhile and ambidextry behaviors. Mimeo. GREQAM, Aix-Marseille University.

[26] Soubeyran, A., 2010. Variational rationality and the "unsatisfied man": routines and the course between aspirations, capabilities and beliefs. Mimeo. GREQAM, Aix-Marseille University.

[27] Souza, S., Oliveira, P., da Cruz Neto, J., Soubeyran, A., 2010. A proximal method with separable Bregman distances for quasiconvex minimization over the nonnegative orthant. European Journal of Operational Research 201, 365–376.

[28] Stojmirović, A., 2004. Quasi-metric spaces with measure. In Proceedings of the 18th Summer Conference on Topology and its Applications, vol. 28, pp. 655–671.