Closedness of Integer Hulls of Simple Conic Sets

Diego A. Morán R. * , Santanu S. Dey *

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Abstract

Let $\mathbf{C}$ be a full-dimensional pointed closed convex cone in $\mathbb{R}^m$ obtained by taking the conic hull of a strictly convex set. Given $A \in \mathbb{Q}^{m \times n_1}$, $B \in \mathbb{Q}^{m \times n_2}$ and $b \in \mathbb{Q}^m$, a *simple* conic mixed-integer set (SCMIS) is a set of the form $\{(x, y) \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2} \mid Ax + By - b \in \mathbf{C}\}$. In this paper, we give a complete characterization of the closedness of convex hulls of SCMISs. Under certain technical conditions on the cone $\mathbf{C}$, we show that the closedness characterization can be used to construct a polynomial-time algorithm to check the closedness of convex hulls of SCMISs. Moreover, we also show that the Lorentz cone satisfies these technical conditions. In the special case of pure integer problems, we present sufficient conditions, that can be checked in polynomial-time, to verify the closedness of intersection of SCMISs.

1 Introduction

A mixed-integer convex program is the optimization problem of minimizing a linear function over a set of the form $K \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})$, where K is a closed convex set. In this context, it is of interest to study integer hulls of convex sets. The integer hull of a convex set K is defined as the convex hull of the mixed-integer points contained in K. Understanding the structure of integer hulls has proven to be critical in the design of various algorithms for solving mixed-integer programs. In the case of mixed-integer linear programs, a particularly important result in this direction, due to Meyer [7], states that the integer hull of a rational polyhedron is a rational polyhedron.

In this paper, we study properties of integer hulls of simple nonlinear sets of the form:

$$\mathcal{P}_{\mathbf{C}} := \{(x, y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \mid Ax + By - b \in \mathbf{C}\},$$

where A and B are rational matrices of suitable dimensions, b is a rational vector and $\mathbf{C} \subseteq \mathbb{R}^m$ is a full-dimensional closed convex cone obtained by taking the conic hull of a strictly convex set. In contrast to the case of mixed-integer linear programs, the integer hull of $\mathcal{P}_{\mathbf{C}}$ is unlikely a rational polyhedron. We therefore explore a more basic question:

Is the integer hull of $\mathcal{P}_{\mathbf{C}}$ closed?

While the integer hull of $\mathcal{P}_{\mathbf{C}}$ is not always closed, we are able to provide a characterization of when it is closed. Furthermore, if the cone $\mathbf{C}$ satisfies some additional properties, this characterization yields a polynomial-time algorithm to verify whether the integer hull of $\mathcal{P}_{\mathbf{C}}$ is closed or not. We find it interesting that it is possible to construct an algorithm (let alone one that runs in polynomial-time) to test the closedness of integer hulls of unbounded nonlinear sets.

*H. Milton Stewart School of Industrial and Systems Engineering, Georgia Institute of Technology.

2 Main Results

We begin with some definitions. Given a matrix B, we use $\text{Kernel}(B)$ to denote the kernel (null space) of the matrix B and $\langle B \rangle$ to denote the linear subspace generated by the columns of the matrix B. The usual Euclidian norm is denoted by $\|\cdot\|$. For a set X we denote its dimension by $\dim(X)$, its relative interior by $\text{rel.int}(X)$, its interior by $\text{int}(X)$, its closure by $\text{cl}(X)$, its boundary by $\text{bd}(X) := \text{cl}(X) \setminus \text{int}(X)$, its relative boundary by $\text{rel.bd}(X) := \text{cl}(X) \setminus \text{rel.int}(X)$, its recession cone by $\text{rec.cone}(X) := \{d \in \mathbb{R}^n \,|\, x + \lambda d \in X,\ \forall x \in X,\ \forall \lambda \geq 0\}$ and its lineality space by $\text{lin.space}(X) = \{d \in \mathbb{R}^n \,|\, x + \lambda d \in X,\ \forall x \in X,\ \forall \lambda \in \mathbb{R}\}$. The convex hull of X is the set $\text{conv}(X) = \{\sum_{i=1}^{N} \lambda_i x_i \,|\, \lambda_i \geq 0,\ x_i \in X, \forall\ i,\ \sum_i \lambda_i = 1,\ N \in \mathbb{N}\}$. The conic hull of X is the set $\text{cone}(X) = \{\sum_{i=1}^{N} \lambda_i x_i \,|\, \lambda_i \geq 0,\ x_i \in X, \forall\ i,\ N \in \mathbb{N}\}$.

Definition 1. *A strictly convex set is a convex set S such that for all $x, y \in S$, $x \neq y$ and for all $\alpha \in (0, 1)$ we have $\alpha x + (1 - \alpha)y \in \text{rel.int}(S)$.*

We say that a convex cone is pointed if it does not contains lines. In this paper, we will only consider convex cones that are pointed and closed.

Definition 2. *A generator for a pointed closed convex cone $\mathbf{C} \subseteq \mathbb{R}^m$ is a bounded closed convex set $\mathbf{S} \subseteq \mathbb{R}^m$ of dimension $\dim(\mathbf{C}) - 1$ such that $\mathbf{C} = \text{cone}(\mathbf{S})$. We say that $\mathbf{C}$ is generated by $\mathbf{S}$.*

Notice that the previous definition implies that: (1) For all $x \in \mathbf{C} \setminus \{0\}$, there exists a unique $\lambda > 0$ such that $\lambda x \in \mathbf{S}$; and (2) The extreme rays of $\mathbf{C}$ are in one to one correspondence with the extreme points of $\mathbf{S}$, in the sense that $r \in \mathbf{C}$ is an extreme ray of $\mathbf{C}$ if and only if r can be scaled by a positive scalar to be an extreme point of $\mathbf{S}$.

2.1 Characterization of closedness

Our first result is a characterization of the closedness of integer hulls of simple conic sets.

Theorem 1. *Let $\mathbf{C} \subseteq \mathbb{R}^m$ be a full-dimensional pointed closed convex cone that is generated by a strictly closed convex set. Let $A \in \mathbb{Q}^{m \times n_1}$, $B \in \mathbb{Q}^{m \times n_2}$ and $b \in \mathbb{Q}^m$. Let*

$$\mathcal{P}_{\mathbf{C}} := \{(x, y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \,|\, Ax + By - b \in \mathbf{C}\}, \tag{1}$$

$V := \{Ax + By - b \,|\, (x, y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}\}$ and $\mathcal{L} := \{Ax + By \,|\, (x, y) \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}\}$. Then $\text{conv}\left(\mathcal{P}_{\mathbf{C}} \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})\right)$ is closed if and only if one of the following holds:

1. *$b \notin \mathcal{L}$.*
2. *$b \in \mathcal{L}$, and $\dim(\mathbf{C} \cap V) \leq 1$.*
3. *$b \in \mathcal{L}$, $\dim(\mathbf{C} \cap V) = 2$, $\dim(\langle B \rangle) \leq 0$ and the two extreme rays of the cone $\mathbf{C} \cap V$ can be scaled by a non-zero scalar so that they belong to the lattice $\{Ax \,|\, x \in \mathbb{Z}^{n_1}\}$.*
4. *$b \in \mathcal{L}$, $\dim(\mathbf{C} \cap V) \geq 2$ and $\dim(\langle B \rangle) \geq \dim(V) - 1$.*

The proof of Theorem 1 relies on three sets of results: (1) Understanding when affine rational maps preserve closedness. (2) In a recent paper [4] we presented some properties on the closedness of integer hulls of closed convex sets in the pure integer case, with applications to strictly convex sets and cones. We generalize these results from the pure integer case to the mixed-integer case. (3) Geometric properties of cones generated by strictly convex sets. We present a proof of Theorem 1 in Section 3.

2.2 Complexity of checking closedness

For $\mathcal{P}_{\mathbf{C}}$ given by (1) we denote by $\text{size}(\mathcal{P}_{\mathbf{C}})$ the sum of the size of the (usual) binary representation of the matrices A, B and b. (That is for a matrix $M \in \mathbb{R}^{m\times n}$, we have $\text{size}(M) = mn + \sum_{i=1}^{m}\sum_{j=1}^{n}\log(\lceil m_{ij}\rceil + 1)$.)

Definition 3. *A pointed closed convex cone $\mathbf{C} \subseteq \mathbb{R}^m$ is said to be poly-checkable if for all $A \in \mathbb{Q}^{m\times n_1}$, $B \in \mathbb{Q}^{m\times n_2}$ the following can be done in polynomial time with respect to $\text{size}([A\ B])$: (I) To decide whether $\dim(\mathbf{C} \cap \langle [A\ B]\rangle) \leq 1$ or not, and, whenever $\dim(\mathbf{C} \cap \langle [A\ B]\rangle) \geq 2$, to compute $\dim(\mathbf{C} \cap \langle [A\ B]\rangle)$; and (II) Checking condition (3.) of Theorem 1. (Notice that checking condition (3.) of Theorem 1 only depends on $\text{size}(A)$).*

Theorem 1 yields a polynomial-time algorithm to check the closedness of integer hulls of simple conic sets whenever the cone $\mathbf{C}$ is poly-checkable. Formally, we have the following result.

Theorem 2. *Let $\mathbf{C} \subseteq \mathbb{R}^m$ be a full-dimensional pointed closed convex cone that is generated by a strictly closed convex set. Assume that $\mathbf{C}$ is poly-checkable. Let $A \in \mathbb{Q}^{m\times n_1}$, $B \in \mathbb{Q}^{m\times n_2}$, $b \in \mathbb{Q}^m$ and let $\mathcal{P}_{\mathbf{C}}$ be as defined in (1). Then there exists an algorithm that runs in polynomial-time with respect to $\text{size}(\mathcal{P}_{\mathbf{C}})$ to check whether $\text{conv}(\mathcal{P}_{\mathbf{C}} \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$ is closed or not.*

The algorithm in Theorem 2 is constructed by showing that each of the cases in Theorem 1 can be verified in polynomial-time. We present a proof of Theorem 2 in Section 4.

2.3 The Lorentz cone is poly-checkable

The Lorentz cone $\mathbf{L}^m \subseteq \mathbb{R}^m$ is defined as $\mathbf{L}^m := \{(w, z) \in \mathbb{R}^{m-1} \times \mathbb{R} \mid \|w\| \leq z\}$. The following result shows that the class of poly-checkable cones contains the Lorentz cone.

Theorem 3. *The Lorentz cone is poly-checkable.*

Among the two conditions that we need to verify in order to prove that the Lorentz cone is poly-checkable, the most 'interesting' is (II): To check condition (3.) in Theorem 1 in polynomial-time, the key idea is to reduce the verification of this condition to whether a suitable number is a perfect square, via the use of the Hermite normal form algorithm and properties of the Lorentz cone. We present a proof of Theorem 3 in Section 5.

Notice that as a consequence of Theorem 2 and Theorem 3 we obtain that there exist a polynomial-time algorithm to check the closedness of integer hulls of simple second order conic sets.

2.4 A property of integer hulls

In the case of pure integer programs, we prove the following result.

Theorem 4. *Let $K_i \subseteq \mathbb{R}^n$, $i = 1, 2$, be closed convex sets. Assume $\text{conv}(K_i \cap \mathbb{Z}^n)$ is closed, for $i = 1, 2$. If $L = \text{lin.space}(K_1 \cap K_2)$ is generated by integer points, then $\text{conv}\left[(K_1 \cap K_2) \cap \mathbb{Z}^n\right]$ is closed.*

The proof of Theorem 4 uses as its building block a charaterization of closedness of integer hulls of general closed convex sets from [4]. A proof of this result is presented in Section 6. We obtain the following straightforward corollary to Theorem 4.

Corollary 1. *Consider the sets $\mathcal{P}_{\mathbf{C}i} := \{x \in \mathbb{R}^n \mid\ A_i x - b_i \in \mathbf{C}^{m_i}\}$, where for all $i = 1, \ldots, q$, we have $A_i \in \mathbb{Q}^{m_i\times n}$, $b_i \in \mathbb{Q}^{m_i}$, and $\mathbf{C}^{m_i} \subseteq \mathbb{R}^{m_i}$ is a poly-checkable pointed closed convex cone in $\mathbb{R}^{m_i}$ that is generated by a strictly convex set. If the integer hull of $\mathcal{P}_{\mathbf{C}i}$ is closed for all $i = 1, \ldots, q$, then $\text{conv}(\bigcap_{i=1}^{q} \mathcal{P}_{\mathbf{C}i} \cap \mathbb{Z}^n)$ is closed.*

Notice that by the application of Theorem 2 for each $\mathcal{P}_{\mathbf{C}i}$, the sufficient condition of Corollary 1 can be verified in polynomial-time in the size of the input data. We finally note that Theorem 4 does not hold for the mixed-integer case as illustrated in the next example.

Example 1. *Let* $K_1 = \{(x,y) \in \mathbb{R}^2_+ \times \mathbb{R}_+ \,|\, y \geq x_2 - \sqrt{2}x_1\}$ *and* $K_2 = \{(x,y) \in \mathbb{R}^2_+ \times \mathbb{R}_+ \,|\, y \geq \sqrt{2}x_1 - x_2\}$. *It is straightforward to check that* $\text{conv}(K_1 \cap (\mathbb{Z}^2 \times \mathbb{R})) = K_1$ *and that* $\text{conv}(K_2 \cap (\mathbb{Z}^2 \times \mathbb{R})) = K_2$. *Thus, the integer hulls of* K_1 *and* K_2 *are closed. However, we will verify next that* $\text{conv}((K_1 \cap K_2) \cap (\mathbb{Z}^2 \times \mathbb{R}))$ *is not closed. Denote* $X = \{(x,y) \in \mathbb{R}^2_+ \times \mathbb{R}_+ \,|\, y = 0\}$. *Let* $r := \{\lambda(1, \sqrt{2}, 0) \,|\, \lambda \geq 0\} = K_1 \cap K_2 \cap X$. *Thus,* r *is a ray with irrational slope contained in* X. *By the application of Dirichlet Approximation Theorem, we can verify that there are mixed-integer points* $(x,y) \in \mathbb{Z}^2_+ \times \mathbb{R}_+$ *in* $K_1 \cap K_2$ *that are arbitrarily close to the ray* r. *This implies that* r *belongs to the closure of* $\text{conv}((K_1 \cap K_2) \cap (\mathbb{Z}^2 \times \mathbb{R}))$. *On the other hand, since* r *is a face of* $K_1 \cap K_2$ *and* $(0,0,0)$ *is the only mixed-integer point in this face, we obtain that* $r \cap \text{conv}((K_1 \cap K_2) \cap (\mathbb{Z}^2 \times \mathbb{R})) = \{(0,0,0)\}$. *Therefore, we conclude that* $\text{conv}((K_1 \cap K_2) \cap (\mathbb{Z}^2 \times \mathbb{R}))$ *is not a closed set.*

We note here that Example 1 does not exclude the possibility of a result of the form of Corollary 1 for the mixed-integer case when each of the simple second order conic sets are defined using rational data. We have not been able to resolve this question.

We note here that an extended abstract containing some of the results of this paper appeared in [5].

3 Proof of Theorem 1

We first present in Section 3.1 a sketch of the proof of Theorem 1. In particular, we specify the crucial results needed in the proof. Then we present some basic results needed to prove Theorem 1 in Section 3.2 (Properties of convex sets), Section 3.3 (Properties of cones generated by strictly convex sets) and Section 3.4 (Properties of mixed-integer lattices). Next, we present the proofs of the crucial results mentioned in Section 3.1: Proposition 2 is proved in Section 3.5, Proposition 3 and Proposition 5 are proved in Section 3.6 and the proofs of Proposition 4 and Proposition 6 can be found in Section 3.7. The final step of the proof of Theorem 1 is presented in Section 3.8.

3.1 Sketch of Proof of Theorem 1

We present a definition of mixed-integer lattices before presenting the sketch of our proof.

Definition 4 (Mixed-integer lattice [2])**.** *Let* $A = [a_1|\cdots|a_{n_1}] \in \mathbb{R}^{m \times n_1}$ *and* $B = [b_1|\cdots|b_{n_2}] \in \mathbb{R}^{m \times n_2}$, *where* $\{a_1, \ldots, a_{n_1}, b_1, \ldots, b_{n_2}\}$ *is a linearly independent set of* $\mathbb{R}^m$. *Then*

$$\mathcal{L} = \{x \in \mathbb{R}^m \,|\, x = Az + By,\ z \in \mathbb{Z}^{n_1}, y \in \mathbb{R}^{n_2}\}$$

is said to be the mixed-integer lattice generated by A *and* B.

We note here that in the case $A \in \mathbb{Q}^{m \times n_1}$, $B \in \mathbb{Q}^{m \times n_2}$ it can be proved that a set $\mathcal{L}$ defined as above is a mixed-integer lattice, even in the case the set $\{a_1, \ldots, a_{n_1}, b_1, \ldots, b_{n_2}\}$ is not linearly independent (see Proposition 7 in Section 3.4).

An affine subspace W is said to be generated by a mixed-integer lattice $\mathcal{L}$ if $W = \text{aff}(W \cap \mathcal{L})$. In the special case $\mathcal{L} = \mathbb{Z}^n$ we also say that the affine subspace is rational.

Proof Outline:

1. **Simplifying the set** $\mathcal{P}_{\mathbf{C}} := \{(x,y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \,|\, Ax + By - b \in \mathbf{C}\}$**.** To simplify the analysis, we apply the affine map $T : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^m$ defined as $T(x,y) = Ax + By - b$ to the set $(\mathcal{P}_{\mathbf{C}} \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$. The image of $(\mathcal{P}_{\mathbf{C}} \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$ under the map T is the set $((\mathbf{C} \cap V) \cap (\mathcal{L} - b))$ where $V := \{Ax + By - b \,|\, (x,y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}\}$ is an affine subspace and $\mathcal{L} := \{Ax + By \,|\, (x,y) \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}\}$ is a mixed-integer lattice (since A and B are rational matrices). Thus, we obtain the 'simple' set $\mathbf{C} \cap V$ in place of $\mathcal{P}_{\mathbf{C}}$, at the cost of a 'complicated' translated mixed-integer lattice $\mathcal{L} - b$ in place of the mixed-integer lattice $\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$. The closedness of a set is usually not invariant under affine transformations. However, we verify the following result:

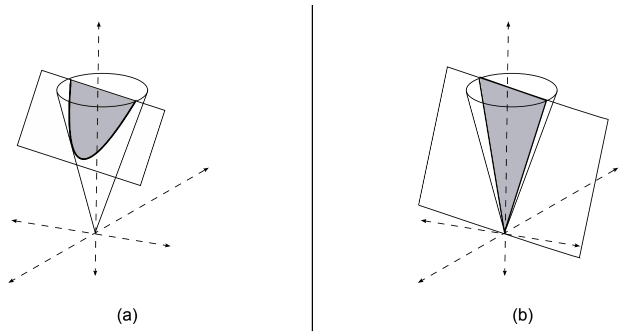

Figure 1: Different cases for $\mathbf{C} \cap V$: (a) Strictly convex set (b) Pointed closed convex cone.

Proposition 2. *Let $K \subseteq \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ be a closed convex set. Let $G : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^m$ defined as $G(x, y) := Ex + Fy - g$ be an affine map, where $E \in \mathbb{R}^{m \times n_1}$, $F \in \mathbb{R}^{m \times n_2}$ and $g \in \mathbb{R}^m$. Assume that E, F satisfy the following:*

(a) $\text{Kernel}([E\ F]) \subseteq \text{lin.space}(K)$,

(b) $\text{Kernel}([E\ F])$ *is generated by points in the mixed-integer lattice* $\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$.

Then

$$\text{conv}(K \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})) \textit{ is closed} \Leftrightarrow \text{conv}\left[G(K) \cap G(\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})\right] \textit{ is closed.}$$

As a consequence of Proposition 2 applied to the affine mapping $T(x, y) = Ax + By - b$ we obtain that

$$\text{conv}\left(\mathcal{P}_{\mathbf{C}} \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})\right) \text{ is closed} \Leftrightarrow \text{conv}\left[(\mathbf{C} \cap V) \cap (\mathcal{L} - b)\right] \text{ is closed.} \tag{2}$$

2. **Case Analysis.** Next we analyze the set $\mathbf{C} \cap V$. Observe that since $\mathbf{C}$ is a cone generated by a closed strictly convex set and V is an affine set, there are two natural cases (see Figure 1):

   (a) **Case 1: $\mathbf{C} \cap V$ is strictly convex set.** If $0 \notin V$, then $\mathbf{C} \cap V$ is a strictly convex set. We verify the following result.

   Proposition 3. *Let $K \subseteq \mathbb{R}^n$ be a closed strictly convex set, $t \in \mathbb{R}^n$ and $\mathcal{L}$ a mixed-integer lattice. Then* $\text{conv}(K \cap [\mathcal{L} + t])$ *is closed.*

   Proposition 3 is a generalization of a result about integer hulls of strictly convex sets from [4]. As a consequence of Proposition 3 and (2) we obtain that $\text{conv}(\mathcal{P}_{\mathbf{C}} \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$ is always closed in this case. Observe that this is case (1.) in Theorem 1 when $0 \notin V$.

   (b) **Case 2: $\mathbf{C} \cap V$ is a cone.** If $0 \in V$, then $V = \langle [A\ B] \rangle = \text{aff}(\mathcal{L})$ and $\mathbf{C} \cap V$ is a closed pointed convex cone. We have two subcases.

   Subcase 1: $b \notin \mathcal{L}$. In this case, $\mathcal{L} - b \neq \mathcal{L}$. Moreover, $\mathcal{L} - b$ is not a mixed-integer lattice. We need the following property.

   Proposition 4. *Let $\mathbf{C} \subseteq \mathbb{R}^m$ be a full-dimensional pointed closed convex cone that is generated by a closed strictly convex set. Let $\mathcal{L} = \{x \in \mathbb{R}^m \mid x = Az + By,\ z \in \mathbb{Z}^{p_1}, y \in \mathbb{R}^{p_2}\}$ be a mixed-integer lattice, where A, B are rational matrices. Denote $V = \text{aff}(\mathcal{L})$, and let $b \in (V \cap \mathbb{Q}^m) \setminus \mathcal{L}$. Then* $\text{conv}((\mathbf{C} \cap V) \cap (\mathcal{L} - b))$ *is closed.*

   This result is a consequence of some properties of the closedness of integer hulls of general closed convex sets from [4]. We can apply Proposition 4 to verify that $\text{conv}\left[(\mathbf{C} \cap V) \cap (\mathcal{L} - b)\right]$ is a

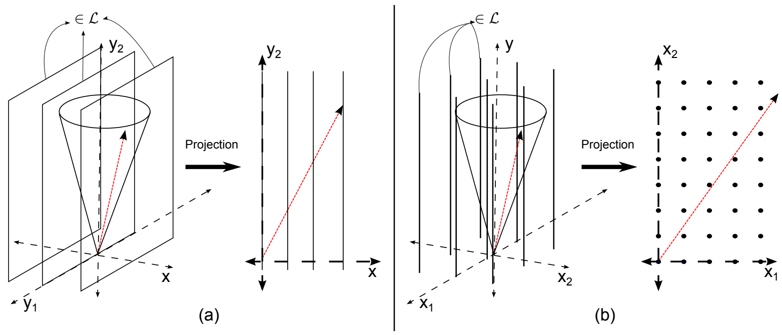

Figure 2: $V = \mathbb{R}^3$. Extreme rays of $\mathbf{C} \cap V$: (a) All scalable to belong to $\mathcal{L} = \mathbb{Z} \times \mathbb{R}^2$ (b) Not all scalable to belong to $\mathcal{L} = \mathbb{Z}^2 \times \mathbb{R}$.

closed set in this case. Notice this is case (1.) in Theorem 1 when $0 \in V$. In particular, this completes the examination of (1.) in Theorem 1.

Subcase 2: $b \in \mathcal{L}$**.** We begin the analysis of this case by verifying the following result.

Proposition 5. *Let C be a closed pointed convex cone in $\mathbb{R}^n$ and let $\mathcal{L}$ be a mixed-integer lattice. Then $\overline{\text{conv}}(C \cap \mathcal{L}) = C \cap W$, where $W = \text{aff}(C \cap \mathcal{L})$. In particular, $\text{conv}(C \cap \mathcal{L})$ is closed if and only if every extreme ray of $C \cap W$ can be scaled by a non-zero scalar to belong to $\mathcal{L}$.*

Proposition 5 is a generalization of a result about integer hulls of cones from [4]. As a consequence of Proposition 5, verifying closedness is equivalent to verifying whether the extreme rays of $\mathbf{C} \cap V$ can be scaled by a non-zero number to belong to $\mathcal{L}$.

When $\dim(\mathbf{C} \cap V) \leq 1$, it is straighforward to verify that this is always the case. This is case (2.) in Theorem 1.

For analyzing the case where $\dim(\mathbf{C} \cap V) > 1$ we need the following additional result.

Proposition 6. *Let $\mathbf{C} \subseteq \mathbb{R}^m$ be a full-dimensional pointed closed convex cone that is generated by the closed strictly convex set. Let $\mathcal{L} = \{x \in \mathbb{R}^m \mid x = Az + By,\ z \in \mathbb{Z}^{n_1}, y \in \mathbb{R}^{n_2}\}$ be a mixed-integer lattice, where A, B are rational matrices. Denote $V = \text{aff}(\mathcal{L})$. Then*

i. Assume $\dim(\mathbf{C} \cap V) = 2$. If $\dim(\langle B \rangle) \geq \dim(V) - 1$, then every extreme ray of $\mathbf{C} \cap V$ can be scaled by a non-zero scalar to belong to $\mathcal{L}$.

ii. Assume $\dim(\mathbf{C} \cap V) \geq 3$. Then $\dim(\langle B \rangle) \geq \dim(V) - 1$ if and only if every extreme ray of $\mathbf{C} \cap V$ can be scaled by a non-zero scalar to belong to $\mathcal{L}$.

The proof of Proposition 6 is based on the cardinality of the set of extreme rays in different dimensions (countable or not) and other geometric properties of the cone $\mathbf{C} \cap V$.

Proposition 6 is essentially stating that when $\dim(\mathbf{C} \cap V) \geq 3$, in order for every extreme ray to be scalable to belong to the mixed-integer lattice $\mathcal{L}$, there should be "sufficient" number of continuous components in the mixed-integer lattice $\mathcal{L}$. See Figure 2 for an illustration. Therefore we obtain that if $\dim(\mathbf{C} \cap V) \geq 3$, then $\text{conv}\,(\mathcal{P}_{\mathbf{C}} \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$ is closed if and only if $\dim(\langle B \rangle) \geq \dim(V) - 1$. Moreover if $\dim(\mathbf{C} \cap V) = 2$ and $\dim(\langle B \rangle) \geq 1$, then $\text{conv}\,(\mathcal{P}_{\mathbf{C}} \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$ is also closed. Together, this constitutes case (4.) in Theorem 1.

The only case that remains is where $\dim(\mathbf{C} \cap V) = 2$ and $\dim(\langle B \rangle) \leq 0$. In this case, we need to explicitly check whether the two extreme rays of $\mathbf{C} \cap V$ can be scaled by a non-zero scalar to belong to the lattice $\mathcal{L}$. This is case (3.) in Theorem 1.

3.2 Properties of convex sets

The proofs of the following lemmas are standard and hence omitted.

Lemma 1. *Let $K \subseteq \mathbb{R}^n$ be a convex set and let $W \subseteq \mathbb{R}^n$ be an affine subspace. If* $\text{rel.int}(K) \cap W \neq \emptyset$*, then*

1. $\text{aff}(K \cap W) = \text{aff}(K) \cap W$.
2. $\text{rel.int}(K \cap W) = \text{rel.int}(K) \cap W$.
3. $\text{rel.bd}(K \cap W) = \text{rel.bd}(K) \cap W$.

Lemma 2. *Let $X \subseteq \mathbb{R}^n$ be a closed strictly convex set and $W \subseteq \mathbb{R}^n$ be an affine subspace. Then $X \cap W$ is a strictly convex set.*

3.3 Properties of cones generated by strictly convex sets

We will use the following lemmas.

Lemma 3. *Let $\mathbf{C} \subseteq \mathbb{R}^m$ be a full-dimensional pointed closed convex cone that is generated by a closed strictly convex set, and let $W \subseteq \mathbb{R}^m$ be an affine subspace. Assume that* $\dim(\mathbf{C} \cap W) \geq 2$*. Then*

1. $\text{int}(\mathbf{C}) \cap W \neq \emptyset$*. Consequently, we obtain that* $\dim(\mathbf{C} \cap W) = \dim(W)$ *and* $\text{rel.bd}(\mathbf{C} \cap W) = \text{bd}(\mathbf{C}) \cap W$.
2. $(\text{bd}(\mathbf{C}) \setminus \{0\}) \cap W \neq \emptyset$.

Proof.

1. Since $\dim(\mathbf{C} \cap W) \geq 2$, there exist three affinely independent points $x, y, z \in \mathbf{C} \cap W$. If $\{x, y, z\} \cap \text{int}(\mathbf{C}) \cap W \neq \emptyset$, we are done. Now, assume that $x, y, z \in \text{bd}(\mathbf{C}) \cap W$. Since $\{x, y, z\}$ is an affinely independent set, we may assume without loss of generality that $x, y \neq 0$ and $x \notin \{\lambda y \mid \lambda \geq 0\}$. Let $\hat{x}, \hat{y} \in \mathbf{S}$ such that $x = \alpha\hat{x}$ and $y = \beta\hat{y}$ for some $\alpha, \beta > 0$. Notice that $\hat{x} \neq \hat{y}$. Since $\mathbf{S}$ is a strictly convex set, we obtain that $\text{conv}(\{\hat{x}, \hat{y}\}) \cap \text{rel.int}(\mathbf{S}) \neq \emptyset$. Let $\lambda\hat{x} + (1-\lambda)\hat{y} \in \text{rel.int}(\mathbf{S})$, where $\lambda \geq 0$. Then we have

$$w := \frac{1}{\left(\frac{\lambda}{\alpha} + \frac{(1-\lambda)}{\beta}\right)}(\lambda\hat{x} + (1-\lambda)\hat{y}) = \frac{\frac{\lambda}{\alpha}x + \frac{(1-\lambda)}{\beta}y}{\frac{\lambda}{\alpha} + \frac{(1-\lambda)}{\beta}}. \tag{3}$$

   Since $\lambda\hat{x} + (1-\lambda)\hat{y} \in \text{rel.int}(\mathbf{S})$, we obtain $w \in \text{int}(\mathbf{C})$. Therefore, by (3) we obtain $w \in \text{conv}(\{x, y\}) \cap \text{int}(\mathbf{C})$. Since $\text{conv}(\{x, y\}) \subseteq W$, we conclude that $\text{int}(\mathbf{C}) \cap W \neq \emptyset$.

   Finally, the facts that $\dim(\mathbf{C} \cap W) = \dim(W)$ and $\text{rel.bd}(\mathbf{C} \cap W) = \text{bd}(\mathbf{C}) \cap W$ are straightforward consequences of Lemma 1.

2. By (1.) we obtain $\dim(\text{int}(\mathbf{C}) \cap W) = \dim(\text{rel.int}(\mathbf{C} \cap W)) \geq 2$. Thus, there exist vectors $w_0, w_1, w_2 \in \text{int}(\mathbf{C}) \cap W$ such that $w_1 - w_0$ and $w_2 - w_0$ are linearly independent vectors. For $i = 1, 2$ consider the line generated by w_0 and w_i, that is $L_i = \{\alpha_i w_0 + (1 - \alpha_i)(w_i - w_0) \mid \alpha_i \in \mathbb{R}\}$. Since $\mathbf{C}$ is a pointed cone we have that $L_i \nsubseteq \text{int}(\mathbf{C})$. Thus, since $L_i \cap \text{int}(\mathbf{C}) \neq \emptyset$, we obtain $L_i \cap \text{bd}(\mathbf{C}) \neq \emptyset$. For $i = 1, 2$ consider $x_i \in L_i \cap \text{bd}(\mathbf{C})$. Since $w_1 - w_0$ and $w_2 - w_0$ are linearly independent vectors, we have that $L_1 \neq L_2$. Hence, since $w_0 \in L_1 \cap L_2$, we obtain that $x_1 \neq x_2$. In particular, without loss of generality we may assume that $x_1 \neq 0$. Since W is an affine subspace, we have that $x_1 \in L_1 \subseteq W$. Therefore, we conclude that $(\text{bd}(\mathbf{C}) \setminus \{0\}) \cap W \neq \emptyset$.

□

The next lemma states the two possible cases for the structure of the set $\mathbf{C} \cap W$, where $\mathbf{C}$ is a full-dimensional pointed closed convex cone that is generated by a closed strictly convex set and W is an affine subspace.

Lemma 4. *Let $\mathbf{C} \subseteq \mathbb{R}^m$ be a full-dimensional pointed closed convex cone that is generated by a closed strictly convex set $\mathbf{S} \subseteq \mathbb{R}^m$, and let $W \subseteq \mathbb{R}^m$ be an affine subspace. Then*

1. *If $0 \notin W$, then $\mathbf{C} \cap W$ is a strictly convex set.*
2. *If $0 \in W$ and $\dim(\mathbf{C} \cap W) \geq 2$, then*
   (a) *$\mathbf{C} \cap W$ is a pointed closed convex cone of dimension $\dim(W)$.*
   (b) *$\mathbf{S} \cap W$ is a strictly convex set and a generator for $\mathbf{C} \cap W$.*
   (c) *If $\dim(W) \geq 3$, then $\mathbf{C} \cap W$ has an uncountable number of extreme rays. Equivalently,* $\text{rel.bd}(\mathbf{S} \cap W)$ *is uncountable.*

Proof.

1. The case $\dim(\mathbf{C} \cap W) \leq 1$ is straightforward. Assume $\dim(\mathbf{C} \cap W) \geq 2$. Let F be a proper face of $\mathbf{C} \cap W$, and let $x, y \in F$. We will show that $F = \{x\}$. By (1.) of Lemma 3 we have that $\text{rel.bd}(\mathbf{C} \cap W) = \text{bd}(\mathbf{C}) \cap W$. Therefore, since F is a proper face of $\mathbf{C} \cap W$, we obtain that x and y belong to a face of $\mathbf{C}$. Since $\mathbf{C}$ is generated by a strictly convex set, then all of its faces have dimension one. Thus, we obtain that there exists $\lambda \geq 0$, such that $y = \lambda x$. Since W is an affine subspace, we have that the set $L := \{x + \alpha(y - x) \,|\, \alpha \in \mathbb{R}\} = \{x + \alpha(\lambda - 1)x \,|\, \alpha \in \mathbb{R}\}$ is a subset of W. This implies we must have $\lambda = 1$, for otherwise, we obtain that $0 \in L \subseteq W$. Thus, $F = \{x\}$. Therefore, we conclude that $\mathbf{C} \cap W$ is a strictly convex set.

2. (a) Since $\mathbf{C}$ and W are cones, we obtain that $\mathbf{C} \cap W$ is also a cone. The fact that $\dim(\mathbf{C} \cap W) = \dim(W)$ follows directly from (1.) of Lemma 3.

   (b) Since $\mathbf{S}$ is a strictly convex set, by Lemma 2 we conclude that $\mathbf{S} \cap W$ is also a strictly convex set.

   In order to prove that $\mathbf{S} \cap W$ is a generator for $\mathbf{C} \cap W$ we need to show that $\mathbf{C} \cap W = \text{cone}(\mathbf{S} \cap W)$ and that $\dim(\mathbf{S} \cap W) = \dim(\mathbf{C} \cap W) - 1$.

   - We prove next that $\mathbf{C} \cap W = \text{cone}(\mathbf{S} \cap W)$. Clearly, $\text{cone}(\mathbf{S} \cap W) \subseteq \mathbf{C} \cap W$. We now prove the inclusion $\mathbf{C} \cap W \subseteq \text{cone}(\mathbf{S} \cap W)$. Let $r \in \mathbf{C} \cap W$ with $r \neq 0$. Then, by definition of $\mathbf{C}$, there exists $\hat{r} \in \mathbf{S}$ such that $r = \alpha \hat{r}$ for some $\alpha > 0$. Notice that since $r \in \mathbf{C} \cap W$, the ray $\{\lambda r \,|\, \lambda \geq 0\} \subseteq \mathbf{C} \cap W$. Thus, we obtain that $\hat{r} \in W$. Since $r = \alpha \hat{r}$ and $\hat{r} \in \mathbf{S} \cap W$, we conclude that $r \in \text{cone}(\mathbf{S} \cap W)$.
   - We now prove that $\dim(\mathbf{S} \cap W) = \dim(\mathbf{C} \cap W) - 1$. Observe first that since $\dim(\mathbf{C} \cap W) \geq 2$, we that that $\dim(\mathbf{C} \cap W) = \dim(W)$. We therefore need to verify that $\dim(\mathbf{S} \cap W) = \dim(W) - 1$.
     Next we claim $\text{rel.int}(\mathbf{S}) \cap W \neq \emptyset$: Since $\dim(\mathbf{C} \cap W) \geq 2$, we obtain by (1.) of Lemma 3 that $\text{int}(\mathbf{C}) \cap W \neq \emptyset$. Let $r \in \text{int}(\mathbf{C}) \cap W$. By definition of $\mathbf{C}$, there exists $\hat{r} \in \mathbf{S}$ such that $r = \alpha \hat{r}$ for some $\alpha > 0$. Since W is a subspace, we obtain $\hat{r} \in W$. Morever, since $r \in \text{int}(\mathbf{C})$, we conclude that $\hat{r} \in \text{rel.int}(\mathbf{S}) \cap W$. Thus, $\text{rel.int}(\mathbf{S}) \cap W \neq \emptyset$.
     Observe that $0 \notin \text{aff}(\mathbf{S})$, for otherwise we would have $\mathbf{C} \subseteq \text{aff}(\mathbf{S})$, a contradiction with the fact $\dim(\mathbf{S}) = \dim(\mathbf{C}) - 1$. Since $\text{rel.int}(\mathbf{S}) \cap W \neq \emptyset$, by Lemma 1 we obtain that $\text{aff}(\mathbf{S} \cap W) = \text{aff}(\mathbf{S}) \cap W$. Therefore, since $0 \in W$ and $0 \notin \text{aff}(\mathbf{S})$, we obtain that $\text{aff}(\mathbf{S} \cap W)$ is strictly contained in W. Thus, $\dim(\mathbf{S} \cap W) \leq \dim(W) - 1$.
     On the other hand, we have
     $$\mathbf{C} \cap W = \text{cone}(\mathbf{S} \cap W) \subseteq \text{aff}((\mathbf{S} \cap W) \cup \{0\}).$$
     Thus, since $0 \notin \mathbf{S} \cap W$, we obtain that $\dim(\mathbf{C} \cap W) \leq \dim(\mathbf{S} \cap W) + 1$. Hence, since $\dim(\mathbf{C} \cap W) = \dim(W)$, we conclude that $\dim(\mathbf{S} \cap W) \geq \dim(W) - 1$.

(c) By (2.), we obtain that $\dim(\mathbf{S}\cap W) = \dim(W)-1$. Thus, $\dim(\mathbf{S}\cap W) \geq 2$. Therefore, since $\mathbf{S}\cap W$ is a bounded and closed convex set of dimension at least 2, we must have that $\text{rel.bd}(\mathbf{S}\cap W)$ is uncountable.

□

3.4 Properties of mixed-integer lattices

The next proposition shows that if the data defining the set $\mathcal{L} = \{x \in \mathbb{R}^m \mid x = Az+By,\ z \in \mathbb{Z}^{n_1}, y \in \mathbb{R}^{n_2}\}$ is rational, then $\mathcal{L}$ is a mixed-integer lattice, even in the case the matrix $[A\ B]$ does not have linearly independent columns.

Proposition 7. *Let $\mathcal{L} = \{x \in \mathbb{R}^m \mid x = Az + By,\ z \in \mathbb{Z}^{n_1}, y \in \mathbb{R}^{n_2}\}$ and denote $W = \langle B\rangle$. Assume that the data defining $\mathcal{L}$ is rational, that is, $A \in \mathbb{Q}^{m\times n_1}$ and $B \in \mathbb{Q}^{m\times n_2}$. Then there exists $p_1 \leq n_1$, a matrix $A' \in \mathbb{Q}^{m\times p_1}$, whose columns are linearly independent and contained in $W^\perp$ and there exists $p_2 \leq n_2$, a matrix $B' \in \mathbb{Q}^{m\times p_2}$, whose columns are linearly independent and $\langle B'\rangle = \langle B\rangle$, such that*

$$\mathcal{L} = \{x \in \mathbb{R}^m \mid x = A'z + B'y,\ z \in \mathbb{Z}^{p_1}, y \in \mathbb{R}^{p_2}\}.$$

Proof.

Denote by $\text{Proj}_{W^\perp}(x)$ the orthogonal projection of $x \in \mathbb{R}^m$ on $W^\perp$. Since the columns of A are rational, the set $\mathcal{M} = \{x \in \mathbb{R}^m \mid x = Az,\ z \in \mathbb{Z}^{n_1}\}$ is a lattice. Moreover, since W is a rational subspace, we have that $\text{Proj}_{W^\perp}(\mathcal{M})$ is a lattice generated by rational vectors. Let $A' \in \mathbb{Q}^{m\times p}$ be matrix whose columns form a basis of $\text{Proj}_{W^\perp}(\mathcal{M})$, that is, the columns of A' are linearly independent and

$$\text{Proj}_{W^\perp}(\mathcal{M}) = \{x \in \mathbb{R}^m \mid x = A'z,\ z \in \mathbb{Z}^{p_1}\}.$$

Let B' be the matrix whose columns are a basis of W (for instance these columns could be any maximal linearly independent subset of columns of B).

We obtain that

$$\begin{aligned}\mathcal{L} &= \mathcal{M} + W\\ &= \text{Proj}_{W^\perp}(\mathcal{M}) + W\\ &= \{x \in \mathbb{R}^m \mid x = A'z,\ z \in \mathbb{Z}^{p_1}\} + W\\ &= \{x \in \mathbb{R}^m \mid x = A'z + B'y,\ z \in \mathbb{Z}^{p_1}, y \in \mathbb{R}^{p_2}\}.\end{aligned}$$

Now, observe that the columns of A' are linearly independent. Since we have $\text{Proj}_{W^\perp}(\mathcal{M}) \subseteq W^\perp$ and since the columns of B' are linearly independent and are contained in W, we have that the columns of $[A'|\ B']$ are linearly independent. Thus, $\mathcal{L}$ is a mixed-integer lattice. □

For a mixed-integer lattice $\mathcal{L} = \{x \in \mathbb{R}^m \mid x = Az+By,\ z \in \mathbb{Z}^{n_1}, y \in \mathbb{R}^{n_2}\}$ let $\Psi_\mathcal{L} : \text{aff}(\mathcal{L}) \to \mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$ be defined as $\Psi_\mathcal{L}(Az + By) = (z, y)$, that is, for a vector $x \in \text{aff}(\mathcal{L})$, $\Psi_\mathcal{L}(x)$ are the coordinates of x in the basis of the linear subspace $\text{aff}(\mathcal{L})$ formed by the columns of $[A\ B]$. We obtain the following straightforward lemma.

Lemma 5. *Let $\mathcal{L} = \{x \in \mathbb{R}^m \mid x = Az + By,\ z \in \mathbb{Z}^{n_1}, y \in \mathbb{R}^{n_2}\}$ be a mixed-integer lattice. Then $\Psi_\mathcal{L} : \text{aff}(\mathcal{L}) \to \mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$ is an invertible mapping and $\Psi_\mathcal{L}(\mathcal{L}) = \mathbb{Z}^{n_1}\times\mathbb{R}^{n_2}$.*

Let $\mathcal{L} \subseteq \mathbb{R}^m$ be a mixed-integer lattice and $W \subseteq \mathbb{R}^m$ be a linear subspace. The following technical lemma states that $W \cap \mathcal{L}$ is a mixed-integer lattice.

Lemma 6. *Let $\mathcal{L} = \{x \in \mathbb{R}^m \mid x = Az + By,\ z \in \mathbb{Z}^{n_1}, y \in \mathbb{R}^{n_2}\}$ be a mixed-integer lattice. Let $W \subseteq \mathbb{R}^m$ be a linear subspace. Then $W \cap \mathcal{L}$ is a mixed-integer lattice.*

Proof. By Lemma 5 and replacing W by $W \cap \text{aff}(\mathcal{L})$, we may assume that $\mathcal{L} = \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$ and that $W \subseteq \mathbb{R}^{n_1+n_2}$ by applying the linear mapping $\Psi_{\mathcal{L}}$ to $\mathcal{L}$ and $W \cap \text{aff}(\mathcal{L})$, respectively. Let $\mathcal{L}' = W \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})$. We want to prove that $\mathcal{L}'$ is a mixed-integer lattice. Denote $M = W \cap (\{0\}^{n_1} \times \mathbb{R}^{n_2})$ and $\Lambda = (W \cap M^{\perp}) \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})$. We will prove that (a) Λ is a lattice and that (b) $\mathcal{L}' = \Lambda + M$. Since $\Lambda \subseteq M^{\perp}$, (a) and (b) together imply that $\mathcal{L}'$ is a mixed-integer lattice.

First we prove (a). It suffices to prove that Λ is a discrete additive subgroup (Theorem 1.4 [1]). Since $W \cap M^{\perp}$ and $\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$ are additive subgroups, we conclude Λ is an additive subgroup. On the other hand, let $(z, y_1), (z, y_2) \in \Lambda$. Then $(z, y_1) - (z, y_2) = (0, y_1 - y_2) \in W \cap M^{\perp} \subseteq M^{\perp}$ and also, by definition of M, we have $(0, y_1 - y_2) \in M$. Thus, we must have $y_1 = y_2$. So, we obtain that if $(z_1, y_1), (z_2, y_2) \in \Lambda$ are distinct, then $z_1 \neq z_2$. Therefore, we have $\|(z_1, y_1) - (z_2, y_2)\|^2 \geq (z_1 - z_2)^2 \geq 1$. We conclude Λ is a discrete set. Therefore, Λ is a lattice.

We next verify (b). The inclusion $\mathcal{L}' \supseteq \Lambda + M$ is easy, since $\mathcal{L}' = W \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})$, $\Lambda \subseteq \mathcal{L}'$, $M \subseteq \mathcal{L}'$, and $\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$, W are additive subgroups. For the inclusion $\mathcal{L}' \subseteq \Lambda + M$, let $(z, y) \in \mathcal{L}'$. We can write

$$(z, y) = (z, u) + (0, v),$$

where (z, u) is the projection on $M^{\perp}$ of (z, y) and $(0, v)$ is the projection on M of (z, y). Since $(z, y) - (0, v) \in W \cap \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$, we conclude $(z, u) \in \Lambda$. Thus, $(z, y) \in \Lambda + M$, as desired.

□

Proposition 8. *Let $\mathcal{L} = \{x \in \mathbb{R}^m \,|\, x = Az + By,\ z \in \mathbb{Z}^{n_1}, y \in \mathbb{R}^{n_2}\}$ be a mixed-integer lattice. Let $K \subseteq \mathbb{R}^m$ be a closed convex set such that $K \cap \mathcal{L} \neq \emptyset$. Then there exist $p_1, p_2 \in \mathbb{Z}_+$ with $p_1 \leq n_1, p_2 \leq n_2$, $p_1 + p_2 = \dim(K \cap \mathcal{L})$, and a full-dimensional closed convex set $K' \subseteq \mathbb{R}^{p_1} \times \mathbb{R}^{p_2}$ such that*

$$\text{conv}(K \cap \mathcal{L}) \text{ is closed if and only if } \text{conv}(K' \cap (\mathbb{Z}^{p_1} \times \mathbb{R}^{p_2})) \text{ is closed.}$$

Moreover, K' can be taken as $K' = \Phi(K \cap \text{aff}(K \cap \mathcal{L}))$, where $\Phi : \text{aff}(K \cap \mathcal{L}) \to \mathbb{R}^{p_1} \times \mathbb{R}^{p_2}$ is an invertible affine mapping such that $\Phi(\mathcal{L} \cap \text{aff}(K \cap \mathcal{L})) = \mathbb{Z}^{p_1} \times \mathbb{R}^{p_2}$. When $0 \in K$, we can chose Φ to be a linear mapping.

Proof. We first show that by translating K by a vector in $k \in K \cap \mathcal{L}$ we may assume $0 \in K$. Let $k \in K \cap \mathcal{L}$. Observe that since $k \in \mathcal{L}$, we have $\mathcal{L} - k = \mathcal{L}$. Hence, we obtain

$$\begin{aligned} \text{conv}(K \cap \mathcal{L}) &= \text{conv}(K \cap \mathcal{L}) + (k - k) \\ &= \text{conv}((K - k) \cap (\mathcal{L} - k)) + k. \end{aligned} \tag{4}$$

Thus, by (4) we conclude that $\text{conv}(K \cap \mathcal{L})$ is closed if and only if $\text{conv}((K - k) \cap \mathcal{L})$ is closed. Therefore, we may assume $0 \in K$.

Denote $W = \text{aff}(K \cap \mathcal{L})$. By Lemma 6, since W is a linear subspace, we obtain that $\mathcal{L}' = W \cap \mathcal{L}$ is a mixed-integer lattice, that is, there exists $p_1 \leq n_1, p_2 \leq n_2$, $A' \in \mathbb{R}^{m \times p_1}$, and $B' \in \mathbb{R}^{m \times p_2}$ such that $[A'\ B']$ has linearly independent columns, and

$$\mathcal{L}' = \{x \in \mathbb{R}^m \,|\, x = A'z + B'y,\ z \in \mathbb{Z}^{p_1}, y \in \mathbb{R}^{p_2}\}.$$

Let $\Psi_{\mathcal{L}'} : \text{aff}(\mathcal{L}') \to \mathbb{R}^{p_1} \times \mathbb{R}^{p_2}$ be the invertible linear mapping in Lemma 5 for the particular case of the mixed-integer lattice $\mathcal{L}'$, and let $K' = \Psi_{\mathcal{L}'}(K \cap W)$. We have

$$\begin{aligned} \text{conv}(K \cap \mathcal{L}) &= \text{conv}((K \cap W) \cap (\mathcal{L} \cap W)) \\ &= \text{conv}((K \cap W) \cap \mathcal{L}') \\ &= {\Psi_{\mathcal{L}'}}^{-1}[\text{conv}(\Psi_{\mathcal{L}'}(K \cap W) \cap \Psi_{\mathcal{L}'}(\mathcal{L}'))] \\ &= {\Psi_{\mathcal{L}'}}^{-1}[\text{conv}(K' \cap (\mathbb{Z}^{p_1} \times \mathbb{R}^{p_2}))] \end{aligned}$$

The first equality uses the fact that $K \cap \mathcal{L} \subseteq W$, the third equality uses the fact that $\Psi_{\mathcal{L}'}$ is an invertible mapping and the last equality uses $\Psi_{\mathcal{L}'}(\mathcal{L}') = \mathbb{Z}^{p_1} \times \mathbb{R}^{p_2}$. Therefore, since $\Psi_{\mathcal{L}'}$ is an homeomorphism, we conclude $\operatorname{conv}(K \cap \mathcal{L})$ is closed if and only if $\operatorname{conv}(K' \cap (\mathbb{Z}^{p_1} \times \mathbb{R}^{p_2}))$ is closed.

Finally, we show that K' can be taken as $K' = \Phi(K \cap \operatorname{aff}(K \cap \mathcal{L}))$, where $\Phi : \operatorname{aff}(K \cap \mathcal{L}) \to \mathbb{R}^{p_1} \times \mathbb{R}^{p_2}$ is an invertible affine mapping such that $\Phi(\mathcal{L} \cap \operatorname{aff}(K \cap \mathcal{L})) = \mathbb{Z}^{p_1} \times \mathbb{R}^{p_2}$. If $0 \in K$, it suffices to take $\Phi = \Psi_{\mathcal{L}'}$ (hence Φ is a linear mapping). If $0 \notin K$, then for an arbitrary $k \in K \cap \mathcal{L}$, we can define the following invertible affine mapping $\Phi : \operatorname{aff}(K \cap \mathcal{L}) \to \mathbb{R}^{p_1} \times \mathbb{R}^{p_2}$, where for all $x \in \operatorname{aff}(K \cap \mathcal{L})$, $\Phi(x) = \Psi_{\mathcal{L}'}(x - k)$. Then, by the previous arguments in the proof, it is clear we can take $K' = \Phi(K \cap \operatorname{aff}(K \cap \mathcal{L}))$. Since $k \in K \cap \mathcal{L}$, we have $\mathcal{L} \cap \operatorname{aff}(K \cap \mathcal{L}) = \mathcal{L} \cap \operatorname{aff}(K \cap \mathcal{L}) + k$. Thus, $\Phi(\mathcal{L} \cap \operatorname{aff}(K \cap \mathcal{L})) = \Psi_{\mathcal{L}'}(\mathcal{L}') = \mathbb{Z}^{p_1} \times \mathbb{R}^{p_2}$. $\square$

3.5 Affine Maps Preserving Closedness and Proof of Proposition 2

For a linear subspace $L \subseteq \mathbb{R}^n$ we denote by $\operatorname{Proj}_{L^\perp}(\cdot)$ the orthogonal projection onto $L^\perp \subseteq \mathbb{R}^n$, the subspace orthogonal to L.

Lemma 7. *Let $\mathcal{L}'$ be a mixed-integer lattice, let $K \subseteq \mathbb{R}^n$ be a closed convex set, and let $L \subseteq \operatorname{lin.space}(K)$ be a linear subspace. Then we have*

$$\operatorname{Proj}_{L^\perp}(K \cap \mathcal{L}') = \operatorname{Proj}_{L^\perp}(K) \cap \operatorname{Proj}_{L^\perp}(\mathcal{L}').$$

Proof. The inclusion $\operatorname{Proj}_{L^\perp}(K \cap \mathcal{L}') \subseteq \operatorname{Proj}_{L^\perp}(K) \cap \operatorname{Proj}_{L^\perp}(\mathcal{L}')$ is always true. Now, to prove the other inclusion, let $x \in \operatorname{Proj}_{L^\perp}(K) \cap \operatorname{Proj}_{L^\perp}(\mathcal{L}')$. Then there exist $l_1, l_2 \in L$ such that $x + l_1 \in K$ and $x + l_2 \in \mathcal{L}'$. Since $x + l_1 \in K$, $(l_2 - l_1) \in L$ and $L \subseteq \operatorname{lin.space}(K)$, we obtain that $x + l_1 + (l_2 - l_1) \in K$. Thus, $x + l_2 \in K$. Therefore, we conclude that $x \in \operatorname{Proj}_{L^\perp}(K \cap \mathcal{L}')$. $\square$

Proposition 9. *Let $\mathcal{L}'$ be a mixed-integer lattice, let $K \subseteq \mathbb{R}^n$ be a closed convex set, and let $L \subseteq \operatorname{lin.space}(K)$ be a linear subspace. If L is generated by points in $\mathcal{L}'$, then*

$$\operatorname{conv}(K \cap \mathcal{L}') = \operatorname{conv}((K \cap L^\perp) \cap \operatorname{Proj}_{L^\perp}(\mathcal{L}')) + L.$$

In particular, $\operatorname{conv}(K \cap \mathcal{L}')$ *is closed* $\Leftrightarrow$ $\operatorname{conv}((K \cap L^\perp) \cap \operatorname{Proj}_{L^\perp}(\mathcal{L}'))$ *is closed.*

Proof. Let $\{l_1, \ldots, l_q\} \subseteq \mathcal{L}'$ be a basis of L. Denote $\mathcal{M} = \{x \in \mathbb{R}^p \mid x = \sum_{i=1}^q z_i l_i,\ z_i \in \mathbb{Z},\ \forall\, i = 1, \ldots, q\}$. Since $\mathcal{M} \subseteq (\operatorname{lin.space}(K) \cap \mathcal{L}')$, we obtain that

$$K \cap \mathcal{L}' = (K \cap \mathcal{L}') + \mathcal{M}. \tag{5}$$

Recall that for all $X, Y \subseteq \mathbb{R}^p$, we have $\operatorname{conv}(X + Y) = \operatorname{conv}(X) + \operatorname{conv}(Y)$. Thus, by (5) we have

$$\operatorname{conv}(K \cap \mathcal{L}') = \operatorname{conv}(K \cap \mathcal{L}') + L. \tag{6}$$

Therefore, as a consequence of (6) we can write

$$\begin{aligned}
\operatorname{conv}(K \cap \mathcal{L}') &= \operatorname{Proj}_{L^\perp}(\operatorname{conv}(K \cap \mathcal{L}')) + L \\
&= \operatorname{conv}(\operatorname{Proj}_{L^\perp}(K \cap \mathcal{L}')) + L \\
&= \operatorname{conv}(\operatorname{Proj}_{L^\perp}(K) \cap \operatorname{Proj}_{L^\perp}(\mathcal{L}')) + L \\
&= \operatorname{conv}((K \cap L^\perp) \cap \operatorname{Proj}_{L^\perp}(\mathcal{L}')) + L,
\end{aligned}$$

where the second equality is by lineality of $\operatorname{Proj}_{L^\perp}(\cdot)$, the third equality uses Lemma 7, and the last equality uses the fact that $L \subseteq \operatorname{lin.space}(K)$.

Finally, the fact that $\operatorname{conv}(K \cap \mathcal{L}')$ is closed $\Leftrightarrow$ $\operatorname{conv}((K \cap L^\perp) \cap \operatorname{Proj}_{L^\perp}(\mathcal{L}'))$ is closed is a straightforward consequence of the identity $\operatorname{conv}(K \cap \mathcal{L}') = \operatorname{conv}((K \cap L^\perp) \cap \operatorname{Proj}_{L^\perp}(\mathcal{L}')) + L$ and the fact that $\operatorname{conv}((K \cap L^\perp) \cap \operatorname{Proj}_{L^\perp}(\mathcal{L}')) \subseteq L^\perp$. $\square$

For a function $G : \mathbb{R}^n \to \mathbb{R}^m$ we denote by G_W its restriction to the linear subspace $W \subseteq \mathbb{R}^n$, that is, $G_W : W \to \mathbb{R}^m$ and $G_W(x) = G(x)$ for all $x \in W$. The following Lemma is straightforward to verify.

Lemma 8. *Let $G : \mathbb{R}^n \to \mathbb{R}^m$ defined as $G(x) = Mx - g$, where $M \in \mathbb{R}^{m\times n}, g \in \mathbb{R}^m$. Let $W = \text{Kernel}(M)$. Let $G_{W^\perp} : W^\perp \to \mathbb{R}^m$ the restriction of G to $W^\perp$. Then*

1. *$G_{W^\perp}$ is injective.*
2. *For all $X, Y \subseteq W^\perp$, we have $G_{W^\perp}(X \cap Y) = G_{W^\perp}(X) \cap G_{W^\perp}(Y)$.*
3. *For all $X \subseteq \mathbb{R}^n$, we have $G_{W^\perp}(\text{Proj}_{W^\perp}(X)) = G(X)$.*

Proposition 2. *Let $K \subseteq \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ be a closed convex set. Let $G : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^m$ defined as $G(x, y) := Ex + Fy - g$ be an affine map, where $E \in \mathbb{R}^{m\times n_1}$, $F \in \mathbb{R}^{m\times n_2}$ and $g \in \mathbb{R}^m$. Assume that E, F satisfy the following:*

1. $\text{Kernel}([E\ F]) \subseteq \text{lin.space}(K)$,
2. $\text{Kernel}([E\ F])$ *is generated by points in the mixed-integer lattice $\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$.*

Then

$$\text{conv}(K \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})) \text{ is closed} \Leftrightarrow \text{conv}\left[G(K) \cap G(\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})\right] \text{ is closed.}$$

Proof. Lets denote $W = \text{Kernel}([E\ F])$, and let $G_{W^\perp}$ be the restriction of G to $W^\perp$. Observe that since $G_{W^\perp}$ is an affine map we obtain that

$$G_{W^\perp}\left[\text{conv}[(K \cap W^\perp) \cap \text{Proj}_{W^\perp}(\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})]\right] = \text{conv}\left[G_{W^\perp}[(K \cap W^\perp) \cap \text{Proj}_{W^\perp}(\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})]\right]. \quad (7)$$

Now, by using (2.) and (3.) of Lemma 8 and the fact that $W \subseteq \text{lin.space}(K)$ we have

$$\begin{aligned}\text{conv}\left[G_{W^\perp}[(K \cap W^\perp) \cap \text{Proj}_{W^\perp}(\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})]\right] &= \text{conv}\left[G_{W^\perp}(K \cap W^\perp) \cap G_{W^\perp}(\text{Proj}_{W^\perp}(\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))\right] \\ &= \text{conv}\left[G_{W^\perp}(\text{Proj}_{W^\perp}(K)) \cap G_{W^\perp}(\text{Proj}_{W^\perp}(\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))\right] \\ &= \text{conv}\left[G(K) \cap G(\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})\right].\end{aligned} \quad (8)$$

By combining (7) and (8), we obtain

$$G_{W^\perp}\left[\text{conv}[(K \cap W^\perp) \cap \text{Proj}_{W^\perp}(\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})]\right] = \text{conv}\left[G(K) \cap G(\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})\right]. \quad (9)$$

Moreover, by (1.) of Lemma 8, the mapping $G_{W^\perp}$ is an homeomorphism from $W^\perp$ to $G(\mathbb{R}^n)$. Hence, as a consequence of (9) we obtain that

$$\text{conv}[(K \cap W^\perp) \cap \text{Proj}_{W^\perp}(\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})] \text{ is closed} \Leftrightarrow \text{conv}\left[G(K) \cap G(\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})\right] \text{ is closed.} \quad (10)$$

On the other hand, since $W \subseteq \text{lin.space}(K)$ and W is generated by points in the mixed-integer lattice $\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$, by Proposition 9 we have that

$$\text{conv}[(K \cap W^\perp) \cap \text{Proj}_{W^\perp}(\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})] \text{ is closed} \Leftrightarrow \text{conv}(K \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})) \text{ is closed.} \quad (11)$$

Therefore, by putting together identities (10) and (11) we conclude that

$$\text{conv}(K \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})) \text{ is closed} \Leftrightarrow \text{conv}\left[G(K) \cap G(\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})\right] \text{ is closed.}$$

□

3.6 Proofs of Proposition 3 and Proposition 5

In [4] some properties of closedness of mixed-integer hulls are presented for the case $\mathcal{L} = \mathbb{Z}^n$. In this section we show the extension of some of these results to the case of a general mixed-integer lattice $\mathcal{L}$. Since the proof techniques of the results of this section are a simple generalization of those in [4], we only present the proofs of Proposition 3 and Proposition 5. We begin with some definitions and preliminary results.

Theorem 5 (Theorem 18.5 [8])**.** *Let $K \subseteq \mathbb{R}^n$ be a closed convex set not containing a line. Let S be the set of extreme points of K and let D be the set of extreme rays of* $\text{rec.cone}(K)$*. Then $K = \text{conv}(S) + \text{cone}(D)$.*

Definition 5 ($u(K,\mathcal{L})$)**.** *Given a convex set $K \subseteq \mathbb{R}^n$ and $u \in K \cap \mathcal{L}$, we define $u(K,\mathcal{L}) = \{d \in \mathbb{R}^n \,|\, u + \lambda d \in \text{conv}(K \cap \mathcal{L})\ \forall \lambda \geq 0\}$.*

The following result, modified from [4], is a characterization of closedness of mixed-integer hulls for general closed convex sets.

Theorem 6. *Let $K \subseteq \mathbb{R}^n$ be a closed convex set. If* $\text{conv}(K \cap \mathcal{L})$ *is closed, then $u(K,\mathcal{L})$ is identical for all $u \in K \cap \mathcal{L}$. Conversely, if $u(K,\mathcal{L})$ is identical for all $u \in K \cap \mathcal{L}$ and K contains no lines, then* $\text{conv}(K \cap \mathcal{L})$ *is closed.*

The following lemma, a generalization of a result from [4], is also crucial.

Lemma 9. *Let $K \subseteq \mathbb{R}^n$ be a full-dimensional closed convex set and let $u \in K \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})$. If $\{u + \lambda d \,|\, \lambda > 0\} \subseteq \text{int}(K)$, then $\{u + \lambda d \,|\, \lambda \geq 0\} \subseteq \text{conv}(K \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$.*

Now we present the proofs of Proposition 3 and Proposition 5.

Proposition 3. *Let $K \subseteq \mathbb{R}^n$ be a closed strictly convex set, $t \in \mathbb{R}^n$ and $\mathcal{L}$ a mixed-integer lattice. Then* $\text{conv}(K \cap [\mathcal{L} + t])$ *is closed.*

Proof. **Case 1:** $t = 0$**.** Since $\text{conv}(K \cap \mathcal{L}) = \text{conv}([K \cap \text{aff}(K \cap \mathcal{L})] \cap \mathcal{L})$ and since by Lemma 2 we have that $K \cap \text{aff}(K \cap \mathcal{L})$ is a strictly convex set, we may assume that $K = K \cap \text{aff}(K \cap \mathcal{L})$. Moreover, since invertible affine functions map closed strictly convex sets to closed strictly convex sets, by Proposition 8, we may assume that K is a full-dimensional strictly convex set and that $\mathcal{L} = \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$.

First note that if K is bounded or if $K \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}) = \emptyset$, then $\text{conv}(K \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$ is closed. Therefore, we assume that K is unbounded and $K \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}) \neq \emptyset$.

We first verify that K does not contain a line. Assume by contradiction that K contains a line in the direction $r \neq 0$. Examine $x \in \text{bd}(K)$. Then points of the form $x + \lambda r$ and $x - \lambda r$ belong to K, where $\lambda > 0$. In particular, $x + \lambda r, x - \lambda r \in \text{bd}(K)$ since $x \in \text{bd}(K)$. However, this contradicts the fact that K is strictly convex.

For ease of notation, for $u \in K \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})$ lets denote $u(K) := u(K, \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})$. We show next that $u(K) = \text{rec.cone}(K)$ for all $u \in K \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})$. Consider a point $u \in K \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})$. Clearly, as K is a closed set and $\text{conv}(K \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})) \subseteq K$, we obtain $u(K) \subseteq \text{rec.cone}(K)$. We now show the other inclusion. Let $r \in \text{rec.cone}(K)$. Since K is strictly convex, we obtain that that set $\{u + \lambda r \,|\, \lambda > 0\}$ is contained in the interior of K. Therefore, by Lemma 9 we obtain that the set $\{u + \lambda r \,|\, \lambda \geq 0\}$ in contained in $\text{conv}(K \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$, so $r \in u(K)$. Thus, $u(K) = \text{rec.cone}(K)$ for all $u \in K \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})$. Therefore, by Theorem 6 we obtain that $\text{conv}(K \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$ is closed.

Case 2: $t \neq 0$**.** Observe that

$$\begin{aligned} \text{conv}(K \cap (\mathcal{L} + t)) &= \text{conv}([(K - t) \cap \mathcal{L}] + t) \\ &= \text{conv}([(K - t) \cap \mathcal{L}]) + t. \end{aligned} \tag{12}$$

On the other hand, since K is a strictly convex set, we obtain that $K - t$ is also a strictly convex set. Thus, by **Case 1**, we have that $\text{conv}([(K - t) \cap \mathcal{L}])$ is closed. Therefore, by equation (12), we conclude that $\text{conv}(K \cap (\mathcal{L} + t))$ is a closed set.

□

Proposition 5. *Let C be a closed pointed convex cone in $\mathbb{R}^n$ and let $\mathcal{L}$ be a mixed-integer lattice. Then $\overline{\text{conv}}(C \cap \mathcal{L}) = C \cap W$, where $W = \text{aff}(C \cap \mathcal{L})$. In particular,* $\text{conv}(C \cap \mathcal{L})$ *is closed if and only if every extreme ray of $C \cap W$ can be scaled by a non-zero scalar to belong to $\mathcal{L}$.*

Proof. Since $\overline{\text{conv}}(C \cap \mathcal{L}) = \overline{\text{conv}}((C \cap W) \cap \mathcal{L})$ and $C \cap W$ is a pointed closed convex cone, we may assume that $C = C \cap W$. Furthermore, by Proposition 8 we can map C to a full-dimensional closed pointed convex cone $C' = \Phi(C) \subseteq \mathbb{R}^{p_1} \times \mathbb{R}^{p_2}$, where Φ is an affine map that also satisfies $\Phi(\mathcal{L} \cap W) = \mathbb{Z}^{p_1} \times \mathbb{R}^{p_2}$. Since Φ is an invertible affine mapping, we have that $\overline{\text{conv}}(C \cap \mathcal{L}) = C$ is equivalent to $\overline{\text{conv}}(C' \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})) = C'$, and that the closedness of $\text{conv}(C \cap \mathcal{L})$ is equivalent to the closedness of $\text{conv}(C' \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$. Therefore, in order to prove the result, we may assume that C is a full-dimensional closed pointed convex cone and that $\mathcal{L} = \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$.

We need to show that $\overline{\text{conv}}(C \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})) = C$ and that $\text{conv}(C \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$ is closed if and only if every extreme ray of C can be scaled by a non-zero scalar to belong to $\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$.

We first verify that $\overline{\text{conv}}(C \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})) = C$. By convexity of C, we obtain that $\text{conv}(C \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})) \subseteq C$. Since C is also closed, we obtain that $\overline{\text{conv}}(C \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})) \subseteq C$. This shows one inclusion. To show the other inclusion, let $r \in \text{int}(C)$. Clearly, we have $\{0 + \lambda r \,|\, \lambda > 0\} \subseteq \text{int}(C)$. So, by Lemma 9 we obtain $\{0 + \lambda r \,|\, \lambda \geq 0\} \subseteq \text{conv}(C \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$. Hence, $\text{int}(C) \subseteq \text{conv}(C \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$. Since C is a full-dimensional closed convex set, we have $C = \overline{\text{int}(C)}$. Thus, by taking the closure on both sides of the inclusion $\text{int}(C) \subseteq \text{conv}(C \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$, we obtain $C \subseteq \overline{\text{conv}}(C \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$.

We now verify that $\text{conv}(C \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$ is closed if and only if every extreme ray of C can be scaled by a non-zero scalar to belong to $\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$. Suppose $\text{conv}(C \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$ is closed. Then $\text{conv}(C \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})) = C$. If r is any extreme ray of C, then observe that $C \setminus \{\lambda r \,|\, \lambda > 0\}$ is a convex set. Since $\{\lambda r \,|\, \lambda > 0\} \subseteq \text{conv}(C \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$, there must be a point $x \in (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})$ in the set $\{\lambda r \,|\, \lambda > 0\}$. In other words, r can be scaled by a non-zero scalar to belong to $\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$.

Now assume that every extreme ray of C can be scaled by a non-zero scalar to belong to $\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$. Let R be the set of all extreme rays of C. Then observe that

$$C = \text{cone}(R) \subseteq \text{conv}(C \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})) \subseteq C,$$

where the first equality follows from Theorem 5. Thus, $\text{conv}(C \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})) = C$ or equivalently $\text{conv}(C \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$ is closed. □

3.7 Proofs of Proposition 4 and Proposition 6

We start with some preliminary results.

Lemma 10. *Let $\mathbf{C} \subseteq \mathbb{R}^m$ be a full-dimensional pointed closed convex cone. Let $t \in \mathbb{R}^m$, $u \in \text{bd}(\mathbf{C}+t) \setminus \{t\}$ and let $r \in \mathbf{C}$. Then*

$$u + r \in \text{bd}(\mathbf{C} + t) \Leftrightarrow r = \lambda(u - t), \text{ for some } \lambda \geq 0.$$

Proof.

Without loss of generality, we may assume that $t = 0$. We need to prove that $u + r \in \text{bd}(\mathbf{C}) \Leftrightarrow r = \lambda u$, for some $\lambda \geq 0$. Assume that $\mathbf{C}$ is generated by the $(m-1)$-dimensional bounded closed strictly convex set $\mathbf{S} \subseteq \mathbb{R}^m$.

($\Leftarrow$) If $r = \lambda u$, for some $\lambda \geq 0$, then $u+r = (1+\lambda)u$. Thus, since $u \in \text{bd}(\mathbf{C})$, we obtain that $u+r \in \text{bd}(\mathbf{C})$

($\Rightarrow$) The case $r = 0$ is straighforward. Lets assume that $r \neq 0$. Observe that, since $u, r \in \mathbf{C} \setminus \{0\}$, we have that there exists $\alpha, \beta > 0$, $\hat{u}, \hat{r} \in \mathbf{S}$ such that $\hat{u} = \alpha u$ and $\hat{r} = \beta r$. We obtain that

$$u + r = \left(\frac{\alpha + \beta}{\alpha\beta}\right) \frac{\beta\hat{u} + \alpha\hat{r}}{\alpha + \beta}. \tag{13}$$

Lets assume for contradiction that $r \neq \lambda u$, for all $\lambda \geq 0$. Hence, the definition of generator ($\mathbf{C} = \text{cone}(\mathbf{S})$ and $\mathbf{S}$ is a $(m-1)$-dimensional bounded closed strictly convex set) implies that $\hat{u} \neq \hat{r}$.

Since $\hat{u}, \hat{r} \in \mathbf{S}$, $\hat{u} \neq \hat{r}$ and $\mathbf{S}$ is a strictly convex set, we have that $\frac{\beta\hat{u}+\alpha\hat{r}}{\alpha+\beta} \in \text{rel.int}(\mathbf{S})$. Therefore, by equation (13) we conclude that $u + r \in \text{int}(\mathbf{C})$, a contradiction.

□

Lemma 11 (Corollary 8.3.1 in [8])**.** *Let $K \subseteq \mathbb{R}^n$ be a convex set. Then*

$$\text{rec.cone}(\text{rel.int}(K)) = \text{rec.cone}(\overline{K}) \supseteq \text{rec.cone}(K).$$

We need the following corollary to Lemma 9.

Corollary 2. *Let $K \subseteq \mathbb{R}^n$ be a closed convex set and let $\mathcal{L}$ be a mixed-integer lattice. Let $u \in K \cap \mathcal{L}$. Assume that $\text{aff}(K) = \text{aff}(K \cap \mathcal{L})$. If $\{u + \lambda d \,|\, \lambda > 0\} \subseteq \text{rel.int}(K)$, then $\{u + \lambda d \,|\, \lambda \geq 0\} \subseteq \text{conv}(K \cap \mathcal{L})$.*

Proof. Let $\Phi : \text{aff}(K \cap \mathcal{L}) \to \mathbb{R}^{p_1} \times \mathbb{R}^{p_2}$ be an invertible affine mapping as in Proposition 8 such that $K' = \Phi(K \cap \text{aff}(K \cap \mathcal{L}))$ is a full-dimensional set and $\Phi(\mathcal{L} \cap \text{aff}(K \cap \mathcal{L})) = \mathbb{Z}^{p_1} \times \mathbb{R}^{p_2}$. The properties of K' and Φ imply that $\{u+\lambda d | \lambda \geq 0\} \subseteq \text{conv}(K \cap \mathcal{L})$ is equivalent to $\{\Phi(u)+\lambda\Phi(d) \,|\, \lambda \geq 0\} \subseteq \text{conv}(K' \cap (\mathbb{Z}^{p_1} \times \mathbb{R}^{p_2}))$. We will show next the inclusion $\{\Phi(u) + \lambda\Phi(d) \,|\, \lambda \geq 0\} \subseteq \text{conv}(K' \cap (\mathbb{Z}^{p_1} \times \mathbb{R}^{p_2}))$.

Since by assumption $\text{aff}(K) = \text{aff}(K \cap \mathcal{L})$, we obtain that $K' = \Phi(K)$. This implies that $\text{int}(K') = \Phi(\text{rel.int}(K))$. Therefore, since $\{u + \lambda d \,|\, \lambda > 0\} \subseteq \text{rel.int}(K)$, we obtain $\{\Phi(u) + \lambda\Phi(d) \,|\, \lambda > 0\} \subseteq \text{int}(K')$. Moreover, since $\Phi(u) \in \mathbb{Z}^{p_1} \times \mathbb{R}^{p_2}$, by Lemma 9, we conclude $\{\Phi(u) + \lambda\Phi(d) \,|\, \lambda \geq 0\} \subseteq \text{conv}(K' \cap (\mathbb{Z}^{p_1} \times \mathbb{R}^{p_2}))$. □

Lemma 12. *Let $\mathcal{L}$ be a mixed-integer lattice and let $V = \text{aff}(\mathcal{L})$. Let $K \subseteq V$ be a non-empty closed convex set such that $\text{aff}(\text{rec.cone}(K)) = V$. Then $\text{aff}(\text{conv}(K \cap \mathcal{L})) = V$.*

Proof. First notice that since $K \neq \emptyset$ and $\text{aff}(\text{rec.cone}(K)) = V$, we obtain that $\text{aff}(K) = V$. Let $\Psi_{\mathcal{L}} : V \to \mathbb{R}^{p_1} \times \mathbb{R}^{p_2}$ be an invertible mapping as in Lemma 5 such that $\Psi_{\mathcal{L}}(\mathcal{L}) = \mathbb{Z}^{p_1} \times \mathbb{R}^{p_2}$ and $K' = \Psi_{\mathcal{L}}(K) \subseteq \mathbb{R}^{p_1+p_2}$ is a full-dimensional closed convex set. Also by construction we have that $\text{aff}(\text{conv}(K \cap \mathcal{L})) = V$ if and only if $\text{aff}(\text{conv}(K' \cap (\mathbb{Z}^{p_1} \times \mathbb{R}^{p_2}))) = \mathbb{R}^{p_1+p_2}$. Moreover, $\text{aff}(\text{rec.cone}(K')) = \mathbb{R}^{p_1+p_2}$, that is $\text{rec.cone}(K')$ is full-dimensional. Let $T \subseteq \text{rec.cone}(K')$ be a rational polyhedral full-dimensional cone. Let $v \in K' \cap \mathbb{Q}^{p_1+p_2}$, that is v be a rational point in K'. Then $v + T \subseteq K'$ and $v + T$ is a rational polyhedron with a full-dimensional recession cone. In particular, $(v + T) \cap (\mathbb{Z}^{p_1} \times \mathbb{R}^{p_2}) \neq \emptyset$. Therefore $\text{rec.cone}(\text{conv}((v + T) \cap (\mathbb{Z}^{p_1} \times \mathbb{R}^{p_2}))) = T$. Since $K' \supseteq v + T$, we obtain that $\text{rec.cone}(\text{conv}(K' \cap (\mathbb{Z}^{p_1} \times \mathbb{R}^{p_2}))) \supseteq T$ or equivalently $\text{aff}(\text{rec.cone}(\text{conv}(K' \cap (\mathbb{Z}^{p_1} \times \mathbb{R}^{p_2})))) = \mathbb{R}^{p_1+p_2}$. Thus, $\text{aff}(\text{conv}(K' \cap (\mathbb{Z}^{p_1} \times \mathbb{R}^{p_2}))) = \mathbb{R}^{p_1+p_2}$, as required. □

We now present the proofs of Proposition 4 and Proposition 6.

Proposition 4. *Let $\mathbf{C} \subseteq \mathbb{R}^m$ be a full-dimensional pointed closed convex cone that is generated by a closed strictly convex set. Let $\mathcal{L} = \{x \in \mathbb{R}^m \,|\, x = Az + By,\ z \in \mathbb{Z}^{p_1}, y \in \mathbb{R}^{p_2}\}$ be a mixed-integer lattice, where A, B are rational matrices. Denote $V = \text{aff}(\mathcal{L})$, and let $b \in (V \cap \mathbb{Q}^m) \setminus \mathcal{L}$. Then $\text{conv}((\mathbf{C} \cap V) \cap (\mathcal{L} - b))$ is closed.*

Proof. Observe that

$$\begin{aligned} \text{conv}((\mathbf{C} \cap V) \cap (\mathcal{L} - b)) &= \text{conv}((\mathbf{C} \cap V) \cap (\mathcal{L} - b)) + (b - b) \\ &= \text{conv}([(\mathbf{C} \cap V) + b] \cap [(\mathcal{L} - b) + b]) - b \\ &= \text{conv}([(\mathbf{C} \cap V) + b] \cap \mathcal{L}) - b. \end{aligned} \tag{14}$$

Thus, we conclude that $\text{conv}((\mathbf{C} \cap V) \cap (\mathcal{L} - b))$ is closed if and only if $\text{conv}([(\mathbf{C} \cap V) + b] \cap \mathcal{L})$ is closed. Denote $\mathbf{C}_V := \mathbf{C} \cap V$.

If $\dim(\mathbf{C}_V) \leq 1$, then it is straighforward to verify that $\text{conv}(\mathbf{C}_V \cap (\mathcal{L} - b))$ is closed. Therefore, we assume $\dim(\mathbf{C}_V) \geq 2$. We claim that $\text{aff}(\mathbf{C}_V + b) = \text{aff}((\mathbf{C}_V + b) \cap \mathcal{L})$: By the application of (1.) of Lemma 3 and Lemma 1 we obtain that $\text{aff}(\mathbf{C}_V) = V$. Let $K := (\mathbf{C} + b) \cap V = \mathbf{C}_V + b$. Then $\text{rec.cone}(K) = \mathbf{C}_V$. Thus $\text{aff}(\text{rec.cone}(K)) = V$. Therefore by applying Lemma 12 with the above defined K, we obtain that $\text{aff}((\mathbf{C}_V + b) \cap \mathcal{L}) = \text{aff}(\text{conv}((\mathbf{C}_V + b) \cap \mathcal{L})) = V$. Thus, since $\text{aff}((\mathbf{C}_V + b) \cap \mathcal{L}) \subseteq \text{aff}(\mathbf{C}_V + b) \subseteq V$, we obtain that $\text{aff}(\mathbf{C}_V + b) = V = \text{aff}((\mathbf{C}_V + b) \cap \mathcal{L})$.

We will prove that $\text{conv}((\mathbf{C}_V + b) \cap \mathcal{L})$ is closed by using Theorem 6 and showing that for all $u \in (\mathbf{C}_V + b) \cap \mathcal{L}$ we have that $u(\mathbf{C}_V + b, \mathcal{L}) := \{d \in \mathbb{R}^m \,|\, u + \lambda d \in \text{conv}(\mathbf{C}_V + b) \cap \mathcal{L})\}$ is the same cone. In particular, we will show that $u(\mathbf{C}_V + b, \mathcal{L}) = \mathbf{C}_V$ for all $u \in (\mathbf{C}_V + b) \cap \mathcal{L}$. To simplify the notation, we will write $u(\mathbf{C}_V + b)$ instead of $u(\mathbf{C}_V + b, \mathcal{L})$ for the rest of the proof.

Notice that since $\text{conv}((\mathbf{C}_V + b) \cap \mathcal{L}) \subseteq \mathbf{C}_V + b$, $\mathbf{C}_V + b$ is a closed set and $\text{rec.cone}(\mathbf{C}_V + b) = \mathbf{C}_V$, the inclusion $u(\mathbf{C}_V + b) \subseteq \mathbf{C}_V$ is straightforward to obtain.

To prove the other inclusion we consider two cases.

Case 1: $u \in \textbf{rel.int}(\mathbf{C}_V + b) \cap \mathcal{L}$**.** By using Lemma 11 we obtain that $\mathbf{C}_V = \text{rec.cone}(\mathbf{C}_V + b) = \text{rec.cone}(\text{rel.int}(\mathbf{C}_V + b))$. Thus, we have $u + \mathbf{C}_V \subseteq \text{rel.int}(\mathbf{C}_V + b)$. Also we have verified that $\text{aff}(\mathbf{C}_V + b) = \text{aff}((\mathbf{C}_V + b) \cap \mathcal{L})$. Therefore, we obtain by Corollary 2 that $u + \mathbf{C}_V \subseteq \text{conv}((\mathbf{C}_V + b) \cap \mathcal{L})$. Hence, we conclude that $\mathbf{C}_V \subseteq u(\mathbf{C}_V + b)$.

Case 2: $u \in \textbf{rel.bd}(\mathbf{C}_V + b) \cap \mathcal{L}$**.** Let $d \in \mathbf{C}_V$. We want to show that $d \in u(\mathbf{C}_V + b)$. We have two subcases.

- Let $d \in \mathbf{C}_V \setminus \{\alpha(u - b) \,|\, \alpha \geq 0\}$. Since $d \neq \alpha(u - b)$ for all $\alpha \geq 0$, Lemma 10 implies that $u + \lambda d \in \text{int}(\mathbf{C} + b)$, for all $\lambda \geq 0$. Since $\text{rel.int}(\mathbf{C}_V + b) = \text{int}(\mathbf{C} + b) \cap V$ (a consequence of $\dim(\mathbf{C}_V) \geq 2$), and $u, d \in V$ we obtain that $\{u + \lambda d \,|\, \lambda > 0\} \subseteq \text{rel.int}(\mathbf{C}_V + b)$. Also since $\text{aff}(\mathbf{C}_V + b) = \text{aff}((\mathbf{C}_V + b) \cap \mathcal{L})$, by Corollary 2, we have that $d \in u(\mathbf{C}_V + b)$.

- Now, let $d \in \{\alpha(u - b) \,|\, \alpha \geq 0\}$. The case $d = 0$ is straightforward to verify. Lets assume then that $d = \alpha(u - b)$, where $\alpha > 0$. Since $\{u + \lambda\alpha(u - b) \,|\, \lambda \geq 0\} = \{u + \lambda(u - b) \,|\, \lambda \geq 0\}$, to show that $d \in u(\mathbf{C}_V + b)$ it suffices to show that $u - b \in u(\mathbf{C}_V + b)$. As a consequence of $b \in (V \cap \mathbb{Q}^m) \setminus \mathcal{L}$ and $u \in \mathcal{L}$, we obtain that $u - b \in \{x \in \mathbb{R}^m \,|\, x = Az + By,\ z \in \mathbb{Q}^{p_1} \setminus \mathbb{Z}^{p_1}, y \in \mathbb{Q}^{p_2}\}$. Hence, since $u \in \mathcal{L}$ we obtain that $\{u + \lambda(u - b) \,|\, \lambda \geq 0\} \subseteq \text{conv}((\mathbf{C}_V + b) \cap \mathcal{L})$. Thus, we conclude that $u - b \in u(\mathbf{C}_V + b)$.

This finishes the proof of $\mathbf{C}_V \subseteq u(\mathbf{C}_V + b)$. Therefore, we obtain $u(\mathbf{C}_V + b) = \mathbf{C}_V$ for all $u \in (\mathbf{C}_V + b) \cap \mathcal{L}$. We conclude, by Theorem 6 that $\text{conv}((\mathbf{C}_V + b) \cap \mathcal{L})$ is closed. ☐

Proposition 6. *Let* $\mathbf{C} \subseteq \mathbb{R}^m$ *be a full-dimensional pointed closed convex cone that is generated by a closed strictly convex set. Let* $\mathcal{L} = \{x \in \mathbb{R}^m \,|\, x = Az + By,\ z \in \mathbb{Z}^{n_1}, y \in \mathbb{R}^{n_2}\}$ *be a mixed-integer lattice, where* A, B *are rational matrices. Denote* $V = \text{aff}(\mathcal{L})$*. Then*

1. *Assume* $\dim(\mathbf{C} \cap V) = 2$*. If* $\dim(\langle B \rangle) \geq \dim(V) - 1$*, then every extreme ray of* $\mathbf{C} \cap V$ *can be scaled by a non-zero scalar to belong to* $\mathcal{L}$*.*

2. *Assume* $\dim(\mathbf{C} \cap V) \geq 3$*. Then* $\dim(\langle B \rangle) \geq \dim(V) - 1$ *if and only if every extreme ray of* $\mathbf{C} \cap V$ *can be scaled by a non-zero scalar to belong to* $\mathcal{L}$*.*

Proof.

Since $[A\ B] \in \mathbb{Q}^{m \times n}$, by Proposition 7 we obtain that $\mathcal{L}$ is a mixed integer lattice, that is, for some $A' \in \mathbb{R}^{m \times p_1}$ and $B' \in \mathbb{R}^{m \times p_2}$, such that $[A'\ B']$ has linearly independent columns, $V = \langle [A\ B] \rangle = \langle [A'\ B'] \rangle$, $\langle B' \rangle = \langle B \rangle$, and we have

$$\mathcal{L} = \{x \in \mathbb{R}^m \,|\, x = A'z + B'y,\ z \in \mathbb{Z}^{p_1}, y \in \mathbb{R}^{p_2}\}.$$

1. We will show that whenever $\dim(\mathbf{C} \cap V) \geq 2$ the following implication is true: If $\dim(\langle B \rangle) \geq \dim(V) - 1$, then every extreme ray of $\mathbf{C} \cap V$ can be scaled by a non-zero scalar to belong to $\mathcal{L}$.

The proof for the case $\dim(\langle B'\rangle) = \dim(V)$ is straightforward and so it is omitted. Assume for the rest of the proof that $\dim(\langle B'\rangle) = \dim(V) - 1$. Since $\dim(\langle B'\rangle) = \dim(V) - 1$, we have that $A' = a$, where $a \in \mathbb{Q}^m$. Since $[a\ B']$ is a linearly independent set generating V, we have

$$V = \langle a\rangle + \langle B'\rangle. \tag{15}$$

Now, let r be a extreme ray of $\mathbf{C} \cap V$. We next show that there exists $r' \in \mathcal{L}$, such that r and r' generate the same extreme ray of $\mathbf{C} \cap V$, that is, there exists $\beta > 0$ such that $r' = \beta r$. Using (15) we can write

$$r = \lambda a + b, \tag{16}$$

where $\lambda \in \mathbb{R}$ and $b \in \langle B'\rangle$. If $\lambda = 0$, then we can take $r' = r$. If $\lambda \neq 0$, consider $r' = \frac{1}{|\lambda|} r$. By equation (16) we have $r' = \frac{\lambda}{|\lambda|} a + \frac{1}{|\lambda|} b$. Thus, we conclude that $r' \in \mathcal{L}$, as desired.

2. By (1.), we only need to prove that the following implication is true: if every extreme ray of $\mathbf{C} \cap V$ can be scaled by non-zero to belong to $\mathcal{L}$ and $\dim(\mathbf{C} \cap V) \geq 3$, then $\dim(\langle B'\rangle) \geq \dim(V) - 1$.

   Note that $\mathbf{C}$ is generated by an $(m-1)$-dimensional bounded closed strictly convex set, call this set $\mathbf{S} \subseteq \mathbb{R}^m$.

   We show first that $\dim(\langle B'\rangle) \geq 2$. Let $r \in \text{rel.bd}(\mathbf{S} \cap V)$ be an extreme ray of $\mathbf{C} \cap V$ (Lemma 4). By hypothesis, there exists $\lambda_r \geq 0$, $z_r \in \mathbb{Z}^{p_1}$ such that $r \in \lambda_r[(A'z_r + \langle B'\rangle) \cap \mathbf{C}]$. Since $C_{z_r} := \text{cone}[(A'z_r + \langle B'\rangle) \cap \mathbf{C}] \subseteq \mathbf{C} \cap V$, we obtain that r must define an extreme ray of C_{z_r}. Now, assume for a contradiction that $\dim(\langle B'\rangle) \leq 1$. Then, since $\dim((A'z_r + \langle B'\rangle) \cap \mathbf{C}) \leq 1$, we obtain that $\dim(C_{z_r}) \leq 2$. Thus, we have that every cone C_{z_r} has at most two extreme rays. In particular, no more than two distinct elements of $\text{rel.bd}(\mathbf{S} \cap V)$ can define an extreme ray of a cone of the form C_{z_r}, for some $r \in \text{rel.bd}(\mathbf{S} \cap V)$. Therefore, since $\mathbb{Z}^{p_1}$ is countable and every element in $\text{rel.bd}(\mathbf{S} \cap V)$ defines an extreme ray of a cone of the form C_{z_r}, for some $r \in \text{rel.bd}(\mathbf{S} \cap V)$, we obtain that the set $\text{rel.bd}(\mathbf{S} \cap V)$ is countable. On the other hand, since $\dim(\mathbf{C} \cap V) \geq 3$, by (2.(c)) of Lemma 4 we obtain $\text{rel.bd}(\mathbf{S} \cap V)$ is uncountable, a contradiction.

   We show next that

$$\mathbf{C} \cap V \subseteq \bigcup_{z \in \mathbb{Z}^{p_1}} \text{cone}[A'z + \langle B'\rangle]. \tag{17}$$

   Let $x \in \mathbf{C} \cap V$. We have two cases depending if $x \in \text{rel.bd}(\mathbf{C}) \cap V$ or not. Observe that by Lemma 3, we have $\text{rel.bd}(\mathbf{C} \cap V) = \text{rel.bd}(\mathbf{C}) \cap V$.

   Case 1: $x \in \text{rel.bd}(\mathbf{C} \cap V)$. The case $x = 0$ is straighforward. Lets assume $x \neq 0$. Then, there exists $r \in \text{rel.bd}(\mathbf{S} \cap V)$ and $\lambda > 0$ such that $x = \lambda r$. Since, by hypothesis, there exists $\lambda_r \geq 0$, $z_r \in \mathbb{Z}^{p_1}$ such that $r \in \lambda_r[(A'z_r + \langle B'\rangle) \cap \mathbf{C}]$, we conclude that $x \in \bigcup_{z \in \mathbb{Z}^{p_1}} \text{cone}[A'z + \langle B'\rangle]$.

   Case 2: $x \notin \text{rel.bd}(\mathbf{C} \cap V)$. Consider the affine subspace $x + \langle B'\rangle$. Since $x \in \text{rel.int}(\mathbf{C} \cap V)$ and $\dim(\mathbf{C} \cap V) \geq 2$, we obtain that $x \in \text{int}(C)$ (by Lemma 3 and Lemma 1). This implies that $\dim(\mathbf{C} \cap (x + \langle B'\rangle)) = \dim(\langle B'\rangle) \geq 2$. Thus, by (2.) of Lemma 3 applied to the cone $\mathbf{C}$ and the affine subspace $x + \langle B'\rangle$, we obtain that $[\text{bd}(\mathbf{C}) \setminus \{0\}] \cap [(x + \langle B'\rangle)] \neq \emptyset$. Therefore, since $x + \langle B'\rangle \subseteq V$ we have that there exists $r \in \text{rel.bd}(\mathbf{S} \cap V)$ and $\lambda > 0$ such that $\lambda r \in x + \langle B'\rangle$. We have

$$\begin{aligned} \lambda r \in x + \langle B'\rangle &\Leftrightarrow \lambda r - x \in \langle B'\rangle \\ &\Leftrightarrow x - \lambda r \in \langle B'\rangle \\ &\Leftrightarrow x \in \lambda r + \langle B'\rangle \\ &\Leftrightarrow \frac{x}{\lambda} \in r + \langle B'\rangle. \end{aligned}$$

By hypothesis, let $\lambda_r > 0$, $z_r \in \mathbb{Z}^{p_1}$ such that $r \in \lambda_r[(A'z_r + \langle B'\rangle) \cap \mathbf{C}]$. Since $r \in \lambda_r[A'z_r + \langle B'\rangle]$ and $\lambda_r[A'z_r + \langle B'\rangle] + \langle B'\rangle = \lambda_r[A'z_r + \langle B'\rangle]$ we obtain $r + \langle B'\rangle \subseteq \lambda_r[A'z_r + \langle B'\rangle]$. Thus, we have

$$\frac{x}{\lambda} \in \lambda_r[A'z_r + \langle B'\rangle].$$

Therefore, $x \in \operatorname{cone}[A'z_r + \langle B'\rangle]$. This proves (17).

Since $\dim(\mathbf{C} \cap V) = \dim(V)$ and $\mathbb{Z}^{p_1}$ is a countable set, by equation (17) we obtain that there exists $z \in \mathbb{Z}^{p_1} \setminus \{0\}$ such that $\dim(\operatorname{cone}(A'z + \langle B'\rangle)) = \dim(V)$. Therefore, since $\dim(\operatorname{cone}(A'z + \langle B'\rangle)) \leq \dim(\langle B'\rangle) + 1$, we conclude $\dim(\langle B\rangle) = \dim(\langle B'\rangle) \geq \dim(V) - 1$, as desired.

□

3.8 Final step of the proof of Theorem 1

Recall the set $\mathcal{P}_{\mathbf{C}} := \{(x, y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \mid Ax + By - b \in \mathbf{C}\}$, the affine map $T : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^m$ defined as $T(x, y) = Ax + By - b$, and the affine subspace $V = T(\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})$. We obtain the following corollary to Proposition 2.

Corollary 3. *If* $\operatorname{lin.space}(\mathcal{P}_{\mathbf{C}}) = \operatorname{Kernel}([A\ B])$ *is a rational subspace, then*

$$\operatorname{conv}(\mathcal{P}_{\mathbf{C}} \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})) \text{ is closed} \Leftrightarrow \operatorname{conv}[(\mathbf{C} \cap V) \cap T((\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))] \text{ is closed.}$$

In the rest of this section, we will present the proof of Theorem 1.

Theorem 1. *Let* $\mathbf{C} \subseteq \mathbb{R}^m$ *be a full-dimensional pointed closed convex cone that is generated by a closed strictly convex set. Let* $A \in \mathbb{Q}^{m\times n_1}$, $B \in \mathbb{Q}^{m\times n_2}$ *and* $b \in \mathbb{Q}^m$. *Let*

$$\mathcal{P}_{\mathbf{C}} := \{(x, y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \mid Ax + By - b \in \mathbf{C}\},$$

$V := \{Ax+By-b \mid (x, y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}\}$ *and* $\mathcal{L} := \{Ax+By \mid (x, y) \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}\}$. *Then* $\operatorname{conv}(\mathcal{P}_{\mathbf{C}} \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$ *is closed if and only if one of the following holds:*

1. $b \notin \mathcal{L}$.
2. $b \in \mathcal{L}$, *and* $\dim(\mathbf{C} \cap V) \leq 1$.
3. $b \in \mathcal{L}$, $\dim(\mathbf{C} \cap V) = 2$, $\dim(\langle B\rangle) \leq 0$ *and the two extreme rays of the cone* $\mathbf{C} \cap V$ *can be scaled by a non-zero scalar so that they belong to the lattice* $\{Ax \mid x \in \mathbb{Z}^{n_1}\}$.
4. $b \in \mathcal{L}$, $\dim(\mathbf{C} \cap V) \geq 2$ *and* $\dim(\langle B\rangle) \geq \dim(V) - 1$.

Proof. First, since A and B are rational by Corollary 3, the closedness of $\operatorname{conv}[\mathcal{P}_{\mathbf{C}} \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})]$ is equivalent to the closedness of $\operatorname{conv}[\mathbf{C} \cap V \cap T(\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})]$. Therefore, it suffices to show that conditions (1.),(2.),(3.), and (4.) of Theorem 1 are equivalent to the closedness of $\operatorname{conv}[(\mathbf{C} \cap V) \cap T(\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})]$.

Notice that $\mathcal{L}$ is a mixed-integer lattice, $T(\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}) = \mathcal{L} - b$ and that if $b \in \mathcal{L}$, then $\mathcal{L} = T(\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})$.

(⇒) By contrapositive.

Case 1: If $b \in \mathcal{L}$ and $\dim(\mathbf{C} \cap V) = 2$, $\dim(\langle B\rangle) \leq 0$ and not all the extreme rays of the cone $\mathbf{C} \cap V$ belong to the lattice $\{Ax \mid x \in \mathbb{Z}^n\}$, then by Proposition 5 we obtain that $\operatorname{conv}[(\mathbf{C} \cap V) \cap T(\mathbb{Z}^n)]$ is not closed.

Case 2: Now assume that $b \in \mathcal{L}$, $\dim(\mathbf{C} \cap V) \geq 3$ and $\dim(\langle B\rangle) < \dim(V) - 1$. Then, by (2.) of Proposition 6, we obtain that not all of the extreme rays of the cone $\mathbf{C} \cap V$ are generated by points of the mixed-integer lattice $\mathcal{L} = T(\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})$. Therefore, by Proposition 5 we conclude that $\operatorname{conv}((\mathbf{C} \cap V) \cap T(\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$ is not closed.

(⇐)

Case 1: If $b \notin \mathcal{L}$ we consider two subcases.

- Assume $0 \notin V$. Then by (1.) of Lemma 4 , we obtain that $\mathbf{C} \cap V$ is a strictly convex set. Notice that $T(\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}) = \mathcal{L} - b$. Therefore, by Proposition 3, we conclude that $\operatorname{conv}((\mathbf{C} \cap V) \cap T(\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$ is closed.
- Assume $0 \in V$. Then since $b \in \langle [A\ B] \rangle$, and A, B, b are rational we conclude that $b \in \{Ax + By \,|\, (x, y) \in (\mathbb{Q}^{n_1} \setminus \mathbb{Z}^{n_1}) \times \mathbb{Q}^{n_2}\}$. Therefore, since $T(\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}) = \mathcal{L} - b$ and $V = \operatorname{aff}(\mathcal{L})$, by Proposition 4 we conclude $\operatorname{conv}((\mathbf{C} \cap V) \cap T(\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$ is a closed set.

Case 2: If $b \in \mathcal{L}$, then $\mathbf{C} \cap V$ is a cone. We consider three subcases.

- Assume $b \in \mathcal{L}$ and $\dim(\mathbf{C} \cap V) \leq 1$. In this case $\mathbf{C} \cap V$ is either a point or a ray, thus $\operatorname{conv}((\mathbf{C} \cap V) \cap T(\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$ is a closed set.
- Assume $b \in \mathcal{L}$ and $\dim(\mathbf{C} \cap V) = 2$, $\dim(\langle B \rangle) \leq 0$ and that all the extreme rays of the cone $\mathbf{C} \cap V$ belong to the lattice $\{Ax \,|\, x \in \mathbb{Z}^n\}$. Then, by Proposition 5, we obtain that $\operatorname{conv}[(\mathbf{C} \cap V) \cap T(\mathbb{Z}^n)]$ is closed.
- Assume $b \in \mathcal{L}$, $\dim(\mathbf{C} \cap V) \geq 2$ and $\dim(\langle B \rangle) \geq \dim(V) - 1$. Then, by Proposition 6, we obtain that all of the extreme rays of $\mathbf{C} \cap V$ are generated by points of $\mathcal{L} = T(\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})$. Therefore, by Proposition 5 we conclude that $\operatorname{conv}((\mathbf{C} \cap V) \cap T(\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$ is closed.

□

4 Proof of Theorem 2

In this section we prove the following result.

Theorem 2. *Let $\mathbf{C} \subseteq \mathbb{R}^m$ be a full-dimensional pointed closed convex cone that is generated by a closed strictly convex set. Assume that $\mathbf{C}$ is poly-checkable. Let $A \in \mathbb{Q}^{m \times n_1}$, $B \in \mathbb{Q}^{m \times n_2}$, $b \in \mathbb{Q}^m$ and let $\mathcal{P}_{\mathbf{C}}$ be as defined in (1). Then there exists an algorithm that runs in polynomial-time with respect to* $\operatorname{size}(\mathcal{P}_{\mathbf{C}})$ *to check whether* $\operatorname{conv}(\mathcal{P}_{\mathbf{C}} \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$ *is closed or not.*

That is, this result states that the closedness of $\operatorname{conv}(\mathcal{P}_{\mathbf{C}} \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$ can be checked in polynomial-time, whenever the cone $\mathbf{C}$ is poly-checkable. To prove Theorem 2 we need to verify that all the conditions of Theorem 1 can be checked in polynomial time with respect to the size of the data. The definition of a poly-checkable cone already implies that some statements of these conditions can be verified in polynomial time. Specifically, in the case $V := \langle [A\ B] \rangle - b$ is a linear subspace, we can decide whether $\dim(\mathbf{C} \cap V) \leq 1$ or not, and in the case $\dim(\mathbf{C} \cap V) \geq 2$ we can compute $\dim(\mathbf{C} \cap V)$ and also we can check condition (3.) of Theorem 1 in polynomial time. In order to check the rest of conditions, we use the fact that the dimension of linear subspaces generated by rational matrices can be computed in polynomial time, and the following well-known result.

Lemma 13. *Let $b \in \mathbb{Q}^m$ and $\mathcal{L} = \{Ax + By \,|\, (x, y) \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}\}$ be a mixed-integer lattice, where $A \in \mathbb{Q}^{m \times n_1}$ and $B \in \mathbb{Q}^{m \times n_2}$. Then the condition $b \in \mathcal{L}$ can be checked in polynomial-time with respect to the size of A, B, b.*

Proof of Theorem 2. First observe that by Lemma 13, we can check if $b \in \mathcal{L}$ or not in polynomial time with respect to $\operatorname{size}(\mathcal{P}_{\mathbf{C}})$. This implies that we can distinguish whether we are in the case defined by condition (1.) of Theorem 1 or not. In particular, condition (1.) of Theorem 1 can be checked in polynomial time.

Now assume that $b \in \mathcal{L}$. Then $b \in \langle [A\ B] \rangle$, and so $V := \langle [A\ B] \rangle - b$ is a linear subspace. Thus, we can compute the dimension of $\langle B \rangle$ and V in polynomial time (by the Gaussian algorithm [6]). Since $\mathbf{C}$ is poly-checkable, we obtain that deciding whether $\dim(\mathbf{C} \cap V) \leq 1$ or not, and in the case $\dim(\mathbf{C} \cap V) \geq 2$, computing $\dim(\mathbf{C} \cap V)$ can be done in polynomial time with respect to $\operatorname{size}(\mathcal{P}_{\mathbf{C}})$. This implies that we can identify which case among the ones defined by conditions (2.), (3.) or (4.) we need to analyze. Moreover, since we have already computed $\dim(\langle B' \rangle)$ and $\dim(V)$, we conclude that we can check conditions (2.) and (4.) of Theorem 1 in polynomial time. Finally, in the case given by condition (3.) of Theorem 1, since $\mathbf{C}$ is poly-checkable, we conclude that checking this condition can be done in polynomial time with respect to $\text{size}(\mathcal{P}_{\mathbf{C}})$.

□

5 The Lorentz cone is poly-checkable

Recall that the Lorentz cone $\mathbf{L}^m \subseteq \mathbb{R}^m$ is defined as $\mathbf{L}^m := \{(w, z) \in \mathbb{R}^{m-1} \times \mathbb{R} \mid \|w\| \leq z\}$. A generator for $\mathbf{L}^m$ is given by $\mathbf{S}^m := \{(w, 1) \in \mathbb{R}^{m-1} \times \mathbb{R} \mid \|w\| \leq 1\}$. In this section, we prove the following result.

Theorem 3. *The Lorentz cone is poly-checkable.*

In order to prove Theorem 3, we need to verify that for all $A \in \mathbb{Q}^{m \times n_1}$, $B \in \mathbb{Q}^{m \times n_2}$ the following conditions are satisfied:

(I) To decide whether $\dim(\mathbf{L}^m \cap \langle [A\ B] \rangle) \leq 1$ or not, and, whenever $\dim(\mathbf{L}^m \cap \langle [A\ B] \rangle) \geq 2$, to compute $\dim(\mathbf{L}^m \cap \langle [A\ B] \rangle)$ can be done in polynomial-time with respect to $\text{size}([A\ B])$.

(II) To check condition (3.) of Theorem 1 in polynomial time can be done in polynomial-time with respect to $\text{size}([A\ B])$.

In Section 5.1 we present all the required results to prove that the Lorentz cone satisfies Condition (I). Checking that the Lorentz cone satisfies Condition (II) is more involved, and is presented in Section 5.2.

5.1 Verifying the validity of Condition (I)

In order to verify Condition (I) we will show that for an arbitrary rational matrix $[A\ B] \in \mathbb{Q}^{m \times n}$, we can decide whether $\dim(\mathbf{L}^m \cap \langle [A\ B] \rangle) \leq 1$ or not in polynomial time with respect to the size of A, B and that whenever $\dim(\mathbf{L}^m \cap \langle [A\ B] \rangle) \geq 2$ we can compute $\dim(\mathbf{L}^m \cap \langle [A\ B] \rangle)$, also in polynomial time with respect to the size of A, B.

For a linear subspace $W \subseteq \mathbb{R}^n$, recall that Proj_W denotes the orthogonal projection onto the linear subspace W.

Lemma 14. *Let $[A\ B] \in \mathbb{Q}^{m \times n}$ be a rational matrix and let $a \in \mathbb{Q}^m$ be a vector of polynomial size with respect to the size of A, B. Denote $W = \langle [A\ B] \rangle$. Then $\text{Proj}_W(a)$ can be computed in polynomial-time with respect to the size of A, B.*

The following lemma can be use to decide whether $\dim(\mathbf{L}^m \cap W) \leq 1$ or not, for an arbitrary linear subspace W[1].

Lemma 15. *Let $W \subseteq \mathbb{R}^m$ be a linear subspace. Then*

1. $\dim(\mathbf{L}^m \cap W) \leq 1$ *if and only if* $(\text{int}(\mathbf{L}^m) \cap W = \emptyset$ *or* $\dim(W) \leq 1)$.
2. *Lets denote $a := (0, 1) \in \mathbb{R}^{m-1} \times \mathbb{R}$. Assume $\dim(W) \geq 2$. Then*

$$\dim(\mathbf{L}^m \cap W) \geq 2 \text{ if and only if } \text{Proj}_W(a) \in \text{int}(\mathbf{L}^m).$$

Proof.

[1]We are grateful to Arkadi Nemirovski for a preliminary version of this idea.

1. ($\Rightarrow$) If $\text{int}(\mathbf{L}^m) \cap W \neq \emptyset$ and $\dim(W) \geq 2$, then we obtain by a standard result in convex analysis (see Lemma 1) that $\text{aff}(\mathbf{L}^m \cap W) = W$. Therefore, $\dim(\mathbf{L}^m \cap W) \geq 2$.

   ($\Leftarrow$) If $\dim(W) \leq 1$, then clearly $\dim(\mathbf{L}^m \cap W) \leq 1$. On the other hand, if $\text{int}(\mathbf{L}^m) \cap W = \emptyset$, then by (1.) of Lemma 3, we obtain that $\dim(\mathbf{L}^m \cap W) \leq 1$.

2.

   ($\Leftarrow$) If $\text{Proj}_W(a) \in \text{int}(\mathbf{L}^m)$, then $\text{int}(\mathbf{L}^m) \cap W \neq \emptyset$. Therefore, since $\dim(W) \geq 2$, by (1.) we conclude that $\dim(\mathbf{L}^m \cap W) \geq 2$.

   ($\Rightarrow$) Now assume $\dim(\mathbf{L}^m \cap W) \geq 2$. Therefore, by (1.) of Lemma 3, we obtain that $\text{int}(\mathbf{L}^m) \cap W \neq \emptyset$. Let $(l, l_m) \in \text{int}(\mathbf{L}^m) \cap W$, where $l \in \mathbb{R}^{m-1}$ and $l_m \in \mathbb{R}$. Since W is a linear subspace, without loss of generality we may assume that $l_m = 1$. Hence $(l, l_m) \in \mathbf{S}^m$. By (2.) of Lemma 4, we have that $\mathbf{S}^m \cap W$ is a generator of the cone $\mathbf{L}^m \cap W$. Thus, since $(l, l_m) \in \text{int}(\mathbf{L}^m)$, we must have $r := \|l\| < 1$. Let $K' = \text{cone}(\{(x, 1) \in \mathbb{R}^{m-1} \times \mathbb{R} \,|\, \|x\| = r\})$. Observe that since $r < 1$, we have $K' \cap \mathbf{S}^m \subseteq \text{rel.int}(\mathbf{S}^m)$. Thus we obtain that $K' \setminus \{0\} \subseteq \text{int}(\mathbf{L}^m)$. Let d be the distance between a and the ray $R := \{\lambda(l, l_m) \,|\, \lambda \geq 0\}$. Since $R \subseteq W$, we obtain that $\text{Proj}_W(a) \in B(a, d) := \{x \in \mathbb{R}^m \,|\, \|x - a\| \leq d\}$. Using the symmetry of K', a simple two dimensional argument shows that $B(a, d) \subseteq K' \setminus \{0\}$. Therefore, we conclude that $\text{Proj}_W(a) \in (K' \setminus \{0\}) \subseteq \text{int}(\mathbf{L}^m)$.

□

In the following proposition we verify that Condition (I) is satisfied in the case of the Lorentz cone.

Proposition 10. *Let $[A\ B] \in \mathbb{Q}^{m \times n}$ be a rational matrix, and denote $W = \langle [A\ B] \rangle$.*

1. *The condition $\dim(\mathbf{L}^m \cap W) \leq 1$ can be checked in polynomial-time with respect to the size of A, B.*
2. *If $\dim(\mathbf{L}^m \cap W) \geq 2$, then $\dim(\mathbf{L}^m \cap W)$ can be computed in polynomial-time with respect to the size of A, B.*

Proof. Observe that $\dim(W) = \dim(\langle [A\ B] \rangle)$ and thus, since A, B are rational matrices, it can be computed in polynomial-time with respect to the size of A, B, b by the Gaussian algorithm [6].

1. Since $\dim(W)$ can be computed in polynomial-time, we can check whether $\dim(W) \leq 1$ or not in polynomial-time with respect to the size of A, B. Now, to verify whether $\dim(\mathbf{L}^m \cap W) \leq 1$ or not we use Lemma 15 as follows. If $\dim(W) \leq 1$, by (1.) of Lemma 15 we conclude that $\dim(\mathbf{L}^m \cap W) \leq 1$. If $\dim(W) \geq 2$, then by (2.) of Lemma 15 to verify whether $\dim(\mathbf{L}^m \cap W) \geq 2$ or not, we need to check if $\text{Proj}_W(a) = (u, u_m) \notin \text{int}(\mathbf{L}^m)$ or not. Thus, we only need to compute $\|u\|^2$, and compare it with u_m^2. By Lemma 14 the size of (u, u_m) is polynomial in the size of A, B, therefore we obtain that this comparison also can be done in polynomial-time with respect to the size of A, B.
2. Since $\dim(\mathbf{L}^m \cap W) \geq 2$, then by (1.) of Lemma 3 we obtain $\dim(\mathbf{L}^m \cap W) = \dim(W)$. By previous claim, $\dim(W)$ can be computed in polynomial-time.

□

5.2 Verifying the validity of Condition (II)

In this section we assume $V := \{Ax \,|\, x \in \mathbb{R}^{n_1}\}$, $\mathcal{L} := \{Ax \,|\, x \in \mathbb{Z}^{n_1}\}$, $b \in \mathcal{L}$ and $\dim(\mathbf{L}^m \cap V) = 2$. Since A is a rational matrix, a basis for $\mathcal{L}$ can be found in polynomial-time with respect to size(A) by computing the Hermite normal form of A (and the size of these basis vectors are bounded by a polynomial function of size(A)). Let the vectors $(A_1, a_1), (A_2, a_2) \in \mathbb{Q}^{m-1} \times \mathbb{Q}$ form a basis of $\mathcal{L}$. We denote $S_V := \mathbf{S}^m \cap V$.

The following lemma characterizes the extreme rays of $\mathbf{L}^m \cap V$ in terms of the basis of the lattice $\mathcal{L}$.

Lemma 16. *Let $V := \{Ax \,|\, x \in \mathbb{R}^n\}$. Assume that $\dim(\mathbf{L}^m \cap V) = 2$. Then*

1. *The numbers a_1 and a_2 cannot be both zero.*

2. *The relative boundary of S_V is given by the solutions of the following system of two equations:*

$$\begin{aligned} \|\alpha_1 A_1 + \alpha_2 A_2\|^2 &= 1 \\ \alpha_1 a_1 + \alpha_2 a_2 &= 1. \end{aligned} \tag{18}$$

3. *Let (α_1, α_2) and (α_1', α_2') be the solutions of the system of equations (18). Then the two extreme rays of $\mathbf{L}^m \cap V$ can be written as*

$$\alpha_1 \begin{pmatrix} A_1 \\ a_1 \end{pmatrix} + \alpha_2 \begin{pmatrix} A_2 \\ a_2 \end{pmatrix} \quad \text{and} \quad \alpha_1' \begin{pmatrix} A_1 \\ a_1 \end{pmatrix} + \alpha_2' \begin{pmatrix} A_2 \\ a_2 \end{pmatrix}.$$

Proof. Observe that

$$\{(A_1, a_1), (A_2, a_2)\} \text{ is a basis of } V. \tag{19}$$

1. Since $\dim(\mathbf{L}^m \cap V) = 2$, by (1.) of Lemma 3 we obtain that $\mathrm{int}(\mathbf{L}^m) \cap V \neq \emptyset$. Thus, since $\mathrm{int}(\mathbf{L}^m) \cap \mathbb{R}^{m-1} \times \{0\} = \emptyset$, we have that $\mathbf{L}^m \cap V \not\subseteq \mathbb{R}^{m-1} \times \{0\}$. In particular, $V \not\subseteq \mathbb{R}^{m-1} \times \{0\}$. Therefore, by (19), we conclude that a_1 and a_2 cannot be both zero.

2. Since $S_V = \mathbf{S}^m \cap V \subseteq V$ and by (19), we obtain that

$$S_V = \left\{ \alpha_1 \begin{pmatrix} A_1 \\ a_1 \end{pmatrix} + \alpha_2 \begin{pmatrix} A_2 \\ a_2 \end{pmatrix} \mid \|\alpha_1 A_1 + \alpha_2 A_2\|^2 \leq 1;\ \alpha_1 a_1 + \alpha_2 a_2 = 1 \right\}.$$

3. By (2.) of Lemma 4 we have $\mathbf{L}^m \cap V = \mathrm{cone}(S_V)$ and that r is an extreme ray of $\mathbf{L}^m \cap V$ if and only if r can be scaled to belong to the relative boundary of S_V. Therefore, the extreme rays of $\mathbf{L}^m \cap V$ can be found using equation (18).

□

Notice that by (1.) of Lemma 16 we have that either $a_1 \neq 0$ or $a_2 \neq 0$. Thus, we may assume without loss of generality throughout the rest of this section that $a_2 \neq 0$.

Lemma 17. *Let (α_1, α_2) and (α_1', α_2') be the solutions of the system of equations (18). Then the extreme rays of $\mathbf{L}^m \cap V$ can be scaled by a non-zero scalar to belong to $\mathcal{L}$ if and only if $\alpha_1, \alpha_1' \in \mathbb{Q}$.*

Proof.
($\Rightarrow$) We use (3.) of Lemma 16 to characterize the extreme rays of $\mathbf{L}^m \cap V$ in terms of (α_1, α_2) and (α_1', α_2'). First we consider the extreme ray associated to (α_1, α_2). There exists $\lambda > 0$ and $\gamma \in \mathbb{Q}^m$ such that

$$\lambda \left[\alpha_1 \begin{pmatrix} A_1 \\ a_1 \end{pmatrix} + \alpha_2 \begin{pmatrix} A_2 \\ a_2 \end{pmatrix} \right] = \gamma. \tag{20}$$

Since $\alpha_1 a_1 + \alpha_2 a_2 = 1$, by considering the last constraint in (20) we obtain that $\lambda \in \mathbb{Q} \setminus \{0\}$. Thus, we obtain that (α_1, α_2) is the unique solution to a system of linear equations with rational data and thus α_1, α_2 are rational. Similarly α_1', α_2' are also rational.

($\Leftarrow$) Observe first that since (α_1, α_2) and (α_1', α_2') are the solutions to (18), we obtain that

$$\alpha_1 a_1 + \alpha_2 a_2 = 1 \quad \text{and} \quad \alpha_1' a_1 + \alpha_2' a_2 = 1.$$

If $\alpha_1 = 0$, then α_2 is rational. If $\alpha_1 \neq 0$, then α_2 is rational if and only if α_2 is rational, since a_1 and a_2 are rational. Therefore in general α_1 is rational if and only if α_2 is rational. Thus by hypothesis we obtain that $(\alpha_1, \alpha_2), (\alpha_1', \alpha_2') \in \mathbb{Q}^2$. Hence, there exists $\lambda, \lambda' > 0$ such that $\lambda(\alpha_1, \alpha_2), \lambda'(\alpha_1', \alpha_2') \in \mathbb{Z}^2$. Therefore, by (3.) of Lemma 16 we obtain that the extreme rays of $\mathbf{L}^m \cap V$ can be scaled to belong to $\mathcal{L}$. □

The following proposition verifies the validity of Condition (II) for the Lorentz cone.

Proposition 11. *If* $\dim(\mathbf{L}^m \cap V) = 2$*, then whether the two extreme rays of the cone* $\mathbf{L}^m \cap V$ *can be scaled by a non-zero scalar to belong to* $\mathcal{L}$ *can be checked in polynomial-time.*

Proof. Let (α_1, α_2) and (α_1', α_2') be the solutions of the system of equations (18). Since $a_2 \neq 0$, we can write $\alpha_2 = \frac{1-\alpha_1 a_1}{a_2}$ and $\alpha_2' = \frac{1-\alpha_1' a_1}{a_2}$. Therefore, by Lemma 17, in order to check whether the extreme rays of the cone $\mathbf{L}^m \cap V$ can be scaled to belong to $\mathcal{L}$, we only need to verify if the solutions α_1, α_1' to the quadratic equation

$$\left\| \alpha A_1 + \frac{1-\alpha a_1}{a_2} A_2 \right\|^2 = 1 \tag{21}$$

belong to $\mathbb{Q}$. We will show that this can be done in polynomial-time with respect to the data $A_1, A_2 \in \mathbb{Q}^{m-1}$, $a_1, a_2 \in \mathbb{Q}$. Since the size of the product of all the denominators of the components of the vectors and the scalars appearing in (21) is polynomial with respect to the size of the original data, without loss of generality we obtain the following equivalent equation

$$\|\alpha p + q\|^2 = r, \tag{22}$$

where $p, q \in \mathbb{Z}^{m-1}$, $r \in \mathbb{Z}$ and size(p), size(q) and size(r) are polynomial with respect to the size of the original data. Notice that equation (22) can be written as

$$\left(\sum_{i=1}^{m-1} p_i^2\right)\alpha^2 + \left(\sum_{i=1}^{m-1} 2p_i q_i\right)\alpha + \sum_{i=1}^{m-1} q_i - r = 0. \tag{23}$$

Let $c_1 = \sum_{i=1}^{m-1} p_i^2$, $c_2 = \sum_{i=1}^{m-1} 2p_i q_i$ and $c_3 = \sum_{i=1}^{m-1} q_i - r$. Observe that size$(c_1)$, size$(c_2)$ and size(c_3) are polynomial with respect to the size of the original data. Using this notation, and by solving the quadratic equation (23) we obtain

$$\alpha_1 = \frac{-c_2 + \sqrt{c_2^2 - 4c_1c_3}}{2c_1} \quad \text{and} \quad \alpha_1' = \frac{-c_2 - \sqrt{c_2^2 - 4c_1c_3}}{2c_1}.$$

Therefore, $\alpha, \alpha' \in \mathbb{Q}$ if and only if $c_2^2 - 4c_1c_3$ is a perfect square. Since the latter can be checked in polynomial-time with respect to the size of c_1, c_2, c_3 (see, for example, Section 1.7 of [3]), we conclude that we can determine if $\alpha, \alpha' \in \mathbb{Q}$ in polynomial-time with respect to size of the original data. □

6 Invariance of Closedness of Integer Hulls Under Finite Intersection in the Pure Integer Case

The proof of Theorem 4 relies on a characterization of closedness of integer hulls that we proved in a recent paper [4] (see Section 3.6).

We need the following straightforward corollary to Lemma 9 (see also Corollary 2).

Corollary 4. *Let* $K \subseteq \mathbb{R}^n$ *be a closed convex set such that* aff(K) *is a rational subspace. Let* $u \in K \cap \mathbb{Z}^n$*. If* $\{u + \lambda d \,|\, \lambda > 0\} \subseteq \text{rel.int}(K)$*, then* $\{u + \lambda d \,|\, \lambda \geq 0\} \subseteq \text{conv}(K \cap \mathbb{Z}^n)$*.*

We now prove the main result in this section.

Theorem 4. *Let* $K_i \subseteq \mathbb{R}^n$*,* $i = 1, 2$*, be closed convex sets. Assume* conv$(K_i \cap \mathbb{Z}^n)$ *is closed, for* $i = 1, 2$*. If* $L = \text{lin.space}(K_1 \cap K_2)$ *is generated by integer points, then* conv$\left[(K_1 \cap K_2) \cap \mathbb{Z}^n\right]$ *is closed.*

Proof. If $(K_1 \cap K_2) \cap \mathbb{Z}^n = \emptyset$, then we are done. Assume $(K_1 \cap K_2) \cap \mathbb{Z}^n \neq \emptyset$.

We may assume that $K_1 = \text{conv}(K_1 \cap \mathbb{Z}^n)$ and $K_2 = \text{conv}(K_2 \cap \mathbb{Z}^n)$. By Theorem 6 we know that $u(K_i) = U_i$ for all $u \in K_i \cap \mathbb{Z}^n$, $i = 1, 2$.

We have two cases:

Case 1: $L = \{0\}$, that is, $(K_1 \cap K_2)$ does not contain lines.

By Theorem 6, to prove that $\text{conv}((K_1 \cap K_2) \cap \mathbb{Z}^n)$ is closed it is sufficient to show that for all $u \in (K_1 \cap K_2) \cap \mathbb{Z}^n$ we have $u(K_1 \cap K_2) = U_1 \cap U_2$.

We first verify $u(K_1 \cap K_2) \subseteq U_1 \cap U_2$. Since $\text{conv}\,[(K_1 \cap K_2) \cap \mathbb{Z}^n] \subseteq \text{conv}(K_1 \cap \mathbb{Z}^n) \cap \text{conv}(K_2 \cap \mathbb{Z}^n)$, we have $u(K_1 \cap K_2) \subseteq u(K_1) \cap u(K_2) = U_1 \cap U_2$.

Now we verify that $u(K_1 \cap K_2) \supseteq U_1 \cap U_2$. Let $u \in (K_1 \cap K_2) \cap \mathbb{Z}^n$ and let $d \in U_1 \cap U_2$. Since K_1 is a closed convex set, there exists a face F_1 of K_1 (F_1 may be K_1) such that $u \in F_1$ and $\{u + \lambda d | \lambda > 0\} \subseteq \text{rel.int}(F_1)$. Similarly, let F_2 be the face of K_2 such that $u \in F_2$ and $\{u + \lambda d | \lambda > 0\} \subseteq \text{rel.int}(F_2)$. Let $Q = F_1 \cap F_2$. Observe that $\{u + \lambda d | \lambda > 0\} \subseteq \text{rel.int}(F_1) \cap \text{rel.int}(F_2)$, thus we have $\text{rel.int}(Q) = \text{rel.int}(F_1) \cap \text{rel.int}(F_2)$. Hence, by a standard result in convex analysis, we obtain that $\text{aff}(Q) = \text{aff}(F_1) \cap \text{aff}(F_2)$. Thus, since $\text{aff}(F_1)$ and $\text{aff}(F_2)$ are rational affine subspaces, we obtain that $\text{aff}(Q)$ is a rational affine subspace. Therefore, by Corollary 4, $\{u + \lambda d \,|\, \lambda \geq 0\} \subseteq \text{conv}(Q \cap \mathbb{Z}^n) \subseteq \text{conv}\,[(K_1 \cap K_2) \cap \mathbb{Z}^n]$ and so, $d \in u(K_1 \cap K_2)$.

Therefore, for all $u \in (K_1 \cap K_2) \cap \mathbb{Z}^n$, $u(K_1 \cap K_2) = u(K_1) \cap u(K_2) = U_1 \cap U_2$.

Case 2: $L \neq \{0\}$, that is, $(K_1 \cap K_2)$ contains lines.

Since L is generated by integer points, by the Hermite normal form algorithm, there exists an unimodular matrix U such that $UL = \mathbb{R}^p \times \{0\}^{n-p}$. Thus, since $U\mathbb{Z}^n = \mathbb{Z}^n$ and the invertible linear mapping defined by U preserves closedness, we may assume that $L = \mathbb{R}^p \times \{0\}^{n-p}$. For $i = 1, 2$ let $K_i' \subseteq \mathbb{R}^{n-p}$ be the convex set such that $K_i \cap L^\perp = \{0\}^p \times K_i'$. Notice that by Proposition 9 we only need to show that $\text{conv}((K_1 \cap K_2 \cap L^\perp) \cap \text{Proj}_{L^\perp}(\mathbb{Z}^n)) = \text{conv}(\{0\}^p \times (K_1' \cap K_2' \cap \mathbb{Z}^{n-p}))$ is closed. This is equivalent to show that $\text{conv}(K_1' \cap K_2' \cap \mathbb{Z}^{n-p})$ is closed. Observe that for $i = 1, 2$ we have that $\text{conv}(K_i \cap \mathbb{Z}^n)$ is closed. Hence, by Proposition 9 we obtain that $\text{conv}((K_i \cap L^\perp) \cap \text{Proj}_{L^\perp}(\mathbb{Z}^n)) = \text{conv}(\{0\}^p \times (K_i' \cap \mathbb{Z}^{n-p}))$ is closed, $i = 1, 2$. Equivalently, $\text{conv}(K_i' \cap \mathbb{Z}^{n-p})$ is closed, $i = 1, 2$. Now, notice that the set $(K_1' \cap K_2')$ does not contain lines. Thus, by Case 1 applied to the sets K_1' and K_2', we obtain that $\text{conv}(K_1' \cap K_2' \cap \mathbb{Z}^{n-p})$ is closed, as desired. □

As observed in Example 1 in Section 2.4, Theorem 4 is not necessarily true when we replace $\mathbb{Z}^n$ by an arbitrary mixed-integer lattice $\mathcal{L}$. In order to see what is going wrong in the general case observe that in the proof of Theorem 4 we use the following implication: "If $\text{aff}(F_1)$ and $\text{aff}(F_2)$ are rational affine subspaces, then $\text{aff}(F_1) \cap \text{aff}(F_2)$ is also a rational affine subspace". We show next that this implication is not true when we consider affine subspaces that are generated by a general mixed-integer lattice $\mathcal{L}$. Recall the sets $K_1 = \{(x, y) \in \mathbb{R}^2_+ \times \mathbb{R}_+ \,|\, y \geq x_2 - \sqrt{2}x_1\}$ and $K_2 = \{(x, y) \in \mathbb{R}^2_+ \times \mathbb{R}_+ \,|\, y \geq \sqrt{2}x_1 - x_2\}$ from Example 1. Let $F_1 = \{(x, y) \in K_1 \,|\, y = x_2 - \sqrt{2}x_1\}$ and let $F_2 = \{(x, y) \in K_2 \,|\, y = \sqrt{2}x_1 - x_2\}$. Then $\text{aff}(F_1)$ and $\text{aff}(F_2)$ are affine subspaces that are generated by the mixed-integer lattice $\mathbb{Z}^2 \times \mathbb{R}$, but the affine subpace $\text{aff}(F_1) \cap \text{aff}(F_2)$ is not generated by $\mathbb{Z}^2 \times \mathbb{R}$.

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References

[1] A. Barvinok, *A course in convexity*, Graduate studies in mathematics, American Mathematical Society, 2002.

[2] D Bertsimas and R Weismantel, *Optimization over integers*, vol. 13, Dynamic Ideas, 2005.

[3] H. Cohen, *A course in computational algebraic number theory*, Graduate Texts in Mathematics, Springer-Verlag, 1993.

[4] S. S. Dey and D. A. Morán R., *Some properties of convex hulls of integer points contained in general convex sets*, Mathematical Programming, 1–20, 10.1007/s10107-012-0538-7.

[5] S. S. Dey and D. A. Morán R., *A polynomial-time algorithm to check closedness of simple second order mixed-integer sets*, Integer Programming and Combinatorial Optimization (Michel Goemans and José Correa, eds.), Lecture Notes in Computer Science, vol. 7801, Springer Berlin Heidelberg, 2013, pp. 266–277.

[6] J. Edmonds, *Systems of distinct representatives and linear algebra*, Journal of Research of the National Bureau of Standards (B) **71** (1967), 241–245.

[7] R. R. Meyer, *On the existence of optimal solutions of integer and mixed-integer programming problems*, Mathematical Programming **7** (1974), 223–225.

[8] G. T. Rockafeller, *Convex analysis*, Princeton University Press, New Jersey, NJ, 1970.